Structural analysis of complex ecological economic optimal control problems

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This thesis demonstrates the importance and effectiveness of methods of bifurcation theory applied to studying non-convex optimal control problems. It opens up a new methodological approach to investigation of parameterized economic models. While standard analytical methods are not efficient and sometimes impossible to apply to non-convex problems, the numerical geometrical methods developed in the thesis allow to solve and analyze such problems quickly.

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Her research mainly focuses on application of deterministic and stochastic optimal control methods and bifurcation theory to complex dynamical models.
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Structural Analysis of Complex Ecological Economic Optimal Control Problems

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Chapter 1

Introduction

This thesis investigates parameterized infinite-horizon non-convex dynamic optimization problems on one-dimensional state spaces. Such problems often occur in environmental economics, see Brock and Starrett (2003), Mäler et al. (2003), Scheffer et al. (2001), Scheffer (2009). Non-convex problems may exhibit multiple equilibria, that occur or disappear as the parameters vary. Among those equilibria more than one can be optimal to converge to, depending on the initial state of the system. There is a body of work in different areas of economics devoted to multiplicity of optimal solutions, see for instance Sethi (1977), Skiba (1978), Dechert and Nishimura (1983), Tahvonen and Salo (1996), Sethi and Thompson (2000), Caulkins et al. (2001), Brock and Starrett (2003), Haunschmied et al. (2003), Wagener (2003), Dawid and Deissenberg (2005), Haunschmied et al. (2005), Caulkins et al. (2007), Crepin (2007), Kossioris et al. (2008), Zeiler et al. (2009), Kiseleva and Wagener (2010).

The presence of multiple equilibria can make solving a non-convex optimal control problem quite complicated. In problems with linear or convex dynamics small changes have small effects on the solution structure. In contrast, for problems with non-convex dynamics slight modifications of the system parameters can change this structure not only quantitatively but also qualitatively. In this thesis methods are developed for non-convex optimal management problems that allow to obtain the global solution structure of the problem and also to indicate the critical parameter values that correspond to qualitative changes of this structure.
In the theory of dynamical systems there is a branch - bifurcation theory - focusing on obtaining qualitative information about solutions of parameterized dynamical systems. The basic idea is to determine persistent properties of systems. Examples of such properties are the number and the type of equilibria, the presence of periodic orbits etc. If slight changes of the system parameters cause a change in one of these properties, then the dynamical system is said to undergo a bifurcation.

Bifurcation theory is based on and is a ramification of the theory of structural stability of dynamical systems, which has a long history. It has been anticipated by Poincaré at the end of the 19th century. The concept of structural stability was first introduced as ‘roughness’ by Andronov and Pontrjagin in 1930s. It took off in the 1960s together with the development of the theory of differentiable dynamical systems, and was further developed and popularized by Peixoto, Smale, Thom, Zeeman and Arnol’d in 1960s and 1970s.

In this thesis ideas from dynamical systems theory are used to develop bifurcation theory for infinite horizon optimal control problems. The solution of such a problem can be expressed as an optimal vector field, which gives the state dynamics under the optimal policy. It turns out that qualitative changes of the optimal vector fields are connected to qualitative changes of the canonical state–costate system. This provides a way to link bifurcation theory of dynamical systems to the theory of bifurcating optimal vector fields.

The results of a bifurcation analysis are usually expressed in a bifurcation diagram, which shows the ‘shapes’, or types of the structure, of the optimal vector field for all values of the parameters as well as the bifurcation set where the shape changes. The structure of a generic (i.e. Kupka-Smale) one-dimensional vector field is characterized by the number and the types of its steady states; they can either be attractors or repellers. Optimal vector fields also feature ‘indifference points’ that are particular to control problems. At such a point a system manager is indifferent between two strategies implying two different long run outcomes. An indifference point acts as a repeller in the sense that trajectories starting near it move away, though it is not a steady state of the dynamics.

An example of a bifurcation diagram is given in Figure 1.1 (taken from Chapter 3). It shows
Figure 1.1: Bifurcation diagram for the shallow lake model with respect to the parameters: relative costs of pollution and discount rate. The figure is taken from Chapter 3.

the bifurcations of the shallow lake system (introduced in Chapter 3) that describes the problem of optimal management of water pollution. In the model the social planner limits usage of artificial fertilizers on farmlands surrounding a shallow lake. The phosphorus contained in the fertilizers is washed into the lake by underground flows and is the main pollutant of the water. The optimal policy of the planner depends upon the model parameters: cost of pollution and discount rate. From Figure 1.1 the structure of the optimal solution can immediately be determined for any value of the parameters. The shapes of the optimal vector field are shown as well.

It can be seen from Figure 1.1 that the shallow lake system has three qualitatively different configurations of optimal paths depending on the values of the two parameters: (I) the optimal state dynamics converges to the unique steady state; (II) depending on the initial value the optimal state dynamics converges to one of the two attractors, separated by an indifference point; (III) depending on the initial value the optimal state dynamics converges to one of the two attractors, separated by a repeller. The solid lines that separate regions corresponding to different solution structures are bifurcation curves of the problem. A point on such a curve determines the
parameter values for which dynamics under optimal management of the shallow lake system is structurally unstable; a slight deviation of the parameters yields qualitative changes of the solution structure. The dashed lines in Figure 1.1 correspond to bifurcations in the associated state-costate system of the problem that do not correspond to bifurcations of the optimization problem.

In region (I) there is a unique socially optimal equilibrium. The pollution level at the equilibrium continuously depends on the parameter values: for a fixed discount rate the higher the cost of pollution the lower the equilibrium pollution level. In region (II) the initial pollution level determines which of the two equilibria, ‘clean’ or ‘polluted’, is socially optimal. If initially the lake is clean then it is optimal to keep it clean: the social planner puts a low quota on usage of fertilizers and steers the lake to the ‘clean’ equilibrium. If initially the lake is polluted then the planner lets farmers enjoy high benefits and steers the lake to the ‘polluted’ equilibrium. In region (III) there are three optimal steady states: ‘clean’, ‘intermediate’ and ‘polluted’. The scenarios of convergence to the ‘clean’ and ‘polluted’ equilibria are the same as in region (II). The ‘intermediate’ equilibrium is a repeller, thus it is optimal only for one single initial pollution level - the pollution level at the ‘intermediate’ equilibrium.

The next natural step is to extend the bifurcation methodology for optimal vector fields to stochastic problems. However, several difficulties present themselves. First of all, the corresponding Hamilton-Jacobi-Bellman equation is a singularly perturbed differential equation and requires special solution methods. Chapter 4 develops a method that allows to compute an approximate solution of such an equation. The method is based on considering the stochastic problem as a singular perturbation of the corresponding deterministic one (for singular perturbation theory see Fleming and Souganidis (1986) and Verhulst (2005)). This assumption allows to approximate the value function of the stochastic problem by the quantities computable from the deterministic value function.

Also, concepts like bifurcation and indifference points cannot be adapted from the deterministic context directly. A bifurcation is a qualitative change of persistent properties of a system. For deterministic systems that means for instance a change in the number or the type of
equilibria of the optimal vector field. For stochastic systems optimal vector fields are however not well defined. Instead, qualitative changes in a certain geometrical invariant of the resulting controlled stochastic process, the so-called transformation invariant function (see Wagenmakers et al. (2005)), are considered in Chapter 5. A stochastic bifurcation is then understood as a qualitative change of the shape of this function, such as the change from unimodality to bimodality. Local maxima of the transformation invariant function are then stable steady states of the stochastic process; its local minima are called regime switching thresholds. A regime of a system is an interval in a state space bounded by such thresholds. When the boundaries are crossed due to a large shock, the regime of the system changes. A regime switching threshold is a stochastic analog of a deterministic threshold point in the sense that they both separate basins of attraction of stable steady states of the optimally controlled process. However there is an important difference between them: at a threshold point there can exist multiple optimal controls whereas at a regime switching threshold there exists a unique optimal control.

Figure 1.2: Bifurcation diagram for the stochastic lake problem with respect to the noise intensity (bottom figure) is shown together with the corresponding transformation invariant function. The solid and dotted lines correspond to maxima and local minima of the transformation invariant function respectively. The figure is taken from Chapter 5.
Using the new bifurcation concept, a bifurcation analysis of a stochastic lake model is performed (see Chapter 5). This model is an extension of the deterministic shallow lake model where the pollution dynamics equation is perturbed by a stochastic term. Figure 1.2 displays a bifurcation diagram with respect to the noise intensity as well as the associated transformation invariant function. The figure shows that the transformation invariant function of the problem is bimodal for low levels of noise; it becomes unimodal when the noise level increases. The dashed lines indicate the levels of the noise intensity for which the transformation invariant functions are plotted. The stable steady states of the controlled process are located at solid lines in the parameters plane and correspond to maxima of the transformation invariant function. The regime switching thresholds are located on the dotted line and correspond to local minima of the transformation invariant function. Such a point separates two regimes, ‘clean’ and ‘polluted’, of the pollution dynamics in the shallow lake. When the lake is in ‘clean’(‘polluted’) regime, the pollution fluctuates around the ‘clean’(‘polluted’) stochastic steady state. Transitions between the regimes occur due to large shocks in the pollution stock.

Figure 1.2 shows that as the noise intensity in the lake system increases the transformation invariant function becomes unimodal with a mode at the ‘clean’ equilibrium. It is a demonstration of the precautionary principle: facing the uncertainty the social planner acts to avoid serious or irreversible potential harm to the environment.

The next section provides a short outline of the thesis.

**Outline of the thesis**

The results of this thesis are presented in four chapters, each of which is mostly self-contained. Some basic notions and definitions are briefly restated in subsequent chapters.

In Chapter 2 parameterized families of deterministic optimal control problems with two dimensional state-control space are studied. The concept of an optimal vector field corresponding to such a problem is introduced. It is a one-dimensional multivalued vector field that describes the state dynamics under the optimal policy. The families of optimal vector fields can bifur-
cate. A classification of all possible bifurcations of optimal vector fields up to codimension 2 is obtained.

In Chapter 3 the theory of bifurcations of one-dimensional vector fields developed in Chapter 2 is applied to the shallow lake model. This model serves as a prototype of an optimal management problem with conflicting intertemporal interests, short-term benefits and long-term costs, that features in many economic-ecological problems. A bifurcation analysis of the shallow lake problem is given with respect to all system parameters: natural resilience, relative importance of the resource for social welfare and future discount rate. In particular, it is shown how the increase of the discount rate affects the parameter regions where an oligotrophic steady state, corresponding to low pollution level, is either globally or locally stable under optimal dynamics. A modified version of Chapter 3 constitute a paper published in Journal of Economic Dynamics and Control.

In Chapter 4 stochastic optimal control problems with small noise intensities are studied. A method of constructing approximate solutions to such problems is developed, based on singular perturbation theory.

In Chapter 5 a concept of stochastic bifurcation and a stochastic analog of a deterministic threshold point are introduced. A bifurcation analysis of the stochastic lake model with respect to the noise intensity parameter is performed. Particularly, it is shown that the mode of the transformation invariant function associated with the ‘polluted’ steady state vanishes when the noise intensity in the system increases.
Chapter 2

Bifurcations of one-dimensional optimal vector fields

The investigation of an economic dynamic optimization problem that features a globally attracting steady state reduces mostly to a quantitative quasi-static analysis of this state, determining the rates of change of the position of the steady state and the value of the objective functional as certain key parameters are varied. In contrast, if there are more than one attracting steady state in the system, or more generally, more than one attracting set, the question arises towards which of these the system is driven by the optimal policy. Put differently, in the presence of a single globally attracting steady state, optimal policies can differ only in degree; if there are multiple attracting states, they may also differ in kind.

Since the late 1970’s, optimal policies that are qualitatively different have been found in many economic models: in growth theory they have been used to explain poverty traps (Skiba (1978), Dechert and Nishimura (1983)); in fisheries, they can model the coexistence of conservative versus overexploiting policies (Clark (1976)); there are environmental models where both industry-promoting but polluting as well as ecologically conservative policies are optimal in the same model, depending on the initial state of the environment (Tahvonen and Salo (1996), Mäler et al. (2003), Wagener (2003), Kiseleva and Wagener (2010)); in migration studies, active relocation as well as no action policies occur in the same model (Caulkins et al. (2005));
optimal advertising efforts may depend on the initial awareness level of a product (Sethi (1977, 1979)); the successful containment of epidemics may depend on the initial infection level (Sethi (1978), Rowthorn and Toxvaerd (2011)); in the control of illicit drug use, high law enforcement as well as low enforcement and treatment of drug users (Tragler et al. (2001), Feichtinger and Tragler (2002)) can depend on the initial level of drug abuse; in R&D policies of firms, the optimal decision between high R&D expenditure investment (Hinloopen et al. (2010)) to develop a technology versus low investment to phase a technology out may depend on the initial technology level.

In all such models, there is for certain parameter configurations a critical state where both kinds of policy are simultaneously optimal, and where the decision maker is consequently indifferent between them. These points will be called *indifference points* in the following, though they go by many other names as well.

Usually, the presence of an indifference point is established numerically for a fixed set of parameter values of the model. In order to study the dependence of the qualitative properties of the optimal policies on the system parameters, it is possible in principle to do an exhaustive search over all parameter combinations. Such a strategy, while feasible, would however be very computing intensive.

A different approach is suggested by the theory of bifurcations of dynamical systems: to identify only those parameter configurations at which the qualitative characteristics of the solutions change. For instance, in Wagener (2003) it was shown that indifference points disappear if a heteroclinic bifurcation of the state-costate system occurs. This mechanism, for which we propose the term *indifference-attractor bifurcation*, relates the change of the solution structure of the optimal control problem to a global bifurcation of the state-costate system.

The present article conducts a systematic study of the bifurcations of infinite horizon optimal control problems on the real line that are expected to occur in one- and two-parameter families. The theory developed here has already been applied in several places (Wagener (2003), Caulkins et al. (2007), Graß (2010), Kiseleva and Wagener (2010)).

\footnote{For instance Skiba points, Dechert-Nishimura-Skiba points, Dechert-Nishimura-Sethi-Skiba points, regime switching thresholds, Maxwell points, shocks etc.}
2.1 Setting

2.1.1 Definitions

Let $X \subset \mathbb{R}$ be an open interval, and $U \subset \mathbb{R}^r$ a closed convex set with non-empty interior. Let $\rho > 0$ be a positive constant and $f : X \times U \to \mathbb{R}$, $g : X \times U \to \mathbb{R}$ be infinitely differentiable, or smooth, in the interior of $X \times U$, and such that all derivatives can be extended continuously to a neighborhood of $X \times U$. Finally, let $\xi \in X$.

Set

$$H = g(x, u) + pf(x, u)$$

and assume that

$$\frac{\partial^2 H}{\partial u^2}(x, p, u) < 0$$

(2.1)

for all $(x, p, u) \in X \times \mathbb{R} \times U$.

Consider the problem to maximise

$$J(x, u) = \int_0^\infty g(x(t), u(t))e^{-\rho t}dt$$

(2.2)

over the space of state-control trajectories (or programs) $(x, u)$ that satisfy

1. the function $u : [0, \infty) \to U$ is locally Lebesgue integrable over $[0, \infty)$ and essentially bounded; that is, $u \in L^\infty([0, \infty), U)$

2. the function $x : [0, \infty) \to X$ is absolutely continuous and satisfies

$$\dot{x} = f(x, u)$$

(2.3)

almost everywhere;

3. the initial value of $x$ is given as $x(0) = \xi$.

This problem will be referred to as infinite horizon problem in the following. A solution $(x, u)$ to the problem is usually called a maximizer or a maximizing trajectory.
**Assumption 2.1.1.** In the infinite horizon problem, for every $\xi \in X$ there exists at least one maximizer $(x, u)$ satisfying $x(0) = \xi$.

Maximizing trajectories enjoy the following time invariance property, which is commonly known as the dynamic optimization principle.

**Theorem 2.1.1.** If the trajectory $(x(\cdot), u(\cdot))$ solves the infinite horizon problem with initial condition $\xi$, then for any $\tau > 0$, the time shifted trajectory $(x(\tau + \cdot), u(\tau + \cdot))$ solves the infinite horizon problem with initial condition $x(\tau)$.

Define the maximized Hamiltonian as

$$
\mathcal{H}(x, p) = \max_{u \in U} \{g(x, u) + p f(x, u)\}
$$

Assumption (2.1) implies that the maximum is taken at a unique point $u = v(x, p)$, where $v$ depends smoothly on its arguments; consequently, the function $\mathcal{H}$ is smooth as well.

For a maximizing state trajectory $x$, there exists a continuous costate trajectory $p$ satisfying the reduced canonical equations

$$
\dot{x} = F_1 = \mathcal{H}_p, \quad \dot{p} = F_2 = \rho p - \mathcal{H}_x,
$$

(2.4)

which define the reduced canonical vector field $F = (F_1, F_2)$. Moreover, $x$ and $p$ satisfy the transversality condition

$$
\lim_{t \to \infty} e^{-\rho t} p(\tilde{x} - x) \leq 0
$$

(2.5)

for all admissible trajectories $\tilde{x}$. Trajectories of the state-costate equations (2.4) are classically called *extremal*. Extremal trajectories that satisfy the transversality condition (2.5) will be called *critical* in the following. Note that a noncritical trajectory cannot be a maximizer.

Recall that the power set $\mathcal{P}(S)$ of a set $S$ is the set of all subsets of $S$.

**Definition 2.1.1.** The *optimal costate rule* is the set valued map $p^\circ : X \to \mathcal{P}(\mathbb{R})$ with the property that if $\eta \in p^\circ(\xi)$, then the solution of the reduced canonical equations with initial...
value
\[(x(0), p(0)) = (\xi, \eta)\]

maximizes the integral $J$. Associated to it are the **optimal feedback rule**

\[u^o(x) = v(x, p^o(x)),\]

and the **optimal vector field**

\[f^o(x) = \mathcal{H}_p(x, p^o(x)) = f(x, u^o(x))\]

which are both set-valued as well.

A map $x : [0, \infty) \to X$ is a trajectory of an optimal vector field if

\[\dot{x}(t) \in f^o(x(t))\]

for all $t \geq 0$.

The solution trajectories of an optimal vector field solve the associated maximization problem. Note that an optimal vector field is commonly called a ‘regular synthesis’ in the literature.

**Theorem 2.1.2.** The sets $p^o(x(t))$ and $f^o(x(t))$ are single-valued for all $t > 0$.


2.1.2 Indifference points

The following definition is one of the possible interpretations of the notion of ‘Skiba point’.

**Definition 2.1.2.** If $\xi \in X$ is such that there are maximisers $x_1, x_2$ of the infinite horizon problem with $x_1(0) = x_2(0) = \xi$ and $x_1(t) \neq x_2(t)$ for some $t \in [0, \infty)$, then $\xi$ is called an **indifference point**. The totality of indifference points form the **indifference set**; its complement in $X$ is the **domain of uniqueness**.
In one-dimensional problems an indifference point is an initial point of two trajectories that have necessarily different long run behavior. It is worthwhile to note that this is not true for problems with higher dimensional state spaces, or for discrete time problems (see Moghayer and Wagener (2009)).

**Definition 2.1.3.** The $\omega$-limit set $\omega(x)$ of a state trajectory $x$ is given as

$$\omega(x) = \{\xi \in X : x(t_i) \to \xi \text{ for some increasing sequence } t_i \to \infty\}.$$  

Using $\omega$-limit sets, threshold points can be defined.

**Definition 2.1.4.** A point $\xi \in X$ is a threshold point, if in every neighborhood $N$ of $\xi$ there are two states $\xi_1, \xi_2 \in N$ that are initial states to state trajectories $x_1, x_2$ such that the respective $\omega$-limit sets are different: $\omega(x_1) \neq \omega(x_2)$.

Threshold points are boundary points of basins of attractions.

**Definition 2.1.5.** A set $B$ is the basin of attraction of another set $A$, the attractor, if for every $x \in B$ the $\omega$-limit set of $x$ is equal to $A$: $\omega(x) = A$ for all $x \in B$.

Unlike the situation for ‘ordinary’ dynamical systems, a threshold point can be an element of one or more basins of attraction, and basins can overlap.

**Definition 2.1.6.** A point $\xi \in X$ is an indifference threshold if it is both an indifference point and a threshold point.

Equivalently, an indifference threshold is a point that is contained in more than one basin of attraction. In the literature, both threshold and non-threshold indifference points have been called ‘Skiba points’. A more precise terminology seems to be desirable.

Dynamical systems on a one-dimensional state space that are defined by a vector field have typically two kinds of ‘special’ points: attractors and repellers, which are both steady states; the knowledge of these special points is sufficient to reconstruct the flow of the system qualitatively.

Analogously, an optimal one-dimensional vector field has optimal attractors and optimal repellers, which are both optimal equilibria; in addition it has indifference points. Again, the knowledge of the optimal equilibria and the indifference points is sufficient to reconstruct the qualitative features of the solution structure of the infinite horizon problem.
2.2 Bifurcations of optimal vector fields

The analysis of bifurcations of a parameterized family of optimal vector fields is performed in terms of the reduced canonical vector field, but it is perhaps worthwhile to point out that the latter is an auxiliary construct.

The optimal vector field defines a continuous time evolution on the state space, that is well defined for all positive times. When the state space is one-dimensional, the evolution has certain special properties: trajectories sweep out intervals that are bounded by optimal attractors and optimal repellers or indifference points. At a bifurcation, the qualitative structure of these trajectories changes. For instance, in a saddle-node bifurcation, an attractor and a repeller coalesce and disappear, together with the trajectory that joins them. Analogously, in an indifference-attractor bifurcation, an indifference point and an attractor coalesce and disappear, again together with the trajectory joining them. It is clearly impossible that a repeller and an indifference point coalesce, for the trajectory which should be joining them could have no $\omega$-limit point. However, there is a third possible bifurcation scenario: a repeller may turn into an indifference point. This also changes the solution structure, the constant solution that remains in the repelling state has no equivalent in the situation with the indifference point.

The indifference-attractor bifurcation and the different kinds of indifference-repeller bifurcations have obviously no counterpart in the theory of dynamical systems: they are typical for optimization problems. Instances of indifference-attractor bifurcations have been analysed in Wagener (2003, 2006).

2.2.1 Preliminary remarks.

If $N$ is a bounded interval of $\mathbb{R}$ with endpoints $a < b$, let the outward pointing ‘vector’ $\nu(x)$ be defined as

$$ \nu(a) = -1, \quad \nu(b) = 1. \quad (2.6) $$
Notions from optimal control theory

The reduced canonical vector field $F$ of the infinite horizon problem under study is given as

$$F = (F_1, F_2) = (\mathcal{H}_p, \rho p - \mathcal{H}_x).$$

Assumption (2.1) implies that the strong Legendre-Clebsch condition

$$\mathcal{H}_{pp}(x, p) > 0 \quad (2.7)$$

holds for all $(x, p)$.

