Structural analysis of complex ecological economic optimal control problems

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Chapter 1

Introduction

This thesis investigates parameterized infinite-horizon non-convex dynamic optimization problems on one-dimensional state spaces. Such problems often occur in environmental economics, see Brock and Starrett (2003), Mäler *et al.* (2003), Scheffer *et al.* (2001), Scheffer (2009). Non-convex problems may exhibit multiple equilibria, that occur or disappear as the parameters vary. Among those equilibria more than one can be optimal to converge to, depending on the initial state of the system. There is a body of work in different areas of economics devoted to multiplicity of optimal solutions, see for instance Sethi (1977), Skiba (1978), Dechert and Nishimura (1983), Tahvonen and Salo (1996), Sethi and Thompson (2000), Caulkins *et al.* (2001), Brock and Starrett (2003), Haunschmied *et al.* (2003), Wagener (2003), Dawid and Deissenberg (2005), Haunschmied *et al.* (2005), Caulkins *et al.* (2007), Crepin (2007), Kossioris *et al.* (2008), Zeiler *et al.* (2009), Kiseleva and Wagener (2010).

The presence of multiple equilibria can make solving a non-convex optimal control problem quite complicated. In problems with linear or convex dynamics small changes have small effects on the solution structure. In contrast, for problems with non-convex dynamics slight modifications of the system parameters can change this structure not only quantitatively but also qualitatively. In this thesis methods are developed for non-convex optimal management problems that allow to obtain the global solution structure of the problem and also to indicate the critical parameter values that correspond to qualitative changes of this structure.

In the theory of dynamical systems there is a branch - bifurcation theory - focusing on obtaining qualitative information about solutions of parameterized dynamical systems. The basic idea is to determine persistent properties of systems. Examples of such properties are the number and the type of equilibria, the presence of periodic orbits etc. If slight changes of the system parameters cause a change in one of these properties, then the dynamical system is said to undergo a bifurcation.

Bifurcation theory is based on and is a ramification of the theory of structural stability of dynamical systems, which has a long history. It has been anticipated by Poincaré at the end of the 19th century. The concept of structural stability was first introduced as 'roughness' by Andronov and Pontrjagin in 1930s. It took off in the 1960s together with the development of the theory of differentiable dynamical systems, and was further developed and popularized by Peixoto, Smale, Thom, Zeeman and Arnol'd in 1960s and 1970s.

In this thesis ideas from dynamical systems theory are used to develop bifurcation theory for infinite horizon optimal control problems. The solution of such a problem can be expressed as an optimal vector field, which gives the state dynamics under the optimal policy. It turns out that qualitative changes of the optimal vector fields are connected to qualitative changes of the canonical state–costate system. This provides a way to link bifurcation theory of dynamical systems to the theory of bifurcating optimal vector fields.

The results of a bifurcation analysis are usually expressed in a bifurcation diagram, which shows the 'shapes', or types of the structure, of the optimal vector field for all values of the parameters as well as the bifurcation set where the shape changes. The structure of a generic (i.e. Kupka-Smale) one-dimensional vector field is characterized by the number and the types of its steady states; they can either be attractors or repellers. Optimal vector fields also feature 'indifference points' that are particular to control problems. At such a point a system manager is indifferent between two strategies implying two different long run outcomes. An indifference point acts as a repeller in the sense that trajectories starting near it move away, though it is not a steady state of the dynamics.

An example of a bifurcation diagram is given in Figure 1.1 (taken from Chapter 3). It shows

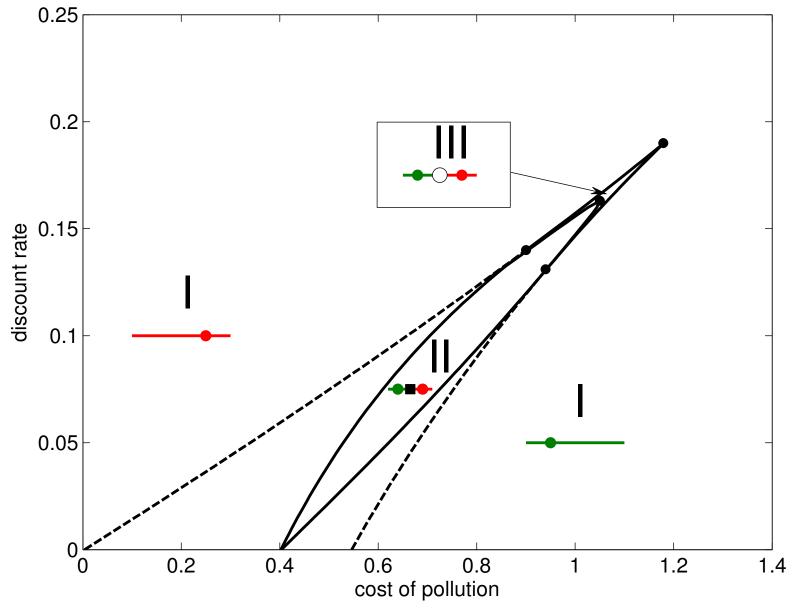

Figure 1.1: Bifurcation diagram for the shallow lake model with respect to the parameters: relative costs of pollution and discount rate. The figure is taken from Chapter 3.

