Bifurcations of indifference points in discrete time optimal control problems

Moghayer, S.M.

Publication date
2012

Chapter 1

Introduction

Dynamic optimisation deals with economic problems that call for an optimal sequential schedule of actions. In a dynamic optimisation problem we are looking for an optimising schedule of a choice variable in each period of time (discrete-time case) or at each point of time (continuous-time case) over a finite or an infinite planning horizon. Regardless of whether the time dependence is discrete or continuous, a typical dynamic optimisation problem contains the following: initial and end point conditions; a set of admissible paths satisfying these conditions; a performance measure such as profit or cost; and a specified objective to maximise or minimise the performance measure by choosing the path optimally.

Historically, the first dynamic optimisation problems were solved using the calculus of variations. In the second half of the 20^{th} century, two new approaches came to prominence: optimal control theory, culminating in the development of the maximum principle by Pontryagin and his co-authors (Pontryagin et al. (1962)), and the Bellman dynamic programming approach based on the value function (Bellman (1952)).

In the control formulation, the set of admissible paths is expressed as the set of solutions to a state evolution equation that depends on one or more admissible control schedules. This seemingly simple reformulation lends itself much better to applications. Dorfman (1969) gives an economic interpretation of optimal control theory by assigning to each element of the maximum principle an economic interpretation. He refers to capital theory, which is used to explain how (and whether) a production factor can be expected to contribute more to the value of output during its lifetime than it costs to produce. In optimal control problems of capital theory, the stock variables are interpreted as physical or environmental capital, controls as some kind of investment in capital, and the main problem is to obtain an optimal investment schedule. Capital accumulation is usually limited by technological constraints on production, which are described by a production function. In addition, for an economy as a whole or even for a corporation, unlike an individual, the assumption of permanent existence is reasonable. Therefore a common specification of such economic problems is the assumption of an infinite planning horizon.

Many models consider situations where technology is convex and the optimal path always converges to a single steady state. Dechert and Nishimura (1983) were among the first to investigate one-sector optimal growth models for which technology is not convex. In this context they showed that the time dependence of the capital stock is necessarily monotonic for an optimal investment schedule, and that depending on the discount rate three situations can occur: the stock converges for all initial values to some positive steady state value, or for all initial values it converges to zero, or finally it depends on the initial state whether the capital stock converges to a positive steady state or to zero. In the last situation, there can be an initial state for which there are two optimal solutions, one tending to the positive steady state, the other tending to zero.

Intermediate states of this type, which occur in the so-called non-convex class of optimal control problems, subsequently have been called 'Skiba', 'Dechert-Nishimura-Skiba' (DNS) or 'Dechert-Nishimura-Sethi-Skiba' (DNSS) states (see Grass et al., 2008), recognising the contributions of Sethi (1977) and Skiba (1978). In this thesis the designation *indifference state* is preferred for a state from which several different optimal solutions originate, possibly converging to the same long-run steady state or long-run dynamics, and the designation *indifference threshold* for an indifference state for which the originating optimal solutions converge to different long-run steady states or long-run dynamics.

The study of indifference thresholds in dynamic optimisation problems, initiated in the late 1970's, took off only comparatively recently. Especially in the context of environmental economics, where natural systems often feature non-convexities, indifference thresholds have been studied by several authors; see for instance Tahvonen (1995); Brock and Starrett (2003); Mäler et al. (2003); Wagener (2003); Kiseleva and Wagener (2010); other studies include Grüne and Semmler (2004); Steindl and Feichtinger (2004); Dawid and Deissenberg (2005); and Caulkins et al. (2007).

The following example will recur throughout this thesis. Consider a lake that is affected by some human activities, for instance through pollution that is a by-product of agricultural activities around the lake. The terminology of Brock and Starrett (2003) is used to distinguish two groups of lake users with conflicting interests: 'affecters' of the lake, who indirectly benefit from polluting the lake by using fertilisers; and 'enjoyers' of the lake, like fishermen, tourists or water companies, who benefit from high quality of the lake water. There is the problem of assessing the relative interests of these two groups. For this consider a social planner who tries to maximise society's welfare, which is composed of a weighted sum of benefits that are derived from the lake by both groups. This would be an entirely classical economic problem, if the dynamics of the pollution stock were convex. It being non-convex changes matters, as there may be more than one optimal steady state.

In problems featuring multiple steady states there can be changes in the qualitative (or topological) structure of the set of solutions if parameters are varied, such as the occurrence of single steady states for some sets of parameter values and multiple steady states for others.

In the lake pollution problem qualitative changes can occur in the optimal policy if the weight assigned to benefits of each group of agents is varied. Consequently, in the analysis of this problem there is a need to classify the solutions of the optimisation problem by their qualitative characteristics: are the interests of the producers to be preferred and is the ecosystem to function mainly as a waste dump, or is it better for society to restrict production and enjoy the ecosystem? If the weight of the benefits of enjoyers in society's welfare is low, the former situation prevails, whereas if the value of this parameter is high, the latter occurs. Therefore, there should be in-between these ranges a critical value of the weight where the solution structure changes qualitatively. To find such critical values *bifurcation theory* is an appropriate tool, as it provides the mathematics of qualitative changes. Its central aim is to study the dependence of solution structures on model parameters.