One of the implications of this condition is that eigenspaces of equilibria of the reduced canonical vector field are never vertical. More precisely, the following lemma holds.

**Lemma 2.2.1.** If the strong Legendre-Clebsch condition holds, all eigenvectors $v$ of $DF$ can be written in the form $v = (1, w)$.

**Proof.** The lemma is implied by the statement that if $\mathcal{H}_{pp} \neq 0$, then $e_2 = (0, 1)$ cannot be an eigenvector of

$$DF = \begin{pmatrix} \mathcal{H}_{px} & \mathcal{H}_{pp} \\ -\mathcal{H}_{xx} & \rho - \mathcal{H}_{px} \end{pmatrix}.$$

This is easily verified. \(\Box\)

**Lemma 2.2.2.** Assume that the strong Legendre-Clebsch condition holds. If $v_1 = (1, w_1)$ and $v_2 = (1, w_2)$ are two eigenvectors of $DF$ with $\lambda_1 < \lambda_2$, then $w_1 < w_2$.

**Proof.** The first component of the vector equation $DFv_i = \lambda_i v_i$ reads as

$$\mathcal{H}_{px} + \mathcal{H}_{pp} w_i = \lambda_i.$$

As $\mathcal{H}_{pp} > 0$, the lemma follows. \(\Box\)

The value of the objective $J$ over an extremal trajectory can be computed by evaluating the maximized Hamiltonian at the initial point (see for instance Skiba (1978), Wagener (2003)).
Theorem 2.2.1. Let \((x(t), p(t))\) be a trajectory of the reduced canonical vector field \(F\) that satisfies \(\lim_{t \to \infty} \mathcal{H}(x(t), p(t))e^{-\rho t} = 0\), and let \(u(t) = v(x(t), p(t))\) be the associated control function. Then

\[ J(x, u) = \frac{1}{\rho} \mathcal{H}(x(0), p(0)). \]

Notions from dynamical systems

Recall the following notions from the theory of dynamical systems: two vector fields are said to be \textit{topologically conjugate}, if all trajectories of the first can be mapped homeomorphically onto trajectories of the second; that is, by a continuous invertible transformation whose inverse is continuous as well.

An equilibrium \(\bar{z}\) of a vector field \(f\) is called \textit{hyperbolic}, if no eigenvalue of \(Df(\bar{z})\) is situated on the imaginary axis. The sum of the generalized eigenspaces associated to the hyperbolic eigenvalues is the hyperbolic eigenspace \(E^h\), which can be written as the direct sum of the stable and unstable eigenspaces \(E^s\) and \(E^u\), associated to the stable and unstable eigenvalues respectively. The sum of the eigenspaces associated to the eigenvalues on the imaginary axis is the neutral eigenspace \(E^c\). The center-unstable and center-stable eigenspaces \(E^{cu}\) and \(E^{cs}\) are the direct sums \(E^c \oplus E^u\) and \(E^c \oplus E^s\) respectively.

The center manifold theorem (see Hirsch et al. (1977)), ensures the existence of invariant manifolds that are tangent to the stable and unstable eigenspaces.

Theorem 2.2.2 (Center Manifold Theorem). Let \(f\) be a \(C^k\) vector field on \(\mathbb{R}^m\), \(k \geq 2\), and let \(f(\bar{z}) = 0\). Let \(E^s, E^u, E^c, E^{cu}\) and \(E^{cs}\) denote the generalized eigenspaces of \(Df(\bar{z})\) introduced above. Then there are \(C^k\) manifolds \(W^s\) and \(W^u\) tangent to \(E^s\) and \(E^u\) at \(\bar{z}\), and \(C^{k-1}\) invariant manifold \(W^c\), \(W^{cu}\) and \(W^{cs}\) tangent to \(E^c\), \(E^{cu}\) and \(E^{cs}\) respectively at \(\bar{z}\). These manifolds are all invariant under the flow of \(f\); the manifolds \(W^s\) and \(W^u\) are unique, while \(W^c, W^{cu}\) and \(W^{cs}\) need not be.

Invariant manifolds can be used to choose convenient coordinates around an equilibrium point of a vector field. For instance, let \(f(0) = 0\), let \(E^1\) and \(E^2\) be two linear subspaces such
that
\[ E^1 \oplus E^2 = \mathbb{R}^m, \]
and let \( W^1 \) and \( W^2 \) be two invariant manifolds that are locally around 0 parameterized as the graphs of functions
\[ w^1 : E^1 \to E^2, \quad w^2 : E^2 \to E^1, \]
satisfying \( Dw^1(0) = 0, Dw^2(0) = 0 \). For a sufficiently small neighborhood \( N \) of 0 and for \((z_1, z_2) \in U \subset E^1 \times E^2\), define the coordinate transformation
\[ (\zeta_1, \zeta_2) = (z_2 - w^1(z_1), z_1 - w^2(z_2)). \]

In the new coordinates, the vector field has necessarily the form
\[
f(\zeta) = \begin{pmatrix}
A_1\zeta_1 + \zeta_1\varphi_1(\zeta) \\
A_2\zeta_2 + \zeta_2\varphi_2(\zeta)
\end{pmatrix},
\]
where \( \varphi_i(\zeta) \to 0 \) as \( \zeta \to 0 \).

For a hyperbolic equilibrium of a vector field on the plane, a much stronger result is available, the \( C^1 \) linearization theorem of Hartman (see Hartman (1960, 1964), Palis and Takens (1993)).

**Theorem 2.2.3** (Hartman’s \( C^1 \) linearization theorem). Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^2 \) vector field in the plane, and let \( z = 0 \) be a hyperbolic equilibrium of \( f \). Then there is a neighborhood \( N \) of 0 and coordinates \( \zeta \) on \( N \), such that
\[ f(\zeta) = Df(0)\zeta \]
in these coordinates.

### 2.2.2 Codimension one bifurcations

In this subsection, the codimension one bifurcations of optimal vector fields are treated: these are the bifurcations that cannot be avoided in one-parameter families. These are the indifference-
repeller, the indifference-attractor and the saddle-node bifurcation.

It turns out that there are two configurations of the state co-state system that can give rise to indifference-repeller bifurcations of the optimal vector field; they are referred to as type 1 and type 2, respectively.

A general remark on notation: the codimension of a bifurcation will be denoted by a subscript, whereas the type is indicated, if necessary, by additional information in brackets. For instance, the abbreviation IR\(_1\)(2) denotes a codimension one indifference repeller bifurcation of type 2.

**IR\(_1\)(1) bifurcation**

Consider the situation that the reduced canonical vector field \( F \) has an equilibrium \( e = (x_e, p_e) \in \mathbb{R}^2 \) with eigenvalues \( 0 < \lambda^u < \lambda^{uu} \). Let \( E^{uu} \) denote the eigenspace associated to \( \lambda^{uu} \). As this eigenspace is invariant under the linear flow \( DF(0)z \), by Hartman’s linearization theorem there is a one-dimensional differentiable curve \( W^{uu} \), the strong unstable invariant manifold, that is invariant under \( F \) and tangent to \( E^{uu} \) at \( e \).

**Definition 2.2.1.** A point \( e = (x_e, p_e) \) is a (codimension one) indifference repeller singularity of type 1, notation IR\(_1\)(1), of an optimization problem with reduced canonical vector field \( F \), if the following conditions hold.

1. The eigenvalues \( \lambda^u, \lambda^{uu} \) of \( DF(e) \) satisfy \( 0 < \lambda^u < \lambda^{uu} \).

2. On some compact interval neighborhood \( N \) of \( x_e \), there is defined a continuous function \( p : N \to \mathbb{R} \) such that

\[
p^0(x) = \{ p(x) \}.
\]

for all \( x \in N \), and such that \( p_e = p(x_e) \).

3. Let \( W^{uu} \) denote the strong unstable manifold of \( F \) at \( e \), parameterized as the graph of \( w : N \to \mathbb{R} \). Also, let \( \nu(x) \) be the outward pointing vector of \( N \). There is exactly one \( \bar{x} \in \partial N \) such that

\[
p(\bar{x}) = w(\bar{x}),
\]

(2.8)
whereas for $x \in \partial N$ and $x \neq \bar{x}$, we have that

$$\nu(x)(p(x) - w(x)) < 0.$$  \hfill (2.9)

The definition is illustrated in Figure 2.1b.

**Theorem 2.2.4.** Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_1(1)$ singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.

1. For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu(e_\mu)$ satisfy $0 < \lambda^u_\mu < \lambda^{uu}_\mu$. Let the strongly unstable manifold $W^{wu}_\mu$ of $e_\mu$ be parameterized as the graph $p = w(x, \mu)$ of a differentiable function $w : N \times \Gamma \rightarrow \mathbb{R}$.

2. There is a function $p : \partial N \times \Gamma \rightarrow \mathbb{R}$, differentiable in its second argument, such that

$$p_\mu^p(x) = \{p(x, \mu)\}$$

for all $x \in \partial N$ and all $\mu \in \Gamma$.

3. The function

$$\alpha(\mu) = \nu(\bar{x})(p(\bar{x}, \mu) - w(\bar{x}, \mu)),$$

for which $\alpha(0) = 0$ by (2.8), is defined on $N$ and satisfies

$$D\alpha(0) \neq 0.$$

Then the optimal vector field $f^*$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \{x\}.$$
whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = \begin{cases} 
\{-1\}, & x < 0, \\
\{-1, 1\}, & x = 0, \\
\{1\}, & x > 0, 
\end{cases}$$

The theorem is illustrated in Figure 2.1. Shown is a neighborhood of a repelling equilibrium of the state-costate equation. The dotted lines are the linear unstable eigenspaces of the equilibrium; the strongly unstable eigenspace corresponds to the line with the largest gradient. Approaching the equilibrium are two phase curves, drawn as solid black lines. The thick part of these curves denote the optimal costate rule.

The indifference point is marked as a vertical dashed line. At the top of the diagrams, the corresponding situation in the state space is sketched; solid black circles correspond to equilibria of the optimal vector field, squares to indifference points. In this case, all equilibria of the optimal vector field are repelling.

At the bifurcation, the relative position of the optimal trajectories and the strongly unstable manifold changes: for $\alpha(\mu) < 0$ the backward extension of the optimal trajectories are tangent to $E^u$ at either side of the equilibrium. This ensures that the equilibrium itself corresponds to an optimal repeller. For $\alpha(\mu) > 0$, the backward extensions are tangent to $E^u$ at the same side of the equilibrium. One of them necessarily intersects the line $x = x_e$, which implies that $e$ cannot be an optimal trajectory.

**Proof.** Let $E^{uu} = \mathbb{R} v^{uu}$ and $E^u = \mathbb{R} v^u$ be the eigenspaces spanned by the eigenvectors $v^{uu} = (1, w^{uu})$ and $v^u = (1, w^u)$ of $DF(r)$ corresponding to the eigenvalues $\lambda^{uu}$ and $\lambda^u$ respectively. Note that $w^{uu} > w^u$ as a consequence of lemma 2.2.2.

For a sufficiently small neighbourhood of $e$ introduce $C^1$ linearizing coordinates $\zeta = \zeta(z)$, with $C^1$ inverse $z = z(\zeta) = (x(\xi, \eta), p(\xi, \eta))$, such that $\zeta(e) = 0$, such that the linear map $D\zeta(0)$ maps $v^{uu}$ to $(1, 0)$ and $v^u$ to $(0, -1)$, and such that in these coordinates the vec-
Figure 2.1: Before, at and after the indifference-repeller bifurcation point.
tor field $F$ takes the form
\[ \dot{\zeta} = \begin{pmatrix} \lambda^{uu} & 0 \\ 0 & \lambda^u \end{pmatrix} \zeta. \]

As a consequence of these choices, the map $\zeta$ is orientation preserving and
\[ x_\xi(0, 0) = \pi_1 Dz(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_1 v^{uu} = 1 > 0, \]
\[ x_\eta(0, 0) = \pi_1 Dz(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\pi_1 v^u = -1 < 0, \]
where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the first component, and where $x_\xi$ denotes partial derivation with respect to $\xi$ etc. By continuity, there is a neighbourhood $V$ of 0 such that
\[ x_\xi > 0, \quad x_\eta < 0, \quad \text{and} \quad \det Dz > 0 \]
on $V$.

Let $\bar{x}_i$, $i = 1, 2$ be such that
\[ N = [\bar{x}_1, \bar{x}_2] \quad (2.10) \]
and set
\[ \bar{z}_i = (\bar{x}_i, \bar{p}_i) = (\bar{x}_i, p(\bar{x}_i, \mu)), \quad (2.11) \]
as well as
\[ \bar{\zeta}_i = (\bar{\xi}_i, \bar{\eta}_i) = \zeta(\bar{x}_i, \bar{p}_i). \]
Assume that $\bar{x} = \bar{x}_2$, that is
\[ \alpha(\mu) = p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu); \]
the proof in the case $\bar{x} = \bar{x}_1$ is similar.
The trajectory \( z_1(t) = (x_1(t), y_1(t)) \) of \( F \) through \( \bar{z}_1 \) has in linearizing coordinates the form

\[
\zeta_1(t) = (\xi_1 e^{\lambda u^u t}, \eta_1 e^{\lambda u^u t}),
\]

with \( \xi_1 < 0 \) and \( \eta_1 > 0 \) for all \( \mu \). Note that it satisfies

\[
\dot{x}_1 = \frac{d}{dt} x(\zeta_1(t)) = x_1 \xi_1 e^{\lambda u^u t} + x_\eta \eta_1 e^{\lambda u^u t} u^u
\]

\[
= e^{\lambda u^u} (\eta_1 \lambda^u x_\eta + \bar{\xi}_1 \lambda^u e^{(\lambda u^u - \lambda u^u)t} x_\xi)
\]

As \( \eta_1 > 0, x_\eta < 0 \) and

\[
\lim_{t \to -\infty} e^{(\lambda u^u - \lambda u^u)t} x_\xi(\zeta_1(t)) = 0,
\]

it follows that there is a constant \( T < 0 \) such that \( \dot{x}_1 < 0 \) for all \( t < T \) and all \( \mu \) in a small neighbourhood \( \Gamma \) of \( 0 \). If necessary by choosing \( \varepsilon > 0 \) smaller, it may be assumed that \( T = 0 \).

By assumption, the point

\[ \bar{z}_2 = z(\bar{\xi}_2, \bar{\eta}_2) \]

can be written as

\[ x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = \bar{p} = w(\bar{x}) + \alpha. \]  

(2.12)

Note that for \( \alpha = 0 \), the point \( \bar{p} \) is on \( W^{uu} \), and therefore \( \bar{\eta}_2 = 0 \). To establish the dependence of \( \eta \) on \( \alpha \), derive first (2.12) with respect to \( \alpha \) to obtain

\[
x_\xi (\bar{\xi}_2) + x_\eta (\bar{\eta}_2) = 0, \quad p_\xi (\bar{\xi}_2) + p_\eta (\bar{\eta}_2) = 1.
\]

Solving for \( (\bar{\eta}_2)_\alpha \) yields

\[
(\bar{\eta}_2)_\alpha = \frac{x_\xi}{\det Dz}
\]

from which it follows that

\[
(\bar{\eta}_2)_\alpha > 0.
\]
The trajectory $z_2(t)$ through $\bar{z}_2$ has in linearizing coordinates the form

$$\zeta_2(t) = (\bar{\xi}_2 e^{\lambda u t}, \bar{\eta}_2 e^{\lambda u t}) .$$

Note that $\bar{\xi}_2 > 0$ for all $\mu$, and that

$$\bar{\eta}_2 = \bar{\eta}_2(\alpha(\mu))$$

with $\bar{\eta}_2(0) = 0$. As before

$$\dot{x}_2 = e^{\lambda u t} \left( \lambda^u \bar{\eta}_2 x_\eta + \bar{\xi}_2 \lambda^u e^{(\lambda^u - \lambda) t} x_\xi \right) .$$

If $\alpha \leq 0$, then $\bar{\eta}_2 < 0$. From $x_\eta < 0$ it then follows that $\dot{x}_2 > 0$ for all $t$. By continuity, it follows also that $\dot{x}_2(0) > 0$ if $\alpha > 0$ is sufficiently small. Note however that

$$\lim_{t \to -\infty} \dot{x}_2 e^{-\lambda u t} = \bar{\xi}_2 \lambda^u x_\eta(0, 0) < 0 .$$

Consequently, for $\alpha > 0$ there is, by the intermediate value theorem, at least one $t < 0$ such that $x_2(t) < \bar{x}$ and $\dot{x}_2(t) = 0$. Let $t_*$ denote the largest of these $t$ if there are several.

Note that for $\alpha \leq 0$, the continuous curve formed by the union of the trajectories $z_1, z_2$ and the point $e$ intersects each leaf $\{x = \text{const}\}$ exactly once, and defines therefore a continuous function $x \mapsto p_\mu^0(x)$, which is necessarily the optimal costate map.

If $\alpha > 0$, then the trajectory $z_2$ is tangent to the leaf $L = \{x = x_2(t_*)\}$ at $z_{2*} = z_2(t_*)$, and $z_2$ cuts all other leaves $\{x = \text{const}\}$ transversally for $t^* < t \leq 0$. The leaf $L$ is cut by $z_1$ at $z_{1*}$. Since $\dot{x} = \mathcal{H}_p = 0$ at $z_{2*}$ and $\mathcal{H}$ is strictly convex in $p$, it follows that

$$\mathcal{H}(z_{2*}) < \mathcal{H}(z_{1*}) .$$

Since $\xi_1(0) = \delta$, there is $t^* \in (t_*, 0)$ such that $x_2(t^*) = 0$. Again by convexity of $\mathcal{H}$ in $p$, it follows that

$$\mathcal{H}(z_2(t^*)) > \mathcal{H}(e) = \lim_{t \to -\infty} \mathcal{H}(z_1(t)) .$$
Consequently there is $\tilde{t} \in (t_*, t^*)$ such that

$$\mathcal{H}(\tilde{z}_1) = \mathcal{H}(\tilde{z}_2),$$

where $\tilde{z}_i = (\tilde{x}, \tilde{p}_i) = (x_i(\tilde{t}), p_i(\tilde{t}))$, and $\tilde{x}$ is an indifference point by theorem 2.2.1. \hfill \Box

**IR$_1$(2) bifurcation**

An indifference-repeller singularity of type 2 occurs in certain situations when the dynamics of the repeller is a Jordan node. Specifically, consider the situation that the vector field $F$ on $\mathbb{R}^2$ has an equilibrium $e = (x_e, p_e)$, that its linearization $DF(e)$ has two equal positive eigenvalues $\lambda_1 = \lambda_2 = \lambda > 0$, and such that its proper eigenspace $E^{pu}$ is only one-dimensional.

By the Hartman theorem, there is a $C^1$ curve $W^{pu}$, the *proper unstable invariant manifold*, which is the image of $E^{pu}$ in general coordinates; trajectories $z(t)$ in $W^{pu}$ are characterized by the requirement that

$$\lim_{t \to -\infty} \sup \|z(t) - e\| e^{-\lambda t} < \infty.$$

**Definition 2.2.2.** A point $e = (x_e, p_e)$ is a (codimension one) indifference repeller singularity of type 2, notation IR$_1$(2), of an optimization problem with reduced canonical vector field $F$, if the following conditions hold.

1. The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1, \lambda_2$ of $DF_0(e)$ satisfy $\lambda_1 = \lambda_2 = \frac{\rho}{2}$.

2. On some compact interval neighbourhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that

$$p^0(x) = \{p(x)\}.$$

for all $x \in N$, and such that $p_e = p(x_e)$.

3. Let $W^{uu}$ denote the strong unstable manifold of $F$ at $e$, parameterized as the graph of $w : N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of $N$. For all $x \in \partial N$, we have
that

\[ \nu(x) \left( w(x) - p(x) \right) > 0. \quad (2.13) \]

This singularity also gives rise to an indifference repeller bifurcation, as in the previous case, but through a different mechanism. See Figure 2.2: at bifurcation, the equilibrium of the reduced canonical vector field is a Jordan node. When the eigenvalues move off the real axis, it turns into a focus. This precludes the possibility of an optimal repeller. When the eigenvalues remain on the real axis but separate, two independent eigenspaces \( E^u \) are generated. Condition (2.13) then implies the existence of an optimal repeller.

**Theorem 2.2.5.** Consider a family of optimization problems, depending on a parameter \( \mu \in \mathbb{R}^q \), that has for \( \mu = 0 \) an IR(2) singularity. Assume that there is a neighbourhood \( \Gamma \subset \mathbb{R}^q \) of 0 such that the following conditions hold.

1. For all \( \mu \in \Gamma \), there is \( e_\mu \in \mathbb{R}^2 \) such that \( F_\mu(e_\mu) = 0 \). Let \( D(\mu) \) and \( T(\mu) \) denote the trace and the determinant of \( DF_\mu(e_\mu) \).

2. The function \( \alpha : \Gamma \rightarrow \mathbb{R} \), defined by

\[ \alpha(\mu) = \frac{T(\mu)^2}{4} - D(\mu) \]

and for which \( \alpha(0) = 0 \), satisfies

\[ D\alpha(0) \neq 0. \]

3. There is a function \( p : \partial N \times \Gamma \rightarrow \mathbb{R} \), differentiable in its second argument, such that

\[ p^\mu(x) = \{p(x, \mu)\} \]

for all \( x \in \partial N \) and all \( \mu \in \Gamma \).
Figure 2.2: Before, at and after the type 2 indifference-repeller bifurcation point.
Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \begin{cases} 
-1 & x < 0, \\
\{-1, 1\} & x = 0, \\
1 & x > 0.
\end{cases}$$

whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = x.$$

**Proof.** There is a linear map $C_0$ such that

$$C_0^{-1}DF_0(e)C_0 = \begin{pmatrix} \frac{\rho}{2} & 1 \\
0 & \frac{\rho}{2} \end{pmatrix}.$$  

Arnol’d’s matrix unfolding theorem (Arnold (1988)) then implies that there is a family of maps $C(\alpha)$, smoothly depending on $\alpha$, such that $C(0) = C_0$ and such that

$$A_\alpha = C(\alpha)^{-1}DF_\mu(e_\mu)C(\alpha) = \begin{pmatrix} \frac{\rho}{2} & 1 \\
\alpha & \frac{\rho}{2} \end{pmatrix},$$

where $\alpha = \alpha(\mu)$. The eigenvalues of $DF_\mu(e_\mu)$ and consequently also those of $A_\alpha$ take the form

$$\lambda^u = \frac{\rho}{2} - \sqrt{\alpha}, \quad \lambda^{uu} = \frac{\rho}{2} + \sqrt{\alpha};$$

the corresponding eigenvectors of $A_\alpha$ take the form

$$v^u = (1, -\sqrt{\alpha}), \quad v^{uu} = (1, \sqrt{\alpha}).$$

Note that for $\alpha > 0$, these eigenvectors have the same ordering as the corresponding eigen-
vectors of $DF_\mu(e)$; cf. lemma 2.2.2. It follows that the matrix $C(\alpha)$ is necessarily orientation preserving for $\alpha > 0$ and, by continuity, for all other values of $\alpha$.

