Structural analysis of complex ecological economic optimal control problems

Kiseleva, T.

Chapter 2

Bifurcations of one-dimensional optimal vector fields

The investigation of an economic dynamic optimization problem that features a globally attracting steady state reduces mostly to a quantitative quasi-static analysis of this state, determining the rates of change of the position of the steady state and the value of the objective functional as certain key parameters are varied. In contrast, if there are more than one attracting steady state in the system, or more generally, more than one attracting set, the question arises towards which of these the system is driven by the optimal policy. Put differently, in the presence of a single globally attracting steady state, optimal policies can differ only in degree; if there are multiple attracting states, they may also differ in kind.

Since the late 1970's, optimal policies that are qualitatively different have been found in many economic models: in growth theory they have been used to explain poverty traps (Skiba (1978), Dechert and Nishimura (1983)); in fisheries, they can model the coexistence of conservative versus overexploiting policies (Clark (1976)); there are environmental models where both industry-promoting but polluting as well as ecologically conservative policies are optimal in the same model, depending on the initial state of the environment (Tahvonen and Salo (1996), Mäler *et al.* (2003), Wagener (2003), Kiseleva and Wagener (2010)); in migration studies, active relocation as well as no action policies occur in the same model (Caulkins *et al.* (2005));

optimal advertising efforts may depend on the initial awareness level of a product (Sethi (1977, 1979)); the successful containment of epidemics may depend on the initial infection level (Sethi (1978), Rowthorn and Toxvaerd (2011)); in the control of illicit drug use, high law enforcement as well as low enforcement and treatment of drug users (Tragler *et al.* (2001), Feichtinger and Tragler (2002)) can depend on the initial level of drug abuse; in R&D policies of firms, the optimal decision between high R&D expenditure investment (Hinloopen *et al.* (2010)) to develop a technology versus low investment to phase a technology out may depend on the initial technology level.

In all such models, there is for certain parameter configurations a critical state where both kinds of policy are simultaneously optimal, and where the decision maker is consequently indifferent between them. These points will be called *indifference points* in the following, though they go by many other names as well[1].

Usually, the presence of an indifference point is established numerically for a fixed set of parameter values of the model. In order to study the dependence of the qualitative properties of the optimal policies on the system parameters, it is possible in principle to do an exhaustive search over all parameter combinations. Such a strategy, while feasible, would however be very computing intensive.

A different approach is suggested by the theory of bifurcations of dynamical systems: to identify only those parameter configurations at which the qualitative characteristics of the solutions change. For instance, in Wagener (2003) it was shown that indifference points disappear if a heteroclinic bifurcation of the state-costate system occurs. This mechanism, for which we propose the term *indifference-attractor bifurcation*, relates the change of the solution structure of the optimal control problem to a global bifurcation of the state-costate system.

The present article conducts a systematic study of the bifurcations of infinite horizon optimal control problems on the real line that are expected to occur in one- and two-parameter families. The theory developed here has already been applied in several places (Wagener (2003), Caulkins *et al.* (2007), Graß (2010), Kiseleva and Wagener (2010)).

[1]For instance Skiba points, Dechert-Nishimura-Skiba points, Dechert-Nishimura-Sethi-Skiba points, regime switching thresholds, Maxwell points, shocks etc.

2.1 Setting

2.1.1 Definitions

Let $X \subset \mathbb{R}$ be an open interval, and $U \subset \mathbb{R}^r$ a closed convex set with non-empty interior. Let $\rho > 0$ be a positive constant and $f : X \times U \to \mathbb{R}$, $g : X \times U \to \mathbb{R}$ be infinitely differentiable, or *smooth*, in the interior of $X \times U$, and such that all derivatives can be extended continuously to a neighborhood of $X \times U$. Finally, let $\xi \in X$.

