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### Bifurcations of indifference points in discrete time optimal control problems

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# Chapter 2

## Phase space methods in discrete time optimal control

In this chapter, a general autonomous discrete time optimisation problem is considered. The necessary first order conditions are formulated in terms of a boundary value problem of the state-costate (or phase) map. If the phase map possesses a saddle point, then all orbits on the stable manifold of this point will solve the boundary value problem. Results are derived that allow to compare the values of these orbits.

### 2.1 First variation and Hamiltonian formalism

Here discrete optimal control theory for systems with  $n$ -dimensional state space is formulated in a way that is suitable for the purpose of this thesis.

**2.1.1 Definitions and problem specification.** To keep notation minimal, for a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f = f(x)$  the following notation is employed for the derivative

$$\frac{df}{dx} = \left( \frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right).$$

Likewise, for a function  $g = g(x, y)$ ,  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ , the notation

$$g_x = \left( \frac{\partial g}{\partial x_1} \quad \cdots \quad \frac{\partial g}{\partial x_n} \right)$$

is used etc.

Time  $t$  is discrete, and takes values  $0, 1, 2, \dots$ . The state space  $\mathcal{X}$  and the control set  $\mathcal{U}$  are open and convex subsets of  $\mathbb{R}^n$ . Note that only interior solutions are considered in the following. On the state space the state dynamics is given as

$$x_t = f(x_{t-1}, u_t), \tag{2.1}$$

for  $t \geq 1$  where  $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$  is *smooth*. A function is smooth if it has as many derivatives as necessary; ordinarily, we shall think of  $C^\infty$  functions, but the reader can substitute  $C^k$  with  $k > 0$  sufficiently large.

For technical convenience the special assumption is made that for all  $(x, u) \in \mathcal{X} \times \mathcal{U}$  we have

$$\det f_u(x, u) \neq 0.$$

Note that this encompasses a large class of optimal control problems, as well as all discrete calculus of variations problems, where  $f(x, u) = u$ .

If  $\mathbf{x} = \{x_t\}_{t=0}^\infty$  and  $\mathbf{u} = \{u_t\}_{t=1}^\infty$  are sequences in  $\mathcal{X}$  and  $\mathcal{U}$  respectively, the pair  $(\mathbf{x}, \mathbf{u})$  is called *weakly admissible* if equation (2.1) holds for all  $t \geq 1$ .

Let  $\mathcal{W}_\alpha = \mathcal{W}$  denote the set of weakly admissible pairs of sequences  $(\mathbf{x}, \mathbf{u})$  with  $x_0 = \alpha$ .

Let  $\rho > 0$  be a positive real number, and let  $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$  be a smooth real-valued function. For each integer  $T \geq 1$ , define a functional  $J_T : \mathcal{W} \rightarrow \mathbb{R}$  by setting

$$J_T(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^T g(x_{t-1}, u_t) e^{-\rho t}. \quad (2.2)$$

A sequence  $\mathbf{a} = \{a_t\}$  of positive real numbers is called *summable* if  $\sum_t a_t < \infty$ . A pair of weakly admissible sequences  $(\mathbf{x}, \mathbf{u}) \in \mathcal{W}$  is called *admissible*, if there is a positive summable sequence  $\mathbf{a}$  such that for all  $t \geq 1$

$$|g(x_{t-1}, u_t) e^{-\rho t}| \leq a_t.$$

The set of admissible pairs  $(\mathbf{x}, \mathbf{u})$  is denoted by  $\mathcal{A}$ . Define a functional  $J : \mathcal{A} \rightarrow \mathbb{R}$  by

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}.$$

Note that  $J$  is well-defined on  $\mathcal{A}$ .

The general infinite horizon autonomous discrete time optimisation problem is formulated as follows: maximise an objective

$$J = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}, \quad (2.3)$$

where  $\rho > 0$ , under the side condition that

$$x_t = f(x_{t-1}, u_t), \quad (2.4)$$

for all  $t \geq 1$ . Note that as  $f$  takes values in  $\mathcal{X}$ , there are no binding state constraints.

**2.1.2 Variations.** In this subsection the necessary first order conditions are derived and formulated in terms of a boundary value problem of the state-costate (or phase) map. Given an admissible pair  $(\mathbf{x}^0, \mathbf{u}^0)$ , consider the *variations*

$$(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)) = (\mathbf{x}^0 + \varepsilon \boldsymbol{\xi}(\varepsilon), \mathbf{u}^0 + \varepsilon \mathbf{v}(\varepsilon)).$$

A variation is called (weakly) admissible, if  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  is (weakly) admissible for every  $\varepsilon \in [0, 1]$ . Throughout, it will be assumed that all variations are weakly admissible for all  $\varepsilon \in [0, 1]$ , and that for all  $t$  the functions

$$\varepsilon \mapsto \xi_t(\varepsilon) \quad \text{and} \quad \varepsilon \mapsto v_t(\varepsilon)$$

are smooth. Write

$$\xi_t^0 = \xi_t(0) \quad \text{and} \quad v_t^0 = v_t(0).$$

Define a function  $j_T : [0, 1] \rightarrow \mathbb{R}$  by setting

$$j_T(\varepsilon) = J_T(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)).$$

where  $J_T$  is defined in (2.2).

To compute the derivative of  $j_T$  at  $\varepsilon = 0$ , note that if the pair  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  is weakly admissible for every  $\varepsilon \in [0, 1]$ , then for all  $t$

$$x_t + \varepsilon \xi_t(\varepsilon) = f(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon v_t(\varepsilon)).$$

Expanding in  $\varepsilon$  around  $(x_t, u_t)$  and solving for  $v_t$  yields

$$v_t(\varepsilon) = f_u^{-1} \left[ \xi_t(\varepsilon) - f_x \xi_{t-1}(\varepsilon) \right] + \varepsilon r, \tag{2.5}$$

where  $|r(x, u, \varepsilon, \xi_1, \xi_2)| \leq C|\xi|^2$ , uniformly in  $(x, u, \varepsilon)$ . Note that in equation (2.5) the arguments  $(x_{t-1}, u_t)$  have been omitted; this will be done

whenever there is no chance for confusion. Taking  $\varepsilon \rightarrow 0$  yields

$$v_t^0 = f_u^{-1} \left[ \xi_t^0 - f_x \xi_{t-1}^0 \right]. \quad (2.6)$$

Moreover

$$\begin{aligned} \frac{j_T(\varepsilon) - j_T(0)}{\varepsilon} &= \frac{J_T(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)) - J_T(\mathbf{x}^0, \mathbf{u}^0)}{\varepsilon} \\ &= \sum_{t=1}^T [g_x \xi_{t-1}^0 + g_u v_t^0] e^{-\rho t} + \mathcal{O}(\varepsilon) \\ &= \sum_{t=1}^T [(g_x - g_u f_u^{-1} f_x) \xi_{t-1}^0 + g_u f_u^{-1} \xi_t^0] e^{-\rho t} + \mathcal{O}(\varepsilon), \end{aligned} \quad (2.7)$$

where in the last equality (2.6) has been used.

Introduce the sequence of costates  $\mathbf{y} = \{y_t\}_{t=0}^\infty$  by setting

$$\begin{cases} y_t = -g_u(x_{t-1}, u_t) f_u^{-1}(x_{t-1}, u_t) & \text{for } t \geq 1, \text{ and} \\ y_0 = e^{-\rho} (g_x(x_0, u_1) + y_1 f_x(x_0, u_1)). \end{cases} \quad (2.8)$$

Note that the  $y_t$  are row vectors. Taking in (2.7) the limit  $\varepsilon \rightarrow 0$ , the following result is obtained.

**Proposition 2.1.1.** *Let  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  be admissible variations and let*

$$j_T(\varepsilon) = J_T(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)).$$

*The right derivative  $D_+ j_T(0)$  at  $\varepsilon = 0$  exists and equals*

$$D_+ j_T(0) = y_0 \xi_0^0 - e^{-\rho T} y_T \xi_T^0 + \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}^0] e^{-\rho t}. \quad (2.9)$$

From this, the necessary conditions for optimality are obtained.