Define $\bar{x}_i, \bar{p}_i$ and $\bar{z}_i$ as in (2.10) and (2.11).

When $\alpha < 0$, the eigenvalues are complex, and the trajectories $z_1$ and $z_2$ emanating from $\bar{z}_1$ and $\bar{z}_2$ respectively spiral towards $e$ as $t \to -\infty$. Let $t_*$ be the largest $t \leq$ such that $\dot{x}_2(t) = 0$. Then necessarily

$$x_* = x_2(t_*) < x_e.$$  

The trajectory $z_2$, restricted to $[t_*, 0]$, can be parameterized as the graph of a continuous function $p_2 : [x_*, \bar{x}_2] \to \mathbb{R}$. In the same way, if $t^* < 0$ is the largest $t$ such that $\dot{x}_1(t) = 0$, then $z_1$ restricted to $[t^*, 0]$ can be parameterized as the graph of the function $p_1 : [\bar{x}_1, x^*] \to \mathbb{R}$, where

$$x^* = x_1(t^*) > x_e.$$  

Moreover, as $\mathcal{H}$ is strictly convex and $\mathcal{H}_p(x_*, p_2(x_*)) = 0$, it follows that

$$\mathcal{H}(x_*, p_2(x_*)) < \mathcal{H}(x_*, p_1(x_*));$$

likewise

$$\mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)).$$

By continuity, there is a point $\tilde{x} \in [x_*, x^*]$ such that

$$\mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})).$$

By Theorem 2.2.1, this is an indifference point.

Point 2 of Definition 2.2.2 implies that for $\mu = 0$, the sets $p^n(x)$ contain a single element $p(x)$ for all $x \in N$. Necessarily, the graph of $p$ is formed by two trajectories of $F_0$ as well as the equilibrium point $e$. These trajectories intersect the lines $\{x = \text{const}\}$ transversally and they are tangent to $E^{pu}$ at $x_e$; put differently, the graph of $p$ is tangent to the proper unstable
manifold $W^m$ at $e$. By point 3 of the definition, $p(x) > w(x)$ if $\bar{x}_1 < x < x_e$ and $p(x) < w(x)$ if $x_e < x < \bar{x}_2$.

For $\alpha(\mu) > 0$, there is a family $W^u_\mu$ of strong unstable manifolds, depending continuously on $\mu$, and parameterized as the graph of a family of $C^1$ functions $w_\mu$ around $x_e$. In particular, if $N$ and $\Gamma$ sufficiently small

$$\nu(x) (w_\mu(x) - p(x, \mu)) > 0$$

for all $x \in \partial N$ and $\mu \in \Gamma$. By continuity, the backward trajectories through $\bar{z}_1$ and $\bar{z}_2$ intersect all lines $\{x = \text{const}\}$ transversally and are tangent to the weak unstable direction $E^u$ at $x_e$. But this implies that they form, together with the equilibrium $e$, the graph of a $C^1$ function $p_\mu$ that is defined on $N$, and for which

$$p_\mu^0(x) = \{p_\mu(x)\}.$$ 

\IA bifurcation

**Definition 2.2.3.** A point $e = (x_e, p_e)$ is a (codimension one) indifference attractor singularity, notation $\IA$, of an optimization problem with reduced canonical vector field $F$, if the following conditions hold.

1. The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda^s, \lambda^u$ of $DF(e)$ satisfy $\lambda^s < 0 < \lambda^u$.

2. On some compact interval neighbourhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that $p^0(x) = \{p(x)\}$ for all $x \in N$, and such that $p_e = p(x_e)$. 

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3. Let $W^s$ and $W^u$ denote respectively the stable and the unstable manifold of $F$ at $e$, parameterized as the graph of functions $w^s, w^u : N \rightarrow \mathbb{R}$. If $\partial N = \{\bar{x}_1, \bar{x}_2\}$, then

$$p(\bar{x}_1) = w^u(\bar{x}_1), \quad p(\bar{x}_2) = w^s(\bar{x}_2).$$ (2.14)

Note that this definition does not require the points $\bar{x}_1$ and $\bar{x}_2$ to be ordered in a certain way.

**Theorem 2.2.6.** Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IA$^1$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.

1. For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu'(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu'(e_\mu)$ satisfy $\lambda^s_\mu < 0 < \lambda^u_\mu$. Let the stable and the unstable manifolds $W^s_\mu$ and $W^u_\mu$ of $e_\mu$ be parameterized as graph $p = w^s(x, \mu)$ and $p = w^u(x, \mu)$ of differentiable functions $w^s, w^u : N \times \Gamma \rightarrow \mathbb{R}$.

2. There is a function $p : \partial N \times \Gamma \rightarrow \mathbb{R}$, differentiable in its second argument, such that

$$p^\mu(x) = \{p(x, \mu)\}$$

for all $x \in \partial N$ and all $\mu \in \Gamma$.

3. The function

$$\alpha(\mu) = \nu(x_1)(p(x_1, \mu) - w(x_1, \mu)),$$

for which $\alpha(0) = 0$ by (2.14), is defined on $\Gamma$ and satisfies

$$D\alpha(0) \neq 0.$$

4. For all $\mu \in \Gamma$ such that $\alpha(\mu) < 0$, the equality

$$p(x_2, \mu) = w^s_\mu(x_2)$$
Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \begin{cases} 
\{-x\}, & x < 1, \\
\{-1, 1\}, & x = 1, \\
\{1\}, & x > 1,
\end{cases}$$

whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = \{1\}$$

The theorem is illustrated in Figure 2.3. As for the IR$_1(1)$ bifurcation, at the bifurcation the relative position of the optimal trajectories and the 'most unstable' invariant manifold changes. If $\alpha(\mu) < 0$, the backward extension of the optimal trajectory through the point $\bar{z}_1 = (\bar{x}_1, p(\bar{x}_1, \mu))$ has a vertical tangent at a certain point. Past this point, the trajectory cannot be optimal, even locally. It follows that $x_e$ is locally optimal. For $\alpha(\mu) > 0$, the trajectory through $\bar{z}_1$ intersects the line $x = x_e$. Theorem 2.2.1 then implies that the constant trajectory $e$ cannot be optimal at all in this case.

In many applications, the optimal trajectory through $\bar{z}_1$ is on the stable manifold of another equilibrium $e'$. For $\alpha(\mu) = 0$, we have also that $\bar{z}_1$ is in the unstable manifold of $e$, and the trajectory of $F$ through $\bar{z}_1$ then forms a heteroclinic connection between $e$ and $e'$. In this form, the indifference-attractor bifurcation was investigated in Wagener (2003). The present formulation in terms of the optimal costate rule is more general as it captures, for instance, also the situation that the optimal trajectory through $\bar{z}_1$ tends to infinity as $t \to \infty$ (cf. Hinloopen et al. (2010)).

Proof. Restricted to a neighbourhood of the saddle, in linearizing coordinates the vector field $F_\mu$ takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^u & 0 \\ 0 & \lambda^s \end{pmatrix} \zeta.$$
Figure 2.3: Before, at and after the indifference-attractor bifurcation point.
The coordinates are chosen such that the coordinate transformation is orientation preserving; moreover, the direction of the axes is chosen such that

\[ x_\xi > 0, \quad x_\eta < 0. \]

Note that the unstable and stable manifolds are in these coordinates equal to the horizontal and vertical coordinate axes respectively.

As in the proof of theorem 2.2.4, set \( \bar{x}_i, \bar{p}_i \) and \( \bar{z}_i \) as in (2.10) and (2.11).

Assume that \( \bar{x} \) of point 2 of Definition 2.2.3 satisfies \( \bar{x} = \bar{x}_2 \); the opposite situation can be handled analogously. If \( \bar{\xi}_2 \) and \( \bar{\eta}_2 \) are defined as

\[ x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = w(\bar{x}) + \alpha, \]

then it follows as in the proof of theorem 2.2.4 that

\[ (\bar{\eta}_2)_\alpha > 0 \]

and \( \bar{\eta}_2 = 0 \) if \( \alpha = 0 \).

The trajectory \( z_2(t) = (x_2(t), p_2(t)) \) through \( \bar{z}_2 \) has in linearizing coordinates the form

\[ \zeta_2(t) = (\xi_1(t), \eta_1(t)) = (\bar{\xi}_2 e^{\lambda_u t}, \bar{\eta}_2 e^{\lambda_s t}). \]

It follows that

\[ \dot{x}_2 = e^{\lambda_s t} \left( \lambda^s_{\bar{\eta}_2} x_\eta + \lambda^u_{\bar{\xi}_2} x_\xi e^{(\lambda_u - \lambda_s)t} \right). \]  \[ \text{(2.15)} \]

If \( \alpha(\mu) > 0 \), then \( \bar{\eta}_2 > 0 \); as both \( \lambda^s x_\eta > 0 \) and \( \lambda^u \bar{\xi}_2 x_\xi > 0 \), it follows from (2.15) that \( \dot{x}_2 > 0 \) for all \( t \). That is, the trajectory \( z_2 \) intersects each line \( x = \text{const} \) exactly once, and therefore defines a \( C^1 \) function \( x \mapsto p(x, \mu) \), which then necessarily satisfies

\[ p^\circ_\mu(x) = \{ p(x, \mu) \} \]
for all $x \in N$.

Consider now the case that $\alpha(\mu) < 0$. By equation (2.15), if $\alpha(\mu)$ and hence $\bar{\eta}_2$ is sufficiently close to 0, then $\dot{x}_2(0) > 0$. Let $T_\mu < 0$ be such that $\eta_2(T_\mu) = -\bar{\xi}_2$. Then

$$T_\mu = \frac{1}{\lambda^s} \log \frac{-\bar{\xi}_2}{(-\bar{\eta}_2)}$$

and equation (2.15) yields

$$\dot{x}_2(T_\mu) e^{-\lambda^s T_\mu} = \lambda^s \bar{\eta}_2 x_\eta + \lambda^u \bar{\xi}_2 x_\xi \left( \frac{-\bar{\eta}_2}{\bar{\xi}_2} \right)^{1+\lambda^u/|\lambda^s|}$$

$$= \lambda^s \bar{\eta}_2 x_\eta + o(\bar{\eta}_2).$$

This is negative if $\bar{\eta}_2$, and hence $\alpha(\mu)$, is sufficiently close to 0. For such values of $\alpha$, there exists $t < 0$ such that $\dot{x}_2(t) = 0$. Let $t_*$ denote the largest value of $t$ with this property, and introduce

$$x_* = x_2(t_*).$$

For $x_* \leq x \leq \bar{x}_2$, the trajectory $(x_2(t), p_2(t))$ parameterizes the graph of a function $p_2(x)$. As $\dot{x} = H_p$ along trajectories, note that

$$H_p(x_*, p_2(x_*)) = 0.$$

Let $p_1 : N \to \mathbb{R}$ be such that its graph parameterizes the stable manifold $W^s$ of $s$. Strict convexity of $H$ implies the inequality

$$H(x_*, p_2(x_*)) < H(x_*, p_1(x_*)). \quad (2.16)$$

Define functions $V_1$ on $N$ and $V_2$ on $[x_*, \bar{x}_2]$ by

$$V_1(x) = \frac{H(x, p_1(x))}{\rho}$$

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and
\[ V_2(x) = \frac{\mathcal{H}(x, p_1(x))}{\rho}. \]

Then
\[ V_2(x_*) < V_1(x_*). \]

To establish the opposite inequality for some \( x^* \in [x_*, \tilde{x}_2] \), consider the situation for \( \alpha(\mu) = 0 \), when \( \tilde{z}_2 \in W^u \). Then \( V_2 \) is defined for all \( x_s < x < \bar{x}_2 \). Moreover,

\[ \lim_{x \downarrow x_s} V_2(x) = V_1(x). \]

Note that since \( V_i'(x) = p_i(x) \) and
\[ p_2(x) > p_1(x) \]
for all \( x_s < x < \bar{x}_2 \), it follows that
\[ V_2(x) - V_1(x) = \int_{x_s}^{x} (p_2(\sigma) - p_1(\sigma)) \, d\sigma > 0 \]
for all \( x > x_s \). This implies in particular that
\[ \mathcal{H}(x, p_2(x)) > \mathcal{H}(x, p_1(x)) \]
for all \( x > x_s \), if \( \alpha(\mu) = 0 \).

Fix \( x^* \in (x_*, \bar{x}_2) \). Then for \( \alpha(\mu) < 0 \) sufficiently close to 0, by continuity
\[ \mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)). \] (2.17)

As a consequence of (2.16) and (2.17), there is \( \tilde{x} \in (x_s, x^*) \) such that
\[ \mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})). \]
By theorem 2.2.1, it follows that $\bar{x}$ is an indifference point.

**The saddle-node bifurcation**

The saddle-node bifurcation of dynamical systems has a natural counterpart as a bifurcation of optimal vector fields.

Recall that a family of vector fields $f_\mu : \mathbb{R}^m \to \mathbb{R}^m$ can be viewed as a single vector field $g : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ by writing

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = g(x, \mu) = \begin{pmatrix} f_\mu(x) \\ 0 \end{pmatrix}.$$ 

Consider the situation that for $\mu = 0$ the point $\bar{z}$ is an equilibrium of $f_0$, and that $Df_0(\bar{z})$ has a single eigenvalue 0. Then $Dg(\bar{z}, 0)$ has two eigenvalues zero and an associated two-dimensional eigenspace $E^c$. The center manifold theorem applied to $g$ implies that there is a differentiable invariant manifold $W^c$ of $g$ that is tangent to $E^c$ at $(\bar{z}, 0)$. The manifold $W^c$ can be viewed as a parameterized family of invariant manifolds $W^c_{\mu}$, which are defined for $\mu$ taking values in a full neighbourhood of $\mu = 0$. Note that the center manifolds need not be unique.

**Definition 2.2.4.** A point $e = (x_e, p_e)$ is a (codimension one) saddle-node singularity, notation $SN_1$, of an optimization problems with reduced canonical vector field $F$, if the following conditions hold.

1. The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1, \lambda_2$ of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.

2. There is a compact interval $N$ of $X$ containing $x_e$ and a function $p : N \to \mathbb{R}$ such that

$$p^\circ(x) = \{p(x)\}$$

for all $x \in N$, and such that the graph of $p$ is a center manifold $W^c$ of $F$ at $e$. 

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3. The restriction

\[ F^c(x) = F_1(x_e + x, p(x_e + x)). \]

of \( F \) to \( W^c \) satisfies

\[ F^c(0) = 0, \quad (F^c)'(0) = 0, \quad (2.18) \]

and

\[ (F^c)''(0) \neq 0. \quad (2.19) \]

**Theorem 2.2.7.** Consider a family of optimization problems, depending on a parameter \( \mu \in \mathbb{R}^q \), that has for \( \mu = 0 \) a \( SN_1 \) singularity. Assume that there is a neighbourhood \( \Gamma \subset \mathbb{R}^q \) of 0 such that the following conditions hold.

1. There is a function \( p : N \times \Gamma \rightarrow \mathbb{R} \) such that

\[ p_\mu(x) = \{p(x, \mu)\} \]

for all \( (x, \mu) \in N \times \Gamma \).

2. For \( \mu \in \Gamma \), the graphs of \( x \mapsto p(x, \mu) \) form a family of center manifolds \( W^c_\mu \) of \( F \) at \( e \).

3. If \( F^c_\mu \) is

\[ F^c_\mu(x) = (F_\mu)_1 (x_e + x, p(x_e + x, \mu)) \]

then the function

\[ \alpha(\mu) = F^c_\mu(0) \]

satisfies

\[ D\alpha(0) \neq 0. \]

Then the optimal vector field \( f^o_\mu \) restricted to \( N \) is for \( \mu \in \Gamma \) topologically conjugate to

\[ Y_\mu(x) = \{\alpha(\mu) - \sigma x^2\} \]
where $\sigma \in \{-1, 1\}$ is given as

$$\sigma = \text{sgn}(F_0'')(0).$$

Proof. This is a direct consequence from the usual saddle-node bifurcation theorem.

2.2.3 Codimension two bifurcations

Most codimension two situations are straightforward extensions of the corresponding codimension one bifurcations. The results in this subsection will in most cases be stated more briefly and less formally. An exception is made for the indifference-saddle-node bifurcation.

A model case: the IR$_2$(1,1) bifurcation

Definition 2.2.5. A point $e = (x_e, p_e)$ is a (codimension two) indifference repeller singularity of type (1,1), notation IR$_2$(1,1), of an optimization problem with reduced canonical vector field $F$, if all conditions of definition 2.2.1 hold, but with (2.8) and (2.9) replaced by the condition that

$$p(x) = w(x)$$

for all $x \in \partial N$.

Theorem 2.2.8. Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_2$(1,1) singularity. Let all the conditions of Theorem 2.2.4 hold, excepting point 3, which is replaced by the following.

The function

$$\alpha(\mu) = \left(\nu(x_1)(p(\bar{x}_1, \mu) - w(\bar{x}_1, \mu)), \nu(x_2)(p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu))\right).$$

for which $\alpha(0) = (0, 0)$, is defined on $\Gamma$ and satisfies

$$\text{ran } D\alpha(0) = 2.$$
Then the optimal vector field $f^o$ restricted to $N$ is topologically conjugate to

$$Y(x) = x$$

if $\alpha_1(\mu) \leq 0$ and $\alpha_2(\mu) \leq 0$, whereas it is conjugate to

$$Y(x) = \begin{cases} 
-1 & x < 0, \\
\{1, 1\} & x = 0, \\
1 & x > 0.
\end{cases}$$

if $\alpha_1(\mu) > 0$ or $\alpha_2(\mu) > 0$. In particular, the curves $\alpha_1(\mu) = 0$, $\alpha_2(\mu) < 0$ and $\alpha_2(\mu) = 0$, $\alpha_1(\mu) < 0$ are codimension one indifference-repeller bifurcation curves.

The proof is a simple modification of the proof of the codimension one case and is therefore omitted.

**Other indifference-repeller and indifference-attractor bifurcations**

Looking at the definition of the IR$_1(2)$ bifurcation, it is clear that bifurcations of higher codimension are obtained when condition (2.13) is violated at a boundary point. If this happens at one of the boundary points, a codimension two situation is obtained where an IR$_1(1)$ and an IR$_1(2)$ curve meet in a IR$_2(1,2)$ point. If it happens at both boundary points, a codimension three situation arises, denoted IR$_3$, where two IR$_1(1)$ and a IR$_1(2)$ surface meet. In order to avoid unnecessary repetitions, the exact definitions for these bifurcations are not formulated; they can all be modeled on Definition 2.2.5 and Theorem 2.2.8. Their bifurcation diagrams are given in Figures 2.4(b) and 2.5.

Likewise, a codimension two bifurcation is obtained if condition 2.14 is replaced by

$$p(x_1) = w^u(x_1), \quad p(x_2) = w^u(x_2).$$

(2.20)
Figure 2.4: Indifference-repeller bifurcations of co-dimension two.

This a two-sided or double indifference attractor bifurcation, denoted DIA₂. Its bifurcation diagram is given in Figure 2.6.

**Degenerate saddle-node bifurcations**

The degenerate saddle-node bifurcations like the cusp (SN₂), the swallowtail (SN₃) etc. can be treated entirely analogously to the saddle-node itself.

**The indifference-saddle-node bifurcation**

The indifference-attractor and indifference-repeller bifurcations correspond to global bifurcations involving hyperbolic equilibria of the reduced canonical vector field; in contrast, the saddle-node bifurcation corresponds to a local bifurcation. The final bifurcation to be considered is the indifference-saddle-node bifurcation, which corresponds to a global bifurcation involving a nonhyperbolic equilibrium.

**Definition 2.2.6.** A point \( e = (x_e, p_e) \) is a (codimension two) indifference-saddle-node singularity, notation ISN₂, of an optimization problem with reduced canonical vector field \( F \), if the following conditions hold.

1. The point \( e \) is an equilibrium of \( F \), such that the eigenvalues \( \lambda_1, \lambda_2 \) of \( DF(e) \) satisfy \( \lambda_1 = 0, \lambda_2 = \rho \).
Figure 2.5: \( IR_3 \) bifurcation diagram.

Figure 2.6: \( DIA_2 \) bifurcation diagram.
2. On some compact interval neighbourhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that

$$p^o(x) = \{p(x)\}$$

for all $x \in N$, and such that $p_e = p(x_e)$.

3. Let $W^u$ denote the unstable manifold of $F$ at $e$, parameterized as the graph of a function $w^u : N \to \mathbb{R}$. There is a unique $\bar{x} \in \partial N$ such that

$$p(\bar{x}) = w^u(\bar{x}).$$

(2.21)

4. There is a center manifold $W^c$ of $F$ at $e$, parameterized as the graph of $w^c : N \to \mathbb{R}$, such that for $x \in \partial N$ and $x \neq \bar{x}$, we have that

$$p(x) = w^c(x).$$

(2.22)

5. The restriction

$$F^c(x) = F_1(x_e + x, w^c(x_e + x))$$

of $F$ to $W^c$ satisfies

$$F^c(0) = 0, \quad (F^c)'(0) = 0,$$

and

$$(F^c)''(0) \neq 0.$$ 

Theorem 2.2.9. Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an ISN$_2$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.
1. There is a function $p : \partial N \times \Gamma \rightarrow \mathbb{R}$, differentiable in the second argument, such that

$$ p_\mu^\partial = \{ p(x, \mu) \} $$

for all $(x, \mu) \in \partial N \times \Gamma$, and such that

$$ \alpha_2(\mu) = p(\bar{x}, \mu) - p(\bar{x}, 0) $$

satisfies

$$ D\alpha_2(0) \neq 0. $$

2. There is a family of center manifolds $W^c_\mu$, parameterized as the graphs of functions $x \mapsto w^c(x, \mu)$, such that $p(x, 0) = w^c(x, \mu)$ if $x \in \partial N \setminus \{\bar{x}\}$. 