Set

$$H = g(x,u) + pf(x,u)$$

and assume that

$$\frac{\partial^2 H}{\partial u^2}(x,p,u) < 0 \tag{2.1}$$

for all $(x,p,u) \in X \times \mathbb{R} \times U$

Consider the problem to maximise

$$J(x,u) = \int_0^\infty g(x(t),u(t))\mathrm{e}^{-\rho t}\mathrm{d}t \tag{2.2}$$

over the space of state-control trajectories (or programs) (x,u) that satisfy

1. the function $u : [0,\infty) \to U$ is locally Lebesgue integrable over $[0,\infty)$ and essentially bounded; that is, $u \in L^\infty([0,\infty),U)$

2. the function $x : [0,\infty) \to X$ is absolutely continuous and satisfies

$$\dot{x} = f(x,u) \tag{2.3}$$

   almost everywhere;

3. the initial value of x is given as $x(0) = \xi$.

This problem will be referred to as *infinite horizon problem* in the following. A solution (x,u) to the problem is usually called a *maximizer* or a *maximizing trajectory*.

Assumption 2.1.1. *In the infinite horizon problem, for every $\xi \in X$ there exists at least one maximizer (x, u) satisfying $x(0) = \xi$.*

Maximizing trajectories enjoy the following time invariance property, which is commonly known as the dynamic optimization principle.

Theorem 2.1.1. *If the trajectory $(x(.), u(.))$ solves the infinite horizon problem with initial condition ξ, then for any $\tau > 0$, the time shifted trajectory $(x(\tau + .), u(\tau + .))$ solves the infinite horizon problem with initial condition $x(\tau)$.*

Define the *maximized Hamiltonian* as

$$\mathcal{H}(x, p) = \max_{u \in U}\{g(x, u) + pf(x, u)\}$$

Assumption (2.1) implies that the maximum is taken at a unique point $u = v(x, p)$, where v depends smoothly on its arguments; consequently, the function $\mathcal{H}$ is smooth as well.

For a maximizing state trajectory x, there exists a continuous costate trajectory p satisfying the reduced canonical equations

$$\dot{x} = F_1 = \mathcal{H}_p, \qquad \dot{p} = F_2 = \rho p - \mathcal{H}_x, \tag{2.4}$$

which define the reduced canonical vector field $F = (F_1, F_2)$. Moreover, x and p satisfy the transversality condition

$$\lim_{t \to \infty} \mathrm{e}^{-\rho t} p(\tilde{x} - x) \leq 0 \tag{2.5}$$

for all admissible trajectories $\tilde{x}$. Trajectories of the state-costate equations (2.4) are classically called *extremal*. Extremal trajectories that satisfy the transversality condition (2.5) will be called *critical* in the following. Note that a noncritical trajectory cannot be a maximizer.

Recall that the *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

Definition 2.1.1. *The* ***optimal costate rule*** *is the set valued map $p^o : X \to \mathcal{P}(\mathbb{R})$ with the property that if $\eta \in p^o(\xi)$, then the solution of the reduced canonical equations with initial* *value*

$$(x(0), p(0)) = (\xi, \eta)$$

maximizes the integral J. Associated to it are the ***optimal feedback rule***

$$u^o(x) = v(x, p^o(x)),$$

and the ***optimal vector field***

$$f^o(x) = \mathcal{H}_p(x, p^o(x)) = f(x, u^o(x))$$

which are both set-valued as well.

A map $x : [0, \infty) \to X$ is a trajectory of an optimal vector field if

$$\dot{x}(t) \in f^o(x(t))$$

for all $t \geq 0$.

The solution trajectories of an optimal vector field solve the associated maximization problem. Note that an optimal vector field is commonly called a 'regular synthesis' in the literature.

Theorem 2.1.2. *The sets $p^o(x(t))$ and $f^o(x(t))$ are single-valued for all $t > 0$.*

Proof. See Fleming and Soner (2006), p. 44, corollary I.10.1. □

2.1.2 Indifference points

The following definition is one of the possible interpretations of the notion of 'Skiba point'.

