Bifurcations of indifference points in discrete time optimal control problems
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1.1 The parameter space of the discrete lake system: ‘economic importance of clean lake’ versus ‘natural robustness of lake’. In the bifurcation diagram, dashed lines are saddle-node bifurcation curves, separating the region of parameters for which there is a unique equilibrium in the state-costate system from the region of multiple equilibria. In the ‘low pollution’ region, solutions tending to the clean equilibrium are optimal; in the ‘high pollution’ region, solutions tending to the polluted equilibrium are optimal. Solid lines indicate heteroclinic bifurcation curves. These curves bound the region for which indifference thresholds exist in the lake optimal control problem. In each of these regions the optimal dynamics are depicted; attractors are marked by a circle, indifference thresholds by a diamond. .................................................. 8

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2.3 Relation between values and area: \( v(\beta) - v(\alpha) = \text{area}(A)/(e^\rho - 1) \). The boundary of \( A \) is the curve \( \alpha \to \beta \to \tilde{\beta} \to \tilde{\alpha} \to \alpha \); it is negatively oriented, consequently the orientation of \( A \) is negative as well and \( \Omega(A) = \text{area}(A) \).

3.1 The stable manifold \( W_s^+ \) of \( z_+ \) (solid) and the unstable manifold \( W_u^- \) of \( z_- \) (dashed). The stable manifold \( W_s^+ \) is composed of all points that are forward asymptotic to \( z_+ \); likewise, \( W_u^- \) is composed of all points backward asymptotic to \( z_- \). A heteroclinic point is an intersection of \( W_s^+ \) and \( W_u^- \), hence a point that is forward asymptotic to \( z_+ \) and backward asymptotic to \( z_- \). As both manifolds contain infinitely many orbits, they do not necessarily coincide (unlike in the continuous time case).

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