Figure 2.7: ISN$_2$ bifurcation diagram.
3. Let $F^c_\mu$ be the restriction

$$F^c_\mu(x) = (F_\mu)_1(x_e + x, w^c(x_e + x, \mu))$$

of $F$ to $W^c$. Then the function

$$\alpha_1(\mu) = F^c_\mu(0)$$

satisfies

$$D\alpha_1(0) \neq 0.$$ 

4. Let $\alpha(\mu) = (\alpha_1(\mu), \alpha_2(\mu))$. Then $\text{ran } D\alpha(0) = 2$.

Then there is a differentiable functions $C(\alpha_2)$ such that $C(0) = C'(0) = 0$ and $C''(0) \neq 0$, and such that the problem has an indifference-attractor bifurcation if

$$\alpha_1 = C(\alpha_2), \quad \alpha_2 > 0,$$

an indifference-repeller bifurcation if

$$\alpha_1 = C(\alpha_2), \quad \alpha_2 < 0,$$

and a saddle-node bifurcation curve if

$$\alpha_1 = 0, \quad \alpha_2 < 0.$$

Proof. Assume without loss of generality that $(F^c)''(0) > 0$.

The system is first put, by an orientation preserving transformation, in coordinates $\zeta = (\xi, \eta)$ such that the center manifold $W^c_\mu$ corresponds to $\eta = 0$ for all $\mu$ close to $\mu = 0$, and the unstable manifold $W^u$ corresponds to $\xi = 0$ at $\mu = 0$. In these coordinates, the system, augmented by
the parameter equation $\dot{\mu} = 0$, takes the form

$$
\dot{\xi} = \alpha_1(\mu) + f_0(\xi, \mu) + \eta f_1(\zeta, \mu),
$$
(2.23)

$$
\dot{\eta} = \rho \eta + \eta g_1(\zeta, \mu),
$$
(2.24)

$$
\dot{\mu} = 0
$$
(2.25)

where by assumption $f_0(\xi, \mu) = c(\mu)\xi^2 + O(\xi^3)$ with $c(0) > 0$, and where $D\alpha_1(0) \neq 0$.

These conditions imply that a saddle-node bifurcation occurs at $(\xi, \eta) = (0, 0)$ if $\alpha_1(\mu) = 0$, generating a family of hyperbolic saddle and one of hyperbolic unstable equilibria of $F$.

The saddle equilibria have associated to them unique unstable invariant manifolds $W^u_\mu$; the unstable equilibria have associated to them strongly unstable manifolds $W^{uu}_\mu$, which are also unique. An indifference-attractor bifurcation occurs if $(x, p(x, \mu)) \in W^u_\mu$; an indifference-repeller bifurcation occurs if $(x, p(x, \mu)) \in W^{uu}_\mu$. The main thing to prove is that the manifolds $W^u_\mu$ and $W^{uu}_\mu$ can be parameterized as graphs of differentiable functions

$$
x \mapsto w^u(x, \mu), \quad x \mapsto w^{uu}(x, \mu).
$$

This is not automatic, for the function $w^u$ and $w^{uu}$ will not be differentiable as functions of $\mu$, having necessarily at $\mu = 0$ a singularity of the order $\sqrt{\mu}$.

In the following, it will however be shown that the closure of the invariant set

$$
W = \bigcup_\mu W^u_\mu \cup W^{uu}_\mu
$$

forms a differentiable manifold. From figure 2.8, it seems likely that $W$ can be described as the level set

$$
W : \quad \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu),
$$

where $w$ is a function yet to be determined. The condition that $W$ is invariant under the flow of (2.23)–(2.25) leads to a first order partial differential equation for the function $w$; this equa-
tion is singular for $\eta = 0$.

To solve this equation using the method of characteristics, introduce $w = w(t)$ as an independent variable by setting

$$
\eta w = \alpha_1 + f_0(\xi, \mu).
$$

Deriving with respect to time and using equations (2.23)–(2.25) yields

$$
\eta \dot{w} = -\dot{\eta} w - \frac{\partial f_0}{\partial \xi} \dot{\xi} = -w(\rho + g_1)\eta + \frac{\partial f_0}{\partial \xi} (w + f_1)\eta.
$$

Dividing out $\eta$ formally, an equation for $\dot{w}$ is obtained. Together with equations (2.23)–(2.25), the following system is obtained:

$$
\begin{align*}
\dot{\xi} &= \eta w + \eta f_1, \\
\dot{w} &= -\rho w - wg_1 - \frac{\partial f_0}{\partial \xi} (w + f_1), \\
\dot{\eta} &= \rho \eta + \eta g_1, \\
\dot{\mu} &= 0.
\end{align*}
$$
Linearizing the new system at \((\xi, \eta, w, \mu) = (0, 0, 0)\) yields
\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta} \\
\dot{w} \\
\dot{\mu}
\end{pmatrix} =
\begin{pmatrix}
0 & -\rho \\
\rho & 0 \\
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
w \\
\mu
\end{pmatrix}
\]

Again invoking the center manifold theorem, we find that there is an invariant center-unstable manifold \(W^{cu}\) that is tangent to the center-unstable eigenspace \(E^{cu} = \{w = 0\}\). Let this manifold be parameterized, in a neighbourhood of the origin, as
\[
W^{cu} : w = w^{cu}(\xi, \eta, \mu).
\]

Then \(w^{cu}\) is the function we have been looking for.

A final note on \(W\): as for \(\mu = 0\) the unstable manifold \(W^u\) is tangent to \(\alpha_1 = 0\) at \(\xi = 0\), the function \(w\) in
\[
W : \quad \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu),
\]  
(2.26)
has to satisfy \(w = \xi^2 \tilde{w}\).

Indifference-attractor or indifference repeller bifurcations occur if \((\bar{x}, p(\bar{x}, \mu)) \in W\). The equations
\[
x = \bar{x}, \quad p = p(\bar{x}, 0) + \alpha_2
\]
take in \((\xi, \eta)\)-coordinates the form
\[
\xi = c_1 \alpha_2 + O(\varepsilon^2 + \alpha_2^2), \quad \eta = c_{21} \alpha_2 + c_{22} \varepsilon + O(\varepsilon^2 + \alpha_2^2).
\]

Note however that if \(\mu = 0\), then \(W\) is given by \(\xi = 0\). Moreover, by assumption \(p(\bar{x}, 0) \in W\); therefore the equations actually read as
\[
\xi = \alpha_2 (c_1 + O(\varepsilon + \alpha_2)), \quad \eta = c_{21} \alpha_2 + c_{22} \varepsilon + O(\varepsilon^2 + \alpha_2^2).
\]
Substitution in equation (2.26) yields the indifference-attractor and indifference-repeller bifurcation curves

\[ \alpha_1 = c(\mu)c_1^2\alpha_2^2 + O(\alpha_2^3). \]

Taken together with the saddle-node curve

\[ \alpha_1 = 0, \]

this yields the bifurcation diagram. Finally, note that if \( p(\bar{x}, \mu) > w^{uu}(\bar{x}) \), the saddle node bifurcation does not correspond to a bifurcation of the optimal vector field.

\[ \square \]

**DISN\textsubscript{3} bifurcation**

It is possible that a ‘double’ ISN singularity, denoted DISN\textsubscript{3}, occurs if conditions (2.21) and (2.22) of definition 2.2.6 are replaced by the condition that

\[ p(x) = w^u(x) \]

for all \( x \in \partial N \). This is clearly a codimension three situation.
Chapter 3

Bifurcations of optimal vector fields in the shallow lake model

In the present chapter\(^1\) ideas from bifurcation theory, developed in Chapter 2, are used to analyze the effects of varying parameters in the shallow lake pollution problem, introduced in Brock and Starrett (2003), Mäler et al. (2003). This is an optimal pollution management problem where a social planner faces a trade-off between interests of farmers, who indirectly benefit from polluting the lake by using fertilizers that are washed into it, and interests of fishermen, tourists and water companies, who benefit from high quality of the lake water. The shallow lake model contains three parameters: \(b\), the rate of loss of pollutant due to sedimentation, representing biological properties of the lake; \(c\), the relative costs of pollution, modeling the trade-off between farmers’ and tourists’ interests; \(\rho\), the discount rate, representing the intertemporal rate of substitution. The main idea of the bifurcation analysis is to study dependence of the solution structure upon the parameters of the model. This model exhibits non-convexities for some parameter values which causes the existence of multiple local optima of the water pollution level and thereby history-dependent optimal pollution policies, see Brock and Starrett (2003), Mäler et al. (2003). In Wagener (2003) the genesis of history-dependent optimal management policies in the shallow lake model has been connected to the occurrence of heteroclinic bifurcations of

\(^1\)A modified version of this chapter constitute a published paper in Journal of Economic Dynamics and Control (see Kiseleva and Wagener (2010)).
the associated state-control system.

This chapter completes the bifurcation analysis of the shallow lake model. The dependence of the solution structure on the system parameters is studied. As a result, two types of planar cuts of the parameter space are obtained: first, the biological properties $b$ of a lake are held fixed while the socio-economic parameters $c$ and $\rho$ vary; second, the discount rate $\rho$ is fixed and $b$ and $c$ vary. This provides a fairly complete picture of how the optimal pollution policy for an eco-system responds to changes in the degree of its resilience, social preferences and economic factors.

With the performed analysis the trade-off between the relative cost of pollution $c$ and the time discount factor $\rho$ is quantified. In particular for several values of $b$, two-parameter bifurcation diagrams with respect to $c$ and $\rho$ are computed; they show that if $\rho$ is decreased, the minimal preference for the environment $c$ that implies that the oligotrophic solution is optimal is decreased proportionally; this solution is characterized by high quality of the lake water and low level of agricultural activity in the long run regardless of the initial pollution level. Thus the oligotrophic solution can be globally optimal in a less environmentally friendly society, if the social planner is sufficiently foresighted.

The outline of the chapter is as follows. Section 3.1 describes the shallow lake model. Section 3.2 introduces the concept of the optimal vector fields and gives examples of optimal vector fields in the shallow lake model. Section 3.3 presents a fairly complete bifurcation analysis of the shallow lake system with respect to all three parameters: the natural rate of pollutant outflow $b$, the relative costs of pollution $c$ and the discount rate $\rho$. Finally, Section 3.4 concludes.

### 3.1 The shallow lake model

The shallow lake problem is an optimal pollution management problem solved by a social planner. This social planner maximizes a social utility functional, which models conflicting interests of two types of lake users: farmers and ‘water users’, such as tourists, water companies and fishermen. Farmers get benefits from using fertilizers that contain phosphorus; the phosphorus runs
off the fields and is eventually washed into the lake. Surplus of phosphorus in water causes growth of aquatic plants that fill the entire water column or that concentrate much of their biomass in the upper water layer. When these plants become dominant the bottom vegetation, which stabilizes the sediment, collapses due to light limitation. As a result surface waves can stir up the sediment - the lake is shallow - and the lake becomes polluted. The drop of water quality leads to losses for the lake users.

The shallow lake model consists of two parts: the pollution dynamics and the social welfare functional. Let $x(t)$ be proportional to the amount of phosphorus in the lake at time $t$. The value of $x(t)$ may change due to the input of more phosphorus due to farming activities, $u(t)$, as well as due to sedimentation and the internal biological processes of production of phosphorus. Mäler et al. (2003) proposed to model the pollution dynamics of a shallow lake as

$$\dot{x} = u - bx + \frac{x^2}{1 + x^2}, \quad x(0) = x_0,$$

where $b \geq 0$ is the coefficient that is proportional to the rate of loss of phosphorus due to sedimentation, and where the last term models the biological production process. For more detailed analysis and the biological background of equation (3.1) see Scheffer (2009).

The second part of the shallow lake model is the social welfare functional. Society in this model consists of lakes users of two types: farmers, who benefit from polluting the lake, and water users, who bear losses when the pollution level increases. The farmers’ benefits, or the farmers’ utility, is assumed to be an increasing concave function of $u$, taken here as $^2 \log u$. The costs of pollution, or the disutility of the water users, is assumed to increase quadratically with the pollution level. Thus the social welfare at time $t$ is $\log u(t) - cx^2(t)$, where $c$ is a nonnegative parameter which models the relative cost of pollution.

The total welfare is given by

$$B[u] = \int_0^\infty \left( \log u - cx^2 \right) e^{-\rho t} dt,$$

where $u$ is the utility function, $\rho$ is the discount rate, and $x$ is the amount of phosphorus in the lake.

Results that are obtained in this chapter are expected to be fairly robust to the choice of this utility function.
where $\rho$ is a discount factor and $x$ is defined by (3.1). The optimal control problem of the social planner is to find the dumping control $u^*$ that maximizes the social welfare functional $B$ given the initial level of pollution $x_0$ and the pollution dynamics (3.1). A solution to this optimization problem is a pair $(\check{x}, \check{u})$, such that $\check{u}$ is an admissible control, meaning that $\check{u}$ is piecewise continuous and $\check{u}(t) \geq 0 \ \forall t \geq 0$, and $\check{x}$ continuous and piecewise continuously differentiable, (3.1) is satisfied and $B(\check{u}) \geq B(u)$ if $(x, u)$ satisfies (3.1).

The standard way of solving such a problem is to introduce the current value Hamiltonian

$$H(x, p, u) = \log(u) - cx^2 + p\left(u - bx + \frac{x^2}{1 + x^2}\right),$$  \hspace{1cm} (3.3)

which has to be maximized with respect to the control variable $u \in \mathbb{R}_+$. The additional variable $p \in \mathbb{R}$ is called the co-state and represents the shadow costs of pollution. According to Pontryagin’s Maximum Principle, if $u : [0, \infty) \to (0, \infty)$ is an optimal solution, then $p(t), x(t)$ and $u(t)$ satisfy three conditions:

1) $u(t)$ maximizes the function $h(u) = H(x(t), p(t), u)$ for each $t$. In the shallow lake model this implies the following one-to-one correspondence between the costate $p$ and the control $u$

$$u = U(p) = -\frac{1}{p},$$  \hspace{1cm} (3.4)

defining the so-called maximized Hamiltonian

$$\mathcal{H}(x, p) = H(x, p, U(p)).$$  \hspace{1cm} (3.5)

2) $x(t)$ and $p(t)$ are solutions of the reduced canonical system

$$\begin{cases}
\dot{x} = \frac{\partial \mathcal{H}}{\partial p} (x, p) \\
\dot{p} = \rho p - \frac{\partial \mathcal{H}}{\partial x} (x, p)
\end{cases}$$  \hspace{1cm} (3.6)
3) the transversality condition

\[ \lim_{t \to \infty} e^{-\rho t} p(t) = 0 \quad \text{if} \quad \lim_{t \to \infty} x(t) > 0. \quad (3.7) \]

is satisfied\(^3\).

Using (3.5) and (3.4) the system (3.6) reads as

\[
\begin{align*}
\dot{x} &= -\frac{1}{p} - bx + \frac{x^2}{1 + x^2} \\
\dot{p} &= 2cx + p\left(\rho + b - \frac{2x}{(1 + x^2)^2}\right).
\end{align*}
\]

(3.8)

The system (3.8) is called the shallow lake system. Due to the one-to-one correspondence (3.4) between the costate \(p\) and the control \(u\) the shallow lake system (3.8) can be rewritten in the state-control form

\[
\begin{align*}
\dot{x} &= u - bx + \frac{x^2}{x^2 + 1} \\
\dot{u} &= -\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right) u + 2cxu^2.
\end{align*}
\]

(3.9)

The shallow lake system (3.9) is a system of parameterized differential equations. Typically in such systems changing the value of the parameters may cause qualitative changes of the solution structure: equilibria may lose stability, new equilibria or attracting sets may appear, etc. Such qualitative changes of the solution structure due to smooth variations of the parameters are called bifurcations. Some bifurcations of the dynamical system (3.9) affect the optimal pollution policy and consequently the long run pollution level \(x\) under the optimal policy. Therefore knowledge of bifurcation values of the system parameters can shed light upon the long run behavior of the system (3.9) when the optimal policy is applied.

In the next section the notion of optimal vector fields is shortly recalled. Also the correspondence between bifurcations of the optimal vector fields and bifurcations of the dynamical system (3.9) is studied.

\(^3\)For trajectories such that \(\lim_{t \to \infty} x(t) = 0\) the transversality condition is given by \(\lim_{t \to \infty} e^{-\rho t} p(t) \geq 0.\)
3.2 Optimal vector fields in the shallow lake model

This section shortly presents, in an improved form, the results of the bifurcation analysis of the shallow lake system obtained in Wagener (2003) and connects them to the concept of optimal vector fields.

Solutions to the problem of maximizing (3.2) subject to (3.1) for fixed $x(0) = x_0$ can be represented as a set of initial costates $p^o(x_0) \subset \mathbb{R}$ such that if $p_0 \in p^o(x_0)$ then $(x(0), p(0)) = (x_0, p_0)$ is an initial condition to an optimal trajectory $(x(t), p(t))$ in the state-costate space. Then the pair $(x(t), u(t)) = (x(t), U(p(t)))$ solves the optimal control problem of the social planner. The set-valued function $p^o(x)$ is called the optimal costate rule. The corresponding set-valued function $u^o(x) = \{U(p) : p \in p^o(x)\}$ is called the optimal policy rule.

For problems with one-dimensional state spaces and infinite time horizons the points $(x_0, p_0)$ with $p_0 \in p^o(x_0)$ are usually situated on the stable manifolds of a steady state of the state-costate system. If $p^o(x_0)$ contains more that one element, then the state is an indifference state. It follows from the principle of optimality that $p^o(x(t))$ is single-valued for all $t > 0$.

**Definition 3.2.1.** The multivalued vector field

$$f^o(x) = \frac{\partial H}{\partial p}(x, p^o(x)), \quad (3.10)$$

is called the optimal vector field.$^4$

The notion of optimal vector field is general and not restricted to the shallow lake problem, see Chapter 2. The optimal vector field determines the direction and the speed of the state flow under the optimal policy. Optimal state trajectories are solutions of

$$\dot{x}(t) = f^o(x(t)), \quad x(0) = x_0. \quad (3.11)$$

Remember though that the optimal policy and consequently the optimal vector field depend upon the system parameters. Hence they may bifurcate when the parameters are varied.

$^4$Boltyanskii (1966) calls these vector fields ‘synthesized’.
The main result of Wagener (2003) is the bifurcation diagram of the shallow lake system with respect to the parameters $b$ and $c$ for the fixed value $\rho = 0.03$ of the discount rate. However this bifurcation diagram is incomplete, as there are bifurcation curves missing. In this chapter the complete version of it is presented in Figure 3.1(a), which shows the parameter plane $(b, c)$ divided into four regions, labeled respectively: Unique equilibrium, Oligotrophic, Region of history dependence and Eutrophic. Those regions correspond to four different types of solutions of the shallow lake problem, differing in number of equilibria of the state-control system \((3.9)\) as well as the long run pollution level under the optimal policy. In Figures 3.1(b)-3.1(h) phase portraits of the state-control system \((3.9)\) for different values of the parameters are given. Optimal trajectories are represented by thick curves, other trajectories by solid curves. In the upper parts of the phase diagrams the phase plots of the optimal vector fields are given. Attractors of the optimal vector fields are represented by bullets, indifference thresholds by black squares. Later, in Figures 3.1(b)-3.1(h), repellers of the optimal vector fields are denoted as circles. These notations are held throughout the chapter.

**Unique equilibrium**

For the values of the parameters $b$ and $c$ in this region the state-control system \((3.9)\) has a unique equilibrium. It is a saddle, see Figure 3.1(b) and 3.1(h). The graph of the optimal solution is always situated on the stable manifold of this saddle\(^5\). The long run pollution level depends then on the values of the parameters $c$ and $b$, changing within the region.

The rate $b$ of natural sedimentation of pollution is relatively high in the region ‘Unique equilibrium’. This means that most of the pollution coming into the water sediments on the bottom of the lake, which implies that the lake can bear even a heavy pollution load without collapsing regardless of its initial pollution level. For any fixed value of $b$ in the region ‘Unique equilibrium’, the water pollution level in the long run smoothly decreases when $c$ increases: higher cost of pollution imply more restrictive pollution policies.

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\(^5\)For the proof see Wagener (2003).
Multiple equilibria

In cases with multiple equilibria of the state-control system (3.9) there are always two saddles, denoted as \( P = (x_P, u_P) \) and \( Q = (x_Q, u_Q) \). The steady state pollution level \( x_P \) in \( P \) is significantly lower than the pollution level \( x_Q \) in \( Q \); they are called oligotrophic and eutrophic steady states of the lake, respectively. The oligotrophic steady state corresponds to a high level of water services and a low level of agricultural activities, whereas the eutrophic steady state corresponds to a high level of agricultural activities and a low level of water services.

It has been proved in Wagener (2003) that the optimal solution of the social planner optimization problem is situated on the stable manifold of one of the saddles. In the case with multiple equilibria of (3.9) the social planner has to choose whether to ‘jump’ to the stable manifold of the oligotrophic equilibrium \( P \) or to the stable manifold of the eutrophic equilibrium \( Q \). Regarding to the choice of the social planner the following three cases are possible:

- the oligotrophic steady state is globally optimal; independently of the initial pollution level of the lake the social planner steers the lake to the clean equilibrium \( P \);

- the eutrophic steady state is globally optimal; independently of the initial pollution level of the lake the social planner steers the lake to the polluted equilibrium \( Q \);

- the oligotrophic steady state and the eutrophic steady state are locally optimal; the long run pollution level depends on the initial level of pollution.

Oligotrophic region  In the oligotrophic case the optimal trajectory is the stable manifold of the saddle \( P \), and the optimal policy is a smooth continuous function of the state, see Figure 3.1(c). The optimal policy steers the lake to the clean equilibrium \( P \) independently of the initial level of the pollution; the clean steady state is globally optimal. The one-dimensional phase diagram of the optimal vector field is drawn in the upper part of the Figure 3.1(c); it has one attractor, denoted by a bullet.
Figure 3.1: Figure 3.1(a) shows the bifurcation diagram of the shallow lake system in the 
(b, c)-parameter space for \( \rho = 0.03 \). Dashed lines represent saddle-node bifurcation curves, 
separating the region of parameters for which there is a unique equilibrium in the system from 
the region of multiple equilibria. Solid lines indicate heteroclinic bifurcation curves. Phase 
portraits of state-control system and of the optimal vector fields are given for \( b = 0.65 \) and 
selected values of \( c \). Optimal trajectories are represented by thick curves, other trajectories are 
represented by solid curves. Optimal solutions are always situated on the stable manifold of one 
of the saddles. In the upper parts of the phase diagrams the phase plots of the optimal vector 
fields are drawn. Attractors of the optimal vector fields are denoted by bullets, indifference 
points by squares.
**Eutrophic region** In the eutrophic case, see Figure 3.1(g), the optimal trajectory is the stable manifold of the saddle $Q$. Regardless of the initial level of pollution, the optimal policy steers the lake to the polluted equilibrium. The optimal vector field, drawn in the upper part of the Figure 3.1(g), has a unique attractor with the whole state space as a basin of attraction; the polluted steady state is globally optimal.

Note that in both cases, oligotrophic and eutrophic, the optimal vector field is single-valued for all initial states.