Definition 2.1.2. *If $\xi \in X$ is such that there are maximisers x_1, x_2 of the infinite horizon problem with $x_1(0) = x_2(0) = \xi$ and $x_1(t) \neq x_2(t)$ for some $t \in [0, \infty)$, then ξ is called an* ***indifference point****. The totality of indifference points form the* ***indifference set****; its complement in X is the* ***domain of uniqueness****.*

In one-dimensional problems an indifference point is an initial point of two trajectories that have necessarily different long run behavior. It is worthwhile to note that this is not true for problems with higher dimensional state spaces, or for discrete time problems (see Moghayer and Wagener (2009)).

Definition 2.1.3. *The* ω***-limit set*** $\omega(x)$ *of a state trajectory* x *is given as*

$$\omega(x) = \{\xi \in X \,:\, x(t_i) \to \xi \textit{ for some increasing sequence } t_i \to \infty\}.$$

Using ω-limit sets, threshold points can be defined.

Definition 2.1.4. *A point* $\xi \in X$ *is a* ***threshold point****, if in every neighborhood* N *of* ξ *there are two states* $\xi_1, \xi_2 \in N$ *that are initial states to state trajectories* x_1, x_2 *such that the respective* ω*-limit sets are different:* $\omega(x_1) \neq \omega(x_2)$*.*

Threshold points are boundary points of basins of attractions.

Definition 2.1.5. *A set* B *is the* ***basin of attraction*** *of another set* A*, the attractor, if for every* $x \in B$ *the* ω*-limit set of* x *is equal to* A*:* $\omega(x) = A$ *for all* $x \in B$*.*

Unlike the situation for ‘ordinary’ dynamical systems, a threshold point can be an element of one or more basins of attraction, and basins can overlap.

Definition 2.1.6. *A point* $\xi \in X$ *is an* ***indifference threshold*** *if it is both an indifference point and a threshold point.*

Equivalently, an indifference threshold is a point that is contained in more than one basin of attraction. In the literature, both threshold and non-threshold indifference points have been called ‘Skiba points’. A more precise terminology seems to be desirable.

Dynamical systems on a one-dimensional state space that are defined by a vector field have typically two kinds of ‘special’ points: attractors and repellers, which are both steady states; the knowledge of these special points is sufficient to reconstruct the flow of the system qualitatively.

Analogously, an optimal one-dimensional vector field has *optimal attractors* and *optimal repellers*, which are both optimal equilibria; in addition it has indifference points. Again, the knowledge of the optimal equilibria and the indifference points is sufficient to reconstruct the qualitative features of the solution structure of the infinite horizon problem.

2.2 Bifurcations of optimal vector fields

The analysis of bifurcations of a parameterized family of optimal vector fields is performed in terms of the reduced canonical vector field, but it is perhaps worthwhile to point out that the latter is an auxiliary construct.

The optimal vector field defines a continuous time evolution on the state space, that is well defined for all positive times. When the state space is one-dimensional, the evolution has certain special properties: trajectories sweep out intervals that are bounded by optimal attractors and optimal repellers or indifference points. At a bifurcation, the qualitative structure of these trajectories changes. For instance, in a saddle-node bifurcation, an attractor and a repeller coalesce and disappear, together with the trajectory that joins them. Analogously, in an indifference-attractor bifurcation, an indifference point and an attractor coalesce and disappear, again together with the trajectory joining them. It is clearly impossible that a repeller and an indifference point coalesce, for the trajectory which should be joining them could have no ω-limit point. However, there is a third possible bifurcation scenario: a repeller may turn into an indifference point. This also changes the solution structure, the constant solution that remains in the repelling state has no equivalent in the situation with the indifference point.

The indifference-attractor bifurcation and the different kinds of indifference-repeller bifurcations have obviously no counterpart in the theory of dynamical systems: they are typical for optimization problems. Instances of indifference-attractor bifurcations have been analysed in Wagener (2003, 2006).

