Bifurcations of indifference points in discrete time optimal control problems

Moghayer, S.M.

Chapter 6

Summary

In this thesis a class of autonomous discrete time infinite horizon optimal control problems with non-convex state dynamics is studied. Methodologically this thesis contributes to the geometrical or phase space analysis of discrete time optimal control problems. Contrary to the situation in the theory of continuous time problems, these methods are not particularly popular in the discrete setting. The main reason for this seems to be that some powerful instruments of the continuous time theory are not readily available.

In Chapter 2 the necessary first order conditions of this class of optimal control problems are formulated in terms of a boundary value problem of the associated phase map φ. If this map possesses a saddle point, then all orbits on its stable manifold will solve the boundary value problem. In the situation that there are several saddle points whose stable manifold cover overlapping parts of the state space of the problem, or in the situation that there are several points on a single stable manifold that project down to the same point in state space, the values of the extremal orbits originating at these points have to be compared. The chapter provides a number of results that allow to make such comparisons.

Systems where the phase map φ has a unique saddle point are encountered in many places in the economic literature (e.g. Ramsey (1928)). Usually in these systems the saddle corresponds to an optimal steady state which is such that all solutions, regardless of the initial state of the system, tend towards it. Whenever there is more than one saddle point present (c.f. Dechert and Nishimura (1983)), or when there is an additional optimal trajectory tending towards infinity (c.f. Hinloopen et al. (2011)), the solution structure is more complicated. If, in the two-saddle case, both correspond to an optimal steady state, there is an indifference threshold (Skiba state) that is initial state to two different optimal trajectories.

In analogous continuous time problems, whether or not a Skiba state occurs in a system depends on the relative position of the stable and unstable manifolds of the saddle equilibria of the phase flow. In particular, the shift from one type of solution to another is characterised by occurrence of a heteroclinic connection, where the stable manifold of one saddle equilibrium coincides with the unstable manifold of the other saddle. In Chapter 3 it has been shown that in discrete time problems the situation is analogous but more complex, due to the fact that, unlike in continuous time, stable and unstable manifolds do not automatically coincide once they have a single point in common. More precisely, the genesis of indifference points in a so-called indifference-attractor bifurcation is linked to heteroclinic bifurcations of the family of phase maps φ, and the consequences for the optimal solutions are analysed. In particular, the bifurcation value at which an indifference threshold appears is characterised by a geometric condition, and it is found that at the bifurcation there are countably infinitely many indifference points that are not indifference thresholds.

In most applications it is impossible to determine analytic expressions for invariant manifolds, so numerical methods are needed: these are discussed in Chapter 4. A simple algorithm is described to compute invariant manifolds numerically. This information is used to determine the locus of the indifference-attractor bifurcation points.

The results and methods developed in this thesis are applied to the lake pollution management problem of Mäler, Xepapadeas and de Zeeuw (Mäler et al. (2003)). In ecological systems such as lakes, internal positive feedbacks may trigger catastrophic shifts. In the lake problem the economics of lake pollution is analysed in terms of tradeoffs between the benefits of agricultural activities, which are responsible for the release of nutrients, and the costs of a polluted lake.

In Chapter 5 a bifurcation analysis of the lake model is performed. The resulting bifurcation diagram summarises the joint effect of the (physical) robustness of the lake and the (economic) importance of the lake on the form of the optimal policy. The diagram is partitioned into four parameter regions: *unique steady state*, *low pollution*, *high pollution*, and *dependent on the initial state*. In the first region, there is a single fixed point of the phase map that corresponds to a globally attracting steady state of the optimal state dynamics. In the other regions, the phase map has two saddle fixed points that can be distinguished as corresponding to a clean or a polluted steady state. The regions correspond to the situations that either the clean steady state or the polluted steady state are globally attracting, or both are locally attracting under the optimal state dynamics. These parameter regions are separated by indifference-attractor and saddle-node bifurcation curves.

In the *low pollution* region, characterised by high robustness and great economic importance of the lake, the optimal policy always steers the lake to the clean steady state independently of the initial state. In the *high pollution* region – low economic importance of the lake – the polluted steady state is eventually reached under optimal management, irrespective of the initial state. For lakes in the *dependent on the initial state* region – lakes that are fragile and of medium to high economic importance fall in this category – the outcome of optimal management is dependent on the initial state: if it is sufficiently low, the clean steady state is reached, otherwise the polluted state results. The two regions in state space are separated by an indifference point.

In Chapter 5 also the 'stiffness' or responsiveness of the lake is varied. A strongly responsive lake exhibits more sudden shifts between clean and polluted regimes. It is found that for a strongly responsive lake the *high pollution* region is much smaller compared to a weakly responsive lake, whereas the *low pollution* region is much larger. That is, it is optimal to avoid regime shift towards the polluted regime if the lake is less economically important, but strongly responsive. Hence, in the pollution management of strongly responsive ecosystems, it is more likely that the optimal policy is 'green'.