**Proposition 2.1.2.** *If  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{W}$  is such that  $J_T(\mathbf{x}^*, \mathbf{u}^*) \geq J_T(\mathbf{x}, \mathbf{u})$  for all  $(\mathbf{x}, \mathbf{u}) \in \mathcal{W}$  with  $x_0 = \alpha$ , then*

$$y_T = 0 \tag{2.10}$$

and

$$e^\rho y_{t-1} = g_x(x_{t-1}, u_t) + y_t f_x(x_{t-1}, u_t) \tag{2.11}$$

for all  $1 \leq t \leq T$ .

Note that (2.11) holds for  $t = 1$  by definition of  $y_0$ . Any admissible pair  $(\mathbf{x}, \mathbf{u})$  that satisfies (2.11) for all  $t \geq 1$  is called *extremal*.

As  $J_T$  depends only on finitely many variables, there are no technical problems in obtaining the first variation formula (2.9). To find the analogous expression for the infinite horizon functional  $J$ , we have to be able to interchange differentiation and infinite summation. This is permitted if the variations are strongly admissible.

**Definition 2.1.1. (Strongly admissible variations)** *An admissible variation*

$$(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$$

*is called strongly admissible, if there is a fixed positive summable sequence  $\mathbf{a} = \{a_t\}$  such that for all  $t \geq 1$  and for all  $\varepsilon \in (0, 1)$*

$$\varepsilon^{-1} \left| g(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon v_t(\varepsilon)) - g(x_{t-1}, u_t) \right| e^{-\rho t} \leq a_t. \tag{2.12}$$

**Proposition 2.1.3.** *Let the variation  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  be strongly admissible, and let*

$$j(\varepsilon) = J(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)).$$

Then the right-hand derivative  $D_+j(0)$  exists. Moreover, there is a positive summable sequence  $\{a_t\}$  and a sequence  $\{R_t\}$  such that

$$D_+j(0) = y_0\xi_0^0 - e^{-\rho T}y_T\xi_T^0 + \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}^0] e^{-\rho t} + R_T$$

and

$$|R_T| \leq \sum_{t=T+1}^{\infty} a_t$$

for every  $T \geq 1$ .

*Proof.* The conditions of strong admissibility precisely guarantee that the series  $j(\varepsilon)$  is uniformly convergent and that we may pass to the limit  $\varepsilon \rightarrow 0$  under the summation sign; see for instance Knopp (1996). ■

Proposition 2.1.3 allows to derive first order necessary conditions for the infinite horizon problem.

**Definition 2.1.2. ( $\delta$ -interior)** *The pair of sequences  $(\mathbf{x}, \mathbf{u})$  is  $\delta$ -interior, if for every sequence  $\boldsymbol{\xi}$  with  $\xi_0 = 0$  and  $|\xi_t| \leq \delta$  for all  $t > 0$ , there is a sequence  $\mathbf{v}(\varepsilon)$  such that the variation  $(\mathbf{x}^* + \varepsilon\boldsymbol{\xi}, \mathbf{u}^* + \varepsilon\mathbf{v}(\varepsilon))$  is strongly admissible.*

**Proposition 2.1.4.** *Let  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{A}$  be such that  $J(\mathbf{x}^*, \mathbf{u}^*) \geq J(\mathbf{x}, \mathbf{u})$  for all  $(\mathbf{x}, \mathbf{u}) \in \mathcal{A}$  with  $x_0 = \alpha$ , and let  $\mathbf{y}^*$  be the associated sequence of costates, given by (2.8). Assume that  $(\mathbf{x}^*, \mathbf{u}^*)$  is  $\delta$ -interior for  $\delta > 0$ . Then*

$$e^\rho y_{t-1}^* = g_x(x_{t-1}^*, u_t^*) + y_t^* f_x(x_{t-1}^*, u_t^*) \quad (2.13)$$

for all  $t \geq 1$  and

$$\lim_{t \rightarrow \infty} e^{-\rho t} y_t^* = 0. \quad (2.14)$$



Equation (2.14) is commonly referred to as the transversality condition. The rather simple form of (2.14) is a consequence of the fact that variations in all directions are assumed to be admissible.

*Proof.* Since  $(\mathbf{x}^*, \mathbf{u}^*)$  maximises  $J$ , necessarily  $D_+j(0) \leq 0$ . Noting that necessarily  $\xi_0 = 0$ , it follows that

$$0 \geq D_+j(0) = \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}] e^{-\rho t} - e^{-\rho T} y_T \xi_T + R_T.$$

Let  $\text{sign}(x)$  denote the sign function

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Setting

$$\begin{cases} \xi_{t-1} &= \delta \text{sign}(g_x + y_t f_x - e^\rho y_{t-1}) & \text{for } 2 \leq t \leq T, \text{ and} \\ \xi_T &= -\delta \text{sign}(y_T) \end{cases}$$

yields that

$$\delta \sum_{t=2}^T |g_x + y_t f_x - e^\rho y_{t-1}| e^{-\rho t} + \delta e^{-\rho T} |y_T| \leq |R_T| \leq \sum_{t=T+1}^{\infty} a_t.$$

Since this inequality has to hold for all  $T \geq 2$ , and since  $\sum_{t=T+1}^{\infty} a_t \rightarrow 0$  as  $T \rightarrow \infty$ , the result follows.  $\blacksquare$

The necessary first order conditions of the optimisation problem considered in this section are that  $(x_t, y_t)$  solves the boundary value problem (2.13) with the initial condition  $x_0 = \alpha$  and the terminal condition (2.14).

**2.1.3 The discrete Hamiltonian.** The results of the previous subsection can be formulated very elegantly if discrete Pontryagin and Hamilton functions are introduced. The former is given as

$$P(x, y, u) = g(x, u) + yf(x, u).$$

In terms of  $P$ , equations (2.1), (2.8) and (2.13) take the form

$$0 = P_u, \quad x_t = P_y, \quad e^{\rho}y_{t-1} = P_x; \quad (2.15)$$

here the argument of the derivatives of  $P$  is  $(x_{t-1}, y_t, u_t)$ .

It will be assumed that  $P_{uu}$  is negative definite. Then the equation  $P_u = 0$  can be solved for  $u = U(x, y)$ , which yields the discrete present value Hamilton function

$$\begin{aligned} H(x, y) &= P(x, y, U(x, y)) \\ &= g(x, U(x, y)) + yf(x, U(x, y)). \end{aligned} \quad (2.16)$$

Note for later reference that, since  $P_u(x, y, U(x, y)) = 0$  identically in  $(x, y)$ , we have

$$g(x, U(x, y)) = H(x, y) - yH_y(x, y). \quad (2.17)$$

The necessary equations can be written in the (present-value) Hamiltonian form

$$x_t = H_y(x_{t-1}, y_t) \quad \text{and} \quad e^{\rho}y_{t-1} = H_x(x_{t-1}, y_t). \quad (2.18)$$

By extension, the pair  $(\mathbf{x}, \mathbf{y})$  is called extremal if equation (2.18) is satisfied for every  $t \geq 1$ . Note that if  $(\mathbf{x}, \mathbf{y})$  is extremal, and if a control sequence  $\mathbf{u}$  is obtained by setting  $u_t = U(x_{t-1}, y_t)$  for  $t \geq 1$ , then the pair  $(\mathbf{x}, \mathbf{u})$  is extremal in the former sense.

**2.1.4 The phase map.** The next step is to solve the present-value Hamiltonian equations (2.18) to obtain a phase map  $\varphi$  that satisfies

$$(x_t, y_t) = \varphi(x_{t-1}, y_{t-1})$$

for every  $t \geq 1$ . Sometimes, the coordinate representation of this relation is used:

$$(x_t, y_t) = (\varphi_1(x_{t-1}, y_{t-1}), \varphi_2(x_{t-1}, y_{t-1})).$$

The domain of definition of  $\varphi$  is  $T^*(\mathcal{X}) = \mathcal{X} \times \mathbb{R}^n$ , called the *phase space*, to distinguish it from the *state space*  $\mathcal{X}$ . The notation indicates that  $T^*(\mathcal{X})$  is, mathematically speaking, the cotangent bundle of  $\mathcal{X}$ .