2.2.1 Preliminary remarks.

If N is a bounded interval of $\mathbb{R}$ with endpoints $a < b$, let the outward pointing 'vector' $\nu(x)$ be defined as

$$\nu(a) = -1, \quad \nu(b) = 1. \tag{2.6}$$

Notions from optimal control theory

The reduced canonical vector field F of the infinite horizon problem under study is given as

$$F = (F_1, F_2) = (\mathcal{H}_p, \rho p - \mathcal{H}_x).$$

Assumption (2.1) implies that the strong Legendre-Clebsch condition

$$\mathcal{H}_{pp}(x, p) > 0 \tag{2.7}$$

holds for all (x, p).

One of the implications of this condition is that eigenspaces of equilibria of the reduced canonical vector field are never vertical. More precisely, the following lemma holds.

Lemma 2.2.1. *If the strong Legendre-Clebsch condition holds, all eigenvectors v of DF can be written in the form $v = (1, w)$.*

Proof. The lemma is implied by the statement that if $\mathcal{H}_{pp} \neq 0$, then $e_2 = (0, 1)$ cannot be an eigenvector of

$$DF = \begin{pmatrix} \mathcal{H}_{px} & \mathcal{H}_{pp} \\ -\mathcal{H}_{xx} & \rho - \mathcal{H}_{px} \end{pmatrix}.$$

This is easily verified. □

Lemma 2.2.2. *Assume that the strong Legendre-Clebsch condition holds. If $v_1 = (1, w_1)$ and $v_2 = (1, w_2)$ are two eigenvectors of DF with $\lambda_1 < \lambda_2$, then $w_1 < w_2$.*

Proof. The first component of the vector equation $DFv_i = \lambda_i v_i$ reads as

$$\mathcal{H}_{px} + \mathcal{H}_{pp} w_i = \lambda_i.$$

As $\mathcal{H}_{pp} > 0$, the lemma follows. □

The value of the objective J over an extremal trajectory can be computed by evaluating the maximized Hamiltonian at the initial point (see for instance Skiba (1978), Wagener (2003)).

Theorem 2.2.1. *Let $(x(t), p(t))$ be a trajectory of the reduced canonical vector field F that satisfies $\lim_{t\to\infty} \mathcal{H}(x(t), p(t))\mathrm{e}^{-\rho t} = 0$, and let $u(t) = v(x(t), p(t))$ be the associated control function. Then*

$$J(x, u) = \frac{1}{\rho}\mathcal{H}(x(0), p(0)).$$

Notions from dynamical systems

Recall the following notions from the theory of dynamical systems: two vector fields are said to be *topologically conjugate*, if all trajectories of the first can be mapped homeomorphically onto trajectories of the second; that is, by a continuous invertible transformation whose inverse is continuous as well.

An equilibrium $\bar{z}$ of a vector field f is called *hyperbolic*, if no eigenvalue of $Df(\bar{z})$ is situated on the imaginary axis. The sum of the generalized eigenspaces associated to the hyperbolic eigenvalues is the hyperbolic eigenspace E^h, which can be written as the direct sum of the stable and unstable eigenspaces E^s and E^u, associated to the stable and unstable eigenvalues respectively. The sum of the eigenspaces associated to the eigenvalues on the imaginary axis is the neutral eigenspace E^c. The center-unstable and center-stable eigenspaces E^{cu} and E^{cs} are the direct sums $E^c \oplus E^u$ and $E^c \oplus E^s$ respectively.

The center manifold theorem (see Hirsch *et al.* (1977)), ensures the existence of invariant manifolds that are tangent to the stable and unstable eigenspaces.

Theorem 2.2.2 (Center Manifold Theorem)**.** *Let f be a C^k vector field on $\mathbb{R}^m$, $k \geq 2$, and let $f(\bar{z}) = 0$. Let E^u, E^s, E^c, E^{cu} and E^{cs} denote the generalized eigenspaces of $Df(\bar{z})$ introduced above. Then there are C^k manifolds W^s and W^u tangent to E^s and E^u at $\bar{z}$, and C^{k-1} invariant manifold W^c, W^{cu} and W^{cs} tangent to E^c, E^{cu} and E^{cs} respectively at $\bar{z}$. These manifolds are all invariant under the flow of f; the manifolds W^s and W^u are unique, while W^c, W^{cu} and W^{cs} need not be.*

