Appendix

Differential forms

Recall from the theory of differential forms the following facts (taken from Spivak, 1965). As the thesis is concerned with 1-forms and 2-forms, the following results are restricted to that case.

A differential 0-form $f$ on $\mathbb{R}^n$ is a real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. A differential 1-form $\eta$ on $\mathbb{R}^n$ is a linear form

$$\eta = \eta_x = \sum_{i=1}^{n} \eta_i(x) \, dx_i,$$

on $\mathbb{R}^n$ with coefficients $\eta_i(x)$ that are differentiable functions; that is, if $v \in \mathbb{R}^n$, then

$$\eta_x(v) = \sum_{i=1}^{n} \eta_i(x)v_i.$$ 

A differential 2-form

$$w = \sum_{i<j} w_{ij} \, dx_i \wedge dx_j$$

is an anti-symmetric bilinear form:

$$\omega_x(v, w) = \sum_{i<j} \omega_{ij}(x)(v_i w_j - v_j w_i)$$
The derivative of a 0-form $f$ is given as $df = \sum_i \frac{\partial f}{\partial x_i} \, dx_i$; the derivative of a 1-form $\eta = \sum_i \eta_i \, dx_i$ is given as

$$d\eta = \sum_{i<j} \left( \frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j} \right) \, dx_i \wedge dx_j.$$  

Second derivatives are always zero: $d^2 \omega = 0$ for any $k$-form $\omega$.

If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a differentiable map, the pull-back $\varphi^* \eta$ of a 1-form $\eta$ is given as

$$(\varphi^* \eta)_x(v) = \eta_{\varphi(x)}(D\varphi(x)v) = \sum_i \eta_i(\varphi(x))(D\varphi(x)v)_i.$$  

This can be written more suggestively as

$$\varphi^* \eta = \sum_i \eta_i(\varphi(x)) \, d\varphi_i(x)$$  

Likewise, the pull-back $\varphi^* \omega$ of a 2-form takes the form

$$(\varphi^* \omega)_x(v, w) = \omega_{\varphi(x)}(D\varphi(x)v, D\varphi(x)w).$$  

A singular $k$-cube $c$ in $\mathcal{C}$ is a continuous map $c : [0, 1]^k \to \mathcal{C}$; a singular $k$-chain is a formal finite sum of singular $k$-cubes with integer coefficients. The standard $k$-cube is $I^k : [0, 1]^k \to \mathbb{R}^k$ given by $I^k(x) = x$. The definition of boundary is given only for 2-chains. Define the edges of $I^2$ as the singular 1-cubes $I^2_{(i,0)}$ and $I^2_{(i,1)}$: $i = 1, 2$ for which

$$I^2_{(1,0)}(x) = I^2(x, 0), \quad I^2_{(1,1)}(x) = I^2(x, 1),$$  

$$I^2_{(2,0)}(x) = I^2(0, x), \quad I^2_{(2,1)}(x) = I^2(1, x).$$  

Define the boundary as the singular 1-chain

$$\partial I^2 = I^2_{(1,0)} - I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}.$$  

120
For a general singular 2-cube \( c \) define first the edges \( c(i,j) \), \( i = 1, 2 \), \( j = 0, 1 \) by setting
\[
c(i,j) = c \circ I(i,j);
\]
then define
\[
\partial c = c(1,0) - c(1,1) + c(2,0) - c(2,1).
\]
The integral of a differential \( k \)-form \( \omega = f(x) \, dx_1 \wedge \cdots \wedge dx_k \) on \([0,1]^k\) is given as
\[
\int_{[0,1]^k} \omega = \int_{[0,1]^k} f(x) \, dx_1 \cdots dx_k.
\]
The integral of a \( k \)-form on \( \mathbb{R}^n \) over a differentiable singular \( k \)-cube \( c \) is given as
\[
\int_c \omega = \int_{[0,1]^k} c^* \omega
\]
The integral over a singular \( k \)-chain \( c = \sum_i \gamma_i c_i \) is defined as
\[
\int_c \omega = \sum_i \gamma_i \int_{c_i} \omega.
\]
For instance, if \( \omega = P(x,y) \, dx + Q(x,y) \, dy \) is a 1-form in \( \mathbb{R}^2 \), and \( c : [0,1] \to \mathbb{R}^2 \) is given as \( c(t) = (c_1(t), c_2(t)) \), then
\[
\int_c \omega = \int_0^1 \left( P(c(t)) \, c'_1(t) + Q(c(t)) \, c'_2(t) \right) \, dt.
\]
This is usually called the line integral of \( \omega \) over \( c \).

The central result about the integration of differentiable forms is Stokes’ theorem, which holds for arbitrary \( k + 1 \)-forms \( \omega \) and \( k \)-chains \( c \), and which states that
\[
\int_{\partial c} \omega = \int_c \partial \omega.
\]
An approximation result

Proposition 6.0.1. Let $\mathbf{a}$ be a summable sequence of positive real numbers. Assume that there is a sequence $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots$ of orbits of the phase map $\varphi$, such that

$$\left| G\left(z_t^{(k)}\right) e^{-\rho t} \right| \leq a_t$$

for all $k \geq 1$ and all $t \geq 1$. Assume moreover that

$$z_0^{(k)} \to z_0^{(\infty)} \quad \text{as} \quad k \to \infty.$$

Then

$$\tilde{J}(z^{(k)}) \to \tilde{J}(z^{(\infty)}).$$

Proof. Choose $\varepsilon > 0$. Then there is an $T > 0$ such that for any $k$ we have

$$\left| \tilde{J}(z^{(k)}) - \tilde{J}_T(z^{(k)}) \right| = \left| \sum_{t=T+1}^{\infty} G(z_{t-1}) e^{-\rho t} \right| \leq \sum_{t=T+1}^{\infty} a_t < \frac{\varepsilon}{3}.$$

Moreover, if $\mathbf{z}$ is an orbit of $\varphi$, note that $J_T(\mathbf{z})$ only depends on the initial segment

$$\mathbf{z}_T = (z_0, \cdots, z_T) = (z_0, \varphi(z_0), \varphi^2(z_0), \cdots, \varphi^T(z_0)),$$

which is a continuous function of $z_0$. Therefore, there is a constant $\delta > 0$ such that

$$|z_0^{(k)} - z_0^{(\infty)}| < \delta \quad \Rightarrow \quad |\tilde{J}_T(z^{(k)}) - \tilde{J}_T(z^{(\infty)})| < \frac{\varepsilon}{3}.$$

Take now $N > 0$ such that $|z_0^{(k)} - z_0^{(\infty)}| < \delta$ for all $n \geq N$. Then

$$|\tilde{J}(z^{(k)}) - \tilde{J}(z^{(\infty)})| \leq |\tilde{J}(z^{(k)}) - \tilde{J}_T(z^{(k)})| + |\tilde{J}_T(z^{(k)}) - \tilde{J}_T(z^{(\infty)})| + |\tilde{J}_T(z^{(\infty)}) - \tilde{J}(z^{(\infty)})| \leq \varepsilon.$$

This proves the claim of the proposition.