Bifurcations of indifference points in discrete time optimal control problems

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Appendix

Differential forms

Recall from the theory of differential forms the following facts (taken from Spivak, 1965). As the thesis is concerned with 1-forms and 2-forms, the following results are restricted to that case.

A differential 0-form f on $\mathbb{R}^n$ is a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$. A differential 1-form η on $\mathbb{R}^n$ is a linear form

$$\eta = \eta_x = \sum_{i=1}^{n} \eta_i(x)\, \mathrm{d}x_i.$$

on $\mathbb{R}^n$ with coefficients $\eta_i(x)$ that are differentiable functions; that is, if $v \in \mathbb{R}^n$, then

$$\eta_x(v) = \sum_{i=1}^{n} \eta_i(x) v_i.$$

A differential 2-form

$$w = \sum_{i<j} w_{ij}\ dx_i \wedge dx_j$$

is an anti-symmetric bilinear form:

$$\omega_x(v, w) = \sum_{i<j} \omega_{ij}(x)(v_i w_j - v_j w_i)$$

The derivative of a 0-form f is given as $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$; the derivative of a 1-form $\eta = \sum \eta_i\,\mathrm{d}x_i$ is given as

$$\mathrm{d}\eta = \sum_{i<j} \left(\frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j} \right) \mathrm{d}x_i \wedge \mathrm{d}x_i.$$

Second derivatives are always zero: $\mathrm{d}^2\omega = 0$ for any k-form ω.

If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a differentiable map, the pull-back $\varphi^*\eta$ of a 1-form η is given as

$$(\varphi^*\eta)_x(v) = \eta_{\varphi(x)}\big(D\varphi(x)v\big) = \sum_i \eta_i(\varphi(x))\big(D\varphi(x)v\big)_i.$$

This can be written more suggestively as

$$\varphi^*\eta = \sum \eta_i(\varphi(x))\,\mathrm{d}\varphi_i(x)$$

Likewise, the pull-back $\varphi^*\omega$ of a 2-form takes the form

$$(\varphi^*\omega)_x(v, w) = \omega_{\varphi(x)}(D\varphi(x)v, D\varphi(x)w).$$

A singular k-cube c in $\mathscr{M}$ is a continuous map $c : [0,1]^k \to \mathscr{M}$; a singular k-chain is a formal finite sum of singular k-cubes with integer coefficients. The standard k-cube is $I^k : [0,1]^k \to \mathbb{R}^k$ given by $I^k(x) = x$. The definition of boundary is given only for 2-chains. Define the edges of I^2 as the singular 1-cubes $I^2_{(i,0)}$ and $I^2_{(i,1)}$, $i = 1, 2$ for which

$$\begin{aligned} I^2_{(1,0)}(x) &= I^2(x,0), \qquad I^2_{(1,1)}(x) = I^2(x,1), \\ I^2_{(2,0)}(x) &= I^2(0,x), \qquad I^2_{(2,1)}(x) = I^2(1,x). \end{aligned}$$

Define the boundary as the singular 1-chain

$$\partial I^2 = I^2_{(1,0)} - I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}.$$

For a general singular 2-cube c define first the edges $c_{(i,j)}$, $i = 1, 2$, $j = 0, 1$ by setting

$$c_{(i,j)} = c \circ I_{(i,j)};$$

then define

$$\partial c = c_{(1,0)} - c_{(1,1)} + c_{(2,0)} - c_{(2,1)}.$$

The integral of a differential k-form $\omega = f(x)\,\mathrm{d}x_1 \wedge \cdots \wedge x_k$ on $[0,1]^k$ is given as

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f(x)\,\mathrm{d}x_1 \cdots\, \mathrm{d}x_k.$$

The integral of a k-form on $\mathbb{R}^n$ over a differentiable singular k-cube c is given as

$$\int_c \omega = \int_{[0,1]^k} c^*\omega$$

The integral over a singular k-chain $c = \sum_i \gamma_i c_i$ is defined as

$$\int_c \omega = \sum_i \gamma_i \int_{c_i} \omega.$$

For instance, if $\omega = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ is a 1-form in $\mathbb{R}^2$, and $c : [0,1] \to \mathbb{R}^2$ is given as $c(t) = (c_1(t), c_2(t))$, then

$$\int_c \omega = \int_0^1 \Big(P\big(c(t)\big)c_1'(t) + Q\big(c(t)\big)c_2'(t) \Big)\,\mathrm{d}t.$$

This is usually called the line integral of ω over c.

The central result about the integration of differentiable forms is Stokes' theorem, which holds for arbitrary $k+1$-forms ω and k-chains c, and which states that

$$\int_{\partial c} \omega = \int_c \partial\omega.$$

An approximation result

Proposition 6.0.1. *Let* $\mathbf{a}$ *be a summable sequence of positive real numbers. Assume that there is a sequence* $\mathbf{z}^{(1)}$, $\mathbf{z}^{(2)}$, $\cdots$ *of orbits of the phase map* φ*, such that*

$$\left| G\left(z_t^{(k)}\right) \mathrm{e}^{-\rho t} \right| \leq a_t$$

for all $k \geq 1$ *and all* $t \geq 1$*. Assume moreover that*

$$z_0^{(k)} \to z_0^{(\infty)} \quad \textit{as} \quad k \to \infty.$$

Then

$$\tilde{J}(\mathbf{z}^{(k)}) \to \tilde{J}(\mathbf{z}^{(\infty)}).$$

Proof. Choose $\varepsilon > 0$. Then there is an $T > 0$ such that for any k we have

$$\left|\tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(k)})\right| = \left| \sum_{t=T+1}^{\infty} G(z_{t-1})\,\mathrm{e}^{-\rho t} \right| \leq \sum_{t=T+1}^{\infty} a_t < \frac{\varepsilon}{3}.$$

Moreover, if $\mathbf{z}$ is an orbit of φ, note that $J_T(\mathbf{z})$ only depends on the initial segment

$$\mathbf{z}_T = (z_0, \cdots, z_T) = (z_0, \varphi(z_0), \varphi^2(z_0), \cdots, \varphi^T(z_0)),$$

which is a continuous function of z_0. Therefore, there is a constant $\delta > 0$ such that

$$|z_0^{(k)} - z_0^{(\infty)}| < \delta \quad \Rightarrow \quad |\tilde{J}_T(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(\infty)})| < \frac{\varepsilon}{3}.$$

Take now $N > 0$ such that $|z_0^{(k)} - z_0^{(\infty)}| < \delta$ for all $n \geq N$. Then

$$\begin{aligned}
|\tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}(\mathbf{z}^{(\infty)})| &\leq |\tilde{J}(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(k)})| + |\tilde{J}_T(\mathbf{z}^{(k)}) - \tilde{J}_T(\mathbf{z}^{(\infty)})| \\
&+ |\tilde{J}_T(\mathbf{z}^{(\infty)}) - \tilde{J}(\mathbf{z}^{(\infty)})| \\
&\leq \varepsilon.
\end{aligned}$$

This proves the claim of the proposition. ∎