Let  $F : T^*(\mathcal{X}) \times T^*(\mathcal{X}) \rightarrow T^*(\mathcal{X})$  be given as

$$F(z, \varphi) = \begin{pmatrix} \varphi_1 - H_y(x, \varphi_2) \\ e^\rho y - H_x(x, \varphi_2) \end{pmatrix},$$

where  $z = (x, y) \in T^*(\mathcal{X})$ . Let the map  $\varphi$  be implicitly defined by the equation

$$F(z, \varphi) = 0.$$

**Proposition 2.1.5.** *If  $H_{xy}$  is invertible, then the equation  $F = 0$  can be solved for  $\varphi = \varphi(z)$ . Moreover,*

$$D\varphi = \begin{pmatrix} H_{xy} - H_{yy}H_{xy}^{-1}H_{xx} & e^\rho H_{yy}H_{xy}^{-1} \\ -H_{xy}^{-1}H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}$$

and

$$\det D\varphi = e^{n\rho}. \tag{2.19}$$

*Proof.* Compute

$$D_\varphi F = \begin{pmatrix} I & -H_{yy} \\ 0 & -H_{xy} \end{pmatrix}.$$

Under the assumption of the lemma, this matrix is invertible at  $(z_0, \varphi_0)$ , and

$$(D_\varphi F)^{-1} = \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix}.$$

By the implicit function theorem, the solution  $\varphi = \varphi(z)$  of  $F(z, \varphi) = 0$  satisfies

$$\begin{aligned} D\varphi &= -(D_\varphi F)^{-1} D_z F \\ &= \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & -e^\rho I \end{pmatrix} \\ &= \begin{pmatrix} H_{xy} - H_{yy}H_{xy}^{-1}H_{xx} & e^\rho H_{yy}H_{xy}^{-1} \\ -H_{xy}^{-1}H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}. \end{aligned}$$

Moreover

$$\det D\varphi = \det \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \det \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & -e^\rho I \end{pmatrix} = e^{n\rho}.$$

■

Summarising, a map  $\varphi$  has been found such that the orbits  $\mathbf{z} = \{z_t\} = \{(x_t, y_t)\}$  of  $\varphi$  are extremal and such that  $\varphi$  satisfies

$$\varphi_1(x, y) = H_y(x, \varphi_2(x, y)) \quad \text{and} \quad e^\rho y = H_x(x, \varphi_2(x, y)). \quad (2.20)$$

**2.1.5 Comparison with continuous time case.** Note that the properties of the phase map have well-known analogues in continuous time. They are sketched briefly here.

The continuous time problem asks to maximise a functional

$$J = \int_0^\infty g(x, u) e^{-\rho t} dt$$

under the condition that

$$\dot{x} = f(x, u).$$

The *continuous time present-value Pontryagin function* takes the form

$$P(x, y, u) = g(x, u) + yf(x, u).$$

An interior optimising orbit satisfies necessarily (cf. equation (2.15))

$$P_u = 0, \quad \dot{x} = P_y, \quad \rho y - \dot{y} = -\frac{d}{dt} (e^{-\rho t} y) \Big|_{t=0} = P_x.$$

If  $P_{uu} < 0$  everywhere,  $u = U(x, y)$  can be solved from  $P_u = 0$ , yielding the *continuous time present-value Hamilton function*

$$H(x, y) = P(x, y, U(x, y)).$$

In terms of  $H$ , the necessary conditions read as (cf. (2.18))

$$\dot{x} = H_y \quad \text{and} \quad \rho y - \dot{y} = H_x.$$

If  $X(x, y) = (H_y, \rho y - H_x)$  denotes the vector field defined by these equations,

$$\operatorname{div} X = n\rho.$$

If moreover  $\Phi_t = e^{tX}$  is the phase map defined by the vector field  $X$ , then

$$\det D\Phi_t = e^{t \operatorname{div} X} = e^{nt\rho}.$$

This equality can be verified easily by differentiation with respect to  $t$ . In particular (cf. equation (2.19))

$$\det D\Phi_1 = e^{n\rho}.$$

## 2.2 Local and associated value functions

In the previous section, the necessary first order conditions of the boundary value problem (2.13) were formulated in terms of the phase map  $\varphi$ : if  $(\mathbf{x}, \mathbf{y})$  is maximising and  $\delta$ -interior for  $\delta > 0$ , then

$$\begin{aligned} (x_t, y_t) &= \varphi(x_{t-1}, y_{t-1}), \\ x_0 &= \alpha \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-\rho t} y_t = 0. \end{aligned} \tag{2.21}$$

If all optimal state trajectories  $\mathbf{x}$  starting in some open sub-region of  $\mathcal{X}$  converge to some steady state, this state corresponds to a fixed point of the phase map  $\varphi$ ; since

$$\det D\varphi = e^{n\rho} > 1,$$

such a fixed point is necessarily a saddle. In proper coordinates around such a saddle  $\bar{z}$ , the linear map  $D\varphi$  takes the form

$$D\varphi(\bar{z}) = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix},$$

where all eigenvalues of  $\Lambda_s$  are inside the unit circle, while all eigenvalues of  $\Lambda_u$  are outside. The union of phase orbits converging to  $\bar{z}$  forms a manifold, the so-called stable manifold  $W^s$  of the fixed point, and orbits on stable manifolds of saddles of  $\varphi$  thus solve the boundary value problem (2.21).

The stable linear eigenspace  $E^s$  of  $\bar{z}$  is tangent to  $W^s$ . The natural projection  $\pi : E^s \rightarrow \mathcal{X}$  is surjective, which implies that locally the stable manifold can be represented as the graph of a function  $\psi$ .

In the following two kinds of value functions are introduced: the orbit value function  $v$  that associates a value to points on  $W^s$  and the value function  $\bar{V}$  associated to the stable manifold  $W^s$ . If the stable manifold of  $\bar{z}$  can be

represented as the graph of a function  $y = \psi(x)$ , then the value function  $\bar{V}$  of the trajectories in the stable manifold  $W^s$  turns out to satisfy  $\partial\bar{V}/\partial x_i = \psi_i$  for all  $i$ , at least locally around  $\bar{z}$ . To recover the value function from  $\psi$  requires an integration. The components of  $\psi$  have therefore to satisfy an integrability condition. To formulate this condition, and to demonstrate that it is satisfied, differential forms are introduced.

These forms will be also used in the proof of some propositions that compare values at different points on a stable manifold which in turn are used to construct the global value function  $V$  of the problem.

**2.2.1 Integrability and symplectic forms.** As mentioned already, a value function  $\bar{V}$  will be associated to the stable manifold  $W^s$ ; for this, some concepts have to be introduced. On  $T^*(\mathcal{X})$ , the canonical 1-form  $\eta = y dx = \sum y_i dx_i$  is defined, as well as its derivative  $\omega = d\eta = dy \wedge dx$ , the *symplectic 2-form*.

The symplectic form  $\omega$  is said to vanish on a submanifold  $\mathcal{M}$  of  $T^*(\mathcal{X})$ , if for any point  $z \in \mathcal{M}$  and any tangent vectors  $v_1, v_2 \in T_z(\mathcal{M})$  to  $\mathcal{M}$  at  $z$  the equality  $\omega_z(v_1, v_2) = 0$  holds. A  $n$ -dimensional submanifold  $\mathcal{M}$  of  $T^*(\mathcal{X})$  is called *Lagrangian* if  $\omega$  vanishes on  $\mathcal{M}$ . Being Lagrangian is an integrability condition: to see this assume that  $\mathcal{M}$  can be represented as the graph  $y = Y(x)$  of a function  $Y : \mathcal{N} \subset \mathcal{X} \rightarrow \mathbb{R}^n$ .

Recall that since  $\mathcal{X}$  is convex, it is topologically trivial. Hence there is a function  $w : \mathcal{X} \rightarrow \mathbb{R}$  that satisfies

$$\frac{\partial w}{\partial x_i} = Y_i$$

for  $i = 1, 2, \dots, n$  if and only if the integrability conditions

$$\frac{\partial Y_j}{\partial x_i} - \frac{\partial Y_i}{\partial x_j} = 0$$

are satisfied for all  $i, j$ .

