Bifurcations of indifference points in discrete time optimal control problems

Moghayer, S.M.

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 5

Application to the lake system

In Mäler et al. (2003) the economics of a lake pollution problem is analysed as an optimal management problem, and in the situation that the lake is a common property resource, as a differential game. The social planner, or in the game context, the players, weigh the conflicting interests of farmers, who indirectly pollute the lake through agricultural activities, and lake users, who are affected as ecosystem services of the lake decrease.

Initially most lakes are in a clear water state. However, due to heavy use of fertilisers by farmers, at some point lakes flip from a clear state to a turbid state that is caused by a dominance of phytoplankton (Carpenter and Cottingham 1997; Scheffer 1997). Lakes are hard to restore to the clear water state in the sense that the nutrient loads have to be reduced far below the level where the flip occurred before the lake returns to a clear state.

When the lake flips to a turbid state, the value of the ecological services of the lake decreases, but there is a high level of agricultural activities. It
depends, of course, on the relative weight attached to these welfare components whether it is better to keep the lake clear or to use it as a waste dump. The complexity of the lake optimal management problem derives from the non-linear dynamics of the lake that leads to a non-convex optimal control problem featuring several system parameters. In such problems, depending on the values of these parameters, there may exist multiple steady states that are the long-run outcome of an optimal management policy. Also, the structure of optimal solutions may change if parameters are varied. The bifurcation analysis developed in Chapter 3 is an appropriate tool to classify the qualitative characteristics of the set of optimal solutions for different values of the model parameters.

5.1 Lake dynamics

Nature is often expected to respond to gradual changes in a smooth way. However, studies of lakes, coral reefs, oceans, forests and arid lands have shown that smooth change can be interrupted by sharp (or catastrophic) shifts to different regimes (Scheffer et al. 2001; Carpenter 2003). One of the best-studied catastrophic shifts is the sudden loss of transparency and vegetation observed in shallow lakes, i.e. lakes with a depth less that 3 meters, as a result of human activities. Initially shallow lakes have clear water and a rich submerged vegetation. However, nutrient loading may change this. For instance nutrients arrive in the lake as a result of the use of artificial fertilisers on surrounding land; they are washed into the lake by rainfall.

Initially water clarity in the lake seems to be hardly affected by the increased amount of nutrients until a critical threshold is reached, at which the lake shifts abruptly from a clear state to a turbid state characterised by
a dominance of phytoplankton. With this increase in turbidity, submerged plants largely disappear, releasing the lake sediment, which causes resuspension of nutrients in the lake. Now, even an instant reduction in nutrient loading is not enough for water plants to regrow easily, as the turbidity of the lake remains high. Longer periods of reduction or an alternative treatment is needed to reduce the nutrient load and hence the turbidity in the lake sufficiently, as the load has to be reduced far below the level where the flip occurred before the lake can flip back to a clear state. The lake is said to show hysteresis. In some cases the turbidity of the lake is even irreversible.

The lake model that is used in the following gives a very simplified representation of these complex ecological feedback mechanisms that are active in a shallow lake. The lake model can also be viewed as a metaphor for general ecological systems with tipping points, so that the analysis developed here will have a wider applicability (cf. Scheffer et al. (2001)).

5.1.1 The lake model. The dynamics of a shallow lake, which was described above, can be modelled as a single non-linear difference equation

\[ x_t = u_t + (1 - b)x_{t-1} + h(x_{t-1}) \]  \hspace{1cm} (5.1)

Here \( x_t \) is the concentration of phosphorus, one of the main nutrients, in the lake. Artificial fertilisers containing phosphorus are used on the fields surrounding the lake. The phosphorus is washed into the lake by rainfall, yields a net inflow \( u_t \) of phosphorus. The parameter \( b \) denotes the sedimentation rate at which phosphorus leaves the water column and enters the sediment at the bottom of the lake. The last term models the internal production of phosphorus in the lake, e.g. through resuspension of the sediment, and is
assumed to be an S-shape function that has its inflection point at the point $x = 1$:

$$h(x) = \frac{x^q}{1 + x^q}.$$  

The exponent $q$, the responsiveness of the lake, is proportional to the steepness of $h$ at $x = 1$. 

For a constant pollution loading $u_t = u$ for all $t$, the fixed points of the lake are solutions of the equation

$$u = g(x) = bx - \frac{x^q}{1 + x^q},$$

which is illustrated for $b = 0.6$, and $q = 2$ and $q = 4$ in Figure 5.1.

![Graph](image)

(a) Weakly responsive lake.  
(b) Strongly responsive lake.