the bifurcations of the shallow lake system (introduced in Chapter 3) that describes the problem of optimal management of water pollution. In the model the social planner limits usage of artificial fertilizers on farmlands surrounding a shallow lake. The phosphorus contained in the fertilizers is washed into the lake by underground flows and is the main pollutant of the water. The optimal policy of the planner depends upon the model parameters: cost of pollution and discount rate. From Figure 1.1 the structure of the optimal solution can immediately be determined for any value of the parameters. The shapes of the optimal vector field are shown as well. It can be seen from Figure 1.1 that the shallow lake system has three qualitatively different configurations of optimal paths depending on the values of the two parameters: (I) the optimal state dynamics converges to the unique steady state; (II) depending on the initial value the optimal state dynamics converges to one of the two attractors, separated by an indifference point; (III) depending on the initial value the optimal state dynamics converges to one of the two attractors, separated by a repeller. The solid lines that separate regions corresponding to different solution structures are bifurcation curves of the problem. A point on such a curve determines the parameter values for which dynamics under optimal management of the shallow lake system is structurally unstable; a slight deviation of the parameters yields qualitative changes of the solution structure. The dashed lines in Figure 1.1 correspond to bifurcations in the associated state-costate system of the problem that do not correspond to bifurcations of the optimization problem.

In region (I) there is a unique socially optimal equilibrium. The pollution level at the equilibrium continuously depends on the parameter values: for a fixed discount rate the higher the cost of pollution the lower the equilibrium pollution level. In region (II) the initial pollution level determines which of the two equilibria, 'clean' or 'polluted', is socially optimal. If initially the lake is clean then it is optimal to keep it clean: the social planner puts a low quota on usage of fertilizers and steers the lake to the 'clean' equilibrium. If initially the lake is polluted then the planner lets farmers enjoy high benefits and steers the lake to the 'polluted' equilibrium. In region (III) there are three optimal steady states: 'clean', 'intermediate' and 'polluted'. The scenarios of convergence to the 'clean' and 'polluted' equilibria are the same as in region (II). The 'intermediate' equilibrium is a repeller, thus it is optimal only for one single initial pollution level - the pollution level at the 'intermediate' equilibrium.

The next natural step is to extend the bifurcation methodology for optimal vector fields to stochastic problems. However, several difficulties present themselves. First of all, the corresponding Hamilton-Jacobi-Bellman equation is a singularly perturbed differential equation and requires special solution methods. Chapter 4 develops a method that allows to compute an approximate solution of such an equation. The method is based on considering the stochastic problem as a singular perturbation of the corresponding deterministic one (for singular perturbation theory see Fleming and Souganidis (1986) and Verhulst (2005)). This assumption allows to approximate the value function of the stochastic problem by the quantities computable from the deterministic value function.

Also, concepts like bifurcation and indifference points cannot be adapted from the deterministic context directly. A bifurcation is a qualitative change of persistent properties of a system. For deterministic systems that means for instance a change in the number or the type of equilibria of the optimal vector field. For stochastic systems optimal vector fields are however not well defined. Instead, qualitative changes in a certain geometrical invariant of the resulting controlled stochastic process, the so-called transformation invariant function (see Wagenmakers *et al.* (2005)), are considered in Chapter 5. A stochastic bifurcation is then understood as a qualitative change of the shape of this function, such as the change from unimodality to bimodality. Local maxima of the transformation invariant function are then stable steady states of the stochastic process; its local minima are called regime switching thresholds. A regime of a system is an interval in a state space bounded by such thresholds. When the boundaries are crossed due to a large shock, the regime of the system changes. A regime switching threshold is a stochastic analog of a deterministic threshold point in the sense that they both separate basins of attraction of stable steady states of the optimally controlled process. However there is an important difference between them: at a threshold point there can exist multiple optimal controls whereas at a regime switching threshold there exists a unique optimal control.