Invariant manifolds can be used to choose convenient coordinates around an equilibrium point of a vector field. For instance, let $f(0) = 0$, let E^1 and E^2 be two linear subspaces such that

$$E^1 \oplus E^2 = \mathbb{R}^m,$$

and let W^1 and W^2 be two invariant manifolds that are locally around 0 parameterized as the graphs of functions

$$w^1 : E^1 \to E^2, \quad w^2 : E^2 \to E^1,$$

satisfying $Dw^1(0) = 0$, $Dw^2(0) = 0$. For a sufficiently small neighborhood N of 0 and for $(z_1, z_2) \in U \subset E^1 \times E^2$, define the coordinate transformation

$$(\zeta_1, \zeta_2) = (z_2 - w^1(z_1), z_1 - w^2(z_2)).$$

In the new coordinates, the vector field has necessarily the form

$$f(\zeta) = \begin{pmatrix} A_1\zeta_1 + \zeta_1\varphi_1(\zeta) \\ A_2\zeta_2 + \zeta_2\varphi_2(\zeta) \end{pmatrix},$$

where $\varphi_i(\zeta) \to 0$ as $\zeta \to 0$.

For a hyperbolic equilibrium of a vector field on the plane, a much stronger result is available, the C^1 linearization theorem of Hartman (see Hartman (1960, 1964), Palis and Takens (1993)).

Theorem 2.2.3 (Hartman's C^1 linearization theorem)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 vector field in the plane, and let $z = 0$ be a hyperbolic equilibrium of f. Then there is a neighborhood N of 0 and coordinates ζ on N, such that*

$$f(\zeta) = Df(0)\zeta$$

in these coordinates.

2.2.2 Codimension one bifurcations

In this subsection, the codimension one bifurcations of optimal vector fields are treated: these are the bifurcations that cannot be avoided in one-parameter families. These are the indifference-repeller, the indifference-attractor and the saddle-node bifurcation.

It turns out that there are two configurations of the state co-state system that can give rise to indifference-repeller bifurcations of the optimal vector field; they are referred to as type 1 and type 2, respectively.

A general remark on notation: the codimension of a bifurcation will be denoted by a subscript, whereas the type is indicated, if necessary, by additional information in brackets. For instance, the abbreviation IR$_1$(2) denotes a codimension one indifference repeller bifurcation of type 2.

IR$_1$(1) bifurcation

Consider the situation that the reduced canonical vector field F has an equilibrium $e = (x_e, p_e) \in \mathbb{R}^2$ with eigenvalues $0 < \lambda^u < \lambda^{uu}$. Let E^{uu} denote the eigenspace associated to λ^{uu}. As this eigenspace is invariant under the linear flow $DF(0)z$, by Hartman's linearization theorem there is a one-dimensional differentiable curve W^{uu}, the *strong unstable* invariant manifold, that is invariant under F and tangent to E^{uu} at e.

Definition 2.2.1. *A point $e = (x_e, p_e)$ is a (codimension one) indifference repeller singularity of type 1, notation IR$_1$(1), of an optimization problem with reduced canonical vector field F, if the following conditions hold.*

1. *The eigenvalues λ^u, λ^{uu} of $DF(e)$ satisfy $0 < \lambda^u < \lambda^{uu}$.*

2. *On some compact interval neighborhood N of x_e, there is defined a continuous function $p : N \to \mathbb{R}$ such that*

$$p^o(x) = \{p(x)\}.$$

   for all $x \in N$, and such that $p_e = p(x_e)$.

3. *Let W^{uu} denote the strong unstable manifold of F at e, parameterized as the graph of $w : N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of N. There is exactly one $\bar{x} \in \partial N$ such that*

$$p(\bar{x}) = w(\bar{x}), \tag{2.8}$$

whereas for $x \in \partial N$ and $x \neq \bar{x}$, we have that

$$\nu(x)\Big(p(x) - w(x)\Big) < 0. \tag{2.9}$$

The definition is illustrated in Figure 2.1b.