Although Dechert and Nishimura (1983) investigated all possible generic configurations of the optimal solution of a non-convex discrete time dynamic optimisation problem where the discount rate is varied, they gave no attention to the precise mechanics of bifurcation. In this thesis the genesis of indifference thresholds is studied for a class of single-state discrete time dynamic optimisation problems as a system parameter changes. The class under consideration contains a range of economic models already treated in the literature, like the optimal growth models studied by Dechert and Nishimura (1983), but also the discrete time version of the lake pollution models introduced by Mäler et al. (2003).

Previously Wagener (2003) showed that in continuous time non-convex optimal control problems, provided that the state variable of the problem is one dimensional, knowledge of the bifurcations of the state-costate dynamical system gives enough information to determine whether or not, for given values of the parameters, there are indifference points in the system: the set of parameters for which indifference points exist was shown to be bounded by heteroclinic bifurcation curves.

In this thesis, the analogous mechanism is investigated for discrete-time problems. Central to this analysis is the state-costate — or *phase* — dynamics that is associated to optimal state orbits. In the systems considered here these are on the stable manifolds of saddle fixed points of the phase system. The main result of the thesis links the genesis of indifference thresholds to the occurrence of heteroclinic orbits in phase space; these are orbits that are forward asymptotic to one saddle fixed point and backward asymptotic to another.

In particular, it is shown that if the phase system goes through a so-called heteroclinic bifurcation scenario, an indifference threshold and a locally optimal steady state are generated in an indifference-attractor bifurcation. Moreover, during the bifurcation scenario, infinitely many indifference points that are not indifference thresholds are generated. The findings are illustrated by computing the indifference threshold and some of the indifference points in a slightly modified version of the lake pollution problem.

Typically, the phase dynamics depends on several parameters. Imagine the case that the phase dynamics depends on a parameter and that there is a set of values of this parameter for which indifference thresholds exist, and another for which there are none. There is a qualitative difference between these type of systems and going from one set to the other entails a qualitative change, or, in technical terms, a bifurcation. The simplest bifurcations that can occur for the phase dynamics studied in this thesis are saddle-node and heteroclinic bifurcations and, as mentioned before, one of the main results of this thesis is that heteroclinic bifurcations may be connected to the genesis of indifference thresholds. Bifurcation theory determines the boundary of regions in parameter space for which indifference thresholds do or do not exist.

In the lake pollution problem that is considered in this thesis, optimal loading policies are classified according to qualitative characteristics; hence the analysis can be based on bifurcation theory. If the discount rate is positive, there are only four types, given in Figure 1.1. An optimal loading policy gives rise to either one or two attracting steady states; in the first case, and if the natural robustness of the lake is not too large, the steady state can be typified as being either 'low pollution' or 'high pollution'.

If there are however two steady states, and if the social planner is mainly concerned with the short run, the optimal policy can exhibit jumps. In the optimal state dynamics, these jumps are associated with either indifference points or indifference thresholds. In an indifference threshold the social planner is indifferent between two types of policies that steer the lake either to a low or high pollution steady state. In an indifference point that is not an indifference threshold the social planer is indifferent between two distinct optimal policies that however steer the lake to the same long-run steady state.

As shown in the lake model, the characteristic feature of this interest conflict is the possibility of qualitatively different outcomes. Social choice often incorporates a qualitative aspect of decision making. Bifurcation theory is therefore a powerful tool to classify these outcomes and to give a graphical overview of the qualitative characteristics of the solution depending on the parameters of the problem. Using bifurcation analysis a full picture, a *bifurcation diagram*, of all qualitatively different optimal pollution policies can be constructed, which describes how these policies depend on the type of ecosystem, on the relative economic weights of the interest groups and on the social discount rate. For the discrete-time version of the lake system, the dependence of the different possibilities on the 'weighting' and 'robustness' parameters are given in Figure 1.1.

In the bifurcation diagram of the lake system boundaries of regions in parameter space that correspond to different types of optimal policies are shown. Each point in the parameter space determines a particular optimisation problem. A certain type of optimal solution corresponds to a particular region, and an intermediate degenerate situation between two types