**Figure 5.1:** Location of fixed points for constant pollution streams $u_t = u$ for all $t$, plotted for $b = 0.6$, and for (a) weakly ($q = 2$) and (b) strongly responsive lakes ($q = 4$). Indicated are stable (solid) and unstable fixed points (dashed).

For both values of $q$ there is a range of $u$-values such that there are multiple steady states. However, the range is bigger for $q = 4$ than for $q = 2$. If the system starts in a low pollution steady state, and if $u$ is then raised very slowly past the first critical value (i.e. the local maximum of $g$) it switches
to a high pollution steady state. A small subsequent decrement of $u$ will not move the system back to the clean branch of steady states. For this, the pollution flow has to be lowered significantly, below the second critical value (the local minimum of $g$).

There is a value $b = b^*$ such that for $b < b^*$ the lake can be trapped in the high pollution steady state of phosphorus. This happens if the first flip, which occurs at $u = \bar{u}$, is irreversible. The critical value is $b^* = \sqrt[4]{27}/4 \approx 0.57$ for $q = 4$ and $b^* = 0.5$ for $q = 2$ (see Figure 5.2 for the case $q = 4$).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.2}
\caption{Irreversibility; location of fixed points for constant pollution streams $u_t = u$ for all $t$, plotted for $b \approx 0.57$ and $q = 4$. Indicated are stable (solid) and unstable fixed points (dashed).}
\end{figure}

5.1.2 Optimal pollution management in lakes. In the lake pollution management problem, a social manager has to weigh the interest of the farmers that derive income from the use of artificial fertilisers against that of the lake users that suffer from pollution damage to the lake. Following
Mäler et al. (2003), the social utility functional is modelled as

\[ J = \sum_{t=1}^{\infty} \left( \log u_t - cx_{t-1}^2 \right) e^{-\rho t}. \]

Here \( c \) is the social preference parameter, and \( \rho > 0 \) the discount rate.

The social manager tries to optimally manage the phosphorus pollution stream

\[ u = \{u_t\}_{t=1}^{\infty} \]

that originates from the use of artificial fertilisers given that the concentration \( x_t \) of phosphorus in the lake follows the lake dynamic (5.1). The optimisation problem is to maximise

\[ J = \sum_{t=1}^{\infty} \left( \log u_t - cx_{t-1}^2 \right) e^{-\rho t}, \quad (5.2) \]

subject to

\[ x_t = u_t + (1 - b)x_{t-1} + \frac{x_{t-1}^q}{1 + x_{t-1}^q}. \quad (5.3) \]

State space and control space are given as \( \mathcal{X} = \mathcal{U} = (0, \infty) \) respectively.

The discrete Pontryagin function is

\[ P = \log u - cx^2 + y \left( u + (1 - b)x + \frac{x^q}{1 + x^q} \right). \]

Note that \( P_{uu} < 0 \) for all \( u > 0 \). The necessary condition \( P_u = 0 \) takes the form

\[ 0 = P_u = \frac{1}{u} + y. \]

Solving for \( u \) yields that

\[ u = U(x, y) = -1/y. \]
Substituting out $u$, the discrete Hamilton function is obtained as

$$H = -\log(-y) - cx^2 - 1 + y \left( (1 - b)x + \frac{x^q}{1 + x^q} \right).$$

Since $H_{yy} = y^{-2} > 0$ and

$$H_{xy} = 1 - b + q \frac{x^{q-1}}{(1 + x^q)^2} > 0,$$

Assumption 3.1.1 is satisfied.

The necessary conditions read as

$$x_t = H_y = -\frac{1}{y_t} + (1 - b)x_{t-1} + \frac{x_{t-1}^q}{1 + x_{t-1}^q},$$

$$e^\rho y_{t-1} = H_x = -2cx_{t-1} + y_t \left( (1 - b) + q \frac{x_{t-1}^{q-1}}{(1 + x_{t-1}^q)^2} \right).$$

Solving the second equation for $y_t$ and substituting into the first yields the phase map

$$\varphi(x, y) = \left( -\frac{(1 - b) + q \frac{x^{q-1}}{1 + x^q}}{e^\rho y + 2cx} + (1 - b)x + \frac{x^q}{1 + x^q} \frac{e^\rho y + 2cx}{(1 - b) + q \frac{x^{q-1}}{(1 + x^q)^2}} \right).$$

Using

$$g(x) = (1 - b)x + x^q/(1 + x^q),$$

this expression can be written as

$$\varphi(x, y) = \left( -\frac{g'(x)}{e^\rho y + 2cx} + g(x), \frac{e^\rho y + 2cx}{g'(x)} \right).$$

### 5.2 Indifference-attractor bifurcations

In the rest of the chapter, the value of $\rho$ is fixed to $\rho = 0.03$. For $b = 0.6$ and $q = 4$, in Figure 5.3 fixed points and their stable and unstable manifolds are plotted for a range of values of $c$; for all these values, the phase map has two saddle fixed points $z_-$ and $z_+$. 

