Bifurcations of indifference points in discrete time optimal control problems

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Citation for published version (APA):

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Chapter 3

The indifference-attractor bifurcation

Systems where the phase map $\varphi$ has a unique saddle fixed point are encountered in many places in the economic literature. Most of the results are about the degree of change of various quantities at that point if parameters are varied (comparative statics). In those systems the unique saddle corresponds to a steady state of the optimal state dynamics which is such that all solutions, regardless of the initial state of the system, tend to this state.

Whenever there is more than one saddle point present, say two, the solution structure is more complicated. It may be that both saddles correspond to an optimal steady state; in that case, Dechert and Nishimura (1983) have showed that there is an intermediate state such that all solutions starting left to this state tend to one steady state and all solutions starting right to it tend to the other. For the intermediate state, there are two possibilities:
either it is a long-term steady state itself, or it is an indifference state\textsuperscript{1}, i.e. a state that is initial state to two different optimal trajectories.

In analogous continuous time problems, whether or not an indifference state occurs in a system depends on the relative position of the stable and unstable manifolds of the saddle equilibria of the phase flow (see Wagener, 2003; Kiseleva and Wagener, 2011). In particular, the shift from one type of solution to another is characterised by the occurrence of a ‘heteroclinic connection’, where the stable manifold of one saddle equilibrium coincides with the unstable manifold of another saddle. The situation in discrete time problems can be more complex due to the fact that, unlike in continuous time problems, stable and unstable manifolds do not automatically coincide once they have a single point in common. This situation will be analysed in the present chapter.

The chapter investigates indifference-attractor bifurcation of the discrete-time optimisation problems with a single state variable. First a class of optimisation problems that have indifference-attractor bifurcation singularities is defined and assumptions that describe the simplest configuration for which the results hold. Then it is shown that this bifurcation is linked to a heteroclinic bifurcation scenario of the phase map $\varphi$. The consequences for the optimal solutions are analysed. In particular, the bifurcation value at which indifference thresholds appear is characterised by a geometric condition, which is stated in Section 3.2 together with the required mathematical concepts. Finally, in the last section, the proofs of the main results are given.

\textsuperscript{1}Often called Skiba, DNS or DNSS state in reference to the contributions of Skiba (1978), Dechert and Nishimura (1983) and Sethi (1977).
3.1 Heteroclinic orbits and indifference-attractor bifurcations

Here a class of optimisation problems that have indifference-attractor bifurcation singularities is defined. This class is characterised in terms of the phase map, which was introduced in the previous chapter. In particular, attention is restricted to the situation that the phase map is defined on a subset of $T^*\mathbb{R} \cong \mathbb{R}^2$. The main characteristic of the class is that the family of phase maps goes through a heteroclinic bifurcation scenario. Throughout the chapter, it will be found that this abstract mathematical condition has a number of powerful implications for the structure of the set of optimising trajectories.

Consider the problem of optimising (2.3) under the side condition (2.4), where $\mathcal{X}$ and $\mathcal{U}$ are open subsets of $\mathbb{R}$. Recall from Chapter 2 that the necessary first order conditions of this problem can be formulated in terms of the phase map $\varphi$, defined on the phase space $T^*\mathcal{X} = \mathcal{X} \times \mathbb{R}$, as follows: if $(x, y)$ maximises (2.3) subject to (2.4), and if $(x, y)$ is $\delta$-interior for $\delta > 0$, then necessarily

\[
(x_t, y_t) = \varphi(x_{t-1}, y_{t-1}), \quad \text{for all } t \geq 1,
\]

\[
x_0 = \alpha, \quad \text{and } \lim_{t \to \infty} e^{-\rho t} y_t = 0. \quad (3.1)
\]

3.1.1 Phase maps. In the following a string of assumptions is made. Their main function is to delineate the simplest configuration for which the results hold; all of them can be checked, at least numerically, for a given system. Moreover, they hold true for a large class of problems of practical
interest. The first assumption implies that the phase maps actually exist.

**Assumption 3.1.1.** The discrete Hamilton function $H = H(x,y)$, which has been introduced in Section 2.1.3, satisfies $H_{xy} > 0$ and $H_{yy} > 0$.

The stronger assumption $H_{xy} = f_x > 0$ is needed to obtain the results of Theorems 3.3.1 and 3.3.2. This assumption requires $f$ as a function of $x \in \mathcal{X}$ to be an orientation-preserving diffeomorphism for each $u \in \mathcal{U}$.

Phase maps originating from optimisation problems of the type given in this section have, like their continuous time counterparts, special geometrical properties: they are conformally symplectic maps; they are even symplectic in the case $\rho = 0$. Symplecticity is an abstract mathematical concept, related to integrability theory; some of its implications have been discussed in Section 2.2.

In the following some concepts from differential geometry are recalled on which the arguments in this chapter are based. For a fuller treatment of this material, especially of differential forms, the reader is referred to the excellent expositions of Spivak (1965) or Arnol’d (1989).

Let

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

In $\mathbb{R}^2$, the **standard symplectic 2-form** is the differential form $\omega = dy \wedge dx$: on a pair $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbb{R}^2$, the form $\omega$ takes the value

$$\omega(v, w) = (v, Ew) = v_2 w_1 - v_1 w_2.$$ 

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be of the form $\psi = (\psi_1(x, y), \psi_2(x, y))$ and let $D\psi$ denote
the $2 \times 2$ Jacobi matrix

$$D\psi = \begin{pmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{pmatrix}.$$ 

The pull-back $\psi^* \omega$ of $\omega$ under $\psi$ is defined as

$$\psi^* \omega(v,w) = \omega(D\psi v, D\psi w). \quad (3.2)$$

The map $\psi$ is called symplectic, if

$$\psi^* \omega = \omega.$$ 

It is called conformally symplectic, if there is a function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\psi^* \omega = \lambda \omega.$$ 

**Proposition 3.1.1.** Let $\mathcal{X} \subset \mathbb{R}$ be an open interval. Then the phase map $\varphi : T^* \mathcal{X} \to T^* \mathcal{X}$, given by (2.20), is conformally symplectic. More precisely, it satisfies

$$\varphi^* \omega = e^\rho \omega.$$ 

This implies that

$$\det D\varphi = e^\rho. \quad (3.3)$$

The proposition is a corollary of the more general Propositions 2.1.5 and 2.2.2, which have been stated and proved in Section 2.

Note that $\det D\varphi = e^\rho$ implies that the map $\varphi$ multiplies phase volume by a factor $e^\rho$. It follows that there are no bounded regions that are invariant under $\varphi$; this implies for instance that $\varphi$ has no invariant circles.
3.1.2 Description of the context. In this subsection a number of assumptions are given (and discussed) which taken together form the definition of a family that has an indifference-attractor singularity. Consider therefore a family of phase maps $\varphi = \varphi_\mu : T^* X \to T^* X$ depending on a parameter $\mu \in \mathbb{R}$.