Theorem 2.2.4. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an $IR_1(1)$ singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu(e_\mu)$ satisfy $0 < \lambda^u_\mu < \lambda^{uu}_\mu$. Let the strongly unstable manifold W^{uu}_μ of e_μ be parameterized as the graph $p = w(x, \mu)$ of a differentiable function $w : N \times \Gamma \to \mathbb{R}$.*

2. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*

$$p^o_\mu(x) = \{p(x, \mu)\}$$

   for all $x \in \partial N$ and all $\mu \in \Gamma$.

3. *The function*

$$\alpha(\mu) = \nu(\bar{x})\big(p(\bar{x}, \mu) - w(\bar{x}, \mu)\big),$$

   for which $\alpha(0) = 0$ by (2.8), *is defined on N and satisfies*

$$D\alpha(0) \neq 0.$$

Then the optimal vector field f^o restricted to N is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \{x\}$$

whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = \begin{cases} \{-1\}, & x < 0, \\ \{-1, 1\}, & x = 0, \\ \{1\}, & x > 0, \end{cases}$$

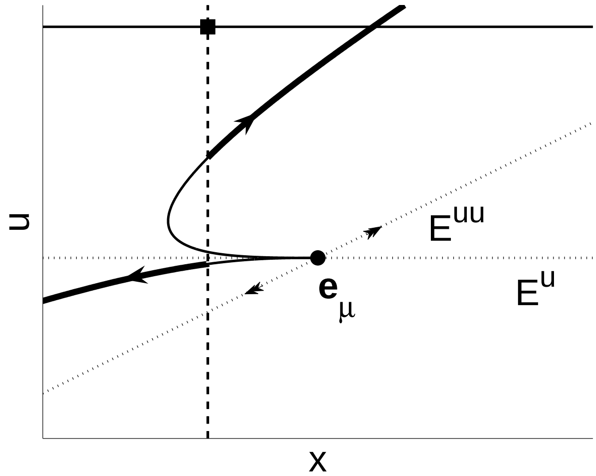
The theorem is illustrated in Figure 2.1. Shown is a neighborhood of a repelling equilibrium of the state-costate equation. The dotted lines are the linear unstable eigenspaces of the equilibrium; the strongly unstable eigenspace corresponds to the line with the largest gradient. Approaching the equilibrium are two phase curves, drawn as solid black lines. The thick part of these curves denote the optimal costate rule.

The indifference point is marked as a vertical dashed line. At the top of the diagrams, the corresponding situation in the state space is sketched; solid black circles correspond to equilibria of the optimal vector field, squares to indifference points. In this case, all equilibria of the optimal vector field are repelling.

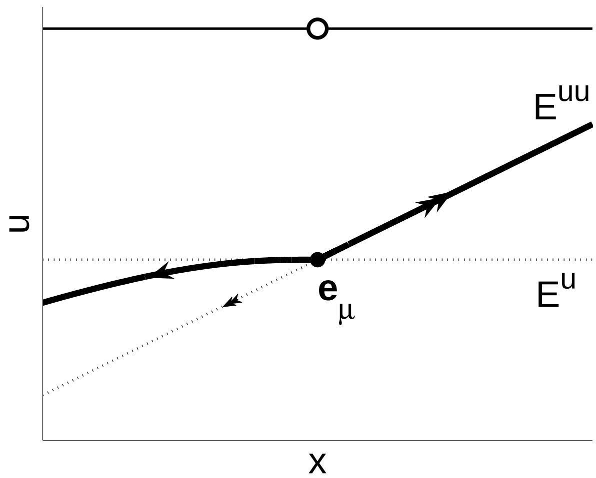
At the bifurcation, the relative position of the optimal trajectories and the strongly unstable manifold changes: for $\alpha(\mu) < 0$ the backward extension of the optimal trajectories are tangent to E^u at either side of the equilibrium. This ensures that the equilibrium itself corresponds to an optimal repeller. For $\alpha(\mu) > 0$, the backward extensions are tangent to E^u at the same side of the equilibrium. One of them necessarily intersects the line $x = x_e$, which implies that e cannot be an optimal trajectory.