**Region of history dependence** History-dependent solutions are distinguished from the other ones by the presence of threshold values of the initial pollution level $x$: if the initial pollution level is below that threshold level then the oligotrophic steady state is optimal, whereas if the initial pollution level is above that threshold level then the eutrophic steady state is optimal. The type of history-dependent solution is determined by the type of the threshold point which can be either a repeller or an *indifference point*.

Indifference points are initial states $x = x^*_0$ for which the social planner is indifferent between steering the lake to the clean or to the polluted state; for these states there exist two optimal controls $u^*_1$ and $u^*_2$, both maximizing the social welfare functional (3.2). In the case when the threshold is an indifference point the optimal policy is a smooth single-valued function everywhere, except from a point where it takes two values. That point is the indifference point. The optimal vector field is also multivalued at that point. The indifference point in Figure 3.1(e) is marked by a black square.

In the case when the threshold is a repeller the optimal policy is a smooth function; it as well as the optimal vector field is everywhere single-valued. This case is not shown in Figure 3.1, it will be illustrated in the next section.

Threshold points separate two basins of attraction of the optimal dynamics: the states below that point constitute the basin of attraction of the clean equilibrium, and the states above that point constitute the basin of attraction of the polluted equilibrium. Note that the indifference

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6Indifference points are also called Skiba points, DNS points or DNSS points. For the naming see Grass et al. (2008) p.238.
point lies in both basins, whereas the repeller lies in neither of them. The history-dependent pollution policy steers the lake to the clean equilibrium only if the lake is initially not very polluted, otherwise it steers the lake to the polluted equilibrium.

One can see from Figure 3.1 that in all cases except the history-dependent case the optimal vector field has only one attractor, whereas in the history-dependence case it has two attractors and one threshold point. Those optimal vector fields correspond to different values of the parameter $c$, hence as the parameter $c$ varies the optimal vector field undergoes a bifurcation. Since the optimal vector fields depend upon the optimal policies, bifurcations of the optimal vector fields are connected with bifurcations of the state-control system. A bifurcation of the state-control system is called *essential* if it corresponds to a bifurcation of the optimal vector field, otherwise it is called *inessential*. In Figure 3.1 the critical parameter values corresponding to essential and inessential bifurcations of the state-costate system are located on the solid and the dashed bifurcation curves respectively.

### 3.3 Bifurcations of optimal vector fields in the shallow lake system

This section studies the dependence of the optimal vector fields upon the system parameters: the natural rate of sedimentation $b$, the relative weight of ecological services (or cost of pollution) $c$ and the discount rate $\rho$. The bifurcation analysis described in the previous section is applied to compute two-parameter bifurcation diagrams. Such a diagram is a partition of the parameter plane into 2D regions of structural stability of a dynamical system, 1D bifurcation curves and bifurcation points. Different regions of the parameter-plane correspond to qualitatively different types of the solution structure of the dynamical system. In the shallow lake system the solution structure defines the type of the optimal solution and therefore the type of the optimal vector field. Hence different regions of bifurcation diagrams correspond to structurally different optimal solutions and structurally different optimal vector fields. Moreover bifurcation curves of the
shallow lake system itself and bifurcation curves of the optimal vector fields are distinguished from each other.

### 3.3.1 No discounting case

First, the shallow lake model without discounting, i.e. \( \rho = 0 \) in (3.6), is considered. In this case, overtaking optimality\(^7\) is used as the optimality criterion. Recall that an admissible control \( u^* \) is called \textit{overtaking optimal} if, for any admissible control \( u \), there is a \( T(u) \) such that for every \( T \geq T(u) \)

\[
B_T(u) \leq B_T(u^*),
\]

where

\[
B_T(u) = \int_0^T (\log u - cx^2) e^{-\rho t} \, dt.
\]

and where \( x \) satisfies (3.1).

For \( \rho = 0 \) the shallow lake system becomes Hamiltonian. Then trajectories of (3.8) are level curves of the maximized Hamiltonian \( \mathcal{H}(x, p) \) given in (3.5) and steady states are critical points of \( \mathcal{H}(x, p) \). Due to the one-to-one state-costate correspondence (3.4) the level curves of \( \mathcal{H}(x, p) \) correspond to the level curves of \( \mathcal{H}(x, -1/u) \) in the state-control space \( (x, u) \).

For solutions on the stable manifolds of \( P \) or \( Q \) the following holds

\[
B_T(u) = \mathcal{H}(P)T + o(T) \text{ as } T \to \infty
\]

and

\[
B_T(u) = \mathcal{H}(Q)T + o(T) \text{ as } T \to \infty.
\]

Since optimal solutions converge to either \( P \) or \( Q \) overtaking optimality is determined by the values \( \mathcal{H}(P) = \log u_P - cx_P^2 \) and \( \mathcal{H}(Q) = \log u_Q - cx_Q^2 \). More precisely, the trajectory converging to \( P \) will be preferable in the sense of overtaking optimality if \( \mathcal{H}(P) > \mathcal{H}(Q) \) and vice versa.

\(^7\)For a detailed discussion of this criterion see Grass et al. (2008).
Figure 3.2 shows the bifurcation diagram of the shallow lake system (3.9) for $\rho = 0$. Due to the fact that the system is Hamiltonian there is only one curve of heteroclinic bifurcations, which ends at the cusp point ‘SN$_2$’. The vertical branch of the indifference-attractor bifurcation curve ‘IA$_1$’ is located on the straight line $b = 1/2$ and separates two regions: the oligotrophic region and the region of history-dependent optimal policies.

Let us compare phase plots of the shallow lake system (3.9) for $b = 0.6$ and $b = 0.4$ while other parameters are held fixed $\rho = 0$, $c = 0.5$. First, fix $b = 0.6$. In this case the shallow lake system has three equilibria with non-zero control: two saddles and one center, see Figure 3.3(left). Since the only candidates for the optimal solution are the stable manifolds of the two saddles $P$ and $Q$, the values $H(P)$ and $H(Q)$ have to be compared to choose the largest one. Note that the stable manifold of $P$ covers the whole state space, therefore it can be presented as a smooth continuous function $u = w^*_P(x)$ with the property $H(x, w^*_P(x)) = H(P)$. Since for $(x, u)$ such that $u < bx - x^2/(1 + x^2)$ the following holds $\partial H(x, u)/\partial u < 0$, which implies that $H(P) = H(x_Q, w^*_P(x_Q)) > H(Q)$ and the stable manifold of $P$ is the optimal trajectory, see Figure 3.3(left).
Figure 3.3: Phase plots of the shallow lake system for $\rho = 0$, $c = 0.5$, $b = 0.6$(left), $b = 0.5$(middle) and $b = 0.4$(right). Solid lines represent invariant manifolds of the two saddles $P$ and $Q$, dashed lines represent the isoclines $\dot{x} = 0$ and $\dot{u} = 0$. The optimal solution is represented by thick curves.

Now let us fix $b = 0.4$. Recall that for $b < 1/2$ the shallow lake system is irreversible, meaning that the lower heteroclinic connections of the saddles are impossible. For this value of $b$ the shallow lake system has two equilibria with non-zero control: the two saddles $P$ and $Q$. The only candidates for the optimal solution are their stable manifolds. The stable manifold of the oligotrophic saddle $P$ does not cover the whole state space. In fact, it covers the interval $[0, \hat{x}]$, where $\hat{x} = (1 - \sqrt{1 - 4b^2}) / (2b)$ is the smallest positive $x$-coordinate of intersection of the isocline $\dot{x} = 0$ and the axis $u = 0$. If the initial level of pollution $x(0) \geq \hat{x}$ then it is not possible to steer the lake to the oligotrophic steady state. In order to find the optimal trajectory

Figure 3.4: Average value flows corresponding to the optimal solutions of the shallow lake model for $\rho = 0$, $c = 0.5$ and $b = 0.6$(left), $b = 0.5$(middle), $b = 0.4$(right). In case of history-dependent optimal policy the average value flow is discontinuous.
the values $\mathcal{H}(P)$ and $\mathcal{H}(Q)$ have to be compared. Analogously to the case $b = 0.6$ these values for $x$ located in the left neighborhood of $x = \hat{x}$ are compared. This implies that $\mathcal{H}(P) > \mathcal{H}(Q)$ implying that the stable manifold of $P$ is the optimal trajectory. However the corresponding optimal policy is available only for $x < \hat{x}$, and for $x \geq \hat{x}$ the optimal solution lies on the stable manifold of $Q$, see Figure 3.3(right). The point $x = \hat{x}$ in this case is called an *irreversibility threshold*. In Figure 3.3 this point is denoted by a black triangle.

It is important to note here that in case of no discounting the value function corresponding to the optimal solution is infinite. Instead the *average value flows* is considered

$$v = \lim_{T \to \infty} \frac{1}{T} B_T(u) = \mathcal{H}(u_\infty).$$

The average value flows corresponding to the optimal solutions are discontinuous if $\rho = 0$ and $(b, c)$ take values in the ‘Region of history dependence’. Figure 3.4 shows the average value flows for the three types of the optimal solution presented in Figure 3.3. For $b = 0.5$ and $b = 0.4$ they are discontinuous at the irreversibility threshold point.

### 3.3.2 Cost of pollution versus discounting

Now the value of the decay rate $b$ is fixed and the cost of pollution $c$ and the discount rate $\rho$ are taken as bifurcation parameters. This kind of analysis allows to study the dependence of the optimal pollution policy upon social preferences, while the biological properties of the lake are assumed to be given. The bifurcation diagram, given in Figure 3.5, displays the results of a ‘comparative dynamics’ analysis of the system, as it indicates how the total solution structure changes with the parameters. All the curves in Figure 3.5 are bifurcations curves of the shallow lake system, but as it is mentioned above some bifurcations of the state-control system are irrelevant to the optimal vector field. To distinguish such irrelevant bifurcation curves they are drawn as dashed curves. The solid bifurcation curves represent curves of essential bifurcations, that is, bifurcations of the optimal vector field. In Figure 3.5 essential curves divide the parameter space into three separate regions. In the outer region, the optimal vector field $f^o$
Figure 3.5: The bifurcation diagram of the shallow lake system in the $(c, \rho)$-parameter space for $b = 0.65$. Solid lines represent bifurcation curves of the optimal vector field, dashed lines represent bifurcation curves of the state-control system, that are not bifurcation curves of the optimal vector field.

has a unique global attractor. For parameters taking values in the lower inner region, $f^\alpha$ has two attractors, separated by an indifference point. In the small upper inner region, there are again two attractors but separated by a repeller. All three steady states are engaged in the cusp bifurcation which marks the point with the largest value $\bar{\rho}(b)$ of $\rho$ in the inner region, which is the supremum of values of $\rho$ such that the optimal vector field can have three equilibria.

The union of the two inner regions is the region where there are multiple long-term steady states: it is called the region of history-dependence. Consider what happens when $\rho = 0.05$ and $c$ decreases from $c = 1$ towards $c = 0$. If $c$ is large, it is always optimal to steer the lake towards a clean ‘oligotrophic’ long-term steady state. Then at $c \approx 0.61$, the region of history-dependence is entered: if the lake is initially sufficiently clean, it is still optimal to steer it towards a clean state. However, if the lake is initially already too polluted, this is not worthwhile any more. Finally, at $c \approx 0.54$, the basin of attraction of the oligotrophic state collapses, and the region is entered where there is again a single long-term optimal steady state,
but now a polluted ‘eutrophic’ one.

Increasing the discount rate $\rho$ has the same effect as decreasing the economic weight $c$ of the lake. This is according to our intuition, since both increasing the discount rate and decreasing the weight of the stock-damage term in the utility functional decreases the importance attached to long-term effects. For parameters in the region $\rho > \bar{\rho}$, there is always a single, globally attracting steady state, which depends continuously on $c$ and $\rho > \bar{\rho}$.

**Effects of varying the natural rate of decay** The parameter space can be divided into four regions, according to the values of $b$: $b \in I_i$, $i = 1, \ldots, 4$, where $I_1 = [0, 1/2]$, $I_2 = (1/2, b^*_1]$, $I_3 = (b^*_1, b^*_2)$, $I_4 = [b^*_2, +\infty)$, with

$$b^*_1 = \frac{(75 - 43\sqrt{3})}{8(-161 + 93\sqrt{3})} \approx 0.5505,$$

$$b^*_2 = \frac{3\sqrt{3}}{8} \approx 0.6495.$$

For $b \in I_4$, the regeneration function $g(x) = -bx + x^2/(1 + x^2)$ in the state dynamics equation $\dot{x} = u - g(x)$ is monotonic. In this case, to every constant loading level $\bar{u}$, satisfying $\bar{u} = g(\bar{x})$ there corresponds a unique pollution level $\bar{x}$, and $\bar{x}$ depends continuously on $\bar{u}$. For $b \in I_1 \cup I_2 \cup I_3$ the regeneration function is not monotonic, and there catastrophic jumps in the pollution level are possible as the constant level $\bar{u}$ gradually increases, see Mäler et al. (2003) and Wagener (2009). However for $b \in I_2 \cup I_3$ catastrophic shifts in the pollution can be reversed by decreasing $\bar{u}$ sufficiently, while for $b \in I_1$ they are not reversible as the self-cleaning ability of the lake is insufficient for these values of $b$.

Figure 3.6 displays the bifurcation diagram of the shallow lake system for $b = 0.55$. With a decrease in $b$ the saddle-node bifurcation lines move away from each other, expanding the lower region of history-dependence. It can be shown that the upper region of history-dependence is unbounded, because there does not exist a cusp point for $b \leq b^*_1$. For the proof see Appendix 3.A. This implies that the region where the optimal vector field $f^o$ has a unique global attractor is now separated into two regions. It can be shown that for any nonnegative value
Figure 3.6: The bifurcation diagram of the shallow lake system in the \((c, \rho)\)-parameter space for \(b = 0.55\). Solid lines represent bifurcation curves of the optimal vector field, dashed lines represent bifurcation curves of the state-control system, that are not bifurcation curves of the optimal vector field.

\(\rho = \rho^*\) of the discount rate the pollution level is lower for \((c, \rho^*)\) in the right lower region than for \((c, \rho^*)\) from the left upper region. Note also that for any positive value of the parameter \(c\) there exists a value of the discount rate \(\rho\) such that the shallow lake system ends up in an equilibrium with a relatively high pollution level.

Finally, the case of an irreversible system, i.e. \(b \in I_1\), is considered. In Figure 3.7 the bifurcation diagram of the shallow lake system for \(b = 0.45\) is displayed. One can immediately notice that the saddle-node bifurcation curve corresponding to the genesis of the eutrophic equilibrium, that is the right ‘\(SN_1\)’ curve in Figure 3.6, and the indifference-attractor bifurcation curve corresponding to the lower heteroclinic connection of the saddles, that is the right ‘\(IA_1\)’ curve in Figure 3.6, are absent in Figure 3.7. The disappearance of the ‘\(SN_1\)’ curve is explained by the following proposition.

**Proposition 3.3.1.** The saddle-node bifurcation curves in the \((b, c)\)-plane that correspond to the genesis of the eutrophic equilibrium have a vertical asymptote \(b = 1/2\) for any positive
Figure 3.7: The bifurcation diagram of the shallow lake system in the \((c, \rho)\)-parameter space for \(b = 0.45\). Solid lines represent bifurcation curves of the optimal vector field, dashed lines represent bifurcation curves of the state-control system, that are not bifurcation curves of the optimal vector field.

value of \(\rho\).

Proof. See Appendix 3.B.

A more detailed discussion of this fact is given in the next subsection. The disappearance of the ‘IA\(_1\)’ curve can be explained by the impossibility of a lower heteroclinic connection of the two saddles. Recall that if the system is reversible it is possible to steer the lake to the clean equilibrium \(P\) starting in a neighborhood of the polluted one \(Q\), as in Figure 3.1(d). However in case of irreversibility the stable manifold of \(P\) cannot be connected to the unstable manifold of \(Q\).

Therefore, for \(b < 1/2\), the eutrophic equilibrium is not involved in any saddle-node bifurcation, meaning that it always exists, nor in any indifference-attractor bifurcation, meaning that it is always locally optimal. This explains why there is only one saddle-node bifurcation curve and only one indifference attractor bifurcation curve in Figure 3.7.
3.3.3 Cost of pollution versus natural rate of decay: the discounted case

Figure 3.8 shows the bifurcation diagram of the optimal vector field \( f^o \) for \( \rho = 0.03 \) and its blow up near the cusp point ‘SN\(_2\)’. The indifference-repeller bifurcation curves ‘IR\(_1\)(1)’ and ‘IR\(_1\)(2)’ are situated in the very corner between the saddle-node bifurcation curves ‘SN\(_1\)’. The ‘IR\(_1\)(1)’ curves are almost coinciding with these in the vicinity of the ‘ISN\(_2\)’ points, see Figure 3.8(right). As \( \rho \) increases the ‘ISN\(_2\)’ points move away from the cusp point ‘SN\(_2\)’ along the saddle-node bifurcation curves.

In fact there exist three saddle-node bifurcation curves of the shallow lake system in Figure 3.8(left): two of them meet at the cusp point ‘SN\(_2\)’. The third one exists only for large values of the parameter \( c \); therefore it is not visible in Figure 3.8. However when \( \rho \) increases it moves down, as shown in Figure 3.9, where it appears in the upper left-hand corner.

Figure 3.9 displays the bifurcation diagram in the \((b, c)\)–plane for \( \rho = 0.245 \). For this value of \( \rho \) the bifurcation curves divide the parameter plane into three regions that correspond to qualitatively different optimal vector fields. The inner region, which is bounded by two ‘IR\(_1\)(1)’, one ‘IR\(_1\)(2)’ and one ‘IA\(_1\)’ curve, corresponds to a history-dependent solution with an indifference point as a threshold. This region, together with two others that are located between

![Figure 3.8: The bifurcation diagram of the optimal vector field \( f^o \) in the \((b, c)\)–plane for \( \rho = 0.03 \) (left) and its blow up near the cusp point (right).](image-url)
Figure 3.9: Bifurcation diagram of the shallow lake system with respect to the parameters $b$ and $c$ for $\rho = 0.245$.

‘SN$_1$’ and ‘IR$_1$ (1)’, ‘IR$_1$ (2)’ curves, and that correspond to a history-dependent solution with a repeller as a threshold, form a region that is called a *region of history dependence*. The outer region is called the *region of uniqueness*. In the region of history dependence the optimal vector field $f^\omega$ has two attractors separated by either a repeller or an indifference point, implying history dependence of the long run pollution level under the optimal policy. In the region of uniqueness the optimal vector field $f^\omega$ has a unique attractor. The bifurcation diagram can be interpreted as follows. Assume that relative cost of pollution $c$ is high, which implies relatively low phosphorus loading. If the natural outflow of the pollutant in a lake is a fast process, i.e. $b$ is high, then the lake is able to sustain low phosphorus loading for any initial level of pollution. But if $b$ is low then the pollutant accumulates in the water, and even an initially clean lake is not able to remain clean in presence of constant phosphorus loading. Now assume that relative cost of pollution $c$ is low, then the initial state of a lake does not affect its long run pollution level due to heavy phosphorus loading.

For the critical parameter value $\rho = 1/4$ the saddle-node bifurcation curves which exist
for $b \leq 1/2$ meet each other, see Figure 3.10(left), and for $\rho > 1/4$ they split again but in

![Bifurcation diagram](image)

Figure 3.10: Bifurcation diagram of the shallow lake system in the parameter plane $(b, c)$ for $\rho = 0.25$(left) and for $\rho = 0.251$(right).

a different manner, see Figure 3.10(right), giving rise to a separate saddle-node bifurcation curve, the dashed curve in the low left-hand corner in Figure 3.10(right). These bifurcations are irrelevant to the optimal vector field, as it is a curve of saddle-node bifurcations of the state-control system (3.9) which do not cause any bifurcations of the optimal vector field $f^\omega$.

The region of history-dependent optimal policies shrinks and moves up as the discount factor $\rho$ increases, see Figure 3.11. One can see that the right boundary of this region hardly moves, whereas the left boundary moves quickly as $\rho$ increases. This fact can be explained as follows. Assume the natural rate of decay to be small, i.e. $b \leq 1/2$, meaning that the lake accumulates most of the pollutant coming in. With an increase in $\rho$ the social planner becomes more myopic. The more myopic optimal policy allows for heavier pollutant loading. Thus, due to the low self-cleaning ability of the lake and the heavy phosphorus loading, the pollution level rapidly converges to a high steady level regardless of its initial value. However, if $c$ is high enough, i.e. the society is sufficiently concerned about the ecosystem quality, the optimal policy becomes history-dependent, and thereby the initial state of the lake determines its long run pollution level. To see this, note that the saddle-node curves that band the region of history-dependence on the left have an asymptote $b = 0$; this is proved in Appendix 3.B. This implies
that for any $0 < b \leq 1/2$ and $\rho > 0$ there is a value of $c$, possible large, such that $(b, c, \rho)$ is in the region of history-dependence.

![Graph showing regions of history-dependent optimal policy for different values of the discount rate $\rho$.](image)

Figure 3.11: Regions of history-dependent optimal policy for different values of the discount rate $\rho$.

### 3.4 Concluding remarks

This chapter applies the tools of exploring dynamic optimization problems with multiple equilibria to the shallow lake model. These tools, the notion of optimal vector fields and the theory of bifurcating optimal vector fields, has been introduced and described in Chapter 2. The present chapter illustrates how they work for a particular optimization problem. These tools however can be applied to a wide class of economic problems.

With the help of the proper bifurcation analysis the analysis of the shallow lake model started in Wagener (2003) has been completed. The full picture of all possible qualitatively different optimal pollution policies depending on the type of an eco-system, social preferences and economic factors has been obtained. Moreover the boundaries of the regions in the parameter space that correspond to different types of optimal policies have been computed. Each point in the
parameter space determines a particular optimization problem. A certain type of the optimal solution corresponds to a point in a particular region, and an intermediate degenerate situation between two types of the optimal solution corresponds to a point on a boundary.

Roughly speaking there are three types of the optimal solution: 1) steering a system to an equilibrium level regardless of its initial state; 2) steering a system to either of two existing equilibria depending on its initial state; 3) steering a system to either of the two equilibria unless the initial state is not at its intermediate steady state level. The last two types of the optimal solution are called history-dependent optimal policies.

The two types of history-dependent optimal policies are distinguished only by the type of the threshold point: it is either an indifference point or a repeller of the optimal vector field. In the shallow lake model, in the first case if the initial pollution level is at the threshold value then the social planner is free to decide which equilibrium, oligotrophic or eutrophic, the lake will be steered to; both policies are optimal. In the second case the threshold pollution level is a repelling equilibrium level; thus the optimal policy keeps that pollution level once started there, otherwise it steers the pollution level away from it.