**Assumption 3.1.2.** For all values of the parameter $\mu$ the map $\varphi = \varphi_\mu$ has two saddle fixed points $z_- = (x_-, y_-)$ and $z_+ = (x_+, y_+)$, such that $x_- < x_+$, and such that there is no saddle point $\tilde{z} = (\tilde{x}, \tilde{y})$ of $\varphi_\mu$ with $x_- \leq \tilde{x} \leq x_+$.

As in the continuous time case, orbits on stable manifolds of saddles are candidates for optimal trajectories, as they satisfy the transversality condition automatically (see Subsection 2.1.5). The next assumption postulates that there is an open subinterval $I \subset X$ with $x_-, x_+ \in I$ such that the optimisation problem has a solution for all initial states, and that orbits on the stable manifolds $W^s_-$ and $W^s_+$ of $z_-$ and $z_+$ respectively are the only candidates for the optimal orbits.

**Assumption 3.1.3.** There is an open interval $I$ such that $x_-, x_+ \in I$ and such that for every $x_0 \in X$ with $x_- \leq x_0 \leq x_+$, the problem to optimise $J$ subject to equation (2.4) has a solution. Moreover, the state-costate trajectory $z$ of such a solution is either on $W^s_-$ or $W^s_+$.

As mentioned above, the genesis of indifference thresholds will turn out to be intimately connected with the occurrence of heteroclinic orbits in the system. A point $z$ is called heteroclinic to $z_-$ and $z_+$, or a heteroclinic intersection of $W^u_-$ and $W^s_+$, if $z \in W^u_- \cap W^s_+$. Note that if $z$ is heteroclinic, so is $\varphi(z)$ and in fact every iterate $\varphi^k(z)$. The orbit $O(z) = \{\varphi^k(z) | k \in \mathbb{Z}\}$ through a heteroclinic point $z$ is called a heteroclinic orbit.
A heteroclinic intersection $z$ is *transversal* if at $z$ the tangent vectors to $W^u_-$ and $W^s_+$ are linearly independent, for instance as in Figure 3.1 or Figure 3.2(c). Invariant manifolds and their tangent spaces depend continuously on parameters: if for a given parameter value $\mu_0$ there is a transversal heteroclinic intersection, then this is the case for all values of $\mu$ sufficiently close to $\mu_0$. A non-transversal heteroclinic intersection is called a heteroclinic tangent (see Figures 3.2(b) and 3.2(d)).

Figure 3.1: The stable manifold $W^s_+$ of $z_+$ (solid) and the unstable manifold $W^u_-$ of $z_-$ (dashed). The stable manifold $W^s_+$ is composed of all points that are forward asymptotic to $z_+$; likewise, $W^u_-$ is composed of all points backward asymptotic to $z_-$. A heteroclinic point is an intersection of $W^s_+$ and $W^u_-$, hence a point that is forward asymptotic to $z_+$ and backward asymptotic to $z_-$. As both manifolds contain infinitely many orbits, they do not necessarily coincide (unlike in the continuous time case).

The family $\varphi_\mu$ is said to go through a heteroclinic bifurcation scenario, involving for instance $W^u_-$ and $W^s_+$, if there is a parameter interval $[\mu_1, \mu_2]$ such that for $\mu < \mu_1$ and $\mu > \mu_2$, the manifolds $W^u_-$ and $W^s_+$ have no points in common, and if for $\mu \in [\mu_1, \mu_2]$ there is at least one heteroclinic orbit.
Necessarily all heteroclinic orbits are tangencies at $\mu = \mu_1$ and $\mu = \mu_2$. Figure 3.2 illustrates the basic scenario. In general the scenario may be more complex, featuring also tangencies for intermediate values of $\mu$.

**Figure 3.2:** Relative position of $W_u^-$ (dashed) and $W^+_s$ at $z_-$, depending on the parameter $\mu$. At $\mu = \mu_1$ and $\mu = \mu_2$, $W_u^-$ and $W^+_s$ exhibit heteroclinic tangencies; for $\mu_1 < \mu < \mu_2$, the manifolds intersect transversally.

The family $\varphi_\mu$ of phase maps is assumed to go through a heteroclinic bifurcation scenario.

**Assumption 3.1.4.** If $\mu < \mu_1$ or $\mu > \mu_2$, then $W^+_s$ and $W_u^-$ have no points in common. On the other hand, if $\mu_1 \leq \mu \leq \mu_2$, then there are heteroclinic
intersections of $W_u^-$ and $W_s^+$. Moreover, if $\mu > \mu_2$, then $W_s^+$ does not intersect the line $x = x_+$, nor does $W_s^-$ intersect $x = x_-$. If $\mu < \mu_1$, then $W_s^+$ intersects the line $x = x_-$. 

Assumption 3.1.3 together with the second half of Assumption 3.1.4 implies that if $\mu > \mu_2$, then both $x_-$ and $x_+$ are locally optimal fixed points. For every initial state $x_0 \in I$, the state-costate trajectory $z$ of a solution to the optimisation problem necessarily lies either on $W_s^+$ or $W_s^-$. However, neither of these two stable manifolds cover the whole state space.

For a given heteroclinic intersection $z \in W_u^- \cap W_s^+$, let $c$ be the curve obtained by taking the part of $W_u^-$ that connects $z_-$ to $z$ and the part of $W_s^+$ that connects $z$ to $z_+$. If $c$ is a curve without self-intersections, then $z$ is called a primary heteroclinic intersection.

The next assumption postulates that the map $\varphi$ has some generic properties.

**Assumption 3.1.5.**

1. For $\mu = \mu_1$ and $\mu = \mu_2$, there is a single orbit of heteroclinic tangencies of $W_u^-$ and $W_s^+$. 

2. There is a finite set $F \subset [\mu_1, \mu_2]$ such that for each $\mu \in [\mu_1, \mu_2] \backslash F$, the manifolds $W_u^-$ and $W_s^+$ have only transversal primary intersection points. If $\mu \in F$, there is one orbit of primary quadratic heteroclinic tangencies of $W_u^-$ and $W_s^+$, as well as at least two orbits of primary heteroclinic transversal intersections.

Remark that the conditions of the assumptions determine an open set of phase maps $\varphi$. Whether this set is also dense in some suitable function
topology is not immediately clear, due to the indirect definition of \( \varphi \). This question is left to a future investigation and instead it is only conjectured that the conditions of Assumption 3.1.5 determine an open and dense set, with respect to the \( C^\infty \) topology, of optimisation problems that satisfy Assumptions 3.1.2–3.1.4.

Note that without the restriction to primary intersection points, the conjecture might well be false, as there may be generically infinitely many values of \( \mu \) for which there is a heteroclinic tangency (cf. Palis and Takens, 1993, Chapter 6).

The next assumption is required since the inverse of \( \varphi \) is not necessarily defined in every part of the phase space \( T^*\mathcal{X} \); consequently, the stable manifold need not be connected.