Proof. Let $E^{uu} = \mathbb{R}v^{uu}$ and $E^u = \mathbb{R}v^u$ be the eigenspaces spanned by the eigenvectors $v^{uu} = (1, w^{uu})$ and $v^u = (1, w^u)$ of $DF(r)$ corresponding to the eigenvalues λ^{uu} and λ^u respectively. Note that $w^{uu} > w^u$ as a consequence of lemma 2.2.2.

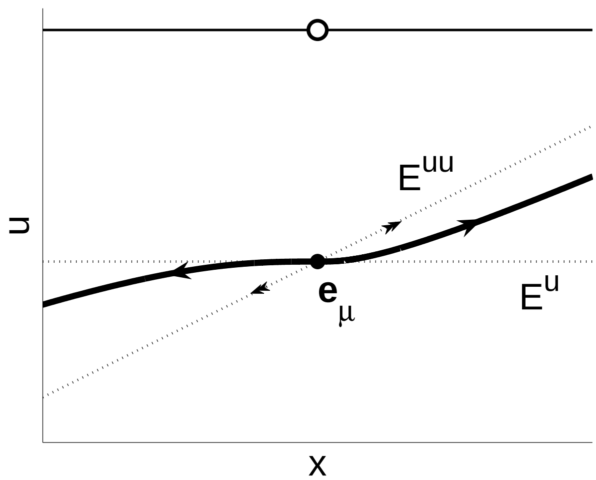
For a sufficiently small neighbourhood of e introduce C^1 linearizing coordinates $\zeta = \zeta(z)$, with C^1 inverse $z = z(\zeta) = (x(\xi, \eta), p(\xi, \eta))$, such that $\zeta(e) = 0$, such that the linear map $D\zeta(0)$ maps v^{uu} to $(1, 0)$ and v^u to $(0, -1)$, and such that in these coordinates the vec-



(a) $\alpha(\mu) < 0$



(b) At bifurcation: $\alpha(\mu) = 0$



(c) $\alpha(\mu) > 0$

Figure 2.1: Before, at and after the indifference-repeller bifurcation point.

tor field F takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^{uu} & 0 \\ 0 & \lambda^{u} \end{pmatrix} \zeta.$$

As a consequence of these choices, the map ζ is orientation preserving and

$$\begin{aligned} x_\xi(0,0) &= \pi_1 Dz(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_1 v^{uu} = 1 > 0, \\ x_\eta(0,0) &= \pi_1 Dz(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\pi_1 v^{u} = -1 < 0, \end{aligned}$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the first component, and where x_ξ denotes partial derivation with respect to ξ etc. By continuity, there is a neighbourhood V of 0 such that

$$x_\xi > 0, \quad x_\eta < 0, \quad \text{and} \quad \det Dz > 0$$

on V.

Let $\bar{x}_i$, $i = 1, 2$ be such that

$$N = [\bar{x}_1, \bar{x}_2] \tag{2.10}$$

and set

$$\bar{z}_i = (\bar{x}_i, \bar{p}_i) = (\bar{x}_i, p(\bar{x}_i, \mu)), \tag{2.11}$$

as well as

$$\bar{\zeta}_i = (\bar{\xi}_i, \bar{\eta}_i) = \zeta(\bar{x}_i, \bar{p}_i).$$

Assume that $\bar{x} = \bar{x}_2$, that is

$$\alpha(\mu) = p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu);$$

the proof in the case $\bar{x} = \bar{x}_1$ is similar.

The trajectory $z_1(t) = (x_1(t), y_1(t))$ of F through $\bar{z}_1$ has in linearizing coordinates the form

$$\zeta_1(t) = \left(\bar{\xi}_1 \mathrm{e}^{\lambda^{uu}t}, \bar{\eta}_1 \mathