Dynamic logic of questions
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Chapter 4
Games with Questioning Moves

We have introduced in the previous chapter a general logical framework for questions, information flow, and their mutual interdependence. In the present chapter we enrich our study with strategic considerations. We will see that adding a strategic aspect to questioning activities is a natural extension of the logical framework which fits with many intuitive scenarios and captures concrete aspects of rational interaction in many practical applications. We will also show that the game theoretic twist enriches with new theoretical content traditional notions used to describe questions and results about epistemic and interactive scenarios.

This enrichment also fulfills several desiderata already encompassing our approach. A dynamic logic of questions is designed to model interactive scenarios in which agents perform questioning moves and aim to achieve epistemic goals. The main modeling desiderata that we will address are: to develop a formal model of questioning actions using an issue relation and to study the epistemic and strategic effects of such questioning actions in a game theoretical setting.

Approaches that combine logical and game theoretic aspects for informative epistemic actions are not new, we start our chapter with a brief account of previous approaches inside the DEL paradigm. Next we will focus on showing how these classical approaches designed to model informative actions can be extended to model questioning actions in interactive scenarios.

We proceed by first introducing strategic games in which moves are question-answer pairs. We give the basic definitions of such games and then proceed to consider results about existence of solution concepts and analysis of concrete examples. Next we introduce extensive games in which moves are questioning actions whose effects are represented by an issue relation. We continue with studying solution concepts and strategic abilities in epistemic games with questioning moves and and general game properties useful to describe them.
4.1 Brief History of Epistemic Games

Several approaches that combine a logic and epistemic approach with a game-theoretic approach for modeling scenarios of rational interaction and communication involving directly or indirectly questioning actions have been previously considered in the literature [50, 77, 80]. We start by briefly surveying the main approaches inside the dynamic epistemic logic paradigm [102], [8], and [2].

4.1.1 Knowledge Games with Epistemic Moves

A first relevant contribution inside the broader DEL paradigm approaches informative epistemic moves in a general game-theoretic setting [102], and defines and studies games with epistemic moves. In brief, this approach starts by considering a set of cards $C$ and a set of agent-labels $A$. Let $d \in A^C$ be a deal of cards, then the initial state of a game for the actual deal of cards $d$ is:

$$\langle (D_{2d}, (\sim_a)_{a \in A}, V), d \rangle,$$

where $\forall a \in A : \forall d_1, d_2 \in D_{2d} : d_1 \sim_a d_2 \iff d_1^{-1}(a) = d_2^{-1}(a)$ and $\forall c \in D_{2d} : \forall c_a \in P : V_c(c_a) = 1 \iff e(c) = a$.

A knowledge game state for cards-deal $d$ is an $S5$ model $\langle (W, (\sim_a)_{a \in A}, V), v \rangle$, such that $v \in W$, $V_d$, and: $\forall w \in W : \exists d' \in D_{2d} : V_w = V_{d'}$, and for all $a \in A : \forall w_1, w_2 \in W : \forall d_1, d_2 \in D_{2d} : (w_1 \sim_a w_2, V_{w_1} = V_{d_1}, V_{w_2} = V_{d_2}) \Rightarrow d_1^{-1}(a) = d_2^{-1}(a)$.

Let $\langle (W, (\sim_a)_{a \in A}, V), d \rangle$ be a knowledge game state. A game action $\mu$ for state $s$ is a quintuple $\mu = \langle q, Q, r, R, pub \rangle$, where $q, r \in A, Q$ is a covering of $W$ that is coarser than $\sim_r$, $R \subseteq Q$ and $pub$ is the identity ‘$=$’ on $Q$.

Let $s = \langle (W, (\sim_a)_{a \in A}, V), w \rangle$ and $\mu = \langle q, Q, r, R, pub \rangle$. The game action $\mu$ is executable in game state $s$ if the answer $R$ contains actual world $w$, i.e., if $r$’s information state $[w]_{\sim_r}$ is contained in the answer-set.

This framework also offers the possibility to describe epistemic actions as they appear to groups of agents in interaction. For instance, when an action is accessible to all agents in a subgroup $B \subseteq A$ this can be represented as $pub_B = (\bigcup_{a \in B} pub_a)^\ast$. The equivalence class of $\sim_B$ that contains the answer $Q$ stands for what subgroup $B$ learns in that game action.

4.1.2 Time-stamped Questions in Communication

The second account that we will consider here was introduced in [8]. Here a general setting of ‘communication acts’ is considered: $[\sigma \phi]_\psi$ saying intuitively that ‘after the communication act $\sigma \phi$ is performed at the current state, sentence $\psi$ will be true at the output state. ‘Abstract dialogues’ or ‘communication sequences’ are defined as words over the alphabet $Act^2$ of all communications acts considered.
4.1. Brief History of Epistemic Games

Interrogatives, or queries are in this setting a particular kind of communication acts. For instance, the ‘public questioning of agent $b$ by agent $a’$ is of type $PubQ_i(a,b)$ having the following structure: $\Sigma_1 = \{PubQ_a\}, PubQ_i(a,b)_c = \{PubQ_i(a,b)\}$ for all agents $c \in Ag$, $PRE_{PubQ_i(a,b)} = \emptyset$, and $CON_{PubQ_i(a,b)} = ?_i(a,b,Ag)$. The resulting communication act is the act of ‘agent $a$ publicly asking $b$ weather $\varphi$ is true or not’. The notable feature here is the fact that such communication acts come with a ‘timestamp $i$’ which will become relevant later on when the communication acts are going to be cast as legal or illegal ‘(communication) moves’ in a ‘dialogue game’. Only moves that answer questions that have been previously time-stamped are going to be legal while moves that also consider the later dynamics are not allowed. Many other kinds of questioning moves can be captured in this setting using more and more complex preconditions for execution and or the publicity of the action, the security of the communication channel, etc. For instance, questions are qualified as ‘normal’ and ‘abnormal’, ‘private’ and ‘public’, ‘secure’ or ‘intercepted’, Socratic, etc.

In this setting, a ‘dialogue game’ is a pair $G = (S_G, Act_G)$ containing a set of initial epistemic states and available moves, both finite. A ‘legal dialogue’ is a string $s_0, \alpha_0, s_1, \alpha_1, \ldots, s_n\alpha_n$ of time-stamped epistemic states and communication moves that have to satisfy various reasonable conditions such as, in the case of questioning moves, responsiveness, sincerity, etc.

### 4.1.3 Public Announcement Games

A more recent approach that models games in which moves are public announcements is [2]. In this framework, the modeling starts from an epistemic structure over two agents. Moves in the game are public announcements. The result of an announcement is the epistemic structure updated with the conjunction of announcement formulae for the two agents. The goals of the agents are represented as epistemic formulae. The payoff for the agents is derived by model checking goal formulae in the output epistemic structures. Both pointed games, which are played locally at a world in the model, and induced games, which are necessary because the agents themselves are uncertain about the actual situation, are defined and analyzed in this setting. The main contribution of this chapter will be to add questioning moves to this formal framework and to study both logical and game theoretic notions and formal properties in settings with questioning moves.

The connection between this approach and question-answer games is very close. While many of the definitions are similar, there are interesting differences already emerging at this level, both conceptually and in terms of basic results. A complete and detailed comparison is beyond the scope of this brief introduction. We include a more detailed comparison starting from and using implementation tools in Chapter 5 and proceed now to introduce definitions for question-answer games in all details as well as illustrations and basic results about them.
4.2 Question-Answer Games

Introduction In general, there is a very close relation between questions and answers. This intimate connection is manifest in many of the historical approaches already discussed and will also be the starting point in the approach we will follow in this chapter. We start with very basic strategic games that have question-answer pairs as constitutive moves. We then explore more complex versions that allow for more realistic features, including more complex questioning actions and sources of information. We then continue our study with a setting of extensive games which have questioning moves as their basic components.

The connection between strategic aspects and information content in questioning activities has been the topic in various studies about games that involve questions and answers such as, for instance: [30, 62]. Moreover, relations between game theory and pragmatic phenomena like the information content of questions and their relevance for solving decision problems have also received attention before, for instance in [110]. Such approaches have revealed close connection between an analysis of questions, information theory and related concepts such as entropy and relevance. However, such approaches usually only consider questioning actions from a single-agent perspective. Following a long tradition, they do not usually involve strategic answer to questions asked to or by other agents. We aim at using an epistemic analysis in game theoretic style in order to integrate such aspects in a broader multi-agent perspective on questioning scenarios.

Questioning Let us give more content to this general perspective by starting from concrete examples. Consider two agents $a$ and $b$, and a relevant propositional symbol $p$. The other alternative, is therefore represented by $\neg p$.

There are several pragmatic preconditions included in the act question ‘$p$?’ being asked by an agent. She should not know the truth about $p$, i.e., she does not know $p$, and she does not know $\neg p$. We are assuming a multi-agent epistemic logic to model questions, where the expression $K_a p$ stands for ‘$a$ knows $p$’, and where the epistemic modality $K_a$ is interpreted with an equivalence relation $\sim_a$. This pragmatic constraint therefore amounts to the precondition $\neg K_a p \land \neg K_a \neg p$.

The agent asking the question also has expectations about the agent that will answer the question, and that constitutes another pragmatic precondition. She considers it possible that he knows the answer, i.e., $\neg K_a \neg (K_b p \lor K_b \neg p)$. (She also considers it likely that he knows the answer, i.e., she believes that tentatively, $B_a (K_b p \lor K_b \neg p)$, where belief and knowledge are combined as in [64]; we may also see that as a combination of preferences and knowledge [99].)

A question $\varphi$? splits the domain in the set of states $[\varphi]$ where $\varphi$ is true and the set of states $[\neg \varphi]$ where $\varphi$ is false, i.e., the question induces a dichotomy on the domain of the model. In the approach from the previous chapter, as well as in [100], a question $\varphi$? is represented by the issue relation $\approx_\varphi$. Such issue relations are natural and intuitive representations for semantic effects of questioning actions.
in a universe of possible worlds, their use goes back to [40, 56]. We will start this chapter by introducing a model that does not use such issue relations and we will return to them later on. To a certain extent a dichotomy can be also represented as two model restrictions corresponding to the possible answers to the questions modeled as announcements, in standard public announcement logic. Some interesting features that are specific for questions and deserve a minute analysis emerge already at this stage, even without considering an issue relation.

**Answers to Questions** If a question is addressed to another agent, which might have his own epistemic limitations, there are three possible answers: ‘Yes’, ‘No’, and ‘I don’t know’. A fourth possibility, which is, however, less relevant as an epistemic action, would be: ‘I decline to answer the question’. Such a response does not convey an epistemic content. Therefore we will focus only on the three different answers that do have an epistemic content. Neither of these is less or more informative than required in a multi-agent context.

When a question is addressed to an agent \(b\) an answer corresponds to the (unique) largest union of \(b\) equivalence classes representing the knowledge that is contained in the \(\varphi\)-states of the model. This is of course exactly the denotation of the formula \(K_b\varphi\). If we make the reasonable assumption that questions are answered truthfully, answers can be encoded as a public announcement \(K_b\varphi!\) [79].

Similarly, if the answer is “No, I don’t,” this can be thought as announcing the formula \(K_b\neg\varphi\) and its denotation is the complement of the (unique) smallest union of equivalence classes that contains the \(\varphi\)-states. The answer “I don’t know” results in the remainder, i.e. the union of all \(\sim_a\) classes that properly intersect with \(\approx_{\varphi}\). The ‘don’t know’ answer can be conceived as announcement of the formula \(\neg K_b\varphi \land \neg K_b\neg\varphi\) and this might also fit into the form of an announcement, namely \(K_b(\neg K_b\varphi \land \neg K_b\neg\varphi)\), so indeed this must also be a union of \(b\)-equivalence classes.

This in fact shows that answers to questions can be seen as rough sets [75]. Given the set \([\varphi]\) (i.e., the subset of the domain consisting of the \(\varphi\)-states), in rough set terms known as the target, take the lower and upper \(\sim_b\) approximation of the target, i.e. \(\sim_b([\varphi])\) and \(\overline{\sim}_b([\varphi])\). If the answer to the question is ‘yes’, the actual state is in the lower approximation. If the answer is ‘no’, the actual state is in the complement of the upper approximation. If the answer is ‘don’t know’, the actual state is in the upper approximation minus the lower approximation. See also Figure 4.1 later in the section for an intuitive illustration.

In dynamic epistemic logic, a public announcement is interpreted as a model restriction. Therefore, answering the question can be seen as executing one of three possible such restrictions, a non-deterministic program so to speak.

**Games with Questions and Answers** Extending all this into a game theoretical scenario is as natural as it can be. The only aspect that needs clarification is the goal of the game. What are the goals of the players? In the case of ask-
ing \( p \), it is most natural to think of the goal as achieving knowledge about \( p \): \( K_a p \lor K_a \neg p \). But in this case it is not so clear if there is an opponent. Clearly if the agent to whom the question is addressed has no interest in the issue he will answer the question, but he will not be playing a game just yet.

But scenarios featuring genuine strategic aspects are frequent in many competitive situations involving questions in a multi-agent setting. Take as a quick illustration a scenario in which you and I are both spies looking to find out a particular secret. The secret is the truth about \( p \). Your goal is to get to know it before me and my goal is to get to know it before you, i.e., each agent has a goal:

\[
\gamma_a = (K_b p \lor K_b \neg p) \rightarrow (K_a p \lor K_a \neg p),
\]

\[
\gamma_b = (K_a p \lor K_a \neg p) \rightarrow (K_b p \lor K_b \neg p),
\]

In other words, it is not a problem if you know it, as long as I already know it too, and vice versa. The epistemic results of asking and answering are new information states wherein we can check whether or not each player’s goal is fulfilled. This determines a payoff function and thus the outcome of the epistemic game.

So in order to play a game with questions and answers the players need a goal, and that goal can be an epistemic formula. Why should a player answer a question if that means giving away information that may make him lose the game? He has no reason whatsoever. However, just like in real life, if you wish the other person to loosen his information strings, you may only expect to obtain that by giving away some information yourself. The procedural version of this expectation is a game where each player may choose between different questions to ask but where the other player addressed by that question is obliged to answer.

**Pointed and induced question-answer games** Given two agents \( a \) and \( b \), an epistemic model \( M = (S, \sim_a, \sim_b, V) \) encodes their uncertainty about facts and about each other; Two formulas \( \gamma_a \) and \( \gamma_b \) in the logical language express what they wish to achieve by their questions. In order to achieve their goals, agent \( a \) asks a question \( \varphi \) to agent \( b \), to which \( b \) is obliged to respond with ‘yes’ (I know that \( \varphi \)), ‘no’ (I know that \( \neg \varphi \)), or ‘don’t know’ (I don’t know whether \( \varphi \)). And similarly for \( b \) asking a question to \( a \). We don’t want to keep saying that all the time, so from now on we may refer to the two agents as \( i \) and \( j \), where \( i \neq j \), and \( i \) may be either \( a \) or \( b \). We assume that both agents ask their question simultaneously, and that subsequently both agents answer the question simultaneously. Of course, more realistic communicative settings would allow for agents to ask questions and respond to them in any order, such generalizations will be modeled as extensive games in a later section of this chapter. The question formulas can be thought of as defining the strategies for the agents.

Executing the strategy \( \varphi \) for agent \( i \) can be thought of as follows. Agent \( i \) asks \( \varphi \) to \( j \). If \( M, s \models K_j \varphi \), then \( j \) answers (announces) “Yes, I know that \( \varphi \)”. If \( M, s \models K_j \neg \varphi \), then \( j \) answers “No, I know that \( \neg \varphi \)”. Otherwise, \( j \) answers “I
4.2. Question-Answer Games

Don’t know whether \( \varphi \)”. The resulting model restriction depends on both answers, e.g., if \( a \) asks \( \varphi ? \) to which \( b \) responds \( K_b \varphi! \) and \( b \) asks \( \psi ? \) to which \( a \) responds \( K_a \neg \psi! \), the result is the restricted model \( M | (K_b \varphi \land K_a \neg \psi) \). We can capture these alternatives with a construct \( K_i \varphi \), for ‘agent \( i \) answers the question \( \varphi ? \)’, defined as follows. Given an epistemic model \( M \) and a state \( s \in M \), if \( M, s \models K_i \varphi \), then \( K_i \varphi \equiv K_i \varphi \); if \( M, s \models K_i \neg \varphi \), then \( K_i \varphi \equiv K_i \neg \varphi \); and else \( K_i \varphi \equiv \neg (K_i \varphi \lor K_i \neg \varphi) \).

Alternatively, we can represent the question by an issue relation \( \approx \varphi \) and the public announcement of answering the question in the link-cutting way from the previous chapters in this thesis going back to [99, 100]. A strategy will be in the current setting a tripartite question, and its execution value consists of the answer to it. As the three epistemically relevant possible answers are mutually exclusive, each world of the model uniquely determines the execution value for each strategy. The values will then be aggregated at the global level of the model.

We associate two different strategic games with these questions, their answers and agents’ goals: pointed question-answer games and not-pointed or global question-answer games. Both are needed: a player may not know what the actual state is, and therefore not know which game he is playing. These definitions are adaptations of similar concepts in [2] with the only notable exception that when modeling questions the pointed game has to use a more complex payoff function.

4.2.1. Definition. [Pointed question-answer game] The state game or pointed question-answer game \( G((M, s), \gamma_a, \gamma_b) \) associated with state \( s \in M \) of goals \( \gamma_a \) and \( \gamma_b \) for agents \( a \) and \( b \) respectively, is the strategic game defined by:

- \( N = \{a, b\} \);
- For \( i = a, b \), \( A_i = \{\varphi ? | \varphi \in \mathcal{L}\} \);
- For \( i = a, b \), \( u_i(\varphi, \psi) = \begin{cases} 1 & \text{if } M, s \models (K_b \varphi \land K_a \psi)! \gamma_i \\ 0 & \text{otherwise} \end{cases} \)

Note that the set of strategies \( A_i \) is the same in all states of the model. The next definition gives a state-independent perspective on question-answer games.

4.2.2. Definition. [Question-answer game] Given state games \( G((M, s), \gamma_a, \gamma_b) \) for each \( s \in M \), the induced game or question-answer game \( G(M, \gamma_a, \gamma_b) = \langle N, \{A'_i : i \in N\}, \{u_i : i \in N\} \rangle \) is the strategic game defined by:

- \( N = \{a, b\} \);
- For \( i = a, b \), \( A'_i \) is the set of uniform functions from \( S \) to \( A_i \);
- For \( i = a, b \), \( a'_a, a'_b \in A'_i \), \( u_i(a'_a, a'_b) = \frac{\sum_{s \in S} u_i(a'_a(s), a'_b(s))}{|S|} \).
Chapter 4. Games with Questioning Moves

In Definition 4.2.2, a strategy \( a_i \) for player \( i \) is \textit{uniform} iff for all \( s, t \in S \): \( s \sim_i t \) implies \( a_i(s) = a_i(t) \). It can be easily shown that this corresponds to a Bayesian game [45], in the sense that it has the same Nash equilibria; we will return to this aspect in more detail later in the section.

As there are countably infinitely many formulas in the basic propositional language, there will be infinitely many strategies in a pointed question-answer game, and therefore also in an induced question-answer game. However, in order to avoid such an explosion and an unnecessary overkill by strategy proliferation we can propose the following major simplification for the notion of strategy.

4.2.3. Definition. [Strategy equivalence] Let an epistemic model \( M \) be given.

Two strategies \( \phi \) and \( \psi \) for a pointed question-answer game played in the model \( M \) are the same (equivalent) for agent \( i \) if:

\[
\{ [K_j \phi]_M, [K_j \neg \phi]_M, [\neg (K_j \phi \lor K_j \neg \phi)]_M \} = \{ [K_j \psi]_M, [K_j \neg \psi]_M, [\neg (K_j \psi \lor K_j \neg \psi)]_M \}
\]

Note that it is common knowledge to \( a \) and \( b \) if two strategies are the same. This is so because we are comparing the denotations of formulas involving \( \phi \) and \( \psi \) in the model, independent of the actual state.

We include a more detailed discussion of this notion of strategy equivalence and additional examples in Section 5.2.

Given the requirement of uniform strategies in the global game, instead of seeing a strategy in the induced game as a function from \( \text{states} \) to formulas, we can also see such a strategy for agent \( i \) as a function from \( i \)-equivalence classes to formulas, and therefore as a function from \( \text{formulas} \) characterizing \( i \)-equivalence classes to formulas. In other words, we can see then as conditional strategies.

With these simplifications there are fewer strategies: given that our starting models are always finite, there are only a finite number of non-equivalent strategies. The next subsection contains an example where both players only have two ‘real’ (i.e., non-equivalent) strategies. Subject to the identification of strategies with the same tri-partition, the number of strategies for \( i \) is a function of the number of the equivalence classes for agent \( j \) in the model \( M \). We discuss further the counting of strategies based on this notion of strategy equivalence in Section 5.2.

We will consider here the simple special case where the ‘don’t know’ alternative is always empty as a first approximation of how strategies can be counted.

Also we will assume that the worlds in the domain of the model can be named either by using nominals or by a characteristic epistemic formula true only at one world. This can always be constructed if the starting epistemic model has been minimized under epistemic bisimulation before the play of the question-answer game begins and the set of formulae representing strategies is determined.

If the strategies for \( i \) have the form \( K_j \phi \), there are only two and not three answers, namely only ‘yes’ and ‘no’, where the second answer contains now both previously negative and unknown answer alternatives. If strategies are required
to have this form for both agents, we call this a \textit{dichotomous question-answer game}. The number of strategies is now even lower. It can be counted as follows, assuming that we work with \{a, b\}-connected models. If we assume that player \(i\) has \(m_i\) equivalence classes and player \(j\) has \(m_j\) equivalence classes. Then the number of pure strategies for player \(i\) will be \(2^{m_j m_i - m_i}\).

There are \(2^{m_j - 1}\) different dichotomies for player \(j\) i.e. coarsenings of player \(j\)’s partition, and for each of \(m_i\) different equivalence classes for the requesting player \(i\), she may choose one of those questions, therefore the total number of pure strategies in the game will be \((2^{m_j - 1})^m = 2^{m_j m_i - m_i}\).

We include further examples and illustrations in the next chapter.

It is time now to take stock of where we stand with our formal model so far. First, our model is based on some implicit but drastic simplifications. These have been useful so far because they provide concreteness to the formal model as a initial approximation. We discuss some of these simplifying assumptions and some further desiderata for games with more realistic features.

\textbf{Simplifications} In the discussion throughout this section, we disregard pragmatic constraints of questions. One may ask a question to which she already knows the answer. In other words, we did not considered that the question \(\varphi?\) is also an informative update / public announcement of the formula:

\[
(\neg K_a \varphi \land \neg K_a \neg \varphi) \land \neg K_a \neg (K_b \varphi \lor K_b \neg \varphi).
\]

However, incorporating such additional constraints does not raise any significant conceptual or technical difficulties. It merely restricts the players’ strategies.

Avoiding to answer the question is not modeled as a move in the version of question answer game considered so far. However, it is not problematic, as that response can be modeled as the trivial announcement.

There are two players only, that ask each other questions. If there are more than two players, one has to specify who is addressed by the question. Again, this is doable, and an answer to the question would still be a public announcement.

So far, we only assume that the two players ask each other a single question, and that they ask the question at the same time; say, by writing down the question on a piece of paper, putting it in an envelope, and then exchanging envelopes. For the answer, they again exchange envelopes.

The most natural interaction between players involving questions and answers is where they ask each other questions in turn not simultaneously, such that a question is answered before the next question is asked. Such scenarios can be modeled but they will require an approach that uses games in extensive form.

However, in order to make sure that our formal model is relevant for realistic applications we have to find the right balance between a naive technical convenience and an approach that can capture a reasonable level of conceptual complexity. For this we list a number of further modeling desiderata.
Modeling Desiderata, Further Extensions  We have shown how epistemic models come with natural games that model interesting phenomena, and suggest interesting logical questions. Our games are very simple, but this starting point itself is an advantage, since well-chosen simple games are a first start for more complex scenarios. Nevertheless, we wish to go beyond that. Most natural from the viewpoint of our general aims are the following extensions:

A first desideratum is to have a richer account of questions that is adequate at least for analyzing questions as possible moves in inquiry. As mentioned in the introductory Section 4.2, a pragmatic constraint on questions is that the interrogator does not know the answer. This amounts to the precondition

$$\neg K_a \varphi \land \neg K_a \neg \varphi \quad (i)$$

for a question $\varphi$?. The agent asking the question also has expectations about the agent that will answer the question, and that constitutes another pragmatic precondition. She considers it possible that he knows the answer, i.e.,

$$\neg K_a \neg (K_b \varphi \lor K_b \neg \varphi) \quad (ii)$$

For a given model, such pragmatic constraints result in a (further) reduction of strategies than the reduction already achieved by strategy equivalence. For instance, given $(i)$, in the example of Section 4.2 the agents would not have had any strategic choice! The trivial question would not be allowed. Also, it makes sense to consider games in which the trivial answer ‘I decline to answer the question’ / announcement of $\top$ is always allowed; or games where this is out-ruled. All such variations can be accommodated in the current model.

Extensive games with longer sequences of moves are an obvious desideratum, and so are logics for them and finding sequential equilibria in extensive games.

An approach that generalizes questioning games by considering more than two agents; multiple or partially ordered goals per agent; a notion of goal equivalence; non-uniform probability distributions over the worlds in the domain.

A final desideratum would be to have multiple goals and also goals that are structured in a richer way, for instance forming a linear order with some of them having a higher priority or relevance over the remaining ones.

4.3 Questioning Games with Oracles

Most of the applications that are relevant for modeling inquiry and scientific research consider alongside strategic questions of rational agents objective answers from Nature or other generic entity like an Oracle or other abstract information sources. The strategic aspects involved in playing a questioning game depend also on what are the sources of information available during the game.
In this section we will address the following modeling desiderata specified earlier: we will consider a setting with more than two players, we will consider a setting in which we can study various sources of information available and a more complex model of questioning: both classical or *dichotomous* questions and epistemically modulated, or *trichotomous* questions. We will show by means of some illustrative examples how these features can be easily integrated in our setting of epistemic games introduced in the previous section. Even more, we will show that the standard approach to questioning actions in an epistemic neutral environment can be captured in an epistemic setting as a particular case, when a perfectly informed source of information is available in the model.

Moreover, will also show how standard results about games in general transfer in the expected way to question-answer epistemic games (QAGs). One property investigated will be the (in)existence of Nash equilibria\(^1\). For games with pure strategies in general there might be the case that no Nash equilibrium exist. We show that this is also the case for QAGs using a setting in which this result can be presented in a compact (even though, admittedly, graphically demanding) way.

However, we will not have questions represented by an issue partition yet, or games with sequential moves, these desiderata will be addressed later on in Section 4.4.2. Let us start by considering a paradigmatic example:

**4.3.1. Example.** The answer to a question \(\varphi?\) can be seen as a rough set. In the figure below, the \((4 \times 4 = 16)\) cells represent the equivalence classes of the agent answering the question. The blue ellipse corresponds to \(\{\varphi\}\), the green region to the answer ‘yes’ and the red region to the answer ‘no’ (and this could have been the other way round), and the white region (necessarily) to ‘don’t know’.  

Figure 4.1: Bipartite versus Tripartite Questioning: adding an epistemic partition (right) enriches the standard approach of formal structure in answering (left).

We can generalize the setting from this example in the following way. Let \(W\) be a set of possible worlds, \(R \in W \times W\) an equivalence relation on \(W\), and \(K\) the modality for \(R\). Let \(P \subseteq W\) be the extension of \(\varphi\). We define \(S3_P \in W^3\) by:

\[
S3_P = \{S0_P, S1_P, S2_P\}
\]

\(^1\)The notion of Nash equilibrium we will use is the standard one, see Definition 6.6.1 in the Theoretical Background appendix 6.6 and/or Definition 4.4.15 later on in this chapter.
where we have: $S_0P = \{w \mid M \models_w K \neg \varphi\}, S_1P = \{w \mid M \models_w K \varphi\},$

$$S_2P = \{w \mid M \models_w \neg K \neg \varphi \land \neg K \varphi\}.$$  

**Discussion**  Intuitively, $K$ represents the query-available information. It can be given by the limitations in the subjective knowledge of the agent or group of agents, that will answer the query in a multi-agent query setting. It can equally represent the objective limits in generic answering capabilities like limitations of measurement instruments, or contingencies of experimental design, restrictions in computing power, or lack of specific resources needed for problem solving. Such limitations can also be encoded by an oracle function.

We can now define an order on questions in the following way:

Let $S$ be the set of tripartite questions expressible over a given domain $W$ and $S3, S3' \in S$ such that $S3 = \{S0, S1, S2\}$, $S3' = \{S0', S1', S2'\}$, we define an order $\leq \in S^2$ in the following way:

$$S3 \leq S3' \iff \forall X \in S3 \exists Y \in S3' : X \subseteq Y$$

This ordering is related to the order over the subsets of the domain, representing assertions, in a natural way. Let $P, P' \subseteq W$ be the extensions of $\varphi, \varphi'$, respectively. We define an order $\leq \in \wp(\wp(W))^2$ in the following way:

$$P? \leq P'? \iff \{P\} \leq \{P'\} \iff S3P \leq S3P'$$

It follows from this definition that $\varphi? \leq \varphi'? \iff P? \leq P'?$.

The two orders can be further lifted in a natural way to questions understood as sets of answers. Let $Q = \{P_1, \ldots, P_n\}, Q' = \{P'_1, \ldots, P'_m\}$, $P_i, P'_j \subseteq W$, we define an order $\leq \in \wp(\wp(\wp(W)))^2$ in the following way:

$$Q? \leq Q'? \iff \{P_1, \ldots, P_n\} \leq \{P'_1, \ldots, P'_m\} \iff \forall P_i \in Q \exists P'_j \in Q' : S3P_i \leq S3P'_j$$

This is a natural ordering of questions which depends on the information available in the epistemic structure considered. However, it turns out that the standard ordering of questions is just a particular case that can be obtained as follows: Let $P_R$ be the partition induced by $R$ on $W$, $|P_R| = c$ be number of equivalence classes in $P_R$ and $|W| = s$ be the size of $W$, i.e. the number of possible worlds. When $R = \text{Id}(W) = \Delta(W^2)$ we have $|P| = c = s = |W|$, and we obtain as a particular case the standard ordering between questions:

$$Q \leq Q' \iff \forall P \in Q, \exists P' \in Q' : P \subseteq P'$$

In the following application we will investigate a scenario in which both kinds of sources of information and their corresponding questions coexist. In this setting it becomes important from a strategic point of view to choose the right source to address the question, or the most efficient oracle, which is also a crucial aspect in designing efficient questioning strategies in inquiry.
Epistemic Games with Subjective Agents and Objective Environment

We start with an epistemic situation where three agents $a, b, c$ have various levels of knowledge about three facts $p, q, r$, as follows: $a$ knows $q$, $b$ knows $p$ and $c$ is fully informed. This epistemic situation can be represented in a relational structure like the one depicted in Figure 4.2.

We specify an epistemic game to be played inside this structure by the following components: the moves that the agents are allowed to make in the game and the goals that each player tries to achieve by making choices during the game. As described in previous sections both of these components are formulas from the specified epistemic language discussed before.

Each questioning move is paired by a truthful and informative answer to it. In the current setting the answer can come from any agent that has the requested information and was addressed by the question. Each player’s payoff is computed in the usual way at a given world inside the epistemic model that results after the initial model is updated with the complete information contained in all the answers for all the asked questions and from every information source.

An intuitive illustration of the game just described is given in Figure 4.3. Although this is not going to be our concern in this section we mention the fact that the given strategies could have been obtained in a uniform way as a result of a meaningful set of pragmatic pre- and/or post-conditions for epistemic actions. For our present purpose it suffices to consider all the game components as primitive. The formal definition is as follows (we use here $p?(!b)$ as a shortcut notation for a sequence in which $a$ asks $p?$ and $b$ answers, and, similarly, $r?(!c)$ for a sequence in which $b$ asks $r?$ and $c$ answers accordingly):

- The set of players is: $N = \{a, b, c\}$,
- The strategy-sets are: $S_a = \{p?(!b), r?(!c)\}, S_b = \{p?(!a), r?(!c)\}, S_c = \{\}$,
- The goal-formulas for each of the players are: $\gamma_a = K_a p, \gamma_b = K_b q, \gamma_c = K_c \perp$
- The payoff for each player is computed by the following function:

$$p^w_i(s_i, s_{-i}) = \begin{cases} 1 & \text{if } M \models s_i \models_w \gamma_i \\ 0 & \text{otherwise} \end{cases}$$
Chapter 4. Games with Questioning Moves

Figure 4.3: Question-answer updates and local games in the epistemic model $M_7$, where $M_??_|\varphi$ for $\varphi := \bigwedge (s_i, s_{-i})$. We denote by $\bigwedge (s_i, s_{-i})$ a strategy profile, in which $s_{-i}$ is the tuple of strategies for players other than $i$. The notation $\bigwedge (s_i, s_{-i})$ is a shortcut for $s_a \land \cdots \land s_c$.

The local state-games that result from this definition in each world of the $M_7$ model are depicted in Figure 4.3. To economize on space we use an abbreviated notation that assumes but does not represent explicitly the strategies and the payoff values for player $c$, which will be 0 in all situations given the goal $\gamma_c$.

The local state-games played in each of the worlds of the epistemic model $M_7$ are represented in Figure 4.4. The Nash equilibria in these games are underlined. We can notice that the distinction between de re and de dicto concepts remains pertinent also for games in which strategies are questions and answers in the same way they were for games in which strategies are public announcements. Note also
that even if the strategies in the normal form for the state-games are questions in
fact the effect of the corresponding answer is the one that changes the epistemic
structure. The semantic effects of questions are not yet modeled at this stage.

Note also that in this setting the same strategy can have different semantic
effects in various worlds of the model. For instance, asking the question $r?$ in
world 7 will lead to a truthful and informative announcement of $r!$ and the cor-
responding update to a model with a domain containing only worlds with odd
numbers: $\{7531\}$. In contrast, asking the same question $r?$ in world 0 will lead
to a truthful and informative announcement of $r!$ and the corresponding update
to a model with a domain containing only worlds with even numbers: $\{6420\}$.
The more general inequality $dom(M \mid \varphi_0?\ldots,\varphi_n?) \neq dom(M \mid \varphi_0?\ldots,\varphi_n?)$ is the reason why the same strategy profile can lead to different payoff values.

We continue with an analysis of the induced Q-A game. Conceptually, the
induced game is the game that is played globally in the whole epistemic model
not just locally in each world of the model. Its formal definition is as follows:

- The set of players is: $N = \{a, b, c\}$,
- The strategy-sets contain all strategies uniform on information cells:

\[
\begin{align*}
S_a = \{S_a^0, S_a^1, S_a^2, S_a^3\} & \quad S_b = \{S_b^0, S_b^1, S_b^2, S_b^3\} \\
S_a^0 = \{(7632, p?), (5410, p?)\} & \quad S_b^0 = \{(7654, q?), (3210, q?)\} \\
S_a^1 = \{(7632, p?), (5410, r?)\} & \quad S_b^1 = \{(7654, q?), (3210, r?)\} \\
S_a^2 = \{(7632, r?), (5410, p?)\} & \quad S_b^2 = \{(7654, r?), (3210, q?)\} \\
S_a^3 = \{(7632, r?), (5410, r?)\} & \quad S_b^3 = \{(7654, r?), (3210, r?)\}
\end{align*}
\]
- The goal-formulas for each of the players are: $\gamma_a = K_a p, \gamma_b = K_b q, \gamma_c = K_c \bot$
- The payoff for each player is computed by the following function:

\[
p_i^M(s_i, s_{-i}) = \frac{\sum_{w \in W} p_i^w(s_i(w), s_{-i}(w))}{|W|}.
\]

If we compute the players’ payoffs in the local state games for each world and
strategy profile according to this definition we obtain the following results:
Epistemic Games with Oracles in which no Nash equilibrium can be found.

Proposition (No Nash Equilibrium).

4.3.2. By putting together all the obtained values we can construct the following payoff matrix for the induced Q-A game (again player $c$’s payoffs are omitted):

<table>
<thead>
<tr>
<th>$(S_a, S_a)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>$(p_a^M, p_b^M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S_0^a, S_0^a)$</td>
<td>0.0, 0.0, 0.1, 0.1, 1.0, 1.0, 1.1,</td>
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<tr>
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<tr>
<td>$(S_0^a, S_0^a)$</td>
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<td>0.1, 0.25, 0.25</td>
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<tr>
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<td>0.1, 0.00, 0.50</td>
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<th>6</th>
<th>7</th>
<th>$(p_a^M, p_b^M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S_0^a, S_0^a)$</td>
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<td>0.0, 0.0, 0.0, 0.25, 0.25</td>
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<tr>
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<td>0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,</td>
<td>0.0, 0.0, 0.0, 0.1, 0.25, 0.25</td>
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<tr>
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<td>0.0, 0.00, 0.25</td>
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<tr>
<th>$(S_a, S_a)$</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>$(p_a^M, p_b^M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S_0^a, S_0^a)$</td>
<td>0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,</td>
<td>0.0, 0.0, 0.0, 0.25, 0.25</td>
<td></td>
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</tr>
<tr>
<td>$(S_0^a, S_0^a)$</td>
<td>0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,</td>
<td>0.0, 0.0, 0.0, 0.1, 0.25, 0.25</td>
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<td></td>
</tr>
<tr>
<td>$(S_0^a, S_0^a)$</td>
<td>0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,</td>
<td>0.0, 0.00, 0.00</td>
<td></td>
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</tbody>
</table>

By putting together all the obtained values we can construct the following payoff matrix for the induced Q-A game (again player $c$’s payoffs are omitted):

<table>
<thead>
<tr>
<th></th>
<th>$S_0^b$</th>
<th>$S_1^b$</th>
<th>$S_2^b$</th>
<th>$S_3^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0^a$</td>
<td>0.50, 0.50, 0.50, 0.25,</td>
<td>0.50, 0.25,</td>
<td>0.50, 0.00</td>
<td></td>
</tr>
<tr>
<td>$S_0^a$</td>
<td>0.25, 0.50, 0.25,</td>
<td>0.25, 0.25,</td>
<td>0.25, 0.00</td>
<td></td>
</tr>
<tr>
<td>$S_0^a$</td>
<td>0.25, 0.50, 0.25,</td>
<td>0.25, 0.25,</td>
<td>0.25, 0.00</td>
<td></td>
</tr>
<tr>
<td>$S_0^a$</td>
<td>0.00, 0.50, 0.00, 0.25,</td>
<td>0.00, 0.25,</td>
<td>0.00, 0.00</td>
<td></td>
</tr>
</tbody>
</table>

As expected, the only Nash equilibrium in this game is given by the strategy profile in which every player asks the question about its goal formula.

We conclude the analysis of the induced game and close this section about Nash equilibria in Question-Answer epistemic games with the following result:

4.3.2. Proposition (No Nash Equilibrium). There exist Question-Answer Epistemic Games with Oracles in which no Nash equilibrium can be found.

---

2The notion of Nash equilibrium we will use here is the standard one, see Definition 6.6.1 in the Theoretical Background appendix 6.6 and/or the previous footnote in this chapter.
4.3. Questioning Games with Oracles

4.3.1. Proof. This fact can be witnessed by changing the following goal formulas in our running example throughout this section:

\[ \gamma_a = (p \leftrightarrow q) \land W_a r \land (W_a p \lor W_b q) \]

and symmetrically \( \gamma_b \) for the second player \( b \), and where \( W_i \varphi = K_i \varphi \lor K_i \overline{\varphi} \).

The rest of the proof consists in building the corresponding game matrix. We include all the details in Section 4.7.

A natural question at this point is whether or not a similar result can be obtained for games with question-answers without oracles that were discussed in Section 4.2. Conceptually there seems to be no difference between these, however, in practice it is more difficult to obtain and display a similar result in the initial setting of question answer games. We have to leave this as an open question, and we conclude this topic with merely the following tentative result:

4.3.3. Conjecture. An example of a Question-Answer epistemic game with no Nash Equilibrium exists, and it will have at least 256 strategy profiles.

Nash Equilibria under Syntactic Restrictions The inexistence result for Nash equilibrium in the previous section can be interpreted as an undesirable situation if a design of an inquiry strategy is meant to offer incentive for cooperation in research. Therefore it would be of interest to have a way of identifying inquiry patterns in which Nash equilibria can be always found under special restrictions.

Positive goals and most informative answers Consider the fragment of the positive formulae of \( L \): \( \varphi ::= p \mid \neg p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid K_i \varphi \mid [\varphi] \varphi \) where \( p \in \Theta \). This notion of positive formulae is found in [103]. It is an extension of [91] who observed that purely epistemic, i.e. without announcement operators, positive formulae are preserved under sub-models. This leads immediately to the result that, if both players have goal formulae that are positive, every pointed question-answer game has a Nash equilibrium. We can easily grasp why this is true. Let \( s \) be the actual state of the Kripke model, for which we play the game. If player \( i \) asks a question to player \( j \) that makes \( j \) reveal all he knows, either \( i \)'s goal is now realized, or there is no way she can realize that goal. This is because if there were a weaker announcement by \( j \) also realizing that goal, a further model restriction would preserve the goal, as it is positive. Therefore, asking the question that

\[ \text{The number of strategy profiles in the next conjecture can be justified as follows: a model constructed starting from only two propositional atoms cannot produce the dynamics described in Proposition 4.3.2. So at least three atoms are needed. A model in which any of the players is fully informed cannot produce the dynamics described in Proposition 4.3.2. So each agent has to have an epistemic partition with at least two equivalence classes. In a model with eight worlds and two equivalence classes an agent has at least } 2^3 \times 2 \text{ strategies in the induced game.} \]
elicits the most informative answer is a weakly dominant strategy for $i$, for the pointed question-answer game for state $s$. And as this holds for both players, that must be a Nash equilibrium of the pointed question-answer game for state $s$. As $s$ was arbitrary, this holds for all pointed games.

4.3.4. Proposition. Whenever both players’ goals are positive formulae, each pointed question-answer game has a Nash equilibrium.

However, what question elicits the most informative answer might not be known to either of the players, as they may be unable to distinguish the actual state from another state in which the same question does not elicit the most informative answer. This makes for another difference between games with questioning moves and games with announcement moves. This is also of more general interest than just for positive goal formulae.

**Most informative answers** We can call a strategy for a given agent weakly dominant *de re*, if the agent knows that it is weakly dominant (i.e., if in all states indistinguishable for that agent, including the current state, that strategy is weakly dominant). The agent has a weakly dominant strategy *de dicto* if he has a weakly dominant strategy in all indistinguishable states – but now it may be a different one in each, so he cannot choose which one to execute. The *de dicto/de re* distinction is well known in the knowledge and action literature [54]. Similarly, we can distinguish a *de dicto* Nash equilibrium from a *de re* Nash equilibrium, wherein all players know what their equilibrium profiles are. We can see how these notions are useful in describing questioning games: the induced question-answer game has an equilibrium if (not necessarily only if) all pointed games have a *de re* equilibrium: each player has a uniform strategy that is weakly dominant.

This aspect reveals a real difference between public announcement games and question-answer games. From a player’s perspective, there is such a thing as a most informative announcement (tell them all you know). If all goals are positive, then the most informative announcement is a weakly dominant strategy in all points of the public announcement game. And because players know what their most informative announcement is, this is therefore an Nash equilibrium strategy of the induced public announcement game.

But the question that elicits the most informative answer from another player cannot be called the most informative question from the questioning player’s point of view. In a different state in the same equivalence class for that player, the question to elicit the most informative answer may be a different question, as the responding player may be in a different equivalence class there. So even when all goals are positive, induced question-answer games may not have an equilibrium. However, finding an example remains an open problem. We will return to the topic of describing strategic abilities for questioning actions in Section 4.5.
4.3. Questioning Games with Oracles

4.3.1 Bayesian Games

The induced question-answer game corresponds to a Bayesian game. This can be shown in an analogous way as the corresponding result for public announcement games in [2]. We merely give an overview of the argument here.

Note that the difference between a game with announcements and one with questions and answers consists only in how the strategies are defined. Once a set of strategies is fixed the remaining overall structure of the argument follows along similar directions. Given an arbitrary question-answer game we can define an associated Bayesian game as follows:

- \( N = \{1, 2\} \),
- \( \Omega = W \),
- \( \tau_a(w) = [w]_a \),
- \( T_a = \{[w]_a : w \in W\} \), where \([w]_a = \{v : w \sim_a v\}\),
- \( S_a = \{\varphi : \{[K_a\varphi]_M, [K_a\neg\varphi]_M, [\neg(K_a\varphi \lor K_a\neg\varphi)]_M\} = \\
\{[K_a\psi]_M, [K_a\neg\psi]_M, [\neg(K_a\psi \lor K_a\neg\psi)]_M\}\},\)
- \( \mu(v, [w]_a) = \begin{cases} 
\frac{1}{|\{w\}_a|} & \text{if } v \in [w]_a \\
0 & \text{otherwise,} \end{cases} \)

where \( u^w_a \) is the utility function of the \( w \)-local question-answer game, and \( a, \neg a \) are variables ranging over \( N \) such that \( a \neq -a \).

A signal \( t_a \) for player \( a \) corresponds to an \( a \)-equivalence class. In a Bayesian game, the combination of a player \( a \) and a signal \( t_a \) defines a virtual player \((a, t_a)\), who has the same strategies \( s_a \in S_a \) at his disposition. But this amounts to the same as our definition involving the same players \( a \) employing strategies \( s'_a \) that are uniform across equivalence classes and that are conditional from states to strategies \( s_a \in S_a \), and therefore can be also seen as conditional at a higher level from equivalence classes to strategies \( s_a \).

In our simplified modeling, all states \( w \) in a given Kripke model get equal a priori probability of \( \mu(w) = \frac{1}{|W|} \), i.e., uniform over the entire domain. (And this is their probability for all agents.) A more general approach would have a given probability distribution as a parameter in the modeling of the game.

However, a uniform distribution is a reasonable assumption from the perspective of the observer or modeler of such a multi-agent system: given common knowledge of the structure of the model, as usual in multi-S5 conditions, there is no reason to prefer one state over another one. For example, if the Kripke model represents uncertainty about card deals, and the cards are shuffled and drawn blindly from the pack by the players, there is no reason to consider any given card deal (possible world / state) more likely than any other card deal.

After receiving their signal, each agent conditionalizes the probability mass over its equivalence class, that is, \( \mu(w|t_a) = \frac{1}{|\{w\}_{-a}|} \) (in fact, we can assume the received signal \( t_a \) to be the corresponding equivalence class \([w]_{-a}\)). This means
Chapter 4. Games with Questioning Moves

that for each state inside the class the probability is non-zero and uniform in that
class, and for each state \( t \) outside that class, the probability is 0. (Of course, we
are now talking about the probability for a given agent.)

For example, continuing our parallel with card games, after the cards have
been dealt and a player has picked up his cards, the agent only considers card deals
possible wherein she holds that hand of cards, but no longer any of the remaining
card deals. Further to this, in the absence of information to the contrary (i.e.,
assuming ‘fair play’) each of the possible deals of cards wherein she holds that
hand of cards will be also considered to be equally likely.

We will consider scenarios in which this is not the case anymore in the setting
with probabilistic extensions to DELQ from Chapter 7.

This provides the key to view an induced question-answer game as a Bayesian
game. In induced games the payoff for agent \( a \) is computed as

\[
u_a(s'_1, s'_2) = \frac{\sum_{w \in W} u^w_a(s'_1(w), s'_2(w))}{|W|}\]

whereas for a Bayesian game, one would get a sum (see [2])

\[
\sum_{w \in W} \mu(w|t_a) u^w_a(s'_1(w), s'_2(w)) = \sum_{w \in W} \frac{u^w_a(s'_1(w), s'_2(w))}{|[w]_{a}}
\]

Although these sums may be different, they induce the same order on payoffs
and thus they induce the same Nash equilibria.

4.3.5. Proposition. Any induced question-answer game has a corresponding
Bayesian game, and for a given model and fixed goal formulae for the players
they have the same Nash equilibria.

So far in this chapter we have introduced a formal model for question-answer
games and we have studied both logical and game theoretical properties that
emerge in such a setting. Such results are very useful as a starting point.

However, we still have two crucial unfulfilled modeling desiderata on our list:
a model that uses genuine questioning actions, represented by the previously
discussed issue relation, and a model of questioning actions as moves in an inquiry
process, which consists of long term sequences of questioning moves.

We will address both these desiderata in further detail during the following
section. These will provide a more general setting adequate for modeling more
realistic questioning scenarios in both inquiry and communication.

4.4 Extensive Questioning Games

Introduction Questions are ubiquitous in communication and social interac-
tion and they are essential for inquiry and scientific research. One possibility to
4.4. Extensive Questioning Games

Approach questions is by their intricate connection with other informative actions in their natural linguistic or scientific environment. This was the perspective followed so far in the current chapter and it was closely related to a more general approach which was initiated by [1, 2] and previous ones going back to [102].

Besides the fact that questions are intertwined with various kinds of informative actions which have been extensively studied inside DEL, there is still place for an approach that takes questioning actions as primitive entities. For all relevant purposes questions cannot be studied completely independently of answers, announcements or other information providing activities, however, it is possible and desirable to identify and separate the information seeking essence of questioning actions, and acknowledge their intrinsic important role in strategic reasoning and rational interaction. This motivates recent approaches that study questions and actions of issue-management inside the DEL paradigm like [100, 105], and previous ones going back to [8].

There are two main desiderata previously discussed that are going to be addressed during this section. The first is to capture genuine questioning using a model based on issue relations, the second one consists in defining a formal model for sequential questioning games.

Questioning actions are knowledge-seeking actions, they do not give information straight away but raise and manage issues or highlight possible alternatives for future informative actions. This will also be the perspective followed in the remaining of this chapter which will study questioning actions in a logical and a game-theoretical framework as long-term rational and interactive activities. The aim of this section is to proceed in this direction by defining and studying formally extensive games in which moves available to players are questioning actions and players seek to achieve epistemic goals.

4.4.1 Some Preliminary Notions

We will use epistemic issue structures as the basis for our investigations:

4.4.1. Definition. [Epistemic-Issue Model] An Epistemic-Issue Model (EIM) is a tuple \((W, N, (\approx_i, \sim_i)_{i \in N}, P, C, V)\) with the following components: \(W\) a set of possible worlds or epistemic alternatives, \(N = \{1, 2, \ldots, n\}\) a set of labels representing agents, \(\approx_i \in W \times W\) a binary issue relation on the domain \(W\), for \(i \in N\), \(\sim_i \in W \times W\) a binary indistinguishability relation on \(W\), for \(i \in N\), \(P\) and \(C\) a sets of symbols representing propositions and nominals, respectively, such that \(P \cap C = \emptyset\) and \(|V(c)| = 1\), for any \(c \in C\), \(V : P \cup C \rightarrow \wp(W)\) a standard propositional valuation function. A pointed EIM \((M, w^*)\) is an EIM together with a designated world \(w \in W\). A set-pointed EIM \((M, Q^*)\) is an EIM with a designated set of worlds \(Q \subseteq W\).
Chapter 4. Games with Questioning Moves

Intuitively, such structures model both the uncertainties agents have about the world and also their agenda for inquiry, or the issues they would like to have resolved by future answers to their questions. Another important feature in our EIMs is the use of nominals. These are propositional symbols which are true in only one world, thus naming it. We will use the symbol \( K_{\text{eim}} \) to refer to the class of all epistemic issue models (EIMs). In this section we assume that \((\sim_i, \approx_i)_{i \in N}\) are equivalence relations on the domain \( W \).