Being Lagrangian expresses the same thing. To see this, let  $\Psi : \mathcal{N} \subset \mathcal{X} \rightarrow T^*(\mathcal{X})$  be given by  $\Psi(x) = (x, Y(x))$ . Set  $\mathcal{M} = \Psi(\mathcal{N})$ . The manifold  $\mathcal{M}$  is Lagrangian if and only if  $\Psi^*\omega = 0$ . Compute

$$\begin{aligned} 0 &= \Psi^*\omega \\ &= \sum_i dY_i(x) \wedge dx_i = \sum_i \sum_j \frac{\partial Y_i}{\partial x_j} dx_j \wedge dx_i \\ &= \sum_i \sum_{j < i} \left( \frac{\partial Y_j}{\partial x_i} - \frac{\partial Y_i}{\partial x_j} \right) dx_i \wedge dx_j; \end{aligned}$$

The classical integrability conditions have been recovered.

The phase map preserves the symplectic form up to a constant factor. Of course, the presence of this factor is an echo of the fact the optimisation problem is formulated in current value variables. Let now  $\varphi$  be the phase map defined in (2.20) by

$$\varphi_1 = H_y(x, \varphi_2(x, y)) \quad \text{and} \quad e^\rho y = H_x(x, \varphi_2(x, y)).$$

**Proposition 2.2.1.** *The equality  $\varphi^*\omega = e^\rho\omega$  holds. Moreover, if  $\psi$  satisfies  $\varphi = e^{\rho/2}\psi$ , then  $\psi^*\omega = \omega$ .*

*Proof.* Using that  $\varphi_1 = H_y(x, \varphi_2)$  (i.e. equation (2.20)), compute

$$\varphi^*\omega = d\varphi_2 \wedge d\varphi_1 = d\varphi_2 \wedge (H_{xy} dx + H_{yy} d\varphi_2) = H_{xy} d\varphi_2 \wedge dx.$$



Analogously, using  $e^\rho y = H_x(x, \varphi_2)$ , it follows that

$$e^\rho dy \wedge dx = dH_x(x, \varphi_2) \wedge dx = (H_{xx} dx + H_{xy} d\varphi_2) \wedge dx = H_{xy} d\varphi_2 \wedge dx.$$

The proof for  $\psi$  runs similarly, using equation (2.20) in the form

$$\psi_1 = e^{-\rho/2} H_y(x, e^{\rho/2} \psi_2) \quad \text{and} \quad e^\rho y = H_x(x, e^{\rho/2} \psi_2).$$

This proves the proposition. ■

Note in particular that if  $\omega_z = 0$ , then  $\omega = 0$  along the orbit of  $\varphi$  through  $z$ .

**Definition 2.2.1. (Symplectic transformation)** *A differential map  $\psi$  that preserves the 2-form  $\omega$ , that is, which is such that  $\psi^*\omega = \omega$ , is called symplectic.*

The fact that  $\psi$  is symplectic has implications for the spectrum of the Jacobian matrix  $D\psi$ .

**Proposition 2.2.2.** (1) *If  $\psi$  is symplectic, and if  $\lambda$  is an eigenvalue of  $D\psi$ , so is  $1/\lambda$ .*

(2) *If  $\varphi$  satisfies  $\varphi^*\omega = e^\rho\omega$  and if  $\lambda$  is an eigenvalue of the phase map  $D\varphi$ , then so is  $e^\rho/\lambda$ .*

*Proof.* See Abraham and Marsden (1978), Proposition 3.1.12, p. 168. ■

**2.2.2 Invariant manifolds.** Let  $\bar{z} = (\bar{x}, \bar{y})$  be a saddle fixed point of  $\varphi$ . The linear stable and unstable manifolds are the eigenspaces  $E^s$  and  $E^u$  associated to the eigenvalues that are respectively lesser and greater than

one in absolute value. The *stable manifold*  $W^s$  and the *unstable manifold*  $W^u$  of  $\bar{z}$  are defined as the set of all points  $z \in T^*(\mathcal{X})$  such that respectively the forward orbit or the backward orbit of  $\varphi$  through  $z$  tends to  $\bar{z}$ :

$$W^s = \{z \in \mathcal{X} \times \mathbb{R}^n \mid \varphi^t(z) \rightarrow \bar{z}, t \rightarrow +\infty\},$$

$$W^u = \{z \in \mathcal{X} \times \mathbb{R}^n \mid \varphi^t(z) \rightarrow \bar{z}, t \rightarrow -\infty\}.$$

The basic result about the sets  $W^s$  and  $W^u$  is the invariant manifold theorem (Hirsch et al., 1977), which states that  $W^s$  and  $W^u$  are smooth manifolds, thus justifying the names.

**Invariant Manifold Theorem.** *Let  $\varphi : T^*(\mathcal{X}) \rightarrow T^*(\mathcal{X})$  be a  $C^k$  invertible map,  $k \geq 1$ , and let  $\bar{z}$  be a saddle fixed point. Then the sets  $W^s$  and  $W^u$  are both  $C^k$ -smooth manifolds, tangent to the corresponding eigenspaces.*

In the following it is shown that the invariant manifolds of a saddle point  $\bar{z}$  are Lagrangian.

**Proposition 2.2.3.** *Let  $\bar{z}$  be a saddle fixed point of the phase map  $\varphi$ , and let  $W^s$  and  $W^u$  be the associated stable and unstable manifolds. Assume that both  $W^s$  and  $W^u$  are  $n$ -dimensional. Then the symplectic form  $\omega$  vanishes on  $W^s$  and  $W^u$ .*

*Proof.* Assume that the theorem does not hold; that is, there are vectors  $v_1, v_2$ , tangent to  $W^s$  at some point  $z \in W^s$  such that  $|z - \bar{z}| \leq \varepsilon$ , for which  $\omega(v_1, v_2) \neq 0$ ; after rescaling it may be assumed that  $\omega(v_1, v_2) = 1$ . Denote, as above, the restriction of  $D\varphi$  to the stable eigenspace  $E^s$  by  $\Lambda_s$ , and let  $|\Lambda_s| = \lambda_s < 1$ , where  $|\Lambda_s|$  is the matrix norm associated to the Euclidean vector norm  $|\cdot|$ .

Note that for  $v$  tangent to  $W^s$  at  $z$ ,

$$|D\varphi^t(z)v_1| < c^t|v_1|$$

for some  $\lambda_s < c < 1$  and for all  $t$ . Consequently

$$1 \leq e^{\rho t} = e^{\rho t} \omega(v_1, v_2) = (\varphi^t)^* \omega(v_1, v_2) = \omega(D\varphi^t v_1, D\varphi^t v_2) \leq c^{2t} |v_1| |v_2|.$$

But for  $t > 0$  sufficiently large, this entails a contradiction.

If  $W^s$  is  $n$ -dimensional, there are  $n$  eigenvalues  $\lambda_i$  of  $D\varphi(\bar{z})$  such that  $|\lambda_i| < 1$ ,  $i = 1, \dots, n$ . Proposition 2.2.2 implies that the other  $n$  eigenvalues then have to satisfy  $|\lambda_{n+i}| > e^\rho$ ,  $i = 1, \dots, n$ . It follows that  $|\Lambda_u| = \lambda_u > e^\rho$ , and that for  $v_1$  tangent to  $W^u$  at  $z$ , the inequality

$$|D\varphi^{-t}(z)v_1| < c^t|v_1|$$

holds for some  $\lambda_u^{-1} < c < e^{-\rho}$ . It follows that for all  $t$

$$\begin{aligned} 1 &= \omega(v_1, v_2) = e^{\rho t} (\varphi^{-t})^* \omega(v_1, v_2) = e^{\rho t} \omega(D\varphi^{-t} v_1, D\varphi^{-t} v_2) \\ &\leq e^{\rho t} c^{2t} |v_1| |v_2| \leq c^t |v_1| |v_2|. \end{aligned}$$

This also leads to a contradiction. ■

Introduce the function  $G : T^*(\mathcal{X}) \rightarrow \mathbb{R}$  by setting

$$G(x, y) = g\left(x, U\left(x, \varphi_2(x, y)\right)\right).$$

Note that with this definition

$$G(z_t) = G(x_t, y_t) = g(x_{t-1}, U(x_{t-1}, y_t)) = g(x_{t-1}, u_t).$$

Let  $\mathbf{y}$  be the sequence of costates associated to the extremal pair  $(\mathbf{x}, \mathbf{u})$ .