105
Figure 5.3: Subfigures (a) and (b) show affecter-friendly configurations (low values of $c$), and subfigure (c) depicts an enjoyer-friendly configuration (high value of $c$). Solid lines indicate stable manifolds, dotted lines unstable manifolds; optimal solutions are marked by thick lines. Note that $y < 0$ throughout, so that the $x$-axis is at the top of the figure. On the $x$-axis, the optimal dynamics are indicated; attractors are marked by a circle, the indifference threshold by a diamond.
It can be shown that Assumption 3.1.3 is satisfied. Taking $\mu = c$ and accepting the geometric evidence from the plots in Figure 5.3, the intermediate value theorem implies that Assumption 3.1.4 is satisfied as well. At least at $c = 0.1541$, geometric evidence also supports Assumptions 3.1.5 and 3.1.6. Granting the assumptions, Theorem 3.1.1 applies.

It should be noted that for $c = 0.14$, that is, in a situation where the interest of farmers weighs relatively heavily, it is for every initial state $x_0 \in \mathcal{X}$ optimal to steer the lake to the high pollution state $x_+$. There is a single upward orbit $p$ such that the value $c = c_\text{IA} \approx 0.1541$ corresponds to the case $\Omega(A_p) = 0$ (cf. Section 3.2); it follows from the Main Theorem that then for $x_0 \leq x_-$ the optimal policy steers the lake to the low pollution state $x_-$, while if $x_0 > x_-$, it is optimal to end at the high pollution state $x_+$.

Moreover, Theorem 3.1.1 implies that for $c = c_\text{IA}$ there is a countable infinity of indifference states. Recall that indifference states are initial states to two distinct optimal policies. At these points the policy function jumps; two of these jumps can be seen in Figure 5.4.

Finally, for $c = 0.17$, both $x_-$ and $x_+$ are locally optimal, and their basins of attraction are separated by an indifference threshold.

At first, the shape of the optimal costate rule shown in Figure 5.4, which is related to the shape of the optimal policy function by the transformation

$$u = -1/y,$$

is completely counter-intuitive, as this function is varying wildly over a relatively small range of state values. However, it should be born in mind that the saddle manifold of $z_+$, of which the graph of the optimal costate function is
Figure 5.4: For \( c = 0.1541 \), the equality \( \Omega(A) = 0 \) holds, and consequently there is an infinity of indifference points.

a part, consists of many orbits. Consider for instance the rightmost indifference point in Figure 5.4. At that point, two different policies are equivalent, one corresponding to relatively high agricultural activity, high pollution and fast convergence to the steady state \( z_+ \), the other characterised by lower pollution and slower convergence to \( z_+ \). Though the asymptotic steady state is equal in both cases, the policies that reach it are vastly different.

5.2.1 The bifurcation diagram. Setting \( q = 4 \) the indifference-attractor bifurcation diagram of \( \varphi \) in Figure 5.5 is plotted. Furthermore, the bifurcation diagrams corresponding to \( q = 2 \) and \( q = 4 \) in Figure 5.6 are depicted and the effect of the change of the responsiveness of the lake on the optimal solution is studied.
Figure 5.5: Bifurcation diagram of the highly responsive lake ($q = 4$).
Figure 5.5(d) shows the bifurcation diagram of the lake system in the 
$(b,c)$-parameter space for $q = 4$ and $\rho = 0.03$. The dashed curve represents 
saddle-node bifurcations, separating the region of values of the parameters for 
which the phase map has a fixed point from the region of multiple fixed points. 
Solid lines indicate indifference-attractor bifurcation curves, separating three 
parameter regions: (i) the low pollution region for which the clean steady 
state is globally optimal, (ii) the high pollution region for which the turbid 
steady state is globally optimal, and (iii) the dependent on the initial state 
region for which both the clean steady state and turbid steady state are 
locally optimal.

Phase portraits of the phase map $\varphi$ and of the optimal map $\varphi^o$ are given 
for $b = 0.7$ in Figure 5.5(b) and $b = 0.6$ for (c) – (f) and for $b = 0.7$ in Figures 
5.5 (a) and (b). The graphs of the optimal costate maps are represented by 
thick curves; these graphs are subsets of either $W^{s}$ or $W^{u}$. The other parts of 
the stable manifolds are indicated by solid curves, and the unstable manifolds 
are given as dotted curves. Note that $y < 0$ throughout, so that the $x$-axis is 
at the top of the figure. On the $x$-axis, the optimal dynamics are indicated; 
attractors are marked by a circle, indifference thresholds by diamonds.