**Assumption 3.1.6.** For each \( \mu \in (\mu_1, \mu_2) \) and every orbit \( O \) of heteroclinic intersections of \( W^s_+ \) and \( W^u_- \), there exists a point \( z \in O \) as well as two smooth curves in \( T^*\mathcal{X} \) that connect \( z \) to \( \varphi(z) \) along \( W^u_- \) and \( \varphi(z) \) to \( z \) along \( W^s_+ \) respectively.

### 3.1.3 The main result and its interpretation.

A state \( \bar{x} \) is called an (optimal) steady state, if the optimal trajectory starting at \( \bar{x} \) is given by \( x_t = \bar{x} \) for all \( t \). An optimal steady state \( \bar{x} \) is **globally optimal**, if every optimal trajectory \( \{ x_t \} \) converges to \( \bar{x} \); the steady state \( \bar{x} \) is **locally optimal**, if for all initial states \( x_0 \) in a neighbourhood of \( \bar{x} \), the optimal trajectory starting at \( x_0 \) converges to \( \bar{x} \).

Now the main result can be stated.
**Theorem 3.1.1.** Let the Assumptions 3.1.2–3.1.6 be satisfied. There is a value

\[ \mu_1 < \mu_c < \mu_2 \]

such that

1. If \( \mu < \mu_c \), the steady state \( x_+ \) is globally optimal;

2. if \( \mu > \mu_c \), both steady states \( x_- \) and \( x_+ \) are locally optimal, and there is a state \( x_- < x_s < x_+ \) such that \( x_s \) is initial state to two optimal solutions of which one converges to \( x_- \) and the other converges to \( x_+ \);

3. if \( \mu = \mu_c \), the steady state \( x_- \) is semi-stable: optimal solutions starting at \( x_0 \leq x_- \) tend to \( x_- \), whereas optimal solutions starting at \( x_0 > x_- \) tend to \( x_+ \);

4. moreover, if \( \mu = \mu_c \), there is an infinite sequence

\[ x_1^{(1)} > x_1^{(2)} > \cdots > x_- , \]

such that

\[ \lim_{k \to \infty} x_1^{(k)} = x_- \]

and such that each \( x_1^{(k)} \) is initial point to two optimal sequences, both converging to \( x_+ \).

The proof is a direct corollary of Theorems 3.2.1–3.2.3 below.

**3.1.4 Optimal maps.** Any optimal state-control trajectory \((x, u)\) corresponds one-to-one with a state-costate trajectory \(z\), which in turn is determined by its initial state \(z_0 = (x_0, y_0)\).
By the principle of optimality, if \( z = \{z_t\}_{t=0}^\infty \) is an optimal state-costate trajectory with initial point \( z_0 \), and if \( n \) is a positive integer, then

\[
\sigma_n z \overset{\text{def}}{=} \{z_{t+n}\}_{t=0}^\infty
\]

is also an optimal trajectory, but with initial state \( z_n \). Therefore,

\[
J(z) = \sum_{t=1}^n e^{-\rho t} g(x_{t-1}, U(x_{t-1}, y_t)) + e^{-\rho n} J(\sigma_n z),
\]

and it is clear that if \( \sigma_n z \) does not maximise \( J \) over the set of admissible trajectories starting at \( z_n \), then \( z \) does not maximise \( J \) over the set of admissible trajectories starting at \( z_0 \).

The set of optimal state-costate trajectories can be described by a set-valued map

\[
Y^o = Y^o(x) \subset \mathbb{R}.
\]

If \( z_0 = (x_0, y_0) \), with \( y_0 \in Y^o(x_0) \), then \( z_0 \) is the initial point of an optimal trajectory. The set-valued map \( Y^o \) is called the \textit{optimal costate map}. In the present context, it follows from Assumption 3.1.3 that \( Y^o(x) \) is a set containing either one or two elements. Analogously, the \textit{optimal state map} is defined

\[
\Psi^o(x) = \varphi_1(x, Y^o(x)) = \{\varphi_1(x, y) \mid y \in Y^o(x)\}.
\]

This is also a set-valued map. The optimal dynamics is given as

\[
x_t \in \Psi^o(x_{t-1}).
\]

Finally, the map

\[
U^o(x) = U(x, Y^o(x)) = \{u(x, y) \mid y \in Y^o(x)\}
\]

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is the *policy* function. A point \( x \) where \( U^\alpha(x) \) consists of two elements is a jump point of the policy function.

The complexity of the optimal state dynamics is not as great as would appear at first sight. Indeed, if a state \( \xi \) is the first iterate of an initial state \( x_0 \) under \( \Psi^o \), that is, if \( \xi \in \Psi^o(x_0) \) for some \( x_0 \), then \( \Psi^o(\xi) \) contains exactly one element; otherwise there would be two optimal state-costate orbits with initial point \((x_0, y_0)\), which contradicts the fact that the phase map \( \varphi \) is well-defined (and hence single-valued).

If \( \Psi^o(x) \) contains only one element, \( \psi^o(x) \) is defined by setting

\[
\Psi^o(x) = \{ \psi^o(x) \}.
\]

Note that an optimal steady state as defined above is just a fixed point of the map \( \psi^o \). If \( \Psi^o(x) \) contains two elements, the state \( x \) is an *indifference state*, as there are two optimal state trajectories starting at \( x \). If these two optimal trajectories have different \( \omega \)-limit sets, then \( x \) is an *indifference threshold*.

The main theorem can be rephrased now in terms of the (parameter-dependent) optimal state dynamics \( \Psi^o_\mu \): if the assumptions are satisfied, and if \( \mu < \mu_c \), then all orbits of the optimal dynamics tend to \( x^+ \), and \( x^+ \) is therefore the global attractor for the optimal dynamics. If \( \mu > \mu_c \), there is an indifference threshold \( x_s \); all orbits starting at \( x_0 < x_s \) tend to \( x_- \), whereas all orbits starting at a point \( x_0 > x_s \) tend to \( x_+ \); both \( x_- \) and \( x_+ \) are local attractors of the optimal dynamics. If \( \mu = \mu_c \), then the orbit \( x = x_- \) is semi-stable: all orbits starting to the left of it converge to \( x_- \), while all orbits starting to the right converge to \( x_+ \). These facts are summarised by saying that at \( \mu = \mu_c \), a locally stable attractor and an indifference
threshold of the optimal dynamics are generated through an *indifference-attractor bifurcation*. The last statement of the main theorem is that for $\mu = \mu_c$ the optimal dynamics has an infinity of indifference points that are not indifference thresholds.

### 3.2 Regions and orientations

In this section, three theorems are formulated that give conditions to determine whether the states $x_-$ and $x_+$ are both locally optimal, or whether $x_+$ is globally optimal. These conditions are formulated in terms of the oriented area of a certain region in phase space.

Recall from Subsection 2.3.1 that a region is a collection of oriented open and bounded sets that are simply connected and that have well-behaved boundaries. Assume that the parts of $W^u_-$ and $W^s_+$ that interact in the heteroclinic bifurcation are parametrised by arclength, starting from the respective fixed points. That is, the parametrisations $\gamma_u(s)$ and $\gamma_s(s)$ satisfy $\gamma_u(0) = z_-$ and $\gamma_s(0) = z_+$ as well as $\|\gamma_u'(s)\| = \|\gamma_s'(s)\| = 1$. Note that this determines an orientation of $W^u_-$ and $W^s_+$.