Set  $z_t = (x_t, y_t)$  for all  $t$ .

Consider a point  $\alpha = (\alpha_1, \alpha_2) \in W^s$  such that in a neighbourhood of  $\alpha$  the manifold  $W^s$  can be represented as the graph of a function  $y : \mathcal{N} \subset \mathcal{X} \rightarrow T^*(\mathcal{X})$ . Then by Proposition 2.2.3 there is a function  $w : \mathcal{N} \rightarrow \mathbb{R}$  such that  $dw = y dx$ . Define  $\tilde{V} : \mathcal{N} \rightarrow \mathbb{R}$  as

$$\tilde{V}(\alpha_1) = \tilde{J}(\mathbf{z}) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = J(\mathbf{x}, \mathbf{u}), \quad (2.22)$$

where  $\mathbf{z}$  is the  $\varphi$ -orbit that originates in  $(x_0, y_0) = (\alpha_1, \alpha_2)$ . The next thing to demonstrate is that up to a constant the function  $w(x)$  is actually equal to the value  $\tilde{V}(\alpha_1)$  for orbits of  $\varphi$  starting at  $(\alpha_1, y(\alpha_1))$ . For this it is sufficient to show that the value function  $\tilde{V}$  for orbits on  $W^s$  is differentiable and satisfies

$$d\tilde{V} = y dx.$$

To formulate this more precisely, choose a smooth parametrisation

$$z : \mathbb{R}^n \rightarrow T^*(\mathcal{X})$$

of the stable manifold  $W^s$  of  $\bar{z}$ . Write

$$z(\sigma) = (x(\sigma), y(\sigma))$$

and assume that  $z(0) = \bar{z}$ . Let

$$\sigma_{t+1} = \psi(\sigma_t)$$

be the smooth map induced by  $\varphi$  on  $\mathbb{R}^n$ . That is, if  $z_t = (x_t, y_t) = (x(\sigma_t), y(\sigma_t))$  is an orbit of  $\varphi$  on  $W^s$ , then

$$\varphi(z(\sigma_t)) = \varphi(x_t, y_t) = (x_{t+1}, y_{t+1}) = z(\sigma_{t+1}) = z(\psi(\sigma_t)).$$

Let  $z(\sigma_0) = (x_0, y_0) \in W^s$ . If  $dx/d\sigma(\sigma_0) \neq 0$ , then on a neighbourhood of  $x(\sigma)$  a function  $y = y(x)$  can be found such that  $y(\sigma) = y(x(\sigma))$  for  $\sigma$  close to  $\sigma_0$ .

**Proposition 2.2.4.** *If  $dx/d\sigma(\sigma_0) \neq 0$ , then  $d\tilde{V}(x) = ydx$  on a neighbourhood of  $x_0$ . Consequently  $\tilde{V} - w$  is constant.*

*Proof.* Let  $z_0 + \varepsilon \zeta_0(\varepsilon)$  be an arbitrary curve of initial points in  $W^s$ ; let  $\mathbf{z} + \varepsilon \boldsymbol{\zeta}(\varepsilon)$  be the trajectories in  $W^s$  defined by these initial points, and let  $(\mathbf{x} + \varepsilon \boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon \mathbf{v}(\varepsilon))$  be the corresponding state-control trajectories.

Set

$$j(\varepsilon) = J(\mathbf{x} + \varepsilon \boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon \mathbf{v}(\varepsilon));$$

then

$$j(\varepsilon) = \tilde{V}(x_0 + \varepsilon \xi_0(\varepsilon)).$$

It will be shown below that  $j$  is differentiable for all curves of initial points through  $z_0$ ; then

$$j'(0) = \frac{d\tilde{V}}{dx}(x_0) \xi_0(0).$$

But it follows from Proposition 2.1.3 that

$$j'(0) = y_0 \xi_0(0).$$

Since  $\xi_0(0)$  is arbitrary, the theorem follows.

It remains therefore to show that  $j$  is differentiable at  $\varepsilon = 0$ ; this will follow from Proposition 2.1.3.

If  $\Lambda_s$  denotes the stable part of  $D\varphi(\bar{z})$ , then it is possible to choose the parametrising coordinate  $\sigma$  of the stable manifold  $W^s$  such that

$$\psi(\sigma) = \Lambda^s \sigma + \mathcal{O}(|\sigma|^2),$$

and

$$D\psi(\sigma) = \Lambda^s + \mathcal{O}(|\sigma|).$$

Define

$$u(\sigma) = U(x(\psi^{-1}(\sigma)), y(\sigma))$$

and note that with this definition  $u(\sigma_t) = U(x_{t-1}, y_t) = u_t$ .

Let  $\mathbf{z} = \{z_t\}$  be the orbit in  $W^s$  starting at  $z_0$ , and let  $\boldsymbol{\sigma} = \{\sigma_t\}$  be its associated orbit of parameters  $z_t = z(\sigma_t)$ . Let moreover  $c_0 : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$  be a smooth curve of the form

$$c_0(\varepsilon) = \sigma_0 + \varepsilon\tau_0,$$

with  $\tau_0 \in \mathbb{R}^n$ . Consider the forward iterates  $c_t = \psi^t(c_0)$ , parametrised as

$$c_t(\varepsilon) = \sigma_t + \varepsilon\tau_t(\varepsilon),$$

where  $\sigma_t = \psi^t(\sigma_0)$ . Note that for all  $t \geq 1$

$$\psi(\sigma_{t-1}) + \varepsilon\tau_t(\varepsilon) = \sigma_t + \varepsilon\tau_t(\varepsilon) = \psi(\sigma_{t-1} + \varepsilon\tau_{t-1}(\varepsilon))$$

and hence that

$$\begin{aligned} \tau_t(\varepsilon) &= D\psi(\sigma_{t-1})\tau_{t-1}(\varepsilon) + \varepsilon\Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) \\ &= \Lambda^s\tau_{t-1}(\varepsilon) + \varepsilon\Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) + \Psi_2(\sigma_{t-1}, \tau_{t-1}), \end{aligned}$$

where  $|\Psi_1(\varepsilon, \sigma, \tau)| \leq C|\tau|^2$  and  $|\Psi_2(\sigma, \tau)| \leq C|\sigma||\tau|$ . Choosing  $T > 0$  sufficiently large and  $\varepsilon_0 > 0$  sufficiently small, this implies for  $t > T$  and  $0 < \varepsilon < \varepsilon_0$  that

$$|\tau_t(\varepsilon)| \leq (\lambda^s + \delta)|\tau_{t-1}(\varepsilon)|,$$

where  $0 < \lambda^s + \delta < 1$ . As a consequence

$$|\tau_t(\varepsilon)| \rightarrow 0,$$

uniformly in  $\varepsilon$ , as  $t \rightarrow \infty$ .

It follows that the curves  $z(c_t)$  in  $W^s$  take the form

$$z(c_t) = z(\sigma_t + \varepsilon\tau_t(\varepsilon)) = z_t + \varepsilon\zeta_t(\varepsilon) = (x_t + \varepsilon\xi_t(\varepsilon), y_t + \varepsilon\eta_t(\varepsilon)),$$

and  $|\zeta_t(\varepsilon)| \rightarrow 0$  uniformly in  $\varepsilon$ . Using the control map  $U$ , it is found that the associated control sequence is also of the form  $u_t + \varepsilon v_t(\varepsilon)$  with  $v_t(\varepsilon) \rightarrow 0$  uniformly in  $\varepsilon$ . As a consequence, the family  $(\mathbf{x} + \varepsilon\boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon\mathbf{v}(\varepsilon))$  is extremal and strongly admissible. Proposition 2.1.3 can be applied to yield differentiability of  $j$  at  $\varepsilon = 0$ .  $\blacksquare$

Consider an arbitrary point  $\alpha = (\alpha_1, \alpha_2) \in W^s$ . A value  $v(\alpha)$  is associated to  $\alpha \in W^s$  by evaluating the objective functional for the point  $\alpha$ :

$$v(\alpha) = \tilde{J}(\mathbf{z}) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = J(\mathbf{x}, \mathbf{u}), \quad (2.23)$$

for  $\varphi$ -orbits on  $W^s$  originating in  $\alpha$ . The function  $v : W^s \rightarrow \mathbb{R}$  is called the orbit value function.

Now the value function  $\bar{V} : \mathcal{X} \rightarrow \mathbb{R}$  associated to the stable manifold  $W^s$  of  $\bar{z}$  can be defined by setting

$$\bar{V}(x) = \sup \left\{ v(\alpha) \mid \alpha_1 = x \right\} \quad (2.24)$$

(recall that  $\sup \emptyset = -\infty$ ). Note that locally around  $\bar{x}$ ,  $\bar{V}(x) = \tilde{V}(x)$ . Therefore, by Proposition 2.2.4

$$d\bar{V} = y dx$$

on a neighbourhood of  $\bar{x}$ .

## 2.3 Value comparison theorems

In this section results are stated that are used to compare values at different points on a stable manifold. These results depend on Stokes' theorem, which relates the integral of a form over the boundary of region to the integral of the derived form over the region itself.

**2.3.1 Regions and area.** Here, the concepts of simple regions and oriented simple regions are introduced. Regions will be the domains of the integral  $\Omega(A) = \int_A \omega$ , which is used extensively in the following.

A *simple region* in  $\mathbb{R}^2$  is any simply connected bounded submanifold  $A$  of  $\mathbb{R}^2$  which is such that its boundary  $\partial A$  is a closed piecewise smooth curve. An *oriented* simple region is a pair  $(A, \mu)$  where  $A$  is a simple region and where  $\mu = \pm 1$  is the orientation; the orientation is often not given explicitly. The oriented simple region  $-A$  is defined by:

$$(-A, \mu) = (A, -\mu).$$

A *region*  $A$  in  $\mathbb{R}^2$  is a formal sum of oriented simple regions  $A_i$ :

$$A = A_1 + A_2 + \cdots + A_k.$$

Moreover, set

$$A - A = 0;$$

that is, equal regions with opposite orientations cancel. It is evident how these concepts generalise to subsets of 2-dimensional oriented manifolds that are diffeomorphic to  $\mathbb{R}^2$ ; these will be called surface regions, if there is a need to stress the difference.



If  $(A, \mu)$  is an oriented simple region, and  $z$  the boundary curve of  $A$ , then  $z$  is oriented *consistently* with  $A$  if the winding number  $n_p(z)$  of  $z$  relative to any point  $p$  in the interior of  $A$  is  $\mu$ ; recall that if  $\vartheta_p$  is the angle of  $z - p$  with the positive horizontal axis, the winding number is defined as

$$n_p(z) = \frac{1}{2\pi} \int_z d\vartheta_p.$$

The boundary curve  $\partial A$  of an oriented region is always chosen consistently with the orientation of  $A$ . Inversely, to any closed curve  $z$  without self-intersection, an oriented region  $A$  is associated such that  $\partial A = z$ . More generally, if  $z$  is a piecewise smooth curve with a finite number of self-intersections then it divides the plane in a finite number of bounded regions  $A_i$  that are such that the boundary of a  $A_i$  is made up of segments of the curve  $z$ . Again, the orientation of  $A_i$  is chosen consistently with that of its boundary arcs. Consider  $A = A_1 + \dots + A_k$  and let  $I \subset \{1, 2, \dots, k\}$  be the index set of positively oriented simple regions  $A_i$ ; that is, if  $i \in I$ , then the orientation of  $A_i$  is positive, otherwise it is negative. Define then the positively and negatively oriented parts of  $A$  by respectively

$$A^+ = \sum_{i \in I} A_i, \quad \text{and} \quad A^- = \sum_{i \notin I} A_i.$$

It is clear how these definitions extend to surface regions.

If  $(A, \mu)$  is an oriented simple region, the area of  $A$  is given as

$$\text{area}(A) = \mu \int_A dx \wedge dy = -\mu \int_A \omega.$$

Introduce for simple regions the map  $\Omega$  as

$$\Omega(A) = \int_A \omega = -\mu \text{area}(A).$$

If the  $A_i$ ,  $i = 1, 2, \dots$ , are simple regions and  $A = \sum A_i$ , then define

$$\Omega(A) = \sum_i \Omega(A_i).$$

The regions  $A_i$  are called the *simple components* of the region  $A$ . In particular

$$\Omega(A) = \sum_i -\mu_i \text{area}(A_i).$$

Recall Stokes' theorem:

### Stokes' theorem

If  $\beta$  is an  $(n - 1)$ -form with compact support on  $\Omega$  and  $\partial\Omega$  denotes the boundary of  $\Omega$  with its induced orientation, then

$$\int_{\Omega} d\beta = \int_{\partial\Omega} \beta. \quad (2.25)$$

For instance, as  $\eta = y \, dx$  implies  $d\eta = \omega$ , Stokes' theorem yields, for a simple region  $A$ , that

$$\text{area}(A) = -\mu \int_A \omega = -\mu \int_{\partial A} \eta.$$

**2.3.2 The area rule.** In this subsection a result is derived that links the location of discontinuities of the derivative of the value function to the geometry of the manifold  $W^s$ .

First some more geometrical facts are recalled. Assume that a saddle fixed point  $\bar{z}$ , a point  $\alpha \in W^s$  and a curve  $z$  with  $z = (x, y) : [0, 1] \rightarrow W^s \subset T^*(\mathcal{X})$  on the stable manifold  $W^s$  of  $\bar{z}$  are given such that  $z(0) = \bar{z}$  and  $z(1) = \alpha$ . As seen before, the value  $v(\alpha)$  is associated to  $\alpha$  by evaluating the objective functional for the phase trajectory starting at  $\alpha$  (cf. (2.23)). At the fixed point the value  $v(\bar{z})$  can be computed. It is given as

$$v(\bar{z}) = \frac{g(\bar{x}, U(\bar{x}, \bar{y}))}{e^\rho - 1}.$$

By Proposition 2.2.4,

$$v(\alpha) = v(\bar{z}) + \int_0^1 y(s)x'(s) ds. \quad (2.26)$$

Note that  $v$  is only defined for points on  $W^s$ . Moreover,  $v(\alpha)$  does not depend on the curve  $z$ ; if  $z_1$  and  $z_2$  are two curves on  $W^s$  that connect  $\bar{z}$  to  $\alpha$ , then  $z_1 - z_2$  is a closed curve that encloses an oriented surface region  $A \subset W^s$ .

By Stokes' theorem,

$$\int_{z_1} y dx - \int_{z_2} y dx = \int_{z_1 - z_2} y dx = \int_A dy \wedge dx = \int_A \omega = 0,$$

since  $W^s$  is Lagrangian.

**Proposition 2.3.1.** *Consider a curve  $z_1 : [0, 1] \rightarrow W^s \subset T^*(\mathcal{X})$  without self-intersections, connecting two distinct phase points  $\alpha = z_1(0)$  and  $\beta = z_1(1)$  that have equal state coordinates  $x_1(0)$ , and satisfying*

$$x_1(\sigma) \neq x_1(0) = x_1(1) \quad \text{for all } 0 < \sigma < 1.$$

*Let moreover  $z_2$  be the straight line joining  $\beta$  to  $\alpha$ , and let  $z = z_1 + z_2$ . The curve  $z$  is closed without self-intersections; let  $A$  be an oriented surface region that is bounded by  $z$ . Then*

$$v(\beta) - v(\alpha) = \int_{z_1} y dx = \int_z y dx = \int_A \omega.$$

*Moreover, the value of  $\int_A \omega$  is independent of the choice of  $z_1$  and  $A$ .*

*Proof.* Using Stokes' theorem and  $\int_{z_2} y dx = 0$  the first result follows. To show the second statement let  $z_1, \tilde{z}_1, A$ , and  $\tilde{A}$  be such that

$$z_1 + z_2 = \partial A \quad \text{and} \quad \tilde{z}_1 + z_2 = \partial \tilde{A}.$$