Another important contribution of the present chapter to the analysis of the shallow lake model is ascertained trade-off between social preferences and economic factors. It can be seen from the bifurcation diagrams with respect to the two parameters: $c$, relative costs of pollution, and $\rho$, the discount factor, see Figures 3.5-3.6. A decrease in $c$ may radically change the long run pollution level. In order to keep it at the same value the social planner has to become less myopic, $\rho$ has to be decreased proportionally.

**Appendix 3.A  Asymptotic behavior of the cusp bifurcation curve**

This Appendix proves that the projections of the cusp bifurcation curve onto the $(b, c)$— and $(b, \rho)$—planes have the vertical asymptote $b = \left(\sqrt{9 + 6\sqrt{3}}\right)/8$ and its projection onto $(c, \rho)$—plane has the asymptote $\rho = Kc + L$, where $K$ and $L$ are given by (3.20)-(3.21).
The shallow lake system in state-control form is given by the following system of differential equations
\[
\begin{align*}
\dot{x} &= u - bx + \frac{x^2}{x^2 + 1} \\
\dot{u} &= -\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right) u + 2cxu^2.
\end{align*}
\]
By solving the following system
\[
\begin{align*}
&u - bx + \frac{x^2}{1 + x^2} = 0 \\
&-\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right) u + 2cxu^2 = 0
\end{align*}
\]
for \( u > 0 \) the manifold of equilibria of the system (3.14) is obtained in the cartesian product \( \mathbb{R} \times \mathbb{R}^3 \) of state space and parameter space
\[
s(x; b, c, \rho) = -\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right) + 2cx \left( bx - \frac{x^2}{1 + x^2} \right) = 0.
\]
From the definition of cusp bifurcation it follows that the cusp bifurcation curve is a solution of the following system
\[
\begin{align*}
s(x; b, c, \rho) &= 0 \\
s_x(x; b, c, \rho) &= 0 \\
s_{xx}(x; b, c, \rho) &= 0.
\end{align*}
\]
The system (3.16) has to be solved with respect to the parameters \( b, c \) and \( \rho \) to obtain an explicit expression for the cusp curve in the parameter space. For that solvability of (3.16) with respect to the parameters has to be checked. The Jacobian of (3.16) is given by
\[
\frac{\partial(s, s_x, s_{xx})}{\partial(b, c, \rho)} = -\frac{8cx^2(-3 + 6x^2 + x^4)}{(1 + x^2)^3}.
\]
For \( c > 0, \ x > 0 \) and \( x \neq \sqrt{2\sqrt{3} - 3} \), which is the only positive root of the equation \(-3 + 6x^2 + x^4 = 0 \), the system (3.16) can be solved with respect to \( b, c \) and \( \rho \). Therefore
the cusp bifurcation curve can be parameterized as the image of the map

$$\Upsilon : x \mapsto (b_{\text{cusp}}(x), c_{\text{cusp}}(x), \rho_{\text{cusp}}(x)),$$

where

$$b_{\text{cusp}}(x) = \frac{-3x - 8x^3 + 9x^5 + 6x^7}{(1 + x^2)^2(-1 - 10x^2 + 15x^4)}$$  \hspace{2cm} (3.17)

$$c_{\text{cusp}}(x) = \frac{1 + 10x^2 - 15x^4}{x^2(1 + x^2)(-3 + 6x^2 + x^4)}$$ \hspace{2cm} (3.18)

$$\rho_{\text{cusp}}(x) = \frac{2x}{(1 + x^2)^3} - \frac{5x}{(1 + x^2)^2} + \frac{15x}{4(1 + x^2)} - b_{\text{cusp}}(x) + \frac{x^3}{4}c_{\text{cusp}}(x)$$ \hspace{2cm} (3.19)

All the parameters in the model are assumed to be nonnegative; the inequalities $b_{\text{cusp}}(x) \geq 0$, $c_{\text{cusp}} > 0$ and $\rho_{\text{cusp}} \geq 0$ imply that $x \in (\bar{x}_1, \bar{x}_2)$, where

$$\bar{x}_1 = \sqrt{2\sqrt{3} - 3} \approx 0.68$$, which is the root of the equation $3 - 6x^2 - x^4 = 0$,

$$\bar{x}_2 \approx 0.8233$$, which is a root of the equation $\rho_{\text{cusp}} = 0$.

For $x \in (\bar{x}_1, \bar{x}_2)$ the functions $b_{\text{cusp}}(x)$, $c_{\text{cusp}}(x)$ and $\rho_{\text{cusp}}(x)$ are monotone functions satisfying the following properties

$$b(\bar{x}_1) = \frac{1}{8}\sqrt{9 + 6\sqrt{3}} \approx 0.5505,$$

$$\lim_{x \downarrow \bar{x}_1} c(x) = +\infty,$$

$$\lim_{x \downarrow \bar{x}_1} \rho(x) = +\infty.$$

That proves that the projection of the cusp bifurcation curve $\Upsilon$ both on $(b, c)$ and $(b, \rho)$-planes has a vertical asymptote $b = \left(\sqrt{9 + 6\sqrt{3}}\right)/8$.

In order to prove that the projection of the cusp bifurcation curve on the $(c, \rho)$-plane has an
inclined asymptote the following limits are computed

\[
K = \lim_{x \to \bar{x}} \frac{\rho(x)}{c(x)} = \frac{1}{4} (2\sqrt{3} - 3)^{\frac{3}{2}},
\] 
(3.20)

\[
L = \lim_{x \to \bar{x}} (\rho(x) - Kc(x)) = \frac{1}{4} \sqrt{2\sqrt{3} - 3}.
\] 
(3.21)

That proves that the cusp bifurcation curve has the asymptote in \((c, \rho)\)–plane, given by

\[
\rho = Kc + L.
\] 
(3.22)

### Appendix 3.B Asymptotic behavior of the saddle-node bifurcation curves

This Appendix proves that the saddle-node bifurcation curves in the \((b, c)\)–plane have two vertical asymptotes \(b = 0\) and \(b = 1/2\).

The shallow lake system in state-control form is given as

\[
\begin{align*}
\dot{x} &= f(x, u; b, c, \rho) = u - bx + \frac{x^2}{x^2 + 1} \\
\dot{u} &= g(x, u; b, c, \rho) = -\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right) u + 2cxu^2.
\end{align*}
\] 
(3.23)
Saddle-node bifurcations occur for points that are solutions of the following system

\[
\begin{align*}
    f(x; u, b, c, \rho) &= 0 \\
    g(x; u, b, c, \rho) &= 0 \\
    D(x; u, b, c, \rho) &= 0
\end{align*}
\] (3.24)

where \( D(x; u, b, c, \rho) = \text{det} \left( \partial(f, g)/\partial(x, u) \right) \). It can be shown that the system (3.24) can always be solved with respect to the parameters \( b, c \) and \( \rho \) and the saddle-node bifurcation surface can be parameterized as image of the map

\[
\Gamma : (x, u) \mapsto (b_{sn}(x, u), c_{sn}(x, u), \rho_{sn}(x, u)).
\]

Solving (3.24) yields

\[
b_{sn}(x, u) = \frac{x^2 + u(1 + x^2)}{x(1 + x^2)}
\] (3.25)

\[
c_{sn}(x, u) = \frac{3x^2 - 1}{(1 + x^2)(2u - x^2 + 4ux^2 + x^4 + 2ux^4)}
\] (3.26)

\[
\rho_{sn}(x, u) = \frac{2x}{(1 + x^2)^2} + 2uxc_{sn}(x, u) - b_{sn}(x, u)
\] (3.27)

Intersection of the saddle-node bifurcation surface with the plane \( \rho = \rho_0 \) can be computed by solving the following equation

\[
\rho_{sn}(x, u) = \rho_0
\] (3.28)

with respect to \( u \) for \( \rho_0 > 0 \). (3.28) is formally equivalent to a quadratic equation in \( u \). Let us denote solutions of (3.28) as \( \hat{u}_1(x, \rho_0) \) and \( \hat{u}_2(x, \rho_0) \). Then the intersection of the saddle-node surface \( \Gamma(x, u) \) with the plane \( \rho = \rho_0 \) is defined as the two following curves

\[
\gamma_1 : x \mapsto \Gamma(x, \hat{u}_1(x, \rho_0)),
\]

\[
\gamma_2 : x \mapsto \Gamma(x, \hat{u}_2(x, \rho_0)).
\]
In order to determine the vertical asymptotes in the \((b, c)\)-plane of the saddle-node curves \(\gamma_1(x)\) and \(\gamma_2(x)\), an \(x^* \geq 0\) has to be computed such that

\[
\lim_{x \to x^*} b_{sn}(x, \hat{u}_i(x, \rho_0)) = b^* < \infty, \tag{3.29}
\]

\[
\lim_{x \to x^*} c_{sn}(x, \hat{u}_i(x, \rho_0)) = \infty, \tag{3.30}
\]

\[
\lim_{x \to x^*} \rho_{sn}(x, \hat{u}_i(x, \rho_0)) = \rho_0. \tag{3.31}
\]

As \(x\) converges to \(x^*\), \(c_{sn}\) diverges to infinity, but \(b_{sn}\) and \(\rho_{sn}\) have finite limits; the equation (3.27) implies that either \(x^* = 0\) or \(\lim_{x \to x^*} \hat{u}_i(x) = 0\) or both together have to hold.

Let us consider the three possible cases:

- **\(x^* = 0\) and \(\lim_{x \to x^*} \hat{u}_i(x) \neq 0\)**

  Together with (3.25) this implies that \(\lim_{x \to x^*} b_{sn}(x, \hat{u}_i) = \infty\), contradicting (3.29).

- **\(\lim_{x \to x^*} \hat{u}_i(x) = 0\) and \(x^* > 0\)**

  Together with (3.26) and (3.30) this implies that \(x^* = 1\). The solution of (3.28) for \(x \approx x^*\) can be written as

  \[
  \hat{u}_1(x) = \left(\frac{1}{4} - \rho_0\right) - \left(\rho_0 + \frac{1}{4(1 - 4\rho_0)}\right) (x - 1) + o((x - 1)^2), \tag{3.32}
  \]

  \[
  \hat{u}_2(x) = \frac{\rho_0}{1 - 4\rho_0} (x - 1) + o((x - 1)^2). \tag{3.33}
  \]

  \- **for \(\rho_0 < 1/4\)**

  \[
  \hat{u}_1(x) = \frac{\rho_0}{1 - 4\rho_0} (x - 1) + o((x - 1)^2), \tag{3.34}
  \]

  \[
  \hat{u}_2(x) = \left(\frac{1}{4} - \rho_0\right) - \left(\rho_0 + \frac{1}{4(1 - 4\rho_0)}\right) (x - 1) + o((x - 1)^2). \tag{3.35}
  \]

- **for \(\rho_0 > 1/4\)**

  \[
  \hat{u}_1(x) = \left(\frac{1}{4} - \rho_0\right) - \left(\rho_0 + \frac{1}{4(1 - 4\rho_0)}\right) (x - 1) + o((x - 1)^2),
  \]

  \[
  \hat{u}_2(x) = \frac{\rho_0}{1 - 4\rho_0} (x - 1) + o((x - 1)^2). \]
For $\rho_0 < 1/4$

$$\lim_{x \to x^*} \hat{u}_1(x) = 1/4 - \rho_0 > 0,$$
contradicting the assumption. Hence the solution $\hat{u}_1(x)$ can be disregarded for $\rho_0 < 1/4$.

This implies that

$$\lim_{x \to x^*} b_{sn}(x, u_2(x)) = b_{sn}(1, 0) = 1/2.$$

For $\rho_0 > 1/4$

$$\lim_{x \to x^*} \hat{u}_2(x) = 1/4 - \rho_0 > 0,$$
contradicting the assumption. Hence the solution $\hat{u}_2(x)$ can be disregarded for $\rho_0 > 1/4$.

This implies that

$$\lim_{x \to x^*} b_{sn}(x, u_1(x)) = b_{sn}(1, 0) = 1/2.$$

For $\rho = 1/4$

$$\lim_{x \to x^*} \hat{u}_1(x) = \lim_{x \to x^*} \hat{u}_2(x) = 0,$$

implying that

$$\lim_{x \to x^*} b_{sn}(x, u_1(x)) = \lim_{x \to x^*} b_{sn}(x, u_2(x)) = b_{sn}(1, 0) = 1/2.$$

- $\lim_{x \to x^*} \hat{u}_i(x) = 0$ and $x^* = 0$

The solutions of the equation (3.28) for $x \approx x^*$ can be written as

$$\hat{u}_1(x, \rho_0) = \frac{1}{2} x^2 - \frac{1}{2 \rho_0} x^3 + o(x^4),$$

$$\hat{u}_2(x, \rho_0) = -\rho_0 x + \frac{1}{2 \rho_0} x^3 + o(x^4).$$

Since $\lim_{x \to x^*} b_{sn}(x, \hat{u}_2(x)) = -\rho_0$ and since $b$ is positive, the solution $\hat{u}_1(x)$ gives the
second asymptote of the saddle-node bifurcation curve $\gamma_1(x)$:

$$\lim_{x \to x^*} b_{sn}(x, \hat{u}_1(x)) = 0.$$ 

The analysis given above implies that the saddle-node bifurcation curves $\gamma_i(x)$ have the vertical asymptotes $b = 0$ and $b = 1/2$ for any $\rho_0 > 0$. Moreover, it can be proven that the saddle-node bifurcation curves have no other asymptotes in the $(b, c)$–plane.
Chapter 4

Stochastic optimal control problems with small noise intensities

In this chapter stochastic optimal control problems with one-dimensional non-convex state dynamics are considered. To solve such a problem at a given noise intensity $\varepsilon$ means to find the value function $V(x_0,\varepsilon)$ and the optimal control $u(x_0,\varepsilon)$ for every initial state $x_0$. The value function $V(x_0,\varepsilon)$ satisfies a second order Hamilton-Jacobi-Bellman equation, which for vanishing noise $\varepsilon = 0$ reduces to a first order differential equation.

Kushner (1967) shows, under suitable assumptions, that the solution of the Hamilton-Jacobi-Bellman equation exists for $\varepsilon > 0$ and is of class $C^2$. Fleming and Souganidis (1986) obtain an asymptotic expansion for $V(x,\varepsilon)$ in regions of strong regularity, that is for subregions of the state space where the value function of the corresponding unperturbed problem $V(x,0)$ is smooth. The convergence of $V(x,\varepsilon)$ to $V(x,0)$ together with its first derivative is shown as $\varepsilon \to 0$. Note that Fleming and Souganidis consider the stochastic control problem over a bounded region with $V(x,\varepsilon)$ satisfying a Dirichlet boundary condition. This chapter considers problems over an unbounded domain with possibly a finite number of interior regularity regions. For each regularity region two conditions are needed to determine the solution of the Hamilton-Jacobi-Bellman equation, which cannot be derived from the Dirichlet conditions of the original problem.
A method is provided of deriving conditions needed to determine the solution of the dynamic programming equation in regularity regions. It is also shown how to obtain an asymptotic series of its solution over the whole state domain.

The structure of the chapter is as follows. In Section 4.1 a stochastic control problem is formulated. Section 4.2 describes the method of approximating its value function when the value function of the corresponding deterministic problem is smooth and there are no irregularity points. In Section 4.3 a method of constructing an asymptotic series for $V(x, \varepsilon)$ is proposed when $V(x, 0)$ is not strongly regular everywhere. Section 4.4 concludes.

### 4.1 Formulation of the problem

Let $x(t)$ and $u(t)$ denote the state of a system and the control applied to the system at time $t$ respectively. It is assumed that $x \in X$ where $X \subset \mathbb{R}$ is an open set and $u \in D$ where $D \subset \mathbb{R}$ is a compact convex set. Let the state evolve according to the following stochastic differential equation

$$dx = f(x, u)dt + \sqrt{2\varepsilon \sigma^2(x)}dw$$

where $w(t)$ is a one dimensional Wiener process, $\varepsilon$ denotes the noise level, and where $f$ and $\sigma$ satisfy the following conditions:

1. $f$ satisfies a uniform Lipschitz condition jointly in $x$ and $u$;

2. $f$ is linear in $u$: $f(x, u) = A(x) + B(x)u$, where the functions $A, B$ are bounded on $X$ together with their first order derivatives;

3. $\sigma$ satisfies a uniform Lipschitz condition in $x$;

4. $\sigma(x) \neq 0 \quad \forall x \in X$.

The reader is referred to Fleming (1971) and Kushner (1967) for a detailed discussion of these conditions.
The state \( x(t) \) is assumed to be completely observable at each time \( t \). Thus the control policy may be represented as a function of the state \( u = u(x) \). The control \( u = u(x) \) is called \textit{admissible} if \( u \) takes values in a compact convex set \( D \) and is locally Lipschitz.

The benefit functional associated with each \( u \) is given by

\[
B[x_0, u] = \mathbb{E}_{x_0}^u \int_0^\infty g(x, u(x)) e^{-\rho t} \, dt
\]  

(4.2)

where the discount rate \( \rho > 0 \), and where \( \mathbb{E}_{x_0}^u \) is the conditional expectation operator given the initial state \( x(0) = x_0 \) and the control \( u \). The integrand \( g(x, u) \) is supposed to satisfy the following conditions:

1. \( g(x, u) \) is bounded in any compact set for any admissible \( u \);

2. \( g(x, u) \) is locally Lipschitz jointly in \( x \) and \( u \);

3. \( \exists c < 0 \) such that

\[
\frac{\partial^2 g(x, u)}{\partial u^2} \leq c
\]  

(4.3)

for all admissible \( u \).

The problem is to find an admissible control \( u^*_\varepsilon = u^*(x, \varepsilon) \) that maximizes the benefit functional (4.2) given the state dynamics equation (4.1) and the initial state \( x(0) = x_0 \). This problem is denoted by \( G(\varepsilon) \).

Introduce the value function

\[
V(x_0, \varepsilon) = \sup_u \mathbb{E}_{x_0}^u \int_0^\infty g(x, u) e^{-\rho t} \, dt
\]  

(4.4)

and the current value Hamiltonian

\[
H(x, p, u) = g(x, u) + pf(x, u),
\]  

(4.5)

where \( p \) is a costate variable.
For each \( x \) and \( p \in \mathbb{R} \) consider the maximized current value Hamiltonian

\[
\mathcal{H}(x, p) = \max_u H(x, p, u) = \max_u [g(x, u) + pf(x, u)] = H(x, p, U(x, p)),
\]

where

\[
U(x, p) = \arg \max_u [g(x, u) + pf(x, u)].
\]

The function \( U(x, p) \) is well defined as \( H \) is strictly concave in \( u \) because of (4.3).

The following theorem of Kushner (1967) gives the existence of an optimal maximizing control.

**Theorem 4.1.1.** Let the functions \( f, \sigma \) and \( g \) satisfy the conditions above. For every \( \varepsilon > 0 \) there exists an optimal control \( u^*(x, \varepsilon) \) for \( G(\varepsilon) \). The corresponding value function \( V(x, \varepsilon) \) has continuous second derivatives w.r.t. \( x \) in any compact set and solves the following equation

\[
\varepsilon \sigma^2(x)V_{xx}(x, \varepsilon) + \mathcal{H}(x, V_x(x, \varepsilon)) - \rho V(x, \varepsilon) = 0.
\]

Equation (4.8) is the Hamilton-Jacobi-Bellman equation of the stochastic optimal control problem \( G(\varepsilon) \). If \( V \) solves this equation then the optimal control policy is given by

\[
u^*(x, \varepsilon) = U(x, V_x(x, \varepsilon)).
\]

According to Theorem 4.1.1 solving \( G(\varepsilon) \) is reduced to solving the second order ordinary differential equation (4.8). The fact that the highest order derivative in (4.8) is multiplied by the perturbation parameter \( \varepsilon \) makes it a singularly perturbed differential equation: for \( \varepsilon = 0 \) this term vanishes and the order of the equation changes. This chapter focuses on constructing approximate solutions to such equations using methods of singular perturbations (see for example Holmes (1995) and Verhulst (2005)).

In the case when the solution of (4.8) for \( \varepsilon = 0 \) is smooth, an approximate solution of (4.8) can be constructed for small \( \varepsilon > 0 \) as in Fleming and Souganidis (1986). However two conditions are needed to determine the solution. Judd (1998) derives the boundary conditions for
such a problem by a Taylor approximation of $V$ in $x$ and $\varepsilon$ at the steady state of the deterministic problem. In contrast, this chapter expands $V(x, \varepsilon)$ as a series of $\varepsilon$. The next section shortly describes the method.

### 4.2 Problems without thresholds

First consider the case that $G(0)$ has a continuously differentiable solution. That is, there exists a $C^1$ function $V(x, 0)$ that satisfies the dynamic programming equation of $G(0)$ for any $x$

$$
\mathcal{H}(x, V_x(x, 0)) - \rho V(x, 0) = 0.
$$

(4.10)

The so-called optimal vector field is then given by

$$
\dot{x} = f^o(x) = \frac{\partial \mathcal{H}}{\partial p}(x, V_x(x, 0)).
$$

(4.11)

It determines the state dynamics under the optimal policy. In this section it is assumed that the optimal vector field $f^o$ of the problem $G(0)$ has a unique global attractor $x = x^*$.

It is assumed (see Fleming and Souganidis (1986)) that for small $\varepsilon$ the solution $V(x, \varepsilon)$ to $G(\varepsilon)$ can be expanded as

$$
V(x, \varepsilon) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \ldots.
$$

(4.12)

Following Fleming and Souganidis (1986), (4.12) is substituted into (4.8) and terms with the same orders of $\varepsilon$ are collected. This yields a series of equations, the first two reading as

$$
\mathcal{H}(x, v^o_0(x)) - \rho v_0(x) = 0,
$$

(4.13)

$$
\sigma^2(x)v''_0(x) + \mathcal{H}_p(x, v'_0(x))v'_1(x) - \rho v_1(x) = 0.
$$

(4.14)

Equation (4.13) is the dynamic programming equation of the unperturbed problem $G(0)$; it is
solved by \( v_0(x) = V(x, 0) \). Any subsequent equation is a first order linear differential equation for the function \( v_k \) with coefficients depending on \( v_{k-1} \). This allows us to determine the \( v_k \) recursively.

As these equations are first-order differential equations, each requires one condition to determine a free parameter in the family of solutions. In order to obtain such a condition for (4.13), the equation is differentiated with respect to \( x \) and terms are rearranged as follows

\[
v''_0(x) = \frac{\rho v'_0(x) - \mathcal{H}_x(x, v'_0(x))}{\mathcal{H}_p(x, v'_0(x))}.
\]  \hspace{1cm} (4.15)

Equation (4.11) implies that equilibria of the optimal vector field \( f^o \) are zeros of \( \mathcal{H}_p(x, v'_0(x)) \). From the assumption about existence of the unique global attractor of \( f^o \), it follows that there exists a unique \( x = x^* \) such that \( \mathcal{H}_p(x^*, v'_0(x^*)) = 0 \). If a solution of (4.13) is to be smooth, it is also required that

\[
\rho v'_0(x^*) - \mathcal{H}_x(x, v'_0(x^*)) = 0.
\]  \hspace{1cm} (4.16)

The equations (4.16) and (4.13) yield the value of \( v_0(x) \) at \( x = x^* \), which gives a condition to determine \( v_0(x) \) in (4.13).