With respect to these parametrisations, a transversal heteroclinic intersection $z = \gamma_u(s_1) = \gamma_s(s_2)$ has intersection number $+1$ (cf. Hirsch, 1976), if

$$\det \begin{pmatrix} \gamma_u'(s_1) & \gamma_s'(s_2) \end{pmatrix} > 0.$$ Intersections of intersection number $-1$ are defined analogously; see Figure 3.3. The intersection number of a quadratic heteroclinic tangency is set to be 0.
Figure 3.3: Orientation of the intersections.

As \( \det D\varphi > 0 \), if \( p_t \) is a transversal heteroclinic intersection with intersection number \( +1 \), then so is \( \varphi(p_t) \). Therefore, the intersection number of a heteroclinic orbit \( p \) is well-defined as the intersection number of any of its elements. Let \( p = \{p_k\}_{k=-\infty}^\infty \) be a transversal heteroclinic intersection of \( W_+^u \) and \( W_-^s \) with intersection number \( +1 \). Heteroclinic orbits of this type will be called **upward** orbits.

If \( p \) is an upward orbit, assume that \( p_0 \) is such that smooth curves \( c^u \), \( c^s \) as postulated in Assumption 3.1.6 exist, connecting \( p_0 \) to \( p_{-1} \). Let \( c \) be the closed curve obtained by first following \( c^s \) from \( p_0 \) to \( p_{-1} \) and then \( c^u \) from \( p_{-1} \) to \( p_0 \). Then \( c \) is the boundary of a region \( A = A_p \), with positively and negatively oriented components \( A^+ \) and \( A^- \) respectively. This situation is illustrated in Figure 3.4.
Differential forms can be integrated over oriented regions (see e.g. Spivak, 1965): for example, if $A$ is an open connected set that has the standard orientation of $T^*X$, then

$$\int_A \omega = -\text{area}(A).$$

This equality is used to define a function $\Omega$ taking regions as arguments, a cochain, by setting

$$\Omega(A) = \int_A \omega.$$ 

Note that

$$\Omega(A^-) = \int_{A^-} \omega = \text{area}(A^-) > 0, \quad \Omega(A^+) = \int_{A^+} \omega = -\text{area}(A^+) < 0,$$

and

$$\Omega(A) = \Omega(A^+) + \Omega(A^-) = \text{area}(A^-) - \text{area}(A^+). \quad (3.4)$$
With these notations Theorems 3.2.1–3.2.3 can be formulated, which imply the main theorem. Theorems 3.2.1 and 3.2.2 describe the generic situations.

**Theorem 3.2.1.** If \( \mu > \mu_2 \) or if \( \Omega(A_p) \geq 0 \) for each upward orbit \( p \), then \( x_- \) is a locally optimal fixed point.

**Theorem 3.2.2.** If \( \mu < \mu_1 \) or if \( \Omega(A_p) < 0 \) for some upward orbit \( p \), then \( x_- \) is not an optimal fixed point and any optimal orbit starting in a neighbourhood of \( x_- \) tends to \( x_+ \).

Theorem 3.2.3 characterises the codimension one situation that separates the two generic cases.

**Theorem 3.2.3.** Let \( \mu_3 \in [\mu_1, \mu_2] \) be such that \( \Omega(A_p) = 0 \) for some upward orbit \( p \). Moreover, assume that \( \Omega(A_q) > 0 \) for all other upward orbits \( q \). Then the fixed point \( x_- \) is optimal. For each \( x_0 > x_- \), the optimal trajectories beginning at \( x_0 \) converge to \( x_+ \). Moreover, there are infinitely many points \( x_{(k)} > x_- \) which are initial point to two distinct optimal trajectories.

Remark that though the magnitude of \( \Omega(A_p) \) depends on the choice of \( p_0 \) of the heteroclinic orbit, the sign of \( \Omega(A_p) \) is independent of that choice, since

\[
\Omega(\varphi(A_p)) = e^{\rho} \Omega(A_p).
\]

The idea of the proof is sketched. Consider an upward heteroclinic orbit \( p = \{p_t\}_{t=-\infty}^\infty \), and let \( \mathcal{N} \) be a small convex open neighbourhood of the fixed point \( z_- \). If \( p_t \in \mathcal{N} \), introduce the set \( W_{+t}^s \) as the largest connected component of \( W^s \cap \mathcal{N} \) that contains \( p_t \); otherwise, let \( W_{+t}^s = \emptyset \). Let moreover \( L_t \) be the line through \( p_t \) and parallel to \( E_- \).
Assume first that $p_t$ is a transversal heteroclinic intersection of $W_s^+$ and $W_u^-$. The inclination lemma from the theory of dynamical systems, which is quoted in Section 3.3, implies that there is $T > 0$, such that for $t < -T$, the set $W_s^{s,t}$ is a curve segment that is $C^1$-close to $L_t$. As $L_t$ intersects the vertical line

$$\ell = \{(x, y) : x = x_-\}$$

through $z_-$, so does $W_s^{s,t}$ if $t < 0$ is sufficiently small. Introduce

$$q_t = \ell \cap W_s^{s,t}.$$

The situation is illustrated in Figure 3.5. An intersection $q_t$ arising in this way from an upward orbit shall be called an upward intersection.

To every point on the stable manifold $W_s^+$, a value can be associated by evaluating the objective functional for the phase trajectory starting at the
point. It follows from Proposition 2.2.4 that the same result is obtained by integrating the form $y \, dx$ along the stable manifold; since the manifold is Lagrangian, the result of the integration is independent of the integration path (see Subsection 2.2.2).

Let $z(s) = (x(s), y(s))$ be the parametrisation of $W^s_+$ by arc length, such that $z(0) = z_+$, and such that heteroclinic points correspond to positive values of the parameters $s$. To a point $\alpha \in W^s_+$ associate the value $v(\alpha)$ given by the phase trajectory $z$ starting at $\alpha$ (c.f. Section 2.2):

$$v(\alpha) = \tilde{J}(z) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = J(x, u).$$

This implies for instance that for the fixed point $z_+$

$$v(z_+) = \frac{g(x_+, U(x_+, y_+))}{e^\rho - 1}.$$

Given a point $\alpha \in W^s_+$, let $s_\alpha$ be such that $\alpha = z(s_\alpha)$. Then, by Proposition 2.2.4,

$$v(\alpha) = v(z_+) + \int_0^{s_\alpha} y(s) x'(s) \, ds. \quad (3.5)$$

Note that $v$ is only defined for points on $W^s_+$.