Let  $D \subset W^s$  be bounded by  $z_1 - \tilde{z}_1$ . Then there is a two dimensional manifold-with-boundary  $S$  such that

$$A - D - \tilde{A} = \partial S.$$

Using Stokes's theorem and the fact that  $d\omega = 0$

$$0 = \int_S d\omega = \int_A \omega - \int_D \omega - \int_{\tilde{A}} \omega.$$

Since  $D \subset W^s$  and  $W^s$  is Lagrangian, it follows that  $\int_D \omega = 0$ , and consequently that

$$0 = \int_A \omega - \int_{\tilde{A}} \omega.$$

■

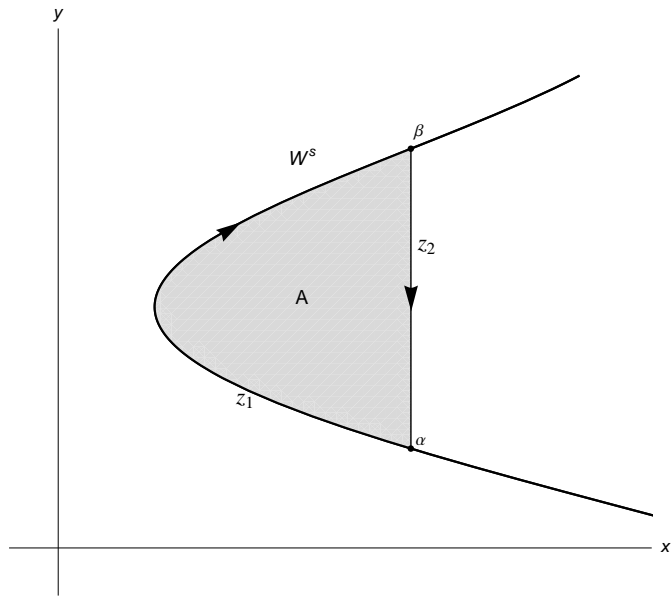
The proposition is illustrated in Figure 2.3.2. In this figure let  $z$  be the curve from  $\alpha$  to  $\beta$  along  $W^s$  and from  $\beta$  to  $\alpha$  along the straight connecting line. If  $z$  surrounds  $A$  negatively, then

$$v(\beta) - v(\alpha) = \text{area}(A);$$

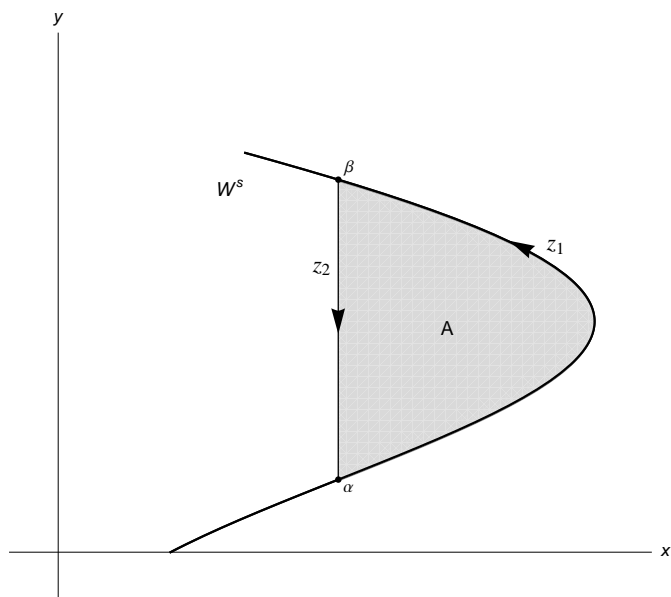
if positively, then

$$v(\beta) - v(\alpha) = -\text{area}(A).$$

**2.3.3 The iterated area rule.** In actual optimisation problems, the phase map  $\varphi$  may not be a diffeomorphism; in particular, it may not be surjective everywhere. This has consequences for the stable manifold: there may be “holes” in it. If an area rule is applied to compare values of orbits, it has to be ascertained that the regions featuring in the rule are actually defined.



(a) Negative orientation,  $v(\beta) > v(\alpha)$



(b) Positive orientation,  $v(\beta) < v(\alpha)$

**Figure 2.1:** *The area rule.*

Proposition 2.2.1 states that the phase map  $\varphi$  leaves the symplectic form  $\omega = dy \wedge dx$  invariant up to a factor. A directly related result can be derived for the 1-form  $\eta = y dx$ .

Recall the function  $G : T^*(\mathcal{X}) \rightarrow \mathbb{R}$ :

$$G(x, y) = g\left(x, U(x, \varphi_2(x, y))\right).$$

Moreover, recall that with this definition

$$G(z_t) = G(x_t, y_t) = g(x_t, U(x_t, y_{t+1})) = g(x_t, u_{t+1}).$$

Let  $\mathbf{y}$  be the sequence of costates associated to the extremal pair  $(\mathbf{x}, \mathbf{u})$ . Set  $z_t = (x_t, y_t)$  for all  $t$ . As seen before,

$$v(z_0) = J(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t}.$$

Using equations (2.17) and (2.20) yields the relation

$$G(z) = G(x, y) = H(x, \varphi_2) - \varphi_1 \varphi_2. \quad (2.27)$$

**Proposition 2.3.2.** *Let  $z : [0, 1] \rightarrow T^*(\mathcal{X})$  be a  $C^1$  curve in  $T^*(\mathcal{X})$ , joining  $\alpha = z(0)$  to  $\beta = z(1)$ , and let  $\varphi_* z = \varphi \circ z$  be its image under  $\varphi$ . Then*

$$e^\rho \eta - \varphi^* \eta = dG. \quad (2.28)$$

*Proof.* This is a simple computation. Deriving equation (2.27) and using (2.20) yields

$$\begin{aligned} dG &= dH - \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 \\ &= H_x dx + H_y d\varphi_2 - \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 \\ &= e^\rho y dx - \varphi_2 d\varphi_1. \end{aligned}$$

Since  $\eta = y dx$  and  $\varphi^* \eta = \varphi_2 d\varphi_1$ , this shows the result. ■

If the curve  $z$  is vertical, that is, if  $dx = 0$  everywhere along the curve, then the form  $\eta = y dx$  vanishes on  $z$ . For such curves proposition 2.3.2 yields the following corollary.

**Proposition 2.3.3.** *Let  $z : [0, 1] \rightarrow T^*(\mathcal{X})$  join  $\alpha = z(0)$  to  $\beta = z(1)$ , and let  $dx = 0$  along  $z$ . Then*

$$-\int_{\varphi_* z} y dx = G(\beta) - G(\alpha).$$

The iterated area rule, which is stated and proved next, is used to formulate a value comparison result with respect to a single fixed region.

Consider the following situation:  $\alpha$  and  $\beta$  are both points on the stable manifold  $W^s$  with the same  $x$ -coordinate and with associated values  $v(\alpha)$  and  $v(\beta)$ , but there is no curve in  $W^s$  joining them. There is, however, a curve

$$\tilde{z}_1 : [a, b] \rightarrow T^*(\mathcal{X})$$

in  $W^s$  that joins  $\tilde{\alpha} = \varphi^T(\alpha) = \tilde{z}_1(a)$  to  $\tilde{\beta} = \varphi^T(\beta) = \tilde{z}_1(b)$ ; for this curve

$$\int_{\tilde{z}_1} \eta = v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \quad (2.29)$$

From the representations

$$v(\alpha) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\alpha)) e^{-\rho t}, \quad v(\beta) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\beta)) e^{-\rho t},$$

it follows easily

$$\begin{aligned} v(\beta) - v(\alpha) &= \sum_{t=1}^T [G(\varphi^{t-1}(\beta)) - G(\varphi^{t-1}(\alpha))] e^{-\rho t} \\ &+ e^{-\rho T} (v(\varphi^T(\beta)) - v(\varphi^T(\alpha))). \end{aligned}$$