For the values of the parameters $b$ and $c$ in the unique equilibrium region 
the phase map $\varphi$ has a unique fixed point. This is a saddle, see Figure 
5.5(a). The optimal orbits are always situated on its stable manifold. The 
long run pollution level depends then on the values of the parameters $c$ and 
$b$, changing within the region.

If the pair $(b, c)$ corresponds to a point of the dependent on the initial state 
region, the phase map $\varphi$ has always two saddle fixed points characterised by
respectively low pollution and high pollution. The clear state of the lake corresponds to a high level of water services and a low level of agricultural activities, whereas the turbid state corresponds to a high level of agricultural activities and a low level of water services. Depending on the initial pollution load, the social planner steers the lake to the clear or to the turbid steady state.

If the pair \((b,c)\) is in the low pollution region the optimal policy steers the lake to the clean steady state independently of the initial state of the lake; the clear state of the lake is globally optimal (see Figure 5.5(a)). Note that the optimal policy imposes very small values of \(u\) for intermediate values of \(x\), until the clear steady state is reached, in order to get rid of the turbidity (Figure 5.5(a)).

If \((b,c)\) is in the high pollution region, see Figure 5.5(e) and (f), the optimal orbit lies on the stable manifold of the polluted equilibrium \(W^sz_+\). Regardless of the initial state of the lake, the optimal policy steers the lake to the turbid state, that is the turbid steady state is globally optimal. Note in Figure 5.5(e) again the equivalence of different policies both leading to \(z_+\). In Figure 5.5(f) the optimal policy prescribes lower values of \(u\) for intermediate values of \(x\), in order to keep the pollution level low as long as is optimally possible.

For a pair \((b,c)\) in the dependent on the initial state region, there exist an indifference threshold, see Figure 5.5(c). If the initial state is below the threshold then the clean steady state is optimal, whereas if the initial state is above the threshold then the turbid steady state is optimal. An indifference threshold separates two basins of attraction of the optimal state map: the
states below the threshold form the basin of attraction of the clear state, and the states above that threshold constitute the basin of attraction of the turbid state.

Therefore, for a pair \((b, c)\) in the dependent on the initial state region the lake is steered to the clear state only if it is initially not very polluted, otherwise it is steered to the turbid state. Note that at the indifference threshold, two different policies are radically different and non-equivalent, one corresponding to high agricultural activity, high pollution and convergence to the steady state \(z_+\), whereas the other is characterised by lower pollution and convergence to \(z_-\).

### 5.2.2 Effects of the responsiveness of the lake.

Recall that the responsiveness of a lake \(q\) is given as the maximum steepness of the function \(h\). Note moreover that the lake system (5.1) can have multiple stable states only if

\[
\max h'(x) > b.
\]

Thus, higher values of \(q\) corresponds to more sudden shifts between the clean and the turbid regimes. This means that the regime shift will be more pronounced in a lake with a higher responsiveness, whereas this shift will be more gradual in a lake with relatively low responsiveness. To illustrate this in Figure 5.6 two bifurcation diagrams in the \((b, c)\)-parameter space are given: for a weakly responsive lake in Figure 5.6(a), and for a strongly responsive lake in Figure 5.6(b).
(a) Weakly responsive lake. (b) Strongly responsive lake.

Figure 5.6: Figure 6(a) and 6(b) show the bifurcation diagram of the discrete time lake system in the \((b, c)\)-parameter space for \(q = 2\) and \(q = 4\) respectively. The dashed curve represents saddle-node bifurcations of the state-costate system, separating the region of values of the parameters for which the phase map has a fixed point from the region of multiple fixed points. Solid lines indicate indifference-attractor bifurcation curves, separating four regions of values of the parameters: (i) the low pollution region for which the clean steady state is globally optimal, (ii) the high pollution region for which the turbid steady state is globally optimal, (iii) the dependent on the initial state region for which both the clean steady state and turbid steady state are locally optimal, and (iv) the unique equilibrium region.

The main difference between these diagrams is that the high pollution region is much smaller for the strongly responsive lake, whereas the low pollution region is much larger. Put differently, for a given value of the physical parameter \(b\) the minimal value of the economic weight \(c\) of the lake for which it is always optimal to steer the lake to the clean steady state is much lower in the strongly responsive lake. This was to be expected, as the
impact of regime shift towards the turbid regime would be felt much earlier in time in the strongly responsive case, hence it is optimal to avoid such a shift for a much larger range of values of $c$. 