It will be established that for every $t < 0$ such that $W^s_{+,t}$ intersects the line $\ell$, there is a region $C_t$ such that

$$v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(A) + e^{\rho t} \text{area } (C_t). \quad (3.6)$$

Consider first the situation that $\Omega(A) > 0$. Since $\text{area } (C_t) > 0$, this implies that $v(q_t)$ increases as $t$ decreases towards minus infinity. It follows from Proposition 6.0.1 in Appendix A that $v(q_t) \uparrow V_-(x_-)$ as $t \to -\infty$. If equation (3.6) holds for every upward orbit $p$, it follows that $V_-(x_-)$ is larger than
any value \( v(z) \) for \( z \in W^+ \cap \ell \), and consequently, that it is optimal to remain in \( z_- \). A similar argument holds if \( \ell \) is replaced by any vertical line through a point sufficiently close to \( z_- \), demonstrating local optimality of \( x_- \). This is Theorem 3.2.1.

Equation (3.6) is also helpful for analysing the case that \( \Omega(A) < 0 \) for some upward orbit \( p \), for it can be shown that

\[
\frac{\text{area}(C_t)}{\Omega(A)} \to 0
\]
as \( t \to -\infty \). This implies that the sequence \( v(q_t), v(q_{t-1}), \ldots \) is eventually decreasing. Note that the limit of the sequence is still \( V_-(x_-) \); therefore, there is some \( T \) such that

\[
v(q_T) > V_-(x_-),
\]
and the steady state \( x_- \) cannot be optimal in this case, implying the result of Theorem 3.2.2.

### 3.3 Proofs of the theorems

In this section, the proofs of Theorems 3.2.1, 3.2.2 and 3.2.3 are given.

#### 3.3.1 Local preliminaries.

**Proposition 3.3.1.** If \( v = (v_1, v_2) \) is a nonzero eigenvector of \( D\varphi \), then \( v_1 \neq 0 \).

**Proof.** The derivative \( D\varphi \) takes the form

\[
D\varphi = \begin{pmatrix}
H_{xy} - \frac{H_{yy} H_{xx}}{H_{xy}} & e^p H_{yy} \\
\frac{H_{xx}}{H_{xy}} & e^p
\end{pmatrix}.
\]

(3.7)
Assume that \( v_1 = 0 \). Then by Proposition 2.1.5 the eigenvalue equation \( \lambda v = (D\varphi)v \) reads as

\[
\begin{pmatrix}
0 \\
\lambda v_2 \\
e^\rho H_{yy}^{-1} v_2 \\
e^\rho H_{xx}^{-1} v_2
\end{pmatrix}.
\]

If \( \lambda = 0 \), then \( H^{-1}_{xy}v_2 = 0 \) and consequently \( v_2 = 0 \); but then \( v \) would be trivial. If \( \lambda \neq 0 \), then \( v_2 = (\lambda/e^\rho)H_{xy}v_2 \). Substituting into the first equation yields that

\[
0 = \lambda H_{yy}v_2.
\]

As \( H_{yy} \) is positive by Assumption 3.1.1, it follows that \( v_2 = 0 \), again implying that \( v \) is trivial.

This proposition implies that there is a neighbourhood \( \mathcal{N} \) of \( z_- \) such that \( W^s \) and \( W^u \) restricted to \( \mathcal{N} \) can be represented as the graphs of functions \( w^s \) and \( w^u \) respectively.

**Proposition 3.3.2.** If \( v^u = (1, v^u_2) \) and \( v^s = (1, v^s_2) \) are stable and unstable eigenvectors of \( D\varphi \), then \( v^u_2 > v^s_2 \).

**Proof.** If \( v = (1, v_2) \) is an eigenvector with eigenvalue \( \lambda \),

\[
(H^2_{xy} - H_{xx}H_{yy}) - \lambda H_{xy} + e^\rho H_{yy}v_2 = 0.
\]

This can be written as

\[
v_2 = \frac{H_{xx}H_{yy} - H^2_{xy}}{e^\rho H_{yy}} + \frac{\lambda H_{xy}}{e^\rho H_{yy}}.
\]

The result now follows from Assumption 3.1.1 that \( H_{xy} > 0 \) and \( H_{yy} > 0 \). \( \blacksquare \)

From Proposition 3.3.2, the following corollary is obtained.
Proposition 3.3.3. Let $\Delta$ be the triangle bounded by the line connecting $(0, 0)$ to $v^u$, followed by the line connecting $v^u$ to $v^u + v^s = (0, v_2^u + v_2^s)$ and the line connecting $v^u + v^s$ to $0$. Then $\Delta$ is positively oriented.

Recall that a map $\Phi$ is symplectic if $\Phi^* \omega = \omega$.

Proposition 3.3.4. There is an open neighbourhood $\mathcal{N} \subset T^* \mathcal{X}$ of $z_-$, an open neighbourhood $\tilde{\mathcal{N}} \subset \mathbb{R}^2$ of $(0, 0)$ and a symplectic coordinate transformation $\Phi : \mathcal{N} \to \tilde{\mathcal{N}}$ of the form

$$\zeta = (\xi, \eta) = \Phi(x, y) = \Phi(z),$$

such that in the new coordinates the map $\varphi$ has the form

$$\varphi(\zeta) = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^u \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \xi \psi_1 \\ \eta \psi_2 \end{pmatrix},$$

where for some $K > 0$, $|\psi_i(\zeta)| \leq K|\zeta|$, $i = 1, 2$ for $\zeta \in \tilde{\mathcal{N}}$.

Proof. The transformation $\Phi$ is the composition of the two successive symplectic coordinate transformations. For the first transformation, the following is taken

$$\tilde{x} = x - x_-, \quad \tilde{y} = y - w^s(x).$$

This transformation is symplectic, as

$$\varphi^* \omega = d\tilde{y} \wedge d\tilde{x} = (d\tilde{y} - (w^s)'(x) \, dx) \wedge dx = dy \wedge dx = \omega.$$

In $(\tilde{x}, \tilde{y})$ coordinates, the fixed point $z_-$ is given by $(\tilde{x}, \tilde{y}) = (0, 0)$, the stable manifold $W_0^s$ by the equation $\tilde{y} = 0$, and the unstable manifold $W_0^u$ by

$$\tilde{y} = w_1^u(\tilde{x}) = w^u(x_- + \tilde{x}) - w^s(x_- + \tilde{x}).$$
The function \( w_1^u \) defined by this equation satisfies

\[
(w_1^u)'(0) = (w^u)'(\tilde{x}) - (w^s)'(\tilde{x}) \neq 0.
\]

For the second transformation, take

\[
\xi = \tilde{x} - (w_1^u)^{-1}(\tilde{y}), \quad \eta = \tilde{y}.
\]

This transformation is well-defined on \( \mathcal{N} \) — possibly the neighbourhood has to be taken smaller to ensure the invertibility of \( w_1^u \) — it is symplectic, it preserves the location \( \eta = 0 \) of the stable manifold, and it maps the unstable manifold to \( \xi = 0 \). In the new coordinates the map \( \varphi \) has then the form given in the proposition.

Recall the inclination lemma or \( \lambda \)-lemma (see Palis and Takens, 1993, p. 155).

