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**Bifurcations of indifference points in discrete time optimal control problems**

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# Appendix

## Differential forms

Recall from the theory of differential forms the following facts (taken from Spivak, 1965). As the thesis is concerned with 1-forms and 2-forms, the following results are restricted to that case.

A differential 0-form  $f$  on  $\mathbb{R}^n$  is a real-valued function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . A differential 1-form  $\eta$  on  $\mathbb{R}^n$  is a linear form

$$\eta = \eta_x = \sum_{i=1}^n \eta_i(x) dx_i.$$

on  $\mathbb{R}^n$  with coefficients  $\eta_i(x)$  that are differentiable functions; that is, if  $v \in \mathbb{R}^n$ , then

$$\eta_x(v) = \sum_{i=1}^n \eta_i(x) v_i.$$

A differential 2-form

$$w = \sum_{i < j} w_{ij} dx_i \wedge dx_j$$

is an anti-symmetric bilinear form:

$$\omega_x(v, w) = \sum_{i < j} \omega_{ij}(x) (v_i w_j - v_j w_i)$$

The derivative of a 0-form  $f$  is given as  $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ ; the derivative of a 1-form  $\eta = \sum \eta_i dx_i$  is given as

$$d\eta = \sum_{i < j} \left( \frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

Second derivatives are always zero:  $d^2\omega = 0$  for any  $k$ -form  $\omega$ .

If  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a differentiable map, the pull-back  $\varphi^*\eta$  of a 1-form  $\eta$  is given as

$$(\varphi^*\eta)_x(v) = \eta_{\varphi(x)}(D\varphi(x)v) = \sum_i \eta_i(\varphi(x)) (D\varphi(x)v)_i.$$

This can be written more suggestively as

$$\varphi^*\eta = \sum \eta_i(\varphi(x)) d\varphi_i(x)$$

Likewise, the pull-back  $\varphi^*\omega$  of a 2-form takes the form

$$(\varphi^*\omega)_x(v, w) = \omega_{\varphi(x)}(D\varphi(x)v, D\varphi(x)w).$$

A singular  $k$ -cube  $c$  in  $\mathcal{M}$  is a continuous map  $c : [0, 1]^k \rightarrow \mathcal{M}$ ; a singular  $k$ -chain is a formal finite sum of singular  $k$ -cubes with integer coefficients. The standard  $k$ -cube is  $I^k : [0, 1]^k \rightarrow \mathbb{R}^k$  given by  $I^k(x) = x$ . The definition of boundary is given only for 2-chains. Define the edges of  $I^2$  as the singular 1-cubes  $I_{(i,0)}^2$  and  $I_{(i,1)}^2$ ,  $i = 1, 2$  for which

$$\begin{aligned} I_{(1,0)}^2(x) &= I^2(x, 0), & I_{(1,1)}^2(x) &= I^2(x, 1), \\ I_{(2,0)}^2(x) &= I^2(0, x), & I_{(2,1)}^2(x) &= I^2(1, x). \end{aligned}$$

Define the boundary as the singular 1-chain

$$\partial I^2 = I_{(1,0)}^2 - I_{(1,1)}^2 + I_{(2,0)}^2 - I_{(2,1)}^2.$$

For a general singular 2-cube  $c$  define first the edges  $c_{(i,j)}$ ,  $i = 1, 2$ ,  $j = 0, 1$  by setting

$$c_{(i,j)} = c \circ I_{(i,j)};$$

then define

$$\partial c = c_{(1,0)} - c_{(1,1)} + c_{(2,0)} - c_{(2,1)}.$$

The integral of a differential  $k$ -form  $\omega = f(x) dx_1 \wedge \cdots \wedge dx_k$  on  $[0, 1]^k$  is given as

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f(x) dx_1 \cdots dx_k.$$

The integral of a  $k$ -form on  $\mathbb{R}^n$  over a differentiable singular  $k$ -cube  $c$  is given as

$$\int_c \omega = \int_{[0,1]^k} c^* \omega$$

The integral over a singular  $k$ -chain  $c = \sum_i \gamma_i c_i$  is defined as

$$\int_c \omega = \sum_i \gamma_i \int_{c_i} \omega.$$

For instance, if  $\omega = P(x, y) dx + Q(x, y) dy$  is a 1-form in  $\mathbb{R}^2$ , and  $c : [0, 1] \rightarrow \mathbb{R}^2$  is given as  $c(t) = (c_1(t), c_2(t))$ , then

$$\int_c \omega = \int_0^1 \left( P(c(t))c_1'(t) + Q(c(t))c_2'(t) \right) dt.$$

This is usually called the line integral of  $\omega$  over  $c$ .

The central result about the integration of differentiable forms is Stokes' theorem, which holds for arbitrary  $k + 1$ -forms  $\omega$  and  $k$ -chains  $c$ , and which states that

$$\int_{\partial c} \omega = \int_c \partial \omega.$$

## An approximation result

**Proposition 6.0.1.** *Let  $\mathbf{a}$  be a summable sequence of positive real numbers. Assume that there is a sequence  $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots$  of orbits of the phase map  $\varphi$ , such that*

$$\left| G\left(z_t^{(k)}\right) e^{-\rho t} \right| \leq a_t$$

for all  $k \geq 1$  and all  $t \geq 1$ . Assume moreover that

$$z_0^{(k)} \rightarrow z_0^{(\infty)} \quad \text{as } k \rightarrow \infty.$$

Then

$$\tilde{J}(\mathbf{z}^{(k)}) \rightarrow \tilde{J}(\mathbf{z}^{(\infty)}).$$

*Proof.* Choose  $\varepsilon > 0$ . Then there is an  $T > 0$  such that for any  $k$  we have

$$\left| \tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(k)}) \right| = \left| \sum_{t=T+1}^{\infty} G(z_{t-1}) e^{-\rho t} \right| \leq \sum_{t=T+1}^{\infty} a_t < \frac{\varepsilon}{3}.$$

Moreover, if  $\mathbf{z}$  is an orbit of  $\varphi$ , note that  $J_T(\mathbf{z})$  only depends on the initial segment

$$\mathbf{z}_T = (z_0, \dots, z_T) = (z_0, \varphi(z_0), \varphi^2(z_0), \dots, \varphi^T(z_0)),$$

which is a continuous function of  $z_0$ . Therefore, there is a constant  $\delta > 0$  such that

$$|z_0^{(k)} - z_0^{(\infty)}| < \delta \quad \Rightarrow \quad |\tilde{J}_T(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(\infty)})| < \frac{\varepsilon}{3}.$$

Take now  $N > 0$  such that  $|z_0^{(k)} - z_0^{(\infty)}| < \delta$  for all  $n \geq N$ . Then

$$\begin{aligned} |\tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}(\mathbf{z}^{(\infty)})| &\leq |\tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(k)})| + |\tilde{J}_T(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(\infty)})| \\ &\quad + |\tilde{J}_T(\mathbf{z}^{(\infty)}) - \tilde{J}(\mathbf{z}^{(\infty)})| \\ &\leq \varepsilon. \end{aligned}$$

This proves the claim of the proposition. ■