Such basic structures will serve the purpose of the present approach but they can be enriched in various ways, for instance, by adding more components to the structure in order to represent beliefs alongside knowledge and issues. This can be done either by using a plausibility relation between states or by introducing a probability distribution over the domain of possibilities.

In order to talk about EIMs we will introduce a logical language that can describe the epistemic-issue structure in a static way:

4.4.2. Definition. [Static Language] The static language of Epistemic Logic of Questions (ELQ), denoted by \( \mathcal{L}_{\text{ELQ}} \), is defined by the following BNF:

\[
\varphi ::= i \mid p \mid \neg \varphi \mid \varphi \land \varphi \mid Q_a \varphi \mid R_a \varphi \mid K_a \varphi
\]

with \( P \) a set of propositional symbols, \( N \) a set of nominal symbols, \( P \cap N = \emptyset \), \( A \) a set of agent-labels, \( p \in P, i \in N \) and \( a \in A \).

Various fragments of this language will be referred to in various places below using the following notation: \( \mathcal{L}_{\text{EL}} \) will denote the language of epistemic logic, which is the fragment without nominals and the issue \( Q \) and intersection \( R \) modalities. \( \mathcal{L}_{\text{HL}} \) will denote the language of hybrid logic, which is the fragment without the knowledge, issue and intersection modalities.

The semantics for our language is the standard modal semantics, using the usual Boolean clauses and the standard relational clauses using \( \approx \) for \( K \). The intersection modality \( R \) is be defined using \( \approx \cap \sim \) as:

\[
M \models_w R \varphi \quad \text{iff} \quad \text{for all } v \in W : w (\sim \cap \approx) v \text{ implies } M \models_v \varphi
\]

This basic language will serve the purpose of this section, it can also be extended if needed. Richer versions usually include the universal modality \( U \varphi \) and group notions for knowledge \( C_G \varphi \) or even issue or intersection.

Validities in EIMs are captured by the axioms given in Definition 4.4.3. This are standard axioms for hybrid logic with nominals and intersection. The intersection axiom makes crucial use of nominals, for the left to right direction. The substitution rule has to keep track of both propositions and nominals.

4.4.3. Definition. [Axiomatization] The ELQ proof system contains:

- minimal modal logic axioms for epistemic and issue modalities:
4.4. **Extensive Questioning Games**

\[-\Box \neg p \leftrightarrow \Diamond p, \Box(p \to q) \to (\Box p \to \Box q), \Box \in \{Q, R, K\},\]

- standard S5 axioms for for epistemic and issue modalities:
  \[p \to \Box \Diamond p, p \to \Diamond p, \Diamond \Diamond p \to \Diamond p, \Diamond \in \{\breve{Q}, \breve{R}, \breve{K}\},\]

- an intersection axiom for the static resolution modality:
  \[\breve{R}i \land \breve{Q}i \leftrightarrow \breve{R}i, \text{where } i \in N \text{ is a nominal},\]

- an axiom governing the behavior of nominals:
  \[\Diamond(i \land p) \to \Box(i \to p), \Box \in \{Q, R, K\}\]

- together with standard derivation rules for hybrid logic:

\[
\begin{align*}
&\vdash \varphi, \sigma_{\text{sort}}(\varphi) = \psi \quad \vdash \varphi, \vdash \varphi \to \psi \\
&\vdash \Box \varphi, \vdash \Box \varphi \to \Box \Box \varphi, \vdash \Box \varphi, i \to \varphi
\end{align*}
\]

for \(\Box \in \{Q, R, K\}, i \text{ not occurring in } \varphi, \text{ and } \sigma \text{ a sorted substitution.}\)

Note that for ease of notation and readability alone we have omitted subscripting the \(Q, R, K\) modalities inside the axioms.

So far we have a logic that can describe knowledge and issues from a static perspective. But for the purpose of a questioning game we also need to be able to describe the way questioning actions change the knowledge of the players. We need a way to describe knowledge and issues from a dynamic perspective. In order to be able to also describe knowledge-change and issue-dynamics and, moreover, the interaction between the two during a game, we will introduce the following model-changing actions:

4.4.4. **Definition.** [Questioning] Let \(M = \langle W, (\overset{\approx}{i}, \overset{\sim}{i})_{i \in N}, V \rangle\) be an arbitrary EIM. A question action, represented as \(\varphi?,\) changes the model \(M\) into a new model \(M \otimes \varphi? = \langle W_{\varphi?}, (\overset{\approx}{i}_{\varphi?}, \overset{\sim}{i}_{\varphi?})_{i \in N}, V_{\varphi?}\rangle,\) in the following way:

\[
\overset{\approx}{i}_{\varphi?} = \overset{\approx}{i} \cap \overset{\approx}{\Xi}_M
\]

while leaving the other components unchanged, where the set of pairs of worlds in \(M\) with equivalent \(\varphi\) value is denoted by \(\overset{\approx}{\Xi}_M = \{(w, v) : \|\varphi\|_w^M = \|\varphi\|_v^M\} .\)

Intuitively, this action says that a questions changes a model by refining the issue partition with the content of a formula \(\varphi\) in a specified language. Such a partition-based approach goes back to [40] where it was used for giving semantics of questions in natural language; here we add a dynamic dimension to questioning actions and place them in a more abstract framework of inquiry and scientific discovery. We also add another dynamic action:
4.4.5. Definition. [Resolution] Let $M = \langle W, (\approx, \sim), i \in N, V \rangle$ be an arbitrary EIM. A resolution action, represented as $!$, changes the model $M$ into a new epistemic-issue model $M \otimes ! = \langle W', (\approx', \sim', i) i \in N, V' \rangle$, in the following way:

$$\approx' = \approx \cap \sim$$

while leaving all the other components in the initial model $M$ unchanged.

Intuitively, the resolution action transforms the model to a situation in which all the questions raised so far would have received one answer or another. This is done by changing the underlying uncertainty to reflect the structure of the issue relation in a given situation.

We will enrich our initial static modal language with two more dynamic modalities which will make reasoning about the flow of information, issue management and question answering possible in a formal way.

4.4.6. Definition. [Dynamic Language] By adding two dynamic modalities: $[\varphi]?\psi$ and $![\psi]$ to the BNF from Definition 4.4.2 we obtain the full language of Dynamic Epistemic Logic of Questions ($DELQ$), denoted by $L_{DELQ}$.

Intuitively, a formula like $[\varphi]?\psi$ says that “after a questioning action about $\varphi$ is performed, $\psi$ holds” or “after asking weather $\varphi$, $\psi$ is the case”. Similarly, $![\psi]$ says that “after performing a resolution action, $\varphi$ holds”.

A note about notation: The $[\varphi]?$ notation is already familiar from Chapter 2 where was used to denote dynamic modalities. Previously in this chapter we used $\varphi?$ to refer to player’s strategies in a game. From now on strategies in a game are also going to be questioning actions, nevertheless, we will try to use the square parenthesis in a consistent way to disambiguate between the two whenever the context might require additional precision.

The semantics of these dynamic modalities is also specified in the standard way using the previously introduced model transformations. We will only need here the minimal system given by the following clauses:

$$M \models_w [\varphi]?\psi \iff M \otimes ? \models_w \psi,$$

$$M \models_w ![\psi] \iff M \otimes ! \models_w \psi,$$

The transformed models are computed as in the action definitions previously introduced. This language can be extended if needed.

4.4.7. Definition. [Reduction Axioms] Formulas in $L_{DELQ}$ can be reduced to equivalent formulas in $L_{ELQ}$ using the following reduction axioms:

- $[\varphi]?a \leftrightarrow a$ (Asking & Atoms), $![a] \leftrightarrow a$ (Resolving & Atoms),
- $[\varphi]?\neg \psi \leftrightarrow \neg[\varphi]?\psi, ![\neg \varphi] \leftrightarrow \neg ![\varphi]$ (Asking/Resolving & Negation),
4.4. Extensive Questioning Games

- \([q](\psi \land \chi) \leftrightarrow [q]\psi \land [q]\chi\), \(q \in \{\varphi, !\}\) (Asking/Resolving & Conjunction),
- \([\varphi?]Q\psi \leftrightarrow (\varphi \land Q(\varphi \rightarrow [\varphi?]\psi)) \lor (\neg \varphi \land Q(\neg \varphi \rightarrow [\varphi?]\psi))\) (Asking & Issue),
- \([\varphi?]R\psi \leftrightarrow (\varphi \land R(\varphi \rightarrow [\varphi?]\psi)) \lor (\neg \varphi \land R(\neg \varphi \rightarrow [\varphi?]\psi))\) (Asking & Intersection),
- \([!]Q\varphi \leftrightarrow Q[!]\varphi, [!]R\varphi \leftrightarrow R[!]\varphi\) (Resolving & Issue/Intersection),
- \([\varphi?]K\psi \leftrightarrow K[\varphi?]\psi, [!]K\varphi \leftrightarrow R[!]\varphi\) (Asking/Resolving & Knowledge).

All these provide a way of reasoning about dynamic actions of asking questions and resolving issues. We will now proceed to defining formally games in which moves are questioning actions as described so far.

4.4.2 Definitions of Main Notions

In this section we give formal definitions for extensive questions games, building on the preliminary notions introduced so far. Next we proceed by giving some intuitive examples and illustrations of EQGs. After that we discuss modeling choices, make some conceptual clarifications, and study solution concepts, strategic abilities and other formal properties of EQGs.

In EQGs, like in any other game or interactive activity, players have objectives or goals and they try to reach these objectives by making appropriate moves during the play of the game. In a setting involving questions such goals will have an epistemic character: players want to acquire some knowledge, but these can also be complex higher-order epistemic situations, say “I want to find out something in a way that prevents you from knowing that I know”, and so on. Players can also have competing, convergent or even incompatible goals. We represent such aspect formally by epistemic formulas.

4.4.8. Definition. [Epistemic-Issue Goal Structure] An Epistemic-Issue Goal Structure (EIGS) is a tuple \(\langle M, G \rangle\) with the following components:

- \(G = \{G_1, G_2, \ldots, G_n\}\) a set of goal-sets, one for each agent \(i \in N\), containing formulas \(G_i = \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{ik}\}\) from \(\mathcal{L}_{EL}\), and \(M\) an EIM.

An EIGS is (set)-pointed if and only if \(M\) is a (set)-pointed EIM.

Achieving such epistemic goals depends on moves made during the game, but it also depends on the structure of the epistemic-issue model in which such a game is played. Therefore, any EQGs will be characterized by the local epistemic perspective of each of the players involved.

Such goal structure need not have the simplest structure considered here, it is realistic to assume that agents might also have their research agenda structured in a certain way, for instance by a priority order over the goals or even a more complex graph structure with formulas as vertices.
Questioning and resolution moves made during the play of the game will change the structure of the initial epistemic situation. Such transformations have an interactive character, they are determined by a combination of choices for each of the players. The resulting model can be computed using the model-changing operations introduced in the previous section.

Finally, we can determine if players achieved their objectives by model checking their goal-formulas in the resulting epistemic-issue structure. We use here a language containing only epistemic modalities for simplicity, but it is conceivable that agents might also have objectives with regard to the structure of the issue at the end of a questioning game, such aspects could be also modeled using a richer language for the goal formulas. We capture all these aspects in a formal way by means of the following definition:

4.4.9. Definition. [Extensive Question Game] The Local Extensive Question Game (LEQG) associated with a pointed EIGS $E = \langle (M, w), G \rangle$ is a tuple $T = \langle E, H, J, F, (U_i)_{i \in N} \rangle$ with the following components:

- $H = \{h_0, h_1, \ldots, h_z\}$ a set of histories such that:
  - $h_i = \emptyset$, for some $i \in \{0, \ldots, z\}$,
  - if $h = (q_0, q_1, \ldots, q_{l-1}, q_l) \in H$, then $l + 1 \leq |N|$,
  - if $h = (q_0, \ldots, q_{l-1}, q_l) \in H$ and $h' = (q_0, \ldots, q_{l-1}, q_l)$, then $h' \in H$,
  - if $h = (q_0, \ldots, q_{l-1}) \in H$ and $h' = (q_0, \ldots, q_{l-1}, q_l)$, then $h' \in H$, for $q_l = \min([\varphi]_M)$, and $[\varphi]_M = \{\psi \in L_{HL} : \|\varphi\|_M = \|\psi\|_M \subseteq W\}$

- $J : H \setminus Z \rightarrow N$ a turn-function assigning players to non-terminal histories, s.t. $J(h) = J(h')$ iff $|h| = |h'|$, with $Z = \{h \in H : |h| = |N|\}$,

- $F : H \rightarrow K_{\text{eim}}$ a model-function assigning EIMs to histories as follows:
  $$F(h) = \begin{cases} M & \text{if } h = \emptyset \\ F((q_0, q_1, \ldots, q_{l-1})) \otimes q_l & \text{otherwise,} \end{cases}$$

- $U_i : Z \rightarrow \mathbb{R}$ a utility-function assigning utilities for player $i \in N$ to terminal histories in the following way:
  $$U_i(h, \varphi) = \begin{cases} 1 & \text{if } M_h \otimes ! =_w \varphi \text{ and } U_i(h) = \sum_{\gamma_k \in G_i} U_i(h, \gamma_k) \text{.} \\ 0 & \text{otherwise} \end{cases}$$

This definition describes the outcome of a questioning game as played in a given possible world. But since the players have epistemic uncertainties, they do not know precisely which is the real situation, hence what is the game that they are playing. In any given state they also consider a number of many other epistemic alternatives. Therefore, we also need a method to aggregate the payoffs obtained in many local games in an expected utility given the entire structure of the initial epistemic-issue model.
4.4. Definition. The Set-Local Extensive Question Game (SEQG) associated to a set-pointed EIGS $E = \langle (M, Q), G \rangle$ is defined as the corresponding LEQG (Definition 4.4.9) with the following modification of utility function: $U_i : Z \to \mathbb{R}$ assigns utilities for $i \in N$ to terminal histories as follows:

$$U_i(h) = \frac{\sum_{w \in Q} U_i(h, \gamma^i_k, w)}{|Q|}$$

where $U_i(h, \gamma^i_k, w) = U_i(h, \gamma^j_k)$ in the LEQG associated with $E = \langle (M, w), G \rangle$. The Global/Induced Extensive Question Game (GEQG) associated with an arbitrary EIGS $E = \langle M, G \rangle$ is the SEQG $E = \langle (M, Q), G \rangle$ in which $Q = W$.

We discuss briefly some of the modeling choices in this formal definitions before we proceed to some concrete examples and illustrations. We already mentioned the importance of the language in which goals are formulated. The same holds for the language in which questioning moves are formulated during the play of a game. The set of available actions at a given history depends on the available questioning actions. It also depends on various pragmatic and epistemic preconditions that one might want to capture in the formal model. We have made here a very general choice by using a very expressive language and by ignoring any pragmatic or epistemic preconditions for question execution. Further, more realistic, options and the consequences that derive from them are discussed in later sections. Also, we assume that an order on the set of formulas is available when choosing the minimal representative from an equivalence class of formulas. We can also think about versions in which resolution actions are available at non-terminal histories, or versions in which players get more than one question during one play, such options are captured by sequential compositions of EQGs as defined here.

Also we make no assumption with regard to the sources of answers that are available in the game. Games in which answers can only come from other players versus versions in which answers can come from nature or an oracle can both be captured by means of pragmatic, epistemic or other arbitrary restrictions on the questioning and resolution actions.

Another important aspect about interactive questioning activities is their cooperative versus competitive character. We can imagine settings in which competing research programs play a questioning game with nature or against each other as well as situations in which inquiry scenarios have a cooperative setting and the only design requirement is the efficiency of acquiring new knowledge in a coalition of convergent research programs. All such aspects can be captured and described formally using our definitions. We discuss such issues in greater detail in later sections.
4.4.3 Examples and Illustrations

We will resort to intuitive examples to illustrate the formal definitions. Consider the very simple concrete example with just two agents and two facts as depicted in Figure 4.5 below. The initial model is represented in the top-left corner. Questioning and resolution moves are indicated with arrows labeled accordingly.

Interesting phenomena can be already described in this simple setting. For instance, we can compare two interrogative scenarios with respect to their fairness as cooperative experimental procedures, depicted in Figure 4.5.

![EQG example of fairness in cooperative experimental procedures](image)

If we describe an interrogative structure using the set of sequences of available moves, or histories in the game, we can notice that, for instance, the following two experimental protocols are very different in terms of fair-learning during the cooperative inquiry process. In the first scenario, represented by the protocol

\[Q_1 = \{p?, q?, p?\top?, q?\top?, p?q?, q?p?\}\]

one player can become fully informed about the world before the other one does, while in the second one

this possibility is ruled out, players become fully informed only simultaneously. For instance, the goal formula

\[ \gamma_b = (K_ap \lor K_a \neg p) \rightarrow (K_b q \lor K_b \neg q) \]

and its symmetric counterpart, are both true in the corners on the main diagonal and are false, respectively, in both corners on the secondary diagonal.

The figure also illustrates another interesting feature: extensive games can be composed to form sequences. Each corner of the second diagonal is the endpoint for the previously described game. However, the resulting model can be the starting point for a new extensive game, starting in the resulting model from previous plays as illustrated and proceeding further by new questioning actions.

Even more complex situations and subtleties about research procedures can be captured using the modeling strength of various procedural restrictions in the available experimental protocol, see [97] for such an account in a setting with informative epistemic actions. All such facts about what players can achieve by playing questioning games can be captured in a systematic way using a logic of strategic ability. We will discuss some of these aspects in a later section. Before that we will focus on some properties of EQGs.

Considering a setting in which questions are represented explicitly by means of an issue relation has further advantages. For instance, we can define a measure of relevance of questions in inquiry. Previous notions of relevance exist in the literature, in the present setting, however, we can capture strategic aspects resulting from both the general goal of inquiry and the available sources of information.

Both can be described as partitions of the epistemic domain. Let \( W \) be a set of possible worlds, \( R, R', R'' \in W \times W \) equivalence relations on \( W \), and \( K, Q, G \) the modalities for \( R, R', R'' \), respectively.

Let \( P_R = \{C_1, \ldots, C_n\} \), \( P_{R'} = \{C_1, \ldots, C_n'\} \), \( P_{R''} = \{C_1, \ldots, C_n''\} \) be the partitions induced in \( W \) by \( R, R', R'' \), respectively. Let \( \text{Par}(W) \) be the set of \( W \)-partitions. For \( P, P' \in \text{Par}(W) \) we define \( \cap : \text{Par}(W)^2 \rightarrow \text{Par}(W) \) by:

\[
S = P \cap P' = \{C_1, \ldots, C_n\} \cap \{C_1', \ldots, C_n'\} = \bigcup\{C_i \in P \mid \exists C_i' \in P' : C_i \subseteq C_i'\}
\]

Let \( P_{R'} = \{C_1, \ldots, C_n'\} \) be the partition induced in \( W \) by a sequence of questioning actions \( Q = \{q_1, \ldots, q_k\} \). We define the source-relevance or evidence-relevance of \( Q \) by:

\[
R_S(Q) = |P_R \cap P_{R'}|
\]

For \( P, P' \in \text{Par}(W) \) we define \( \cap_q : \text{Par}(W)^2 \rightarrow \text{Par}(W) \) as follows:

\[
S = P \cap_q P' = \{C_1, \ldots, C_n\} \cap \{C_1', \ldots, C_n'\} = \bigcup\{C_i \in P \mid C_i \subseteq C_i' \in P'\}
\]
Let $P_S$ be the bipartition $P_S = \{P \cap P', W \setminus (P \cap P')\}$ for $Q = \{q_1, \ldots, q_k\}$. The notion of goal-relevance or inquiry-relevance of $Q$ is defined as follows:

$$R_Q(Q) = |P_{R''} \cap_q P_S|$$

These definitions of goal and source relevance of a question give rise to a natural order on question sequences:

$$Q \leq_s Q' \iff R_S(Q) \leq R_S(Q')$$
$$Q \leq Q' \iff R_Q(Q) \leq R_Q(Q')$$

4.4.11. Example. As an illustration, consider the following situation:

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\},$$
$$P_R = \{\{w_1, w_4, w_7, w_2, w_5, w_8\}, \{w_3, w_6, w_9\}\}$$
$$P_{R''} = \{\{w_1, w_4\}, \{w_9, w_6, w_3\}, \{w_7, w_2, w_5, w_8\}\}$$
$$P_Q = \{\{w_1, w_4\}, \{w_7, w_8, w_2, w_5, w_3, w_6, w_9\}\}$$
$$P_{Q'} = \{\{w_1, w_2, w_5\}, \{w_8, w_7, w_4, w_3\}, \{w_9\}, \{w_6\}\}$$

Here we have:

$$R_S(Q) = 2 < 5 = R_S(Q') \text{ and } R_Q(Q) = 2 > 0 = R_Q(Q')$$

This measure of relevance can express that $Q'$ is more relevant for the available sources of information in the inquiry, but less relevant for the overall goal of the inquiry. The situation is reversed for $Q$, it turns out to be less efficient in using the available sources of information, but more useful for solving the main goal of the inquiry. This is because not all information serving the goal is available.

Other traditional notions that are conceptually related with the notion of relevance like, for example the entropy of a question and the informativity of an answer are suitable for similar epistemic modulations.

4.4.4 Imperfect Information in EQGs

As in games with sequential moves, all actions in an EQG are observable. This means that any player is fully informed about (can distinguish between) any move made previously in the game, either by herself or by other players. This fact is captured in the formalism by the fact that there are no uncertainty lines linking EIMs belonging to different histories.

Even so, it would not be completely accurate to consider that EQGs are games of perfect information due to the fact that agents still have uncertainties during the game, however, not about the moves played but about the worlds in the epistemic model. Because of these particularities a conceptual clarification with regard to the status of imperfect information in EQGs might be of interest.
4.4. Extensive Questioning Games

4.4.12. Fact. Let \( P = \{p_1, \ldots, p_n\} \) and \( A = \{a_1, \ldots, a_m\} \) be sets of propositional atoms and agent-labels, respectively. Any epistemic-issue model can be represented as an imperfect information game in the following way:

- \( N = \{a_1, \ldots, a_m\} \cup \{c\} \), (here \( c \) is the chance player, or Nature),
- \( S_c = \varphi P = 2^{[P]}, S_i = \emptyset \) for \( i \in N \setminus \{c\} \), (Nature chooses a possible world or epistemic alternative, the other players do nothing),
- \( Q_i = \{D_1, \ldots, D_k\} \subset \varphi(\varphi P) \), for \( i \in N \setminus \{c\} \) (the players receive an arbitrary issue partition [the same]),
- \( I_c = \{\{w\} | w \in \varphi P\}, I_i = \{C_1, \ldots, C_k\} \subset \varphi(\varphi P) \) for \( i \in N \setminus \{c\} \) (each agent receives an information partition consistent with the valuation).

In the light of this fact we can say that each game with questioning moves starts from a pre-existing epistemic-issue model \( M \) and proceeds according to the previously introduced definitions. It is then of interest to find the properties that characterize precisely EQGs as games of imperfect information. There are results in the literature that characterize models generated by protocols of dynamic epistemic logic in terms of very general properties:

4.4.13. Fact. [Representation [97]] ETL models generated by state-dependent DEL protocols have the following properties: Propositional Stability, Syncronicity, Perfect Recall, Local No Miracles & Local Bisimulation Invariance. For PAL protocols we also have: Reflexive Events & Distinguished Events.

In the context of EQGs questioning actions resemble PAL and DEL actions but are in addition parametrized by a player which performs the action and the notion of bisimulation should also describe both epistemic and issue relations. We will also discuss the Difraction Property (DP) which is of interest especially for describing strategic abilities in EQGs in a later section.

4.4.5 Strategies and Solution Concepts

In this section we analyze EQGs using standard logical and game-theoretical notions and techniques. Strategies are defined in a standard way, like in [73]:

4.4.14. Definition. [Strategy] In a EQG a strategy for player \( i \in N \) is a function \( S_i : \{h \mid J(h) = i\} \setminus Z \rightarrow \{q \mid h \wedge q \in H\} \) that assigns an available question to each non-terminal history in which it is player \( i \)’s turn to move. A strategy profile \( S = (S_i)_{i \in N} \) is a tuple of strategies, one for each player. The outcome \( O(S) \) of a strategy profile \( S = (S_i)_{i \in N} \) is the history \( h = (q_0, \ldots, q_l) \in Z \) s.t. for \( 0 \leq k \leq l \) we have \( S_{J(q_0, \ldots, q_k)}(q_0, \ldots, q_k) = q_{k+1} \).
Counting strategies. In the very abstract setting used so far the number of strategies at any node is a function of the size of the model $|S_i| = 2^{|W|}$ and the number of strategies for player $i$ moving at history $h$ is $(2^{|W|})^{|h|}$.

Another standard requirement for games with imperfect information is that the players must have uniform strategies, or that the strategies take into account the players’ epistemic situation. There are various ways in which pragmatic and epistemic considerations can be used to capture this aspect in a setting of EQGs. For instance, we can add the following requirement:

$$\text{for } i = J(h') \text{ we have } F(h) \models \neg(K_i q_{l+1} \lor K_i \neg q_{l+1})$$

to item 1-4 in Definition 4.4.9, in an EIM $M$ with $|W| = m$ and where agent $i$ has $n$ equivalence classes, then we are interested in the number of informative questions for player $i$ in $M$. There are $2^{m-1}$ bi-partitions of $W$ and $2^{n-1}$ unions of equivalence classes for $i$ not including $\emptyset$ and $W$. Therefore the following set $R_i = \{ \varphi : M \models_s \varphi \land \widehat{K}_i \neg \varphi, \varphi \in L_{HL}, s \in W \}$ contains all yes/no questions that are informative for $i$ in $M$ and, up to logical equivalence, there are $r_i = |R_i| = 2^{m-1} - 2^{n-1}$ such questions.

Intuitively this says that the criterion for strategy equivalence requires that the player only asks questions about what she does not know at the global level. However we can also require that questions are about what the agent doesn’t know locally, in a given state $w$ in an information cell $C_i$, with: for $i = J(h')$ we have $F(h) \models_w \neg(K_i q_{l+1} \lor K_i \neg q_{l+1})$ in this case the set $R_i = \{ \varphi : M \models_s \varphi \land \widehat{K}_i \neg \varphi, \varphi \in L_{HL}, s \in C_i \}$ of locally informative questions depends on the structure of the information cell $C_i$ of $w$ inside the player’s information partition. And since there are $2^{|C_i|-1}$ subsets of $C_i$, not including $\emptyset$ and $W$, and $2^{|W|-|C_i|}$ subsets of $W \setminus C_i$, we get $|R_i| = (2^{|C_i|-1}) \times 2^{|W|-|C_i|}$ logically non-equivalent strategies.

Note also that the expressive power of the language we use to express the questions is also a parameter in this counting. If we replace $L_{HL}$ by another language, say $L_{EL}$, the numbers might be different, depending on the concrete structure of the model, we get the previous numbers as upper bounds on the total number of logically non-equivalent strategies.

There are multiple other modeling options that could be used here to capture reasonable pragmatic preconditions. For instance, in order to capture a sequential game structure it is very plausible to require the following:

$$\text{for all } q_k \in h \text{ and } q_{l+1} \in L_{HL}, [q_{l+1}]_{F(h)} \neq [q_k]_{F(h)},$$

to capture the idea that players do not repeat a question which, even if informative, was already asked before. For $|M| = m$, and $n$ information cells for agent $i$, the set $R_i = \{ \varphi : M \models_s \varphi \land \widehat{K}_i \neg \varphi, \varphi \in L_{HL}, s \in W \}$ contains, up to logical equivalence, $r_i = |R_i| = 2^{m-1} - 2^{n-1}$ yes/no questions that are informative for $i$ in $M$. But now the history of previously asked questions further restricts available actions, keeping $r_i$ only as an upper bound.
4.4. Extensive Questioning Games

The maximum number of allowed questions for a player $i$ moving at history $h = \{q_0, \ldots, q_l\}$ will be given by $\mathcal{d}_h = r_i \times r_{J(h^{-1})} \times r_{J(h^{-2})} \times \cdots \times r_{J(h^{-|h|})}$ where $h^{-n} = \{q_0, \ldots, q_{l-n}\}$. The total number of strategies for player $i$ at history $h$ will be at most $\binom{\mathcal{d}_h}{g_h} = \mathcal{d}_h! / (g_h! (\mathcal{d}_h - g_h)!)$ and at least $\binom{\mathcal{d}_h}{\mathcal{g}_h} = \mathcal{d}_h! / (\mathcal{g}_h! (\mathcal{d}_h - \mathcal{g}_h)!)$ for all $i \in N$ and $h \in H$ for which $J(h) = i$.

For versions in which agents ask questions to each other not only to Nature, even more pragmatic preconditions make sense, like, for instance, that the questioner considers it possible that the questionee knows the answer.

Even disregarding pragmatic constraints, another aspect that is crucial to the notion of strategy equivalence is the expressive power of the language for goal formulas. If a certain language cannot express some goals then the number of strategies having equivalent effects upon execution will decrease.

Solution concepts. The general framework introduced so far to model EQGs can be adapted to capture a variety of concrete situations. But once a notion of strategy equivalence is fixed, it is even more important to study solution concepts in EQGs. These have standard game-theoretic definitions, like in [73]:

4.4.15. Definition. [Nash Equilibrium] An EQG Nash equilibrium is a strategy profile $S^*$ s.t. for every player $i \in N$ and every strategy $s_i$ of $i$ we have:

$$U_i(O(S^*_{-i}, S^*_i)) \geq U_i(O(S^*_{-i}, S_i)).$$

We have now all the ingredients to obtain an analogous impossibility result analogous to what we had for strategic games:

4.4.16. Fact. There are EQGs in which no pure Nash equilibrium exists.

We only have to construct an appropriate example to establish this fact. We include the details in Section 4.7.

There are other solution concepts which are relevant for games in extensive form. These concepts also provide an adequate analysis for EQGs as particular cases of extensive-form games. One such solution concept, adequate for settings with sequential moves is subgame perfect equilibrium.

We also reserve an extensive discussion regarding the theoretical relevance of such an analysis in the general context of a logical approach to discovery and inquiry for a future occasion and here merely present some basic facts.

Many other interesting problems that are of interest for a general account of inquiry and scientific discovery in which questions play a genuine role emerge at this point. We end this section by listing some of the most relevant ones.
Chapter 4. Games with Questioning Moves

- Explore other solution concepts, considered to be more adequate for games with sequential moves, such as sub-game perfect equilibrium.
- Find systematic connections between existence of solution concepts and syntactic properties of goal formulas and/or strategies.
- Explore a conceptual framework for relevance of questioning and resolution in competitive and/or cooperative interrogative scenarios.

4.5 Strategic Abilities in Questioning

4.5.1 Describing Strategic Abilities

Players’ abilities to achieve certain outcomes during the play of an EQG can be also described using a logical language. Such a language will contain “forcing modalities” like, for instance, in [90]:

\[ M, s \models \langle G, i \rangle \varphi \ \text{“player } i \text{ has a uniform strategy for playing game } G \text{ starting from state } s \text{ which forces a set of outcomes satisfying } \varphi \text{ in } M \} \]

or other modalities expressing strategic abilities as in [34, 53]:

\[ M, q \models \langle \langle A \rangle \rangle \varphi \ \text{“there is a collective strategy } S_A \text{ such that } \varphi \text{ holds for every path } [...] \text{ that may result from agents } A \text{ executing strategy } S_A \text{ from state } q \text{ onward”} \]

Without introducing all the formal details of such frameworks, we mention below some well known facts and discuss their relationship with EQGs.

For perfect information games various fixed-point recursive characterizations of forcing modalities exist in the literature, like the game-logic from [90]:

\[ \langle G, i \rangle \varphi \leftrightarrow (end \land \varphi) \lor (turn_i \land \Diamond \langle G, i \rangle \varphi) \lor (turn_j \land \Box \langle G, i \rangle \varphi) \]

or the ATL fixed-point axiom for strategic ability from [34]:

\[ \langle \langle A \rangle \rangle \Box \varphi \leftrightarrow \varphi \land \langle \langle A \rangle \rangle \circ \langle \langle A \rangle \rangle \Box \varphi \]

However, if we consider imperfect information games in general these fixed-point recursive axioms are known to not be valid anymore. Given the special status of imperfect information in EQGs discussed before, it is natural to ask if such recursive axioms are valid in game structures generated by EQGs. In order to approach this question we need the following fact:

4.5.1. FACT. Any EQG induces a Concurrent Epistemic Game Structure. For EQG \( E = \langle M, G \rangle \) the corresponding CEGS \( S = \langle n, Q, \Pi, \pi, d, \delta, (\Xi)_{a \in \Sigma} \rangle \) is constructed in the following way (cf. Definition 1 in [34], also, [53] p. 440):
4.5. Strategic Abilities in Questioning

- \( n = |N| \), \( Q = W \cup (W \times \wp(W)^n) \), \( \Pi = P \),

- \( \pi(q) = \begin{cases} \{ p \in P \mid q \in V(p) \} & \text{if } q \in W, \\ \{ p \in P \mid \text{fst}(q) \in V(p) \} & \text{otherwise} \end{cases} \)

- \( d_a(q) = |\wp(W)| \), for all \( a \in \Sigma \),

- \( \delta(q, j_1, \ldots, j_n) = \begin{cases} (q, \langle j_1, \ldots, j_n \rangle) & \text{if } q \in W, \\ q & \text{otherwise}. \end{cases} \)

We only describe here one level of questioning and resolution actions but this can be generalized in a similar fashion to any number of such iterations.

**The Difraction Property.** We discussed before some general properties of EQGs, one that becomes particularly relevant in this context because of its relevance in questioning scenarios is the one we will call the *difraction property*. The name is intended to capture the basic intuition behind the notion, namely that the value of the formula diverges in alternative histories. However, the formal definition is completely independent of this intuition, it is the following:

4.5.2. **Definition.** [Difraction Property] We say that an epistemic game structure EGS satisfies the Difraction Property for \( \varphi \) (DP\( \varphi \)) if, for some histories \( h, h' \in H \), the following conditions:

- \( h \sim h', h \models \varphi \) and \( h' \models \varphi \),

- there is some action (transition) \( q \), such that \( h \sim q, h' \sim q \in H \),

imply that the following property obtains:

- \( h \sim q \models \varphi \) and \( h' \sim q \not\models \varphi \) or vice versa.

If an EGS has the DP\( \varphi \) for any formula \( \varphi \in \Gamma \) then it also has DP for \( \Gamma \). If formula \( \varphi \) has factual content (only) we have Factual Difraction (FD), if formula \( \varphi \) has issue/epistemic content, Issue/Epistemic Difraction (I/ED).

Intuitively, the difraction property states that whenever a formula \( \varphi \) has uniform truth value in indistinguishable histories, and the same action \( q \) is available in both nodes/histories (\( q \) might be an issue/learning action [question, resolution, announcement], but does not have to), it follows that, after executing the same
action $q$ in both states/histories, the truth value of $\varphi$ diverges (it is no longer the same in both resulting histories).

We used in Definition 4.5.2 a setting in which actions and transitions are assumed to coincide, such a setting is not specific to ATL but it is commonplace in game-theoretic contexts. However, Definition 4.5.2 can be straightforwardly reformulated even in a setting where actions and transitions are distinct as long as the transition function is deterministic.

DP is interesting because it seems to be essential for the failure of fixed-point axioms for strategic ability in imperfect information games. It can be shown [35] that both directions of the recursive ATL axiom characterizing strategic ability fail in the class of epistemic game structures that satisfy DP.

We show below that, in particular, the recursive axiom characterizing strategic ability holds in the class of non-DP epistemic game structures.

4.5.3. **Fact.** The axiom $\langle\langle A \rangle\rangle \Box \varphi \leftrightarrow \varphi \land \langle\langle A \rangle\rangle \Diamond \langle\langle A \rangle\rangle \Box \varphi$ is valid in the class of epistemic game structures that do not satisfy the difraction property (DP).

We include the details of the proof in the following appendix (Section 4.7).

What we still have to show is that epistemic game structures without DP indeed capture an interesting class of imperfect information games, in particular, that they contain some epistemic games with sequential moves. We show here, in particular, that EQGs do not have DP for positive formulas. The positive fragment of $EL_Q$, denoted $L_{ELQ}^+$, is defined by the following BNF:

\[
\varphi ::= i | p | \neg \varphi | \varphi \land \varphi | \varphi \lor \varphi \\
\chi ::= \chi \land \chi | \chi \lor \chi | Q_a \varphi | R_a \varphi | K_a \varphi
\]

4.5.4. **Fact.** Epistemic game structures generated by extensive question games (EQGs) do not have the difraction property (DP) for positive formulas.

We include the details of the proof in the following appendix (Section 4.7).

We assume in Fact 4.5.4 a setting with questioning and resolution actions but this is not essential for the proof, link-cutting model transformations or non world-eliminating announcements are also captured in such a setting by $[\varphi!] = [\varphi?][!]$.

4.5.2 **Conclusions and Further Research**

In this section we have defined extensive questioning games in a formal way using the general framework of dynamic epistemic logic with questioning and resolution actions and we have also introduced some illustrative examples.

In this general setting we studied solution concepts in EQGs, in particular, we have shown that there are EQGs with no pure Nash equilibrium. The possibility of describing strategic abilities of players in EQGs in a logical language was explored.
In this context we have shown how some strategic ability recursive axioms are preserved in imperfect information games without DP and that EQGs do not have DP for positive formulas.

Many interesting problems and topics for future research emerged, such as: study other solution concepts in the context of EQGs and the relation between existence of solution-concepts and syntactic structure of goal formulas, conceptual clarifications regarding notions of informativity and relevance of questioning moves in scenarios of competitive or cooperative inquiry, find all the remaining valid principles characterizing strategic ability in EQGs.

4.6 Appendix A: Background Definitions

**Public Announcement Logic** The logic of public announcements was one of the earliest developed in the DEL paradigm. It is also one of the most studied and well known members of the DEL family. Because we also used it in the first sections of this chapter we include a brief summary here.

The language $\mathcal{L}$ of public announcement logic (PAL) [79] defined over a set of agents $N = \{1, \ldots, n\}$ and a set of primitive propositions $\Theta$ is given as follows, where $i$ is an agent and $p \in \Theta$ is a propositional symbol:

$$\varphi ::= p \mid K_i \varphi \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid [\varphi_1] \varphi_2$$

We write $\langle \varphi_1 \rangle \varphi_2$ resp. $\widehat{K}_i \varphi$ for the duals $\neg [\varphi_1] \neg \varphi_2$ and $\neg K_i \neg \varphi$.

A Kripke structure or epistemic model over $N$ and $\Theta$ is a tuple $M = (S, \sim_1, \ldots, \sim_n, V)$ where $S$ is a set of states, $\sim_i \subseteq S \times S$ is an epistemic indistinguishability relation that is assumed to be an equivalence relation for each agent $i$, and $V : \Theta \rightarrow S$ assigns primitive propositions to the states in which they are true. A pointed Kripke structure is a pair $(M, s)$ where $s$ is a state in $M$. In this chapter and beyond we will also assume that Kripke structures are finite.

The interpretation of formulae from the public announcement language is defined in a pointed Kripke structure at state as follows:

$$M, s \models p \text{ iff } p \in V(p), \quad M, s \models \neg \varphi \text{ iff not } M, s \models \varphi$$

$$M, s \models K_i \varphi \text{ iff for every } t \text{ such that } s \sim_i t, \quad M, t \models \varphi; \quad \text{and}$$

$$M, s \models \varphi \land \psi \text{ iff } M, s \models \varphi \text{ and } M, s \models \psi$$

$$M, s \models [\varphi!] \psi \text{ iff } M, s \models \varphi \text{ implies that } M|\varphi, s \models \psi,$$

where $M|\varphi = (S', \sim'_1, \ldots, \sim'_n, V')$ such that $S' = \{s' \in S : M, s' \models \varphi\}$, $\sim'_i = \sim_i \cap (S' \times S')$, and $V'(p) = V(p) \cap S'$. For $\{s' \in S : M, s' \models \varphi\}$ we also write $[\varphi]_M$. 

**Strategic Game**  A *strategic game* is a triple \( G = (N, \{A_i : i \in N\}, \{u_i : i \in N\}) \) where: \( N = \{1, \ldots, n\} \) is the finite set of players; for each \( i \in N \), \( A_i \) is the set of strategies (or actions) available to \( i \). \( A = \times_{j \in N} A_j \) is the set of strategy profiles; and for each \( i \in N \), \( u_i : A \to \mathbb{R} \) is the payoff function for \( i \), mapping each strategy profile to a number. Notation \((a_1, \ldots, a_n)\)[\( a_i/a'_i \)] stands for the profile wherein strategy \( a_i \) is replaced by \( a'_i \). A strategy profile is a (pure strategy) *Nash equilibrium* if every strategy is the best response of that agent to the strategies of the other agents, i.e., if the agent can not do any better by choosing a different strategy given that the strategies of the other agents are fixed. Formally, a profile \((a_1, \ldots, a_n)\) is a Nash equilibrium if and only if for all \( i \in N \), for all \( a'_i \neq a_i \), \( u_i((a_1, \ldots, a_n)\)[\( a_i/a'_i \)]]) \leq u_i((a_1, \ldots, a_n)\)[\( a_i/a_i \)]).

**Bayesian game**  The most common model of strategic games with imperfect information is the *Bayesian game* [45]. Our presentation of Bayesian games is as in [73]. A *Bayesian strategic game* \( BG = (N, S, \{A_i : i \in N\}, \{T_i : i \in N\}, \{Pr_i : i \in N\}, \{\tau_i : i \in N\}, \{u_i : i \in N\}) \) has the following components:

\( N \) is the set of players; \( S \) is the finite set of states \( s \) modeling the players’ uncertainty about each other; and for each \( i \in N \): \( A_i \) is the set of strategies or choices in the game; \( T_i \) is the set of signals \( t_i \) that may be observed by player \( i \), \( \tau_i : S \to T_i \) is the *signal function* of player \( i \); \( Pr_i \) is a probability measure on \( S \) (the *prior belief* of player \( i \)) such that \( Pr_i(\tau^{-1}(t_i)) > 0 \) for all \( t_i \in T_i \), that is, each player’s received signal is correct with strictly positive probability; and finally \( u_i \) is a payoff function on the set of probability measures over \( A \times S \) (instead of a payoff function on the set of action profiles \( a \)).

### 4.7  Appendix B: Proofs of Main Results

#### 4.7.1. Proof (Fact 4.3.2).

The following are the answers corresponding to each strategy profile in the game:

<table>
<thead>
<tr>
<th>((S_b, S_o))</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>((a_M, b_M))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((S^a_0, S^q_0))</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{p}, q)</td>
<td>(p, q)</td>
<td>(p, \bar{q})</td>
<td>(p, \bar{q})</td>
<td>(p, q)</td>
<td>(p, q)</td>
<td>((\ , m))</td>
</tr>
<tr>
<td>((S^a_0, S^q_1))</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{p}, \bar{q})</td>
<td>(p, q)</td>
<td>(\bar{p}, q)</td>
<td>(p, \bar{q})</td>
<td>(r, \bar{q})</td>
<td>(r, q)</td>
<td>(p, q)</td>
<td>((\ , ))</td>
</tr>
<tr>
<td>((S^a_0, S^q_2))</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{q}, q)</td>
<td>(r, q)</td>
<td>(p, \bar{q})</td>
<td>(p, \bar{q})</td>
<td>(r, q)</td>
<td>(r, q)</td>
<td>((\ , ))</td>
</tr>
<tr>
<td>((S^a_0, S^q_3))</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{p}, \bar{q})</td>
<td>(\bar{q}, q)</td>
<td>(r, q)</td>
<td>(\bar{q}, \bar{q})</td>
<td>(r, \bar{q})</td>
<td>(r, q)</td>
<td>(r, q)</td>
<td>((m, \ ))</td>
</tr>
</tbody>
</table>
4.7. Appendix B: Proofs of Main Results

123

4.7.2. Proof (4.4.16). This simple fact is witnessed by considering an EQG with two agents $a$ and $b$ informed only about facts $q$ and $p$, respectively, and only aware of propositional atoms $p, q, r$, in which $a$ has the following goal formula:

$$\gamma_a = (p \leftrightarrow \overline{q}) \land W_a r \land (W_a p \lor W_b q)$$

and symmetrically $\gamma_b$ for the second player $b$, and where $W_i = K_i \varphi \lor K_i \overline{\varphi}$. The model described informally so far in Fact 4.3.2 can be precisely specified using in our later Haskell implementation by the following model transformations:

*QAGM* $\mathbf{m}^7 = \text{upd} (\text{upd} (\text{initM \ [a,b] \ [P 0, Q 0, R 0]})$

\[ \text{(info \ [a] \ (P 0))} \]

\[ \text{(info \ [b] \ (Q 0)} \]

and the resulting EQG played in $\mathbf{m}^7$ has the following outcomes:

<table>
<thead>
<tr>
<th>$(S_0, S_0)$</th>
<th>$(S_1, S_0)$</th>
<th>$(S_2, S_0)$</th>
<th>$(S_3, S_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S_0, S_0^a)$</td>
<td>$\overline{p}, \overline{r}$</td>
<td>$\overline{p}, \overline{r}$</td>
<td>$\overline{p}, \overline{r}$</td>
</tr>
<tr>
<td>$(S_1, S_0^a)$</td>
<td>$\overline{r}, r, r$</td>
<td>$\overline{p}, \overline{r}$</td>
<td>$\overline{p}, \overline{r}$</td>
</tr>
<tr>
<td>$(S_2, S_0^a)$</td>
<td>$\overline{p}, \overline{r}, r, r$</td>
<td>$\overline{r}, q, r, q$</td>
<td>$\overline{r}, q, r, q$</td>
</tr>
<tr>
<td>$(S_3, S_0^a)$</td>
<td>$\overline{r}, r, r, r$</td>
<td>$\overline{p}, \overline{r}, r, r$</td>
<td>$\overline{p}, \overline{r}, r, r$</td>
</tr>
</tbody>
</table>

We give next the payoff matrix of the induced game mentioned in Fact 4.3.2.

<table>
<thead>
<tr>
<th>$S_0^b$</th>
<th>$S_1^b$</th>
<th>$S_2^b$</th>
<th>$S_3^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0^a$</td>
<td>0.00, 1.00</td>
<td>0.25, 0.75</td>
<td>0.25, 0.75</td>
</tr>
<tr>
<td>$S_1^a$</td>
<td>0.25, 0.75</td>
<td>0.00, 1.00</td>
<td>0.50, 0.50</td>
</tr>
<tr>
<td>$S_2^a$</td>
<td>0.25, 0.75</td>
<td>0.50, 0.50</td>
<td>0.00, 1.00</td>
</tr>
<tr>
<td>$S_3^a$</td>
<td>0.50, 0.50</td>
<td>0.25, 0.75</td>
<td>0.25, 0.75</td>
</tr>
</tbody>
</table>

$\square$
As in the previous example, we have a cycling pattern of local optima that causes global inexistence of NE. \hfill \Box

4.7.3. **Proof (Fact 4.5.3).** Let $S = \langle n, Q, \Pi, \pi, d, \delta, (\approx_a)_{a \in \Sigma} \rangle$ be an arbitrary epistemic game structure without DP. Suppose that $\langle (A) \rangle \Box \varphi \not\rightarrow \varphi \land \langle (A) \rangle \circ \langle (A) \rangle \Box \varphi$. Then, for some $q \in Q$, we have $q \models \langle (A) \rangle \Box \varphi$ but $q \not\models \varphi \land \langle (A) \rangle \circ \langle (A) \rangle \Box \varphi$. In case $q \not\models \varphi$ we are done. If $q \not\models \langle (A) \rangle \circ \langle (A) \rangle \Box \varphi$ then, by the semantics, for all uniform $A$-strategies $F_A$, for some computation $\lambda \in out(q, F_A)$, we have $\lambda[1] \not\models \langle (A) \rangle \Box \varphi$. This means, by the semantics, that for all uniform $A$-strategies $F_A$, for some computation $\lambda' \in out(\lambda[1], F_A)$, and for some position $i \geq 0$, we have $\lambda'[i] \not\models \varphi$. From $q \models \langle (A) \rangle \Box \varphi$ we have, by the semantics, that there exist a uniform $A$-strategy $F_A$ such that for each computation $\lambda \in out(q, F_A)$ and all positions $i \geq 0$, we have $\lambda[i] \models \varphi$. But we also have, for all $q \in Q$ and $a \in \Sigma$, that $q \not\sim q$, because $\sim$ is an equivalence relation. Therefore, $S$ must satisfy DP, against the assumption. As $q$ and $S$ are arbitrary this holds for all non-DP EGS. The other direction is similar. \hfill \Box

4.7.4. **Proof (Fact 4.5.4).** If for an atomic $\varphi$ we have $h \sim h'$, $h \models \varphi$ and $h' \models \varphi$ then, for any questioning action $q$, we also have both $h^q \models \varphi$ and $h'^q \models \varphi$, because for any EIM $M = \langle W, \sim, \approx V \rangle$ and $M' = M \otimes q = \langle W, \sim', \approx', V' \rangle$ we have, by $[\varphi?], [!]$ definitions, that $V = V'$. If $\varphi$ is a con/(dis)junction of positive formulas $\varphi' \land (\lor)\chi$ then we use the induction hypothesis. If $\varphi$ is a modal formula $Q\psi$ we have to consider two cases. If $h^q \models h'^q$ we are done. If $h^q \not\models h'^q$ then, suppose that for some $k \in H$, $h^q \models k^q$ and $k \models \neg\psi$, and, for all $k' \in H$ such that $h'^q \models k'$, we have $k' \models \psi$. In the first case $\varphi$ is a positive formula only if $\psi$ is factual. But then we also have $k \models \neg \psi$ because questioning actions do not change factual content. Then by the definition of $[q?]$, for any epistemic-issue models $M = \langle W, \sim, \approx V \rangle$ and $M' = M \otimes q = \langle W, \sim', \approx', V' \rangle$, we have $\approx' = \approx \cap q \equiv M$, hence we must also have $h \not\models k$ but then, by the semantics of $Q$, we have $h \not\models \varphi$ and this contradicts the initial assumption. For resolution $[!]$ actions, indistinguishability $\sim$ and formulas with epistemic $K\psi$ or mixed $R\psi$ content the argument is analogous. \hfill \Box