**Inclination lemma.** Let \( M \) be a \( n \)-dimensional manifold and let \( \varphi : M \to M \) be a \( C^k \) diffeomorphism, \( k \geq 1 \), with a hyperbolic fixed point \( z \). Let \( W \subset M \) be a \( C^k \) submanifold such that \( \dim(W) = \dim(W^s(z)) \), and such that \( W \) has a point \( p \) of transversal intersection with \( W^u(z) \).

Then for each \( t \), one can choose a disk \( D_t \subset \varphi^{-t}(W) \), which is a neighbourhood of \( \varphi^{-t}(p) \) in \( \varphi^{-t}(W) \), such that

\[
\lim_{t \to \infty} D_t = D,
\]

where \( D \) is a disk-neighbourhood of \( p \) in \( W^s(z) \). Convergence means here that for \( t \) sufficiently large \( D_t \) and \( D \) are \( C^k \)-near embedded disks.

**3.3.2 Estimating value differences using the area rule.** In this subsection, equation (3.6) is stated, derived, and an estimate of the term \( \text{area}(C_t) \)
is given. Moreover a variant of equation (3.6) needed to prove Theorem 3.2.1 is derived as well.

Take $\xi$ close to $x_-$ and consider the intersection of $W^+_s$ with

$$\ell_\xi = \{(x, y): x = \xi\}$$

as in Figure 3.6. The values at these intersection points can be compared using the area rule. To do this, the following definitions are made.

Let $p$ be an upward heteroclinic orbit. Consider the curve $c_1$ given by the part of $W^+_s$ connecting $p_0$ to $p_{-1}$, followed by the part of $W^-_u$ connecting $p_{-1}$ to $p_0$. Then $c_1$ is the boundary of a region $A$ with positively and negatively oriented components $A^+$ and $A^-$ respectively (see Figure 3.6).

![Figure 3.6: The regions $A^+$ and $A^-$ are respectively positively and negatively oriented.](image)

Let moreover $T > 0$ be such that if $t < -T$, then the part of $W^+_s$ con-
necting \( p_t \) to \( p_{t-1} \) intersects the line \( \ell_\xi \). Let \( q_t^0 \) be the first intersection of \( W_\xi^+ \) with \( \ell_{x_-} \) following \( p_t \) with respect to parametrisation \( z(s) \), that is, let \( q_t^0 \) be such that the segment of \( W_\xi^+ \) connecting \( p_t \) to \( q_t^0 \) has no other points in common with \( \ell_{x_-} \). Let moreover \( q_t(\xi) \) be a continuous function such that \( q_t(z_-) = q_t^0 \) and \( q_t(\xi) \in \ell_\xi \cap W_\xi^+ \). This situation is illustrated in Figure 3.7.

![Figure 3.7: The images of the regions \( A^+ \) and \( A^- \) under \( \varphi^t \), i.e. \( \varphi^t(A^+) \) and \( \varphi^t(A^-) \).](image)

For the sake of notational simplicity, write \( q_t = q_t(\xi) \) for \( \xi \) close to \( x_- \). Define \( \tilde{\ell}_\xi \) as the segment of \( \ell_\xi \) connecting \( q_{t-1} \) to \( q_t \). Then \( \varphi^{-t}\tilde{\ell}_\xi \) is a curve connecting \( \varphi^{-t}(q_{t-1}) \) to \( \varphi^{-t}(q_t) \), which are both located on \( W_\xi^+ \). Consider the curve \( c_2 \) given by the part of \( W_\xi^+ \) connecting \( \varphi^{-t}(q_t) \) to \( \varphi^{-t}(q_{t-1}) \), followed by the curve \( \varphi^{-t}\tilde{\ell}_\xi \). Then \( c_2 \) is the boundary of a region \( B_t \) with positively and negatively oriented components \( B_t^+ \) and \( B_t^- \) respectively.

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Define
\[ C_t^+ = A^+ - B_t^+, \quad C_t^- = A^- - B_t^- , \]
and
\[ C_t = A - B_t = C_t^+ + C_t^- . \]

See Figure 3.8.

**Proposition 3.3.5.** Let \( v \) be as in equation (3.5). If \( t < -T \), then
\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) - e^{\rho t} \Omega(\varphi^t(C_t)) . \]

In particular, if all simple components of \( C_t \) are positively oriented, then
\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) + e^{\rho t} \text{area}(\varphi^t(C_t)) , \]
whereas if all simple components of \( C_t \) are negatively oriented, then
\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) - e^{\rho t} \text{area}(\varphi^t(C_t)) . \]

**Proof.** Recall that \( B_t = A - C_t \). By the iterated area rule
\[ e^{-\rho t} \left( v(q_{t-1}) - v(q_t) \right) = \Omega(B_t) = \Omega(A) - \Omega(C_t) . \]
The result follows. \( \square \)

We now consider the case that \( \ell \) is the line \( x = x_- \). The following proposition states that in that case, all simple components of \( C_t \) are positively oriented (see Figure 3.8), and gives an estimate of \( \Omega(C_t) \).

**Proposition 3.3.6.** If \( \ell \) is the line \( x = x_- \), then all simple components of \( C_t^+ \) and \( C_t^- \) are positively oriented. Moreover, there are constants \( T_0 > 0 \) and \( K > 0 \) such that for all \( t < -T_0 \), the inequality
\[ -K \lambda_u^{2t} \leq e^{\rho t} \Omega(C_t) \leq 0 \]
holds.
Figure 3.8: The regions $B^+_t$ and $C^+_t$. The regions $B^+_t$ and $B^-_t$ are respectively positively and negatively oriented by definition. In the situation depicted in the upper figure, $C^+_t$ and $C^-_t$ are both positively oriented, whereas in the lower figure, only $C^-_t$ is negatively oriented, while $C^+_t$ has both a positively and a negatively oriented component.
Proof. Since \( \det D \varphi = e^\rho > 0 \), the phase map \( \varphi \) preserves orientation. Note that for \( T_0 > 0 \) sufficiently large, if \( t < -T_0 \), then the regions \( \varphi^t(C^\pm_t) \) are contained in the curvilinear triangle \( \tilde{\Delta} \) formed by the part \( W^u_{s,t} \) of \( W^u_s \) connecting \( p_t \) to \( q_t \), the part of \( \ell \) connecting \( q_t \) to \( z_- \), and the part of \( W^u \) connecting \( z_- \) to \( p_t \) (cf. Figure 3.7). By the \( \lambda \)-lemma, for large values of \(-t\) the curve segment \( W^u_{s,t} \) is \( C^1 \)-close to \( W^u_s \). Therefore the curvilinear triangle \( \tilde{\Delta} \) has the same orientation as the triangle \( \Delta \) introduced in Proposition 3.3.3. But that proposition states that \( \Delta \) is positively oriented.