Let now  $z_2 : [a, b] \rightarrow T^*(\mathcal{X})$  be the vertical curve joining  $\beta$  to  $\alpha$ , and let

$$\tilde{z}_2 = \varphi^T \circ z_2.$$

Applying Proposition 2.3.2 repeatedly, the following is obtained

$$\begin{aligned} v(\beta) - v(\alpha) &= G(\beta) - G(\alpha) + e^{-\rho} \left( G(\varphi(\beta)) - G(\varphi(\alpha)) \right) + \dots \\ &\quad + e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right) \\ &= \left( \int_z e^\rho \eta - \int_{\varphi_* z} \eta \right) + e^{-\rho} \left( \int_{\varphi_* z} e^\rho \eta - \int_{\varphi_*^2 z} \eta \right) + \dots \\ &\quad + e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right) \\ &= \int_{z_2} \eta - e^{-\rho T} \int_{\varphi_*^T z_2} \eta + e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right). \end{aligned}$$

Using (2.29), as well as  $\int_{z_2} \eta = 0$  and  $\tilde{z}_2 = \varphi_*^T z_2$ , leads to

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{z}_1 + \tilde{z}_2} \eta.$$

The curve  $\tilde{z} = \tilde{z}_1 + \tilde{z}_2$  is closed; let  $\tilde{A}$  be a surface region that is bounded by this closed curve. Then

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{A}} \omega.$$

Note that by the same argument as used in Proposition 2.3.1, it can be shown that the value of  $\int_{\tilde{A}} \omega$  is independent of the choice of  $\tilde{A}$ . This discussion is summarised in the following proposition.

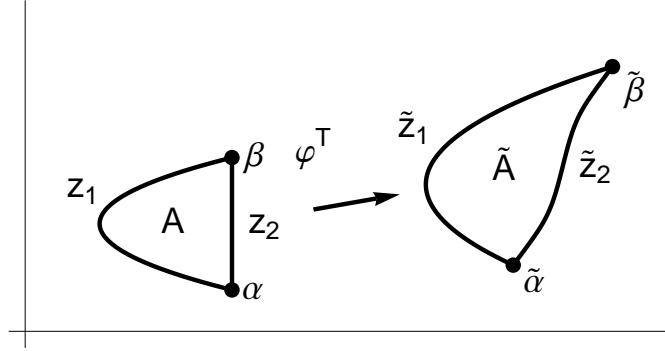
**Proposition 2.3.4.** *Let  $\alpha, \beta \in W^s$  be points with the same  $x$ -coordinate,  $\alpha, \beta$  the associated orbits of  $\varphi$ , and set  $v(\alpha) = J(\alpha)$  and  $v(\beta) = J(\beta)$ . Assume that there is a curve  $\tilde{z}_1$  on  $W^s$  joining  $\varphi^T(\alpha)$  and  $\varphi^T(\beta)$ . Let  $z_2$  be a line connecting  $\beta$  to  $\alpha$ , and set  $\tilde{z}_2 = \varphi^T \circ z_2$ . Let finally  $\tilde{A}$  be a surface region such that  $\partial \tilde{A} = \tilde{z} = \tilde{z}_1 + \tilde{z}_2$ . Then*

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{A}} \omega.$$



This is independent of  $\tilde{z}_1$  and  $\tilde{A}$ .

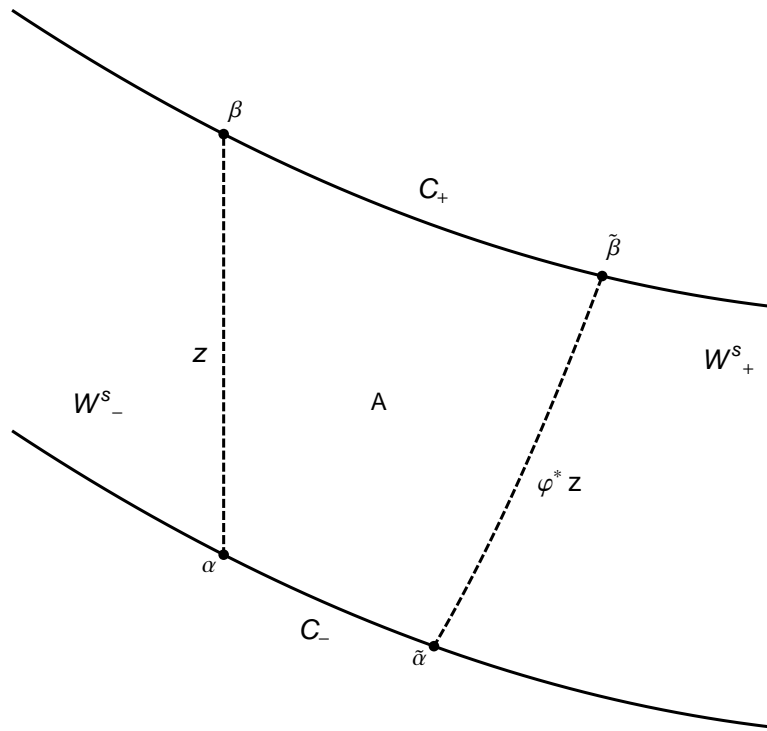
The proposition is illustrated in Figure 2.2.



**Figure 2.2:** The iterated area rule: as  $z_1 + z_2$  is negatively oriented with respect to  $A$ , then  $v(\beta) - v(\alpha) = \text{area}(A)$ . Even if  $z_1$  and hence  $A$  were not defined, still  $v(\beta) - v(\alpha) = e^{-\rho T} \text{area}(\tilde{A})$ .

**2.3.4 Value differences.** Consider now the situation illustrated in Figure 2.3. There are two stable manifolds  $W_-^s$  and  $W_+^s$ , associated to the fixed points  $z_-$  and  $z_+$ , and two points  $\alpha \in W_-^s$  and  $\beta \in W_+^s$ , such that their  $x$ -coordinates are equal (see Figure 2.3). Let  $z$  be the line segment joining  $\alpha$  to  $\beta$  and let  $A$  be the oriented surface region bounded by the concatenation of  $z$ , the part  $w_+$  of  $W_+^s$  joining  $\beta$  to  $\tilde{\beta} = \varphi(\beta)$ , the negative of  $\varphi_* z$  joining  $\tilde{\beta}$  to  $\tilde{\alpha} = \varphi(\alpha)$ , and the negative  $-w_-$  of the part of  $W_-^s$  joining  $\alpha$  to  $\tilde{\alpha}$ . Define finally

$$\Omega(A) = \int_A \omega.$$



**Figure 2.3:** Relation between values and area:  $v(\beta) - v(\alpha) = \text{area}(A)/(e^\rho - 1)$ .  
 The boundary of  $A$  is the curve  $\alpha \rightarrow \beta \rightarrow \tilde{\beta} \rightarrow \tilde{\alpha} \rightarrow \alpha$ ; it is negatively oriented,  
 consequently the orientation of  $A$  is negative as well and  $\Omega(A) = \text{area}(A)$ .

**Proposition 2.3.5.** *In the situation sketched above  $\Omega(A)$  is independent of the choice of the curves and*

$$v(\beta) - v(\alpha) = \frac{\Omega(A)}{e^\rho - 1}.$$

*Proof.* Again by Stokes' theorem:

$$\begin{aligned} \Omega(A) &= \int_A \omega = \int_{\partial A} \eta = \int_z \eta + \int_{c_+} \eta - \int_{\varphi_* z} \eta - \int_{c_-} \eta \\ &= v(\tilde{\beta}) - v(\beta) - \int_{\tilde{z}} \eta - v(\tilde{\alpha}) + v(\alpha). \end{aligned}$$

Using Proposition 2.3.2 yields that

$$- \int_{\tilde{z}} \eta = G(\beta) - G(\alpha).$$

Moreover, using  $v(\alpha) = \sum_1^\infty G(z_t) e^{-\rho t}$  yields

$$v(\tilde{\alpha}) = e^\rho v(\alpha) - G(\alpha).$$

Eliminating with these relations the quantities  $\int_{\tilde{z}} y \, dx$  as well as  $v(\tilde{\alpha})$  and  $v(\tilde{\beta})$ ,

$$\Omega(A) = (e^\rho - 1)(v(\beta) - v(\alpha)),$$

as claimed in the proposition. Independence is shown by the same arguments as employed in the proof of Proposition 2.3.1. ■