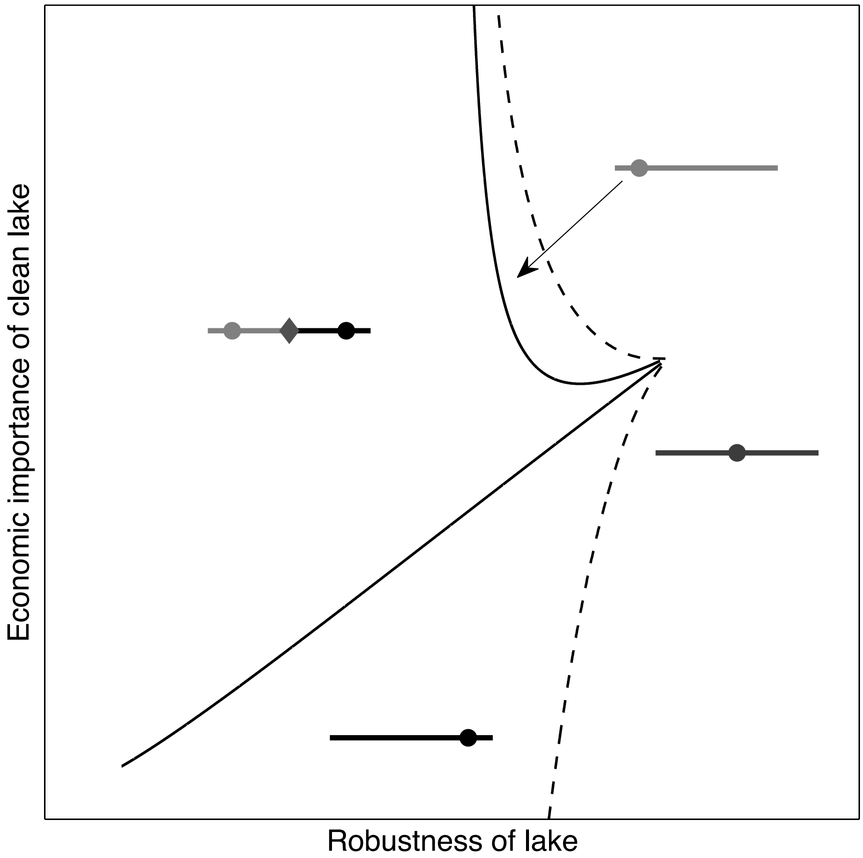

Figure 1.1: *The parameter space of the discrete lake system: 'economic importance of clean lake' versus 'natural robustness of lake'. In the bifurcation diagram, dashed lines are saddle-node bifurcation curves, separating the region of parameters for which there is a unique equilibrium in the state-costate system from the region of multiple equilibria. In the 'low pollution' region, solutions tending to the clean equilibrium are optimal; in the 'high pollution' region, solutions tending to the polluted equilibrium are optimal. Solid lines indicate heteroclinic bifurcation curves. These curves bound the region for which indifference thresholds exist in the lake optimal control problem. In each of these regions the optimal dynamics are depicted; attractors are marked by a circle, indifference thresholds by a diamond.*

of optimal solution corresponds to a boundary. In the lake system there are roughly speaking three types of optimal solution structures: (1) steering a system to a steady state regardless of its initial state; (2) steering a system to either of two steady states depending on its initial state; (3) steering a system to either of the two attracting steady states unless the initial state is exactly at an intermediate repelling steady state. The last two types of optimal solutions are *dependent on initial states*.

Methodologically, this thesis contributes to the geometrical analysis of discrete time dynamic optimisation problems using phase space methods. In particular, extensive use is made of differential forms and geometric integration. Contrary to the continuous time setting, phase space methods are not particularly popular in the discrete time setting (see e.g. Shone (2002)). There are several probable reasons for this: the omnipresence of the Bellman equation, which is well-understood, easy to generalise to stochastic problems, and which has an elegant theory of existence and uniqueness of solutions. Moreover, some powerful instruments of continuous time theory are not readily available: for instance, in a continuous time problem with a one-dimensional state space, knowledge of the isoclines allows to reconstruct the geometry of the phase trajectories to a great extent. In the discrete time setting, there are backward and forward isoclines, and their knowledge does not allow to reconstruct orbits of the phase system as easily. Also in the continuous time setting the value function can be evaluated in terms of the initial state and costate values of an optimal orbit; this is an immediate corollary to the Hamilton-Jacobi equation. In the discrete time setting, there is no such direct way to find the value function, though in Proposition 2.3.5 a partial replacement has been obtained.

In this thesis a geometrical point of view is used. The basic geometrical objects associated with the phase system are the orbits and the stable manifolds of the phase map. These can almost never be computed analytically, hence numerical methods are needed. Therefore to apply phase space methods, accurate algorithms are required to compute invariant manifolds numerically. For computation of approximations to invariant manifolds of maps there are several approaches (see e.g. Simó (1989), Homburg et al. (1995)). The approach used in this thesis, while less sophisticated than these methods, still provides accurate approximations to the stable manifolds; it is however restricted to two-dimensional systems.

This thesis aims to contribute to the application of bifurcation theory in non-convex discrete-time optimal control problems. While the special case of two-dimensional phase space is considered, similar theoretical results to those obtained are expected to hold in the general case of n-dimensional phase space. These results are the starting point of a full bifurcation theory of the solution structure of discrete time dynamic optimisation problems, which will make a general analysis of a wide range of non-convex dynamic economic problems accessible.