Let \( m_0 \) and \( m_{-1} \) be curves through \( p_0 \) and \( p_{-1} \) respectively that intersect \( W^u \) transversally, and which are such that the region bounded by \( m_0, \varphi^{-t} \ell, m_{-1} \) and \( W^u \) contains \( C_t \). Moreover, in local coordinates introduced in Proposition 3.3.4, let

\[ p_t = (0, \eta_t). \]

By the \( \lambda \)-lemma, the iterates of the \( m_i \) have the property that the intersections \( \varphi^t m_i \cap N \) tend to \( W^u_s \cap N \) in the \( C^1 \)-norm as \( t \to -\infty \). That is, given \( \varepsilon > 0 \), there is a \( T > 0 \) such that for \( t < -T \) the intersections in local \((\xi, \eta)\)-coordinates, introduced in Proposition 3.3.4, take the form

\[ \varphi^t m_i \cap N : \eta = \chi^t_i(\xi), \quad i = 0, -1, ... \]

with \( \chi^t_0(0) = \eta_{t+i} \) and

\[ \max_{W^u_s \cap N} |(\chi^t_i)'(\xi)| < \varepsilon. \]

Still in local coordinates, the curve \( \ell \) is represented as the graph of the function

\[ \xi = \kappa(\eta) = -(w^u_1)^{-1}(\eta) \]

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with \( \kappa(0) = 0 \). Moreover, there is \( M > 0 \) with \( -M < \kappa'(0) < 0 \) for all \( \eta \) such that \( (\eta, \kappa(\eta)) \in \mathcal{N} \). The area \( R_t \) bounded by \( W_u, \varphi^t m_0 \) and \( \ell \) contains \( \varphi^t C_t \). Consequently

\[
e^{\rho t} \text{area} (C_t) = \text{area} (\varphi^t C_t) \leq \text{area} (R_t).
\]

The region \( R_t \) itself is contained in the triangle formed by the lines \( \xi = -M\eta, \eta = \eta_t - \varepsilon \xi \) and \( \xi = 0 \); it follows that

\[
\text{area} (R_t) \leq \frac{M}{2(1 - \varepsilon M)} \eta_t^2 = M' \eta_t^2.
\]

The fact that \( \eta_t = M\lambda_u^t + O(\lambda_u^{2t}) \), uniformly in \( t \), proves the proposition. \( \blacksquare \)

As noted in the sketch of the proof, Proposition 3.3.5 shows that if \( \Omega(A) > 0 \), then \( v(q_t) \) increases towards \( V^-(x_-) \) as \( t \to -\infty \). However, not all intersections of \( W_u^s \) with \( \varphi^{-t}\tilde{\ell}_t \) follow directly on an upward intersection of \( W_u^s \) with \( W_u^a \); there may be a configuration as the one depicted in Figure 3.9.

Define \( r_{t,1} = \varphi^{-t}(q_{t,1}), r_{t,2}, \ldots, r_{t,K_t}, r_{t,K_t+1} = \varphi^{-t}(q_{t-1}) \) as the consecutive positive intersections of \( W_u^s \) with \( \varphi^{-t}\tilde{\ell}_t \) that follow \( p_0 \). Set

\[
q_{t,i} = \varphi^i(r_{t,i}).
\]

Denote moreover by \( B_{t,1}^+, B_{t,2}^+ \) the components of the region \( B_t^+ \) that are such that for \( 1 \leq i \leq K_t \), the point \( r_{t,i} \) is contained in the boundary of \( B_{t,i}^+ \). Likewise, denote by \( C_{t,1}^-, C_{t,2}^- \) the components of \( C_t^- \) that are such that for \( 1 \leq i \leq K \) the point \( r_{t,i+1} \) is contained in \( C_{t,i}^- \).

**Proposition 3.3.7.** Let \( 1 \leq k \leq K_t \) be such that \( r_{t,k} = \varphi^{-t}(q_{t,k}) \) follows \( p_0 \), but precedes any other upward intersection of \( W_u^s \) with \( W_u^a \). Then

\[
v(q_{t-1}) - v(q_{t,k}) \geq \Omega(\varphi^t(A)).
\]
Proof. The area rule implies that

\[ v(q_{t,k}) - v(q_{t,1}) = \sum_{i=1}^{k-1} \left\{ \Omega \left( \varphi^t(B_{t,i}^+) \right) + \Omega \left( \varphi^t(C_{t,i}^-) \right) \right\} \]

\[ = \sum_{i=1}^{k-1} \left\{ - \text{area} \left( \varphi^t(B_{t,i}^+ \right) + \text{area} \left( \varphi^t(C_{t,i}^-) \right) \right\}. \]

Using Proposition 3.3.5 and the equality

\[ v(q_{t-1}) - v(q_{t,k}) = v(q_{t-1}) - v(q_t) + v(q_t) - v(q_{t,k}) \]

\[ = v(q_{t-1}) - v(q_t) + \sum_{i=1}^{k-1} \left\{ \text{area} \left( \varphi^t(B_{t,i}^+) \right) - \text{area} \left( \varphi^t(C_{t,i}^-) \right) \right\}. \]
then yields that
\[ v(q_{t-1}) - v(q_{t,k}) = \Omega(\varphi(A)) - \Omega(\varphi(C_t)) \]
\[ - \sum_{i=1}^{k-1} \text{area}(\varphi(C_{t,i}^-)) + \sum_{i=1}^{k-1} \text{area}(\varphi(B_{t,i}^+)) \]  
As for \( \ell = \ell_x \) all regions \( C_{t,i}^- \) and \( C_{t,i}^+ \) are positively oriented,
\[ v(q_{t-1}) - v(q_{t,k}) = \Omega(\varphi(A)) + \sum_{i=1}^{k-1} \text{area}(\varphi(B_{t,i}^+)) \]
\[ + \text{area}(\varphi(C_t)) - \sum_{i=1}^{k-1} \text{area}(\varphi(C_{t,i}^-)) \]
\[ \geq \Omega(\varphi(A)) \]
as \( \bigcup_{i=1}^{k-1} C_{t,i}^- \subset C_t \). The result follows.

3.3.3 Proof of Theorem 3.2.1.

**Proof.** The first part of the Proposition is immediate: if \( \mu > \mu_2 \), then by Assumption 3.1.4, there are open neighbourhoods \( \mathcal{N}_-, \mathcal{N}_+ \) of \( x_- \) and \( x_+ \) respectively, such that \( W^s_+ \cap \mathcal{N}_- \times \mathbb{R} = \emptyset \) and \( W^u_- \cap \mathcal{N}_+ \times \mathbb{R} = \emptyset \). But by Assumption 3.1.3, optimal solutions correspond to trajectories on either \( W^s_+ \) or \( W^u_- \). It follows that all optimal state trajectories starting in \( \mathcal{N}_- \) tend to \( x_- \), and those starting in \( \mathcal{N}_+ \) tend to \( x_+ \).

To prove the second part of the proposition, let as before \( p \) be an upward heteroclinic intersection of \( W^s_+ \) and \( W^u_- \) such that \( p_0 \) and \( A \) be as stated before in this chapter. Let moreover \( \ell \) be the line \( x = x_- \) and as stated in Proposition 3.3.7 let
\[ q_t,i = \varphi(\ell(r_{t,i})), \quad i = 1, \ldots, K_t \]
be the positive intersections of $W^s_+$ with $\ell$ that follow $p_t$ and that precede the next upward intersection of $W^s_+$ and $W^u_-$. Set $q_t = q_{t,1}$.

Using $\Omega(A) \geq 0$ together with Proposition 3.3.5, it is obtained that

$$v(q_t-1) - v(q_t) > 0.$$  

and therefore $v(q_t)$ is an increasing sequence. Since $q_t \to z_-$, it follows from Proposition 6.0.1 that $v(q_t) \to V_-(x_-)$. We conclude that

$$\cdots < v(q_t) < v(q_{t-1}) < v(q_{t-2}) < \cdots < V_-(x_-).$$

Moreover, from Proposition 3.3.7, it follows that for $1 \leq i \leq K_t$, then

$$v(q_{t,i}) < v(q_{t-1}).$$

It is immediate that the remaining intersections of $W^s_+$ and $\ell$ yield even smaller values. But then no orbit on $W^s_+$ yields a value that is as high as $V_-(x_-)$, and the proposition follows.

3.3.4 Proof of Theorem 3.2.2.

Proof. If $\mu < \mu_1$, take $\alpha = z_-$ and $\beta \in W^s_+$ such that $x_\beta = x_-$. Let $E$ be the region bounded by the segment $\bar{\ell}$ of the straight line connecting $\alpha$ to $\beta$, the segment $\Sigma$ of $W^s_+$ connecting $\beta$ to $\varphi(\beta)$ and the image $\varphi(\bar{\ell})$ of $\bar{\ell}$. As $\Sigma$ does not intersect $W^u_-$ or $W^u_-$, it follows that $E$ is negatively oriented (see Figure 3.10). Proposition 2.3.5 then implies that

$$V_+(x_-) > V_-(x_-).$$

But then $x_-$ cannot even be a locally optimal fixed point.
Figure 3.10: Region $E$ is negatively oriented, $\Omega(E) > 0$.

The second part of Theorem 3.2.2, the case that $\mu_s > \mu > \mu_1$ and $\Omega(A) < 0$, follows from Proposition 6.0.1 in Appendix A and Proposition 3.3.5, as a sequence of points $q_t \in W_s^+ \cap \ell x_-$ is found which is such that $q_t \to z_-$ as $t \to -\infty$, implying $v(q_t) \to V_-(x_-)$, and which satisfies for all $t < -T_0$ the inequalities

$$a_t = e^{\rho t} \Omega(A) - C'' \lambda^2 \lambda u \leq v(q_{t-1}) - v(q_t) \leq e^{\rho t} \Omega(A) + C'' \lambda^2 \lambda u = b_t.$$ 

Using $\Omega(A) < 0$ and the fact that $e^\rho = \det D\varphi(z_-) = \lambda u \lambda s$, it follows that

$$e^{-\rho t} b_t = \Omega(A) + C'' \left( \frac{\lambda u}{\lambda s} \right)^t,$$

and there is some $T' > 0$ such that $b_t < 0$ for all $t < -T'$. But then the sequence $v(q_t)$ is eventually decreasing as $t \to -\infty$. Therefore there
is some $t_0$ such that $v(q_{t_0}) > V_-(x_-)$, and the state trajectory remaining
at $x = x_-$ cannot be optimal.

This implies that the optimal solution starting at $x = x_-$ converges to $x = x_+$. Consequently, no solution on $W^s_-$ can be optimal, and therefore, by
Assumption 3.1.3, every optimal solution converges to $x = x_+$. \hfill \Box

3.3.5 Proof of Theorem 3.2.3.

Proof. The optimality of the trajectory $x_t = x_-$ follows from Theorem 3.2.1.

Fix a small neighbourhood of $x_-$, and take $\xi$ in this neighbourhood such
that $\xi > x_-$. Let $\ell_\xi$ be the vertical line $x = \xi$, and denote by $q_t(\xi)$ the
intersection of $W^s_+$ with $\ell_\xi$ defined in Subsection 3.3.2. Moreover, let $t_0 =
t_0(\xi)$ be such that for $t \geq t_0$, the curve segment from $p_t$ to $q_t$ is oriented in
the same way as $W^s_+$, while for $t < t_0$ that curve segment is oriented the
opposite direction.

This implies that orientation of $C_t$ is positive for $t > t_0$, while it is negative
for $t < t_0$. Since by Assumption $\Omega(A) = 0$, Proposition 3.3.5 implies for
all $t > t_0$ that

$$v(q_{t-1}(\xi)) > v(q_t(\xi)),$$

while for $t < t_0$,

$$v(q_{t-1}(\xi)) < v(q_t(\xi)).$$

As $v(q_t(\xi)) \to V_-(\xi)$ as $t \to -\infty$, it follows that

$$v(q_{t_0}(\xi)) > V_-(\xi),$$

and consequently that the optimal state trajectory starting at $\xi$ will tend
to $x_+$. 78
Given a value $\tau$ of $t_0(\xi)$, there are $\xi_1^\tau$ and $\xi_2^\tau$ such that

\[ t_0(\xi) = \tau \text{ if } \xi_1^\tau < \xi < \xi_2^\tau \]

and that

\[ q_{r-1}(\xi_1^\tau) = p_{r-1} \text{ and } q_r(\xi_2^\tau) = p_r. \]

The area rule as illustrated in Figure 3.12 implies that

\[ v(q_{r-1}(\xi_1^\tau)) - v(q_r(\xi_1^\tau)) < 0 \]  \hspace{1cm} (3.8)

and

\[ v(q_{r-1}(\xi_2^\tau)) - v(q_r(\xi_2^\tau)) > 0. \]  \hspace{1cm} (3.9)

Recall that for $t > \tau$

\[ v(q_{t-1}(\xi)) - v(q_t(\xi)) > 0 \]  \hspace{1cm} (3.10)
and that for \( t < \tau \)
\[
v(q_{t-1}(\xi)) - v(q_t(\xi)) < 0. \tag{3.11}
\]

If \( \xi = \xi_1^\tau \), using equations (3.8), (3.10) and (3.11) it is obtained that
\[
\cdots < v(q_{t-2}) < v(q_{t-1}) < v(q_t) > v(q_{t+1}) > \cdots.
\]

Moreover, if \( \xi = \xi_2^\tau \), equations (3.9), (3.10) and (3.11) imply that
\[
\cdots < v(q_{t-2}) < v(q_{t-1}) > v(q_t) > v(q_{t+1}) > \cdots.
\]

That is, if \( \xi = \xi_1^\tau \) then \( v(q_t(\xi)) \) is maximal, whereas if \( \xi = \xi_2^\tau \), then \( v(q_{t-1}(\xi)) \) is maximal. Consequently there is \( \xi_1^\tau < \xi_2^\tau < \xi_3^\tau \) such that
\[
v(q_t(\xi_3^\tau)) = v(q_{t-1}(\xi_2^\tau)).
\]

For the final claim of the theorem, note that \( t_0 = t_0(\xi) \) decreases towards \( -\infty \) as \( \xi \to x_\cdot \).