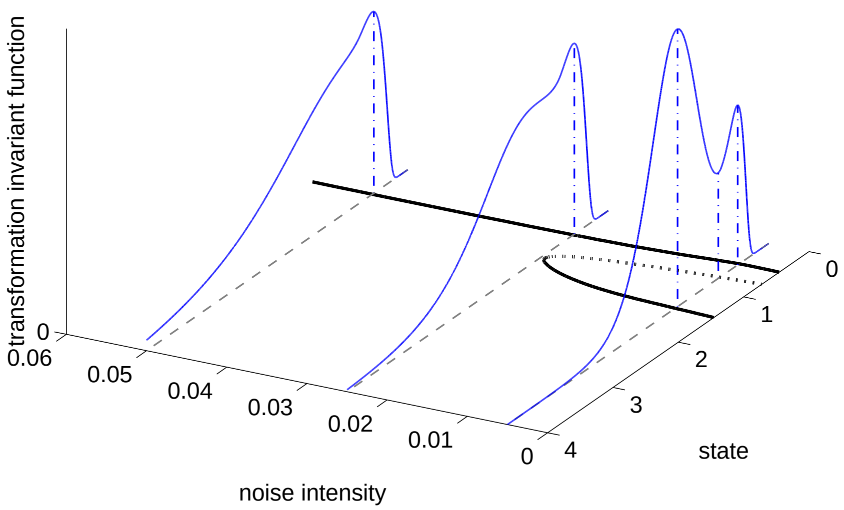

Figure 1.2: Bifurcation diagram for the stochastic lake problem with respect to the noise intensity (bottom figure) is shown together with the corresponding transformation invariant function. The solid and dotted lines correspond to maxima and local minima of the transformation invariant function respectively. The figure is taken from Chapter 5.

Using the new bifurcation concept, a bifurcation analysis of a stochastic lake model is performed (see Chapter 5). This model is an extension of the deterministic shallow lake model where the pollution dynamics equation is perturbed by a stochastic term. Figure 1.2 displays a bifurcation diagram with respect to the noise intensity as well as the associated transformation invariant function. The figure shows that the transformation invariant function of the problem is bimodal for low levels of noise; it becomes unimodal when the noise level increases. The dashed lines indicate the levels of the noise intensity for which the transformation invariant functions are plotted. The stable steady states of the controlled process are located at solid lines in the parameters plane and correspond to maxima of the transformation invariant function. The regime switching thresholds are located on the dotted line and correspond to local minima of the transformation invariant function. Such a point separates two regimes, 'clean' and 'polluted', of the pollution dynamics in the shallow lake. When the lake is in 'clean'('polluted') regime, the pollution fluctuates around the 'clean'('polluted') stochastic steady state. Transitions between the regimes occur due to large shocks in the pollution stock.

Figure 1.2 shows that as the noise intensity in the lake system increases the transformation invariant function becomes unimodal with a mode at the 'clean' equilibrium. It is a demonstration of the precautionary principle: facing the uncertainty the social planner acts to avoid serious or irreversible potential harm to the environment.

The next section provides a short outline of the thesis.

Outline of the thesis

The results of this thesis are presented in four chapters, each of which is mostly self-contained. Some basic notions and definitions are briefly restated in subsequent chapters.

In Chapter 2 parameterized families of deterministic optimal control problems with two dimensional state-control space are studied. The concept of an *optimal vector field* corresponding to such a problem is introduced. It is a one-dimensional multivalued vector field that describes the state dynamics under the optimal policy. The families of optimal vector fields can bifurcate. A classification of all possible bifurcations of optimal vector fields up to codimension 2 is obtained.

In Chapter 3 the theory of bifurcations of one-dimensional vector fields developed in Chapter 2 is applied to the shallow lake model. This model serves as a prototype of an optimal management problem with conflicting intertemporal interests, short-term benefits and long-term costs, that features in many economic-ecological problems. A bifurcation analysis of the shallow lake problem is given with respect to all system parameters: natural resilience, relative importance of the resource for social welfare and future discount rate. In particular, it is shown how the increase of the discount rate affects the parameter regions where an oligotrophic steady state, corresponding to low pollution level, is either globally or locally stable under optimal dynamics. A modified version of Chapter 3 constitute a paper published in Journal of Economic Dynamics and Control.

In Chapter 4 stochastic optimal control problems with small noise intensities are studied. A method of constructing approximate solutions to such problems is developed, based on singular perturbation theory.

In Chapter 5 a concept of stochastic bifurcation and a stochastic analog of a deterministic threshold point are introduced. A bifurcation analysis of the stochastic lake model with respect to the noise intensity parameter is performed. Particularly, it is shown that the mode of the transformation invariant function associated with the 'polluted' steady state vanishes when the noise intensity in the system increases.