Let us rewrite (4.14) as follows

\[
v'_1(x) = \frac{\rho v_1(x) - \sigma^2(x)v'_0(x)}{\mathcal{H}_p(x, v'_0(x))}.
\]  \hspace{1cm} (4.17)

If \( v'_1 \) is to be smooth, it is required that

\[
v_1(x^*) = \frac{\sigma^2(x^*)v''_0(x^*)}{\rho},
\]  \hspace{1cm} (4.18)

which yields a condition for equation (4.14) determining \( v_1(x) \). Proceeding in the same way \( v_2(x), \ldots, v_k(x) \) are obtained iteratively. The \( k \)-th order approximation to the solution of (4.8) is then given by

\[
V_k(x, \varepsilon) = \sum_{j=0}^{k} \varepsilon^j v_j(x).
\]  \hspace{1cm} (4.19)
4.3 Problems with thresholds

Now consider the situation that the optimal vector field $f^o$ of the deterministic problem has multiple equilibria. As every equilibrium of $f^o$ gives rise to a condition for the equations (4.13), (4.14) etc, for every such equilibrium an approximation of the form (4.19) of the solution to the Hamilton-Jacobi-Bellman equation (4.8) is obtained. These approximations are usually defined only on neighborhoods of the corresponding equilibria. In order to get a global approximation they have to be ‘glued’ together in such a way that the resulting function is smooth and approximates the value function $V(x, \varepsilon)$ over the whole state space. This can be done using the method of matched asymptotic expansions, which is described next.

The equation (4.13) is a first order differential equation, but its solution $v_0(x)$ has to satisfy more than one condition. This implies that there exist points of non-differentiability of $v_0(x)$; these are indifference points of the deterministic problem (see Chapter 2). They divide the state domain into outer layers1, that is subdomains where (4.8) is approximated by (4.13), where $v_0(x)$ is differentiable and $V(x, \varepsilon)$ converges to $v_0(x)$ uniformly with its derivative as $\varepsilon \to 0$.

The next step is to introduce an inner layer around each indifference point. Each inner layer contains a point of non-differentiability of $v_0(x)$, therefore the smooth function $V_x(x, \varepsilon)$ needs to change fast in order to approximate the discontinuous function $V_x(x, 0)$. This implies that in an inner layer (4.8) is not well approximated by (4.13) as $V_{xx}(x, \varepsilon)$ will be large and the term $\varepsilon V_{xx}$ in the stochastic Hamilton-Jacobi-Bellman equation (4.8) will not be negligible. To capture this fact, in each inner layer a suitable local variable is introduced. Writing (4.8) in local variables, a second order equation for the nondeterministic correction term which has a solution with two free parameters is obtained. The missing conditions needed to determine the parameters are obtained by matching the approximate solutions of the Hamilton-Jacobi-Bellman equation (4.8) in the inner and outer layers. This is done in the third step by introducing so called transition layers - the intervals where the outer and the inner layers overlap. As a result a collection of approximate solutions to (4.8) in the inner and outer layers is obtained. In the fourth step all

---

1In Fleming and Souganidis (1986) these subdomains are called regions of strong regularity.
approximations are combined in such a way that the resulting function is smooth and solves (4.8) over the whole domain up to a term that tends to 0 as $\varepsilon \to 0$.

**Step 1: Outer layer**

If $f^a$ has two attracting equilibria, $x_L$ and $x_R$, then two approximations of $V(x, \varepsilon)$ are obtained using the method described in Section 4.1. Let $V^{L,1}(x, \varepsilon) = v_0^L(x) + \varepsilon v_1^L(x)$ and $V^{R,1}(x, \varepsilon) = v_0^R(x) + \varepsilon v_1^R(x)$ be the first order approximations of $V(x, \varepsilon)$ with conditions derived at $x = x_L$ and $x = x_R$ respectively. The indifference point $x = \hat{x}$ is determined by the condition

$$v_0^L(\hat{x}) = v_0^R(\hat{x}).$$

(4.20)

Denote $v_0^L(\hat{x}) = v_0^R(\hat{x}) = \hat{v}$. The outer approximation of $V(x, \varepsilon)$ is then given by

$$V^{\text{outer},1}(x, \varepsilon) = \begin{cases} V^{L,1}(x, \varepsilon) & \text{for } x < \hat{x}, \\ V^{R,1}(x, \varepsilon) & \text{for } x \geq \hat{x}. \end{cases}$$

(4.21)

However, the function $V^{\text{outer}}(x, \varepsilon)$ is generally not of class $C^2$; it might be even discontinuous at $x = \hat{x}$. It is therefore not a good approximation of the solution to $G(\varepsilon)$ in a neighborhood of $x = \hat{x}$. Therefore an inner layer is introduced at $x = \hat{x}$ and (4.8) is considered in local variables around $x = \hat{x}$.

**Step 2: Inner layer**

Introduce the local variable

$$\xi_\alpha = \frac{x - \hat{x}}{\varepsilon^\alpha},$$

(4.22)

where $0 \leq \alpha \leq 1$ is a constant. Rewriting (4.8) in terms of $\xi_\alpha$ and denoting the solution in the inner layer as $V^{\text{inner}}(\xi_\alpha, \varepsilon)$ yields

$$\varepsilon^{1-2\alpha} \sigma^2(\hat{x} + \varepsilon^\alpha \xi_\alpha)(V^{\text{inner}})'' + H(\hat{x} + \varepsilon^\alpha \xi_\alpha, \varepsilon^{-\alpha}(V^{\text{inner}})') - \rho V^{\text{inner}} = 0.$$  

(4.23)
Define $V^{inner,\alpha}(\xi, \varepsilon) = \hat{v} + \varepsilon^\alpha W_\alpha(\xi)$. It turns out that the appropriate expansion of $V^{inner}(\xi, \varepsilon)$ is

$$V^{inner}(\xi, \varepsilon) = V^{inner,\alpha}(\xi, \varepsilon) + o(\varepsilon^\alpha). \quad (4.24)$$

Then (4.23) reads as follows

$$\varepsilon^{1-\alpha} \sigma^2 (\hat{x} + \varepsilon^\alpha \xi) W''_\alpha + \mathcal{H} (\hat{x} + \varepsilon^\alpha \xi, W'_\alpha) - \rho \hat{v} - \rho \varepsilon^\alpha W_\alpha = o(\varepsilon^\alpha). \quad (4.25)$$

In order to determine the value of $\alpha$ the following cases are considered:

1) $\alpha = 0$: if $\varepsilon \to 0$ then (4.25) becomes

$$\mathcal{H}(\hat{x} + \xi_0, W'_0) - \rho (\hat{v} + W_0) = 0; \quad (4.26)$$

2) $0 < \alpha < 1$: if $\varepsilon \to 0$ then (4.25) becomes

$$\mathcal{H}(\hat{x}, W'_\alpha) - \rho \hat{v} = 0; \quad (4.27)$$

3) $\alpha = 1$: if $\varepsilon \to 0$ then (4.25) becomes

$$\sigma^2 (\hat{x}) W''_1 + \mathcal{H}(\hat{x}, W'_1) - \rho \hat{v} = 0. \quad (4.28)$$

Note that for $\alpha = 0$ the leading terms of the inner expansion $V^{inner,0}(\xi_0, \varepsilon) = \hat{v} + W_0(\xi_0)$ and in the outer expansion

$$V^{outer,0}(x, \varepsilon) = \begin{cases} v^L_0(x) & \text{for } x < \hat{x}, \\ v^R_0(x) & \text{for } x \geq \hat{x}, \end{cases}$$

coincide, compare (4.13) and (4.26). Note also that for $\alpha = 1$ the inner expansion $V^{inner,1}(\xi_1, \varepsilon)$ approximates the solution $V(x, \varepsilon)$ in the inner layer, as $\alpha = 1$ is the distinguished limit for
Therefore for $0 < \alpha < 1$ the expansion $V^{inner,\alpha}(\xi_\alpha, \varepsilon)$ approximates the solution in the transition layers\(^2\)- two regions (left and right) where the outer layers overlap with the inner layer. This implies that the leading terms in the two approximations must coincide if both are rewritten in $\xi_\alpha$-coordinates and the limit $\varepsilon \rightarrow 0$ is taken. This gives us two conditions that are needed to determine the solution of (4.28): one condition is derived from matching the inner and the outer expansions at the left transition layer, another one from matching them at the right transition layer. The process of matching is done in the next step.

### Step 3: Matching

Outer and inner solutions are matched by writing the two solutions in the transition layers’ coordinates and equating their values. If the outer solutions are rewritten in the transition layers coordinates

$$\xi_\alpha = \frac{x - \hat{x}}{\varepsilon^{\alpha}} = \varepsilon^{-\alpha}\xi_0 = \varepsilon^{1-\alpha}\xi_1$$  \hspace{1cm} (4.29)

then from (4.21) the following is obtained

$$V^{outer}(\xi_0) = V^{outer}(\varepsilon^{\alpha}\xi_\alpha) = \begin{cases} 
    v_0^L(\hat{x} + \varepsilon^{\alpha}\xi_\alpha) + \varepsilon v_1^L(\hat{x} + \varepsilon^{\alpha}\xi_\alpha) & \text{for } \xi_\alpha < 0 \\
    v_0^R(\hat{x} + \varepsilon^{\alpha}\xi_\alpha) + \varepsilon v_1^R(\hat{x} + \varepsilon^{\alpha}\xi_\alpha) & \text{for } \xi_\alpha > 0 
\end{cases}$$

$$= \begin{cases} 
    \hat{v} + (v_0^L)'(\hat{x})\varepsilon^{\alpha}\xi_\alpha + \varepsilon v_1^L(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha < 0 \\
    \hat{v} + (v_0^R)'(\hat{x})\varepsilon^{\alpha}\xi_\alpha + \varepsilon v_1^R(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha > 0 
\end{cases}$$ \hspace{1cm} (4.30)

Rewriting the inner solution in the transition layer coordinates yields, using (4.24) and (4.29)

$$V^{inner,\alpha}(\xi_1) = V^{inner,\alpha}(\varepsilon^{\alpha-1}\xi_\alpha) = \hat{v} + \varepsilon W_1(\varepsilon^{\alpha-1}\xi_\alpha) + o(\varepsilon).$$ \hspace{1cm} (4.31)

\(^2\)This follows from the overlap hypothesis (see Verhulst (2005)), which assumes that if there are two neighboring expansions, then there exists a common subdomain where both expansions are valid. Equation (4.26) is obtained from (4.27) for $\alpha \rightarrow 0$, and equation (4.28) is obtained from (4.27) for $\alpha \rightarrow 1$. 

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The values of the inner and the outer expansions in the transition layers have to be matched, therefore it is required that

\[
\hat{v} + \varepsilon W_1(\varepsilon^{\alpha-1} \xi_\alpha) = \begin{cases} 
\hat{v} + (v_0^L)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^L(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha < 0 \\
\hat{v} + (v_0^R)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^R(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha > 0,
\end{cases}
\]  
(4.32)

which implies for fixed \( x \) two asymptotic conditions for \( W_1 \)

\[
\lim_{\varepsilon \to 0} [W_1(\varepsilon^{\alpha-1} \xi_\alpha) - (v_0^L)'(\hat{x})\varepsilon^{\alpha-1} \xi_\alpha - v_1^L(\hat{x})] = 0, \text{ for } \xi_\alpha < 0 \quad (4.33)
\]

\[
\lim_{\varepsilon \to 0} [W_1(\varepsilon^{\alpha-1} \xi_\alpha) - (v_0^R)'(\hat{x})\varepsilon^{\alpha-1} \xi_\alpha - v_1^R(\hat{x})] = 0, \text{ for } \xi_\alpha > 0. \quad (4.34)
\]

Write \( w = W_1' \). Equation (4.28) takes the form

\[
\sigma^2(\hat{x}) w' + H(\hat{x}, w) - \rho \hat{v} = 0. \quad (4.35)
\]

If \( w(\tau, C) \) designates the general solution of (4.35), where \( w(0, C) = C \), then (4.33), (4.34) can be rewritten as follows

\[
W_1(0) - \int_{-\infty}^{0} \left\{ w(\tau, C) - (v_0^L)'(\hat{x}) \right\} d\tau = v_1^L(\hat{x}), \quad (4.36)
\]

\[
W_1(0) + \int_{0}^{+\infty} \left\{ w(\tau, C) - (v_0^R)'(\hat{x}) \right\} d\tau = v_1^R(\hat{x}). \quad (4.37)
\]

From equalities (4.36), (4.37) the conditions \( W_1(0) \) and \( W_1'(0) = C \) needed to determine the solution of (4.28) are obtained.

**Step 4: Composite expansion**

The description of the solution consists of two parts \( V^{outer} \) and \( V^{inner} \), which now have to be combined to form a **composite expansion**. This is done by adding the expansions and subtracting
common parts. Thus the first order approximation of the solution to (4.8) is given by

\[
V_1(x, \varepsilon) = \begin{cases} 
  v_0^L(x) + \hat{v} + \varepsilon W_1 \left( \frac{x - \hat{x}}{\varepsilon} \right) - \hat{v} - (v_0^L)'(\hat{x})(x - \hat{x}), & \text{for } x < \hat{x} \\
  v_0^R(x) + \hat{v} + \varepsilon W_1 \left( \frac{x - \hat{x}}{\varepsilon} \right) - \hat{v} - (v_0^R)'(\hat{x})(x - \hat{x}), & \text{for } x \geq \hat{x}
\end{cases}
\]

The function \( V_1(x, \varepsilon) \) given by (4.38) is continuous and smooth at \( x = \hat{x} \) as the following is true

\[
\lim_{x \uparrow \hat{x}} V_1(x, \varepsilon) = \lim_{x \downarrow \hat{x}} V_1(x, \varepsilon) = \hat{v}, \quad (4.39)
\]

\[
\lim_{x \uparrow \hat{x}} (V_1)_x(x, \varepsilon) = \lim_{x \downarrow \hat{x}} (V_1)_x(x, \varepsilon) = W_1'(0). \quad (4.40)
\]

### 4.4 Concluding remarks

In this chapter a solution approximation method for stochastic optimal control problems with one-dimensional state space and infinite time horizon has been developed. This method is general and can be applied to problems with small noise intensities. The algorithm of constructing a solution approximation involves solving a number of ordinary differential equations and integral equalities, which can be implemented efficiently numerically. In Chapter 5 this method is applied to a stochastic lake model.
Chapter 5

Regime switching thresholds in stochastic optimal control problems

Recall from Chapter 2 that an indifference point for a deterministic optimal control problem corresponds to an initial state where there are two optimal controls yielding different long run outcomes. This notion of indifference point cannot be extended in a straightforward way to problems with noise as the optimal controls in stochastic problems are always univalued. Therefore a new concept is needed. There has been one attempt in this direction: Dechert and O’Donnell (2006) consider a stochastic control problem in discrete time with a one-dimensional state space and bounded stochastic shocks. They introduce the notion of a stochastic indifference set, which is a stochastic equivalent of a deterministic indifference point, as ‘a transient set between two basins of attraction [...] where there is a positive probability that the dynamics will end up in the lower basin of attraction, and a positive probability they will end up in the upper basin of attraction’. The authors analyze numerically the dependence of such sets on the noise level in the context of a stochastic dynamic game. In Bultmann et al. (2010) the authors adopt this concept and apply it to a model of illicit drug consumption. They show numerically that these indifference sets expand with the noise variance and that the optimal policy becomes a continuous function of the state for sufficiently high levels of the variance. Unfortunately, the concept of stochastic indifference set is not applicable to models with continuous time and
unbounded shocks; for such systems a different notion is needed.

This chapter proposes a concept of stochastic thresholds for one-dimensional continuous
time optimal control models, which is invariant with respect to coordinate transformations. The
presence of stochastic indifference thresholds implies regime switching behavior of the system.
This type of behavior is characterized by persistent state fluctuations around one of the steady
state until a large shock forces the system to pass the threshold between the regimes. After
that the system fluctuates around another equilibrium until the next large shock hits the system
again.

The methodology of obtaining an asymptotic series for the solution of the Hamilton-Jacobi-
Bellman equation, developed in Chapter 4, and the concept of stochastic thresholds developed
in this chapter are applied to a stochastic model of optimal water pollution control.

## 5.1 The concept of regime switching thresholds

This section introduces stochastic bifurcations as bifurcations of a certain geometric invariant
of the optimally controlled process - the transformation invariant function (see Wagenmakers
et al. (2005)). Regime switching thresholds are introduced as its local minima.

### 5.1.1 Processes with constant diffusion

Consider first a stochastic process \( x(t) \) with a constant diffusion satisfying the following stochastic
differential equation

\[
dx = f(x)dt + \sigma dw, \tag{5.1}
\]

where \( w \) is a Wiener process and \( \sigma \) is constant. It is assumed that given an initial distribution \( \varphi_0 \)
of \( x(0) \), \( \varphi_t \) exists for all \( t \geq 0 \). It is well known that \( \varphi_t(x) \) satisfies the Fokker-Plank equation\(^\text{1}\)

\[
\frac{\partial \varphi_t}{\partial t} = -\frac{\partial}{\partial x} [f \varphi_t] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2 \varphi_t]. \tag{5.2}
\]

\(^{1}\)Also known as the Kolmogorov forward equation.
When a stationary solution \( \varphi(x) \) of (5.2) exists, it is unique and given as the solution of

\[
0 = -\frac{d}{dx} [f \varphi] + \frac{1}{2} \frac{d^2}{dx^2} \left[ \sigma^2 \varphi \right].
\]  

(5.3)

By integrating (5.3) once, the following is obtained

\[
-f \varphi + \frac{1}{2} \frac{d}{dx} \left[ \sigma^2 \varphi \right] = C_1.
\]  

(5.4)

To obtain a probability density, \( C_1 \) has to equal 0. Then (5.4) reads as

\[
\frac{\sigma^2}{2} \frac{d\varphi}{d\psi} = f.
\]  

(5.5)

From (5.5) the following is derived

\[
\log \varphi(x) = \frac{2}{\sigma^2} \int_{x_0}^{x} f(\tau) \, d\tau + C_2,
\]  

(5.6)

where the constant \( C_2 \) is determined from the normalization condition \( \int \varphi = 1 \). Thus the probability density function \( \varphi(x) \) of the stationary distribution of the process \( x(t) \) is given by

\[
\varphi(x) = \frac{\exp \left\{ \frac{2}{\sigma^2} \int_{x_0}^{x} f(\tau) \, d\tau \right\}}{\int_{X} \exp \left\{ \frac{2}{\sigma^2} \int_{x_0}^{\theta} f(\tau) \, d\tau \right\} \, d\theta}.
\]  

(5.7)

Equation (5.5) implies that zeros of the drift function \( f \) correspond to critical points of \( \varphi \), i.e. modes and antimodes of the stationary distribution. The modes of \( \varphi \) indicate the most frequent states of the system. Such a state acts as an attractor: if the system slightly deviates from it, the drift \( f \) forces it to come back, as \( f = 0 \) and \( f' < 0 \) at such a point. On the contrary, a local minimum of \( \varphi \), separating two modes, acts as a repeller, as \( f = 0 \) and \( f' > 0 \) at such a point. Therefore it seems reasonable to associate the critical points of \( \varphi \) with the stochastic equilibrium states, see Figure 1. However, critical points of the stationary density function \( \varphi \) are not invariant under transformations of the process \( x(t) \), which will be shown in the next
Figure 5.1: Correspondence between the deterministic drift function $f(x)$ and the probability density function $\varphi(x)$ for a process with a constant diffusion $\sigma$. Stochastic stable steady states correspond to modes of $\varphi(x)$, which are zeros of $f(x)$ where it changes its sign from positive to negative. Whereas stochastic regime switching thresholds (RST) correspond to antimodes of $\varphi(x)$, which are zeros of $f(x)$ where it changes its sign from negative to positive.

subsection.

5.1.2 Transformation invariant function

Now consider a stochastic process $y(t)$ with non-constant diffusion strength satisfying the following equation

$$dy = g(y)dt + \sqrt{2\varepsilon\sigma^2(y)}dw,$$

where the diffusion strength is not necessarily constant. Note that any one-dimensional process with non-constant diffusion strength can be transformed into a process with a constant diffusion by a suitable coordinate transformation. Let $z = r(y)$ be a transformation of the process $y(t)$. Then Itô’s differential rule implies

$$dz = r'(r^{-1}(z))dy + \varepsilon\sigma^2(r^{-1}(z))r''(r^{-1}(z))dt$$

$$= (r'(r^{-1}(z))g(r^{-1}(z)) + \varepsilon\sigma^2(r^{-1}(z))r''(r^{-1}(z))) dt + \sqrt{2\varepsilon}r'(r^{-1}(z))\sigma(r^{-1}(z))dw$$

$$= \tilde{g}(z)dt + \sqrt{2\varepsilon}\tilde{\sigma}(z)dw.$$
Let \( r(y) \) satisfy the following differential equation

\[
 r'(y) = \frac{A}{\sqrt{2\varepsilon \sigma(y)}},
\]  

(5.10)

where \( A > 0 \) is a constant. Then from (5.9) it follows that the process \( z(y) \) has a constant diffusion strength which is equal to \( A \).

Let \( \psi(y) \) and \( \varphi(z) \) denote the stationary solutions of the Fokker-Plank equations corresponding to the process \( y(t) \) and \( z(t) \) respectively. The following then holds

\[
 \varphi(z) = \frac{\psi(r^{-1}(z))}{r'(r^{-1}(z))}.
\]  

(5.11)

From (5.11) it follows that the stationary density function \( \psi \) is not invariant under coordinate transformation. This implies that \( \psi(y) \) and \( \varphi(z) \) can have different number of critical points and therefore different number of stochastic stable steady states and regime switching thresholds. This means that the stationary density function is not a proper measure of the equilibrium points of a stochastic process.

The non-invariance under coordinate transformations of stationary density functions has been pointed out by Zeeman (1988). In Wagenmakers et al. (2005) the authors suggest to consider the transformation invariant function given by

\[
 I(y) = \sigma(y)\psi(y),
\]  

(5.12)

which is invariant under diffeomorphic transformations of coordinates. To see this, let us combine (5.9) and (5.11) and derive the following

\[
 \tilde{\sigma}(r(y))\varphi(r(y)) = r'(y)\sigma(y)\frac{\psi(y)}{r'(y)} = \sigma(y)\psi(y).
\]  

(5.13)

Note, that for processes with a constant diffusion the transformation invariant function is proportional to the stationary density function.
5.1.3 Stochastic bifurcations

The transformation invariant function $I$, introduced above, is a geometrical invariant of the process $y$. The number of critical points of $I$ is therefore a property of the stochastic process $y$ and not only of its representation in a given coordinate system. The following definitions can then be introduced:

**Definition 5.1.1.** Let $y$ be a stochastic process that satisfies (5.8), with transformation invariant function $I(y)$. A local maximum of $I$ is called a stochastic attractor of $y$.

As mentioned above, for systems with constant diffusion the function $I$ is proportional to the stationary density function. Therefore, for such systems local maxima of $I$ indicate most likely states of the process $y$.

**Definition 5.1.2.** Let $y$ be a stochastic process that satisfies (5.8) with the transformation invariant function $I(y)$. A local minimum of $I$ is called a regime switching threshold of the process $y$. Any interval in the state space bounded by such thresholds is called a regime of the process $y(t)$.

Each regime of $y$ contains one attractor and is characterized by fluctuations of the process around it. A large shock may change the regime of the process.

**Definition 5.1.3.** Let $y$ be a stochastic process that satisfies (5.8) with the transformation invariant function $I(y)$. A change in the number of critical points of $I(y)$ indicates a stochastic bifurcation of the process $y$.

5.2 The stochastic lake model

In this section the methods developed in Chapter 4 are applied to analyze a stochastic lake model. The deterministic lake model, introduced in Mäler et al. (2003), is a prototype for an optimal management problem with conflicting intertemporal interests and non-convex feedbacks.
It has been studied extensively: in Wagener (2003) and Kiseleva and Wagener (2010) a bifurcation analysis of the lake model is performed; Heijnen and Wagener (2008) extended the model by adding an industry whose activities increase the pollution; Kossioris et al. (2008) considered a differential lake game on pollution control; Heijdra and Heijnen (2009) studied economic and environmental effects of public abatement in this model; Salerno et al. (2007) considered a model of political economy with the underlying lake dynamics. Dechert and O’Donnell (2006) solved a stochastic lake game numerically; however, as mentioned in the introduction, the authors limited themselves to considering discrete bounded shocks.

In the lake model a social planner controls usage of fertilizers by farmers that pollute a lake close to the farmers’ fields. The planner’s problem is to maximize a social welfare functional which models the conflicting interests of farmers and tourists: farmers indirectly benefit from polluting the lake, and tourists suffer from the polluted lake. If the pollution level at time $t$ is denoted as $x(t)$, and the loading of more pollution due to farming as $u(t)$ then the stochastic lake problem is given by

$$\max_u \mathbb{E}_{x_0} \int_0^\infty \left( \log u(t) - cx^2(t) \right) e^{-\rho t} dt$$  \hspace{1cm} (5.14)$$

$$dx = \left( u - bx + \frac{x^2}{1 + x^2} \right) dt + x\sqrt{2\varepsilon}dW(t), \quad x(0) = x_0.$$  \hspace{1cm} (5.15)$$

The integrand in (5.14) models social utility: the term $\log u$ represents farmers’ profits derived from intensity $u$ of the use of fertilizers, whereas the term $cx^2$ represents tourists’ disutility from the pollution in the lake, where the parameter $c > 0$ models the relative costs of pollution. The social planner maximizes the expected social welfare (5.14) subject to the pollution dynamics equation (5.15), where the parameter $b > 0$ is the coefficient that is proportional to the rate of loss of pollution due to sedimentation and where the last term models the biological production.

---

1. It is assumed that uncertainty enters the lake model via the biological parameter $b$: rewriting equation (5.15) as

$$\frac{dx}{dt} = u - \left( b - \sqrt{2\varepsilon}x^2 \frac{dW(t)}{dt} \right) x + \frac{x^2}{1 + x^2}$$

results in the original lake dynamics equation where the parameter $b$ is perturbed by a white noise term.

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Define the value function of the stochastic lake problem

\[ V(x_0, \varepsilon) = \max_u \mathbb{E}_{x_0} \int_0^\infty \left( \log u(t) - cx^2(t) \right) e^{-\rho t} dt. \]  

(5.16)

It is easy to check that the functions in the model satisfy the conditions of Theorem 4.1.1 if the control \( u \) takes values in a compact interval\(^3\) \([m, M]\). Then Theorem 4.1.1 implies that the value function \( V(x, \varepsilon) \) solves (4.8), which here takes the form

\[ \varepsilon x^2 V_{xx} + \max_u \left[ \log u - cx^2 + V_x \left( u - bx + \frac{x^2}{1 + x^2} \right) \right] - \rho V = 0. \]  

(5.17)

Solving the optimization problem in (5.17) the optimal policy is obtained

\[ u^*(x, \varepsilon) = -\frac{1}{V_x(x, \varepsilon)}. \]  

(5.18)

Then the value function \( V(x, \varepsilon) \) of the stochastic lake problem solves

\[ \varepsilon x^2 V_{xx}(x, \varepsilon) - \log(-V_x) - cx^2 - 1 + V_x \left( -bx + \frac{x^2}{1 + x^2} \right) - \rho V(x, \varepsilon) = 0. \]  

(5.19)

Now, the method developed in Chapter 4 to approximate the solution of (5.19) can be applied. For that the parameter values \( (b, c, \rho) \) have to be fixed. Let us first consider a case with a unique optimum of the corresponding deterministic lake problem: \( b = 0.65, c = 0.7, \rho = 0.03 \).

The corresponding stochastic optimal policy and the long run distribution of the state are shown in Fig. 2. As the noise level \( \varepsilon \) in the model is small, the stochastic optimal policy is almost indistinguishable from the deterministic one. The long run distribution is unimodal.

To consider a case when the deterministic lake problem exhibits an indifference point, the parameters are set at \( b = 0.65, c = 0.5, \rho = 0.03 \). As shown in Fig. 5.4(a) the deterministic optimal policy is discontinuous in this case: a part of the optimal control is formed by a part of

\(^3\)\( u \) is bounded from above by \( M \), where \( M >> 1 \) is an upper limit of the amount fertilizers that land can bear. And \( u \) is bounded from below by \( m \), where \( 0 < m << 1 \), to assure sustainability of the agricultural sector.
Figure 5.2: The graph of the optimal policy function for the stochastic lake problem (dotted) and the corresponding deterministic lake problem (solid). The corresponding long run distribution is shown at the bottom of the figure. It is obtained by running the process (5.15) with \( u = u^*(x, \varepsilon) \) for \( 10^5 \) periods after \( 10^2 \) transient periods. The model parameters are \( b = 0.65 \), \( c = 0.7 \), \( \rho = 0.03 \), \( \varepsilon = 0.001 \).

Let us apply the method described in Section 4.3 to solve the corresponding stochastic lake problem. In Fig. 5.3 the deterministic and stochastic value functions are shown. At the indifference point the deterministic value function has a kink, whereas the stochastic value function is smooth and everywhere differentiable.

It is clear from Figure 5.3(a) that the value function of the stochastic problem is located above the value function of the deterministic problem. It suggests that stochasticity increases the expected social welfare, at least for small noise levels.

The corresponding optimal policy and the long run state distribution are shown in Fig. 5.4(a). The discontinuity of the optimal policy at the indifference point in the deterministic model is smoothed out by uncertainty in the stochastic model and the optimal policy \( u^*(x, \varepsilon) \) is a continuous differentiable function, see Fig. 5.4(a).
Figure 5.3: Figure 5.3(a) shows the deterministic (solid line) and the stochastic (dotted line) value functions for the stochastic lake model. The deterministic value function has a kink at the indifference point, whereas the stochastic value function is smooth. Figure 5.3(b) shows a blow up in a neighborhood of the deterministic indifference point. The model parameters $b = 0.65$, $c = 0.5$, $\rho = 0.03$, $\varepsilon = 0.003$.

Figure 5.4: Figure 5.4(a) shows the graphs of the deterministic optimal policy (solid discontinuous line), the stochastic optimal policy (dotted line) and the long run distribution of the state. The stochastic regime switching threshold is marked as an empty circle. Figure 5.4(b) shows the corresponding times series. It is obtained by running the process (5.15) with $u = u^*(x, \varepsilon)$ for $10^4$ periods after $10^2$ transient periods. Figure 5.4(b) shows that the considered stochastic model exhibits regime switching phenomenon: the system fluctuates around one of the steady states until it is pushed away to another steady state due to a large shock. The basins of fluctuations are separated by the stochastic regime switching threshold. The model parameters are $b = 0.65$, $c = 0.5$, $\rho = 0.03$, $\varepsilon = 0.003$. 
The corresponding long run distribution is bimodal. The stochastic regime switching point acts as a repeller separating two regimes: ‘clean lake’ and ‘turbid lake’, see Fig. 5.4(b).

### 5.2.1 Bifurcation analysis with respect to the noise level

Figure 5.5(a) shows a bifurcation diagram of the stochastic lake problem with respect to the noise level parameter $\varepsilon$ for $b = 0.65$, $c = 0.49$, $\rho = 0.03$. For small noise levels the long run distribution of the state is bimodal with the stochastic regime switching point separating two modes. When the noise level increases the threshold collides with the highest mode and they both disappear via a saddle-node bifurcation. As the noise level increases further the long run distribution remains unimodal. The same scenario is illustrated in Figure 5.5(b) for $c = 0.52$. Larger costs of pollution yield more restrictive policies, and as a result uniqueness of the ‘clean lake’ regime for lower values of $\varepsilon$.

![Bifurcation Diagrams](image)

Figure 5.5: Bifurcation diagrams of the long run distribution in the stochastic lake model with respect to the noise level $\varepsilon$. Solid lines indicate modes of the transformation invariant function $\sigma(x)\varphi(x)$, the dashed lines indicates the antimode of $\sigma(x)\varphi(x)$ - the stochastic regime switching threshold. For small level of noise the long run distribution is bimodal. As the noise parameter $\varepsilon$ increases the eutrophic steady state and the stochastic regime switching point merge in a saddle-node bifurcation and disappear, the long run distribution becomes unimodal. The model parameters are $b = 0.65$, $\rho = 0.03$.

Figure 5.6 shows average time spent in one regime for different values of the noise parameter $\varepsilon$. It exponentially declines with the noise level. Of course, as $\varepsilon$ increases large shocks...
Figure 5.6: Average time (in log) spent by the lake system in a regime before switching to another regime for different noise levels. The model parameters are $b = 0.65$, $c = 0.5$, $\rho = 0.03$.

become more likely, pushing the state process out of one basin of fluctuation to the other.

Figure 5.7(a) shows a collection of transformation invariant functions for different values of $\varepsilon$ and $c = 0.49$. For small $\varepsilon$ the functions are bimodal (see Figure 5.5(a)), the most of the mass is concentrated around the high steady state, so the lake is polluted most of the time. As the noise in the system increases the mass is shifted to the regime of the ‘clean lake’. For large noise the long run distribution is unimodal with a peak at the ‘clean’ equilibrium. In the face of uncertainty the planner acts to avoid serious or irreversible potential harm to the environment. Higher uncertainties in the system yield more restrictive pollution policy when the pollution level is high, see Figure 5.7(c)-5.7(d).

For $c = 0.52$, see Figure 5.7(b), the transformation invariant function is bimodal for small values of $\varepsilon$ (see Figure 5.5(b)), and the mass is concentrated around the ‘clean’ steady state for all values of the noise parameter $\varepsilon$. It shows that high relative costs of pollution $c$ force the decision maker to keep the lake in the ‘clean’ regime.

Figure 5.8 shows the critical values of $c$ for different values of the noise parameter $\varepsilon$ that correspond to the invariant distributions such that the mass concentrated around either of the regimes is equal to $1/2$. For values of $c$ below that curve the lake is turbid most of the time,
and for values of $c$ above it the lake is clean most of the time. As can be seen from Figure 5.8 (right), the larger the noise in the system the lower the critical value of $c$.

### 5.3 Concluding remarks

This chapter introduces a concept of a stochastic regime switching threshold. From the definition it follows that such a point separates two basins of fluctuations, or two regimes of the system. In one regime the system fluctuates around one of the steady states, until it is pushed away by a large shock to another basin, and the regime changes.

Figure 5.7: Transformation invariant functions and optimal control policies for different values of $\varepsilon$ and $c$. The model parameters are $b = 0.65$, $\rho = 0.03$. 

and for values of $c$ above it the lake is clean most of the time. As can be seen from Figure 5.8 (right), the larger the noise in the system the lower the critical value of $c$. 

### 5.3 Concluding remarks

This chapter introduces a concept of a stochastic regime switching threshold. From the definition it follows that such a point separates two basins of fluctuations, or two regimes of the system. In one regime the system fluctuates around one of the steady states, until it is pushed away by a large shock to another basin, and the regime changes.

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Figure 5.8: The bifurcation curve in $(\varepsilon, c)$—parameter space corresponding to the case when the mass concentrated around each of the regimes is equal to 1/2. For the parameter values below this curve the process $\{x_t\}$ spends more time in the ‘turbid lake’ regime, and vice versa. The model parameters are $b = 0.65$, $\rho = 0.03$.

This concept as well as the solution method developed in Chapter 4 are applied to perform a bifurcation analysis of the stochastic lake model. Approximations of the solution for different values of the parameters are obtained, including the case with the indifference point. The optimal policy function is computed, which is smooth for any values of the model parameters. Stochastic regime switching thresholds are computed for different noise levels, and it is shown that as the noise increases the threshold point collides with one of the steady states and disappears via a saddle-node bifurcation.
Chapter 6

Summary

This thesis is devoted to the study of parameterized families of continuous time infinite horizon optimal control problems with one dimensional non-convex state dynamics. Such models often occur in environmental economics (see Tahvonen and Salo (1996), Scheffer et al. (2001), Brock and Starrett (2003), Mäler et al. (2003), Scheffer (2009)). All the theoretical results and methods developed in the thesis are illustrated with the deterministic and stochastic lake model. They are however general and can be applied to any problem of that type.

The main contribution of the thesis is the development of the bifurcation theory of one-dimensional optimal vector fields (see Chapter 2), which allows to obtain the solution structure of parameterized non-convex optimal control problems. Such problems can exhibit multiple local optimal attractors and consequently indifference thresholds separating their basins of attraction. The theory developed in Chapter 2 allows to locate regions in the parameter space for which the controlled system exhibits thresholds. This information may suggest reconsideration of management options or reevaluation of the key system parameters.

Chapter 3 analyzes the shallow lake model using the theory developed in Chapter 2. It is a model of optimal water pollution management, which serves as a prototype of a conflict between ecologic and economic interests. Bifurcation analysis of the model shows the effects of changing the parameter values in the parameter regions where the clean and polluted steady states are globally or locally stable under the optimal dynamics. This information can affect evaluation
of the parameters, such as the discount factor or relative costs of pollution, which are used by the social planner when designing the optimal policy. A slight change of the discount factor $\rho$ can change the optimal policy radically. For instance, for some values of the pollution costs a decrease of $\rho$ can imply that an initially clean lake will be steered to the clean equilibrium, rather than to the polluted equilibrium under the policy with a higher discount factor.

Stochastic optimal control problems with small noise intensities have been studied in Chapter 4. The solution of such a problem reduces to solution of the corresponding Hamilton-Jacobi-Bellman equation. It is a singularly perturbed second order differential equation. When the noise is set to zero, it becomes a first order dynamic programming equation of the corresponding deterministic problem. Chapter 4 develops a method of constructing approximate solutions of the Hamilton-Jacobi-Bellman equation. From these solutions, a geometric invariant - transformation invariant function - is computed. A stochastic bifurcation in the sense of Wagenmakers et al. (2005) is then a qualitative change of this function.

Stochastic optimal control problems exhibiting regime switching behavior are of especial interest. These are the perturbations of deterministic problems exhibiting indifference thresholds. For such stochastic problems multimodality of the transformation invariant function allows to define regime switching thresholds as its local minima. Transition between different regimes is realized by crossing these thresholds due to large shocks. Such a model can explain sudden rapid changes of a state variable, be it an ecosystem or an economy.

This thesis demonstrates the importance and effectiveness of methods of bifurcation theory applied to studying non-convex optimal control problems. It opens up a new methodological approach to investigation of parameterized economic models. While standard analytical methods are not efficient and sometimes impossible to apply to non-convex problems, the numerical geometrical methods developed in the thesis allow to solve and analyze such problems quickly. More and more deterministic and stochastic non-convex optimal control models occur in economics (Caulkins et al. (2001), Brock and Starrett (2003), Stachurski (2003), Crepin (2007), Heijnen and Wagener (2008), Kossioris et al. (2008), Heijdra and Heijnen (2009), Zeiler et al. (2009), Bultmann et al. (2010)), therefore the development of such methods is essential. In
fact, some of the research work already take up the methods of the bifurcation theory of op-
timal vector fields: see Caulkins et al. (2007), Grass et al. (2008), Graß (2010), Hinloopen
et al. (2010) for work on deterministic problems with continuous time, Moghayer and Wagener
(2009) for deterministic one-dimensional discrete time systems, Diks and Wagener (2008) for
stochastic one-dimensional discrete time systems where the authors introduce the dependence
ratio reproducing the transformation invariant function.

Based on the results of this work, possible future research topics include the development of
bifurcation methods for deterministic and stochastic non-convex optimal control problems with
multidimensional state spaces; the generalization of these methods to dynamic games; and the
development of a bifurcation methodology for multidimensional discrete time models.
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Samenvatting (Summary in Dutch)

Dit proefschrift is gewijd aan de studie van geparametriseerde families van continue-tijd optimale sturingsproblemen met oneindige planning horizon en eendimensionale toestandsdynamica. Dergelijke modellen komen vaak voor in de milieu-economie. Alle theoretische resultaten en methoden ontwikkeld in het proefschrift worden geïllustreerd aan de hand van het deterministische en stochastische ‘vlakke-meer model’. Ze zijn echter algemeen en kunnen worden toegepast op elk probleem van dit type.

De belangrijkste bijdrage van dit proefschrift is de ontwikkeling van de bifurcatietheorie van eendimensionale optimale vectorvelden (zie Hoofdstuk 2), die het mogelijk maakt om de oplossingsstructuur van geparametriseerde optimale control problemen te bepalen. De dynamica onder optimale sturing van dergelijke problemen, kort ‘optimale dynamica’, kan meerdere lokale aantrekkers vertonen. De bijbehorende aantrekkingsgebieden worden ofwel door afstotende evenwichten van elkaar gescheiden, of door onverschilligheidspunten, waar er meer dan één optimale oplossing bestaat. De theorie ontwikkeld in Hoofdstuk 2 maakt het mogelijk om de parameterruimte te verdelen in gebieden. Wanneer de parameters variëren binnen een gebied, verandert het optimale beleid alleen kwantitatief. Alleen als de parameters de gebiedsgrenzen overschrijden, veranderen ook kwalitatieve karakteristieken van het optimale beleid.

In Hoofdstuk 3 wordt het ‘vlakke-meer model’ geanalyseerd met behulp van de theorie ontwikkeld in Hoofdstuk 2. Het is een model van optimaal beheer van waterverontreiniging, welk dient als prototype van een conflict tussen ecologische en economische belangen. Bifurcatie-analyse van het model vindt parametergebieden waarover de kwalitatieve structuur van het opti-
male belied niet verandert. Deze informatie kan gebruikt worden in het politieke beslissingsproces welk de parameterwaarden vastlegt die door de sociaalplanner gebruikt moeten worden. Een kleine verandering van de disconteringsvoet kan bijvoorbeeld het optimale beleid radicaal veranderen. In het algemeen betekent een daling van de disconteringsvoet dat een meer eerder naar een niet-vervuild evenwicht gestuurd zal worden.

In Hoofdstuk 4 zijn optimale sturingsproblemen onderzocht onder kleine stochastische verstoringen. De oplossing van een dergelijk probleem wordt gereduceerd tot het oplossen van de bijbehorende Hamilton-Jacobi-Bellman vergelijking, een singulier gestoorde tweede orde differentiaal vergelijking. Zonder verstoringen wordt dit een eerste orde differentiaalvergelijking, de dynamische programmeringsvergelijking van het overeenkomstige deterministische probleem. Hoofdstuk 4 ontwikkelt een methode om benaderende oplossingen van de Hamilton-Jacobi-Bellman vergelijking te vinden. Van deze oplossingen wordt een meetkundige invariant - de transformatie-invariante functie - berekend. Een stochastische bifurcatie in de zin van Wagenmakers et al. (2005) is dan een kwalitatieve verandering van deze invariant.

Stochastische optimale sturingsproblemen die regime-overgangen vertonen, zijn van bijzondere belang. Voor dergelijke stochastische problemen is de transformatie-invariante functie multimodaal; het analogon van een onverschilligheids punt in deze context is een overgangdrempel, gedefinieerd als een lokaal minimum van de invariante functie. Een regime-overgang is dan het overschrijden van een dergelijke drempel; dit is vaak te wijten aan een grote schok. Een model kan plotselinge snelle veranderingen van een toestandsvariabele verklaren.

Dit proefschrift toont het belang en de effectiviteit aan van bifurcatietheorie als deze toegepast wordt op het bestuderen van niet-convexe optimale control problemen. Het opent een nieuwe methodologische benadering voor het onderzoek van geparametriseerde economische modellen. De gebruikelijke analyse methoden beperken zich grotendeels tot het vinden van lokale informatie van een systeem rond een lange-termijn evenwicht, of tot het bepalen van de globale structuur van de oplossingen voor een klein aantal parameterwaarden. Met de numeriek-meetkundige methoden ontwikkeld in dit proefschrift kan de globale oplossingsstructuur voor de hele parameterruimte gevonden worden. Deterministische en stochastische niet-convexe op-
Timale control modellen worden meer en meer in economische analyses gebruikt; derhalve is de ontwikkeling van dergelijke methoden van belang.

De resultaten van dit proefschrift suggereren als mogelijke toekomstige onderzoeksthema’s de ontwikkeling van bifurcatiemethoden voor deterministische en stochastische optimale sturingproblemen met meerdimensionale toestandsruimten, de generalisatie van deze methoden naar dynamische spelen, en de ontwikkeling van een bifurcatiemethodologie voor multidimensionale discrete tijd-modellen.
The Tinbergen Institute is the Institute for Economic Research, which was founded in 1987 by the Faculties of Economics and Econometrics of the Erasmus University Rotterdam, University of Amsterdam and VU University Amsterdam. The Institute is named after the late Professor Jan Tinbergen, Dutch Nobel Prize laureate in economics in 1969. The Tinbergen Institute is located in Amsterdam and Rotterdam. The following books recently appeared in the Tinbergen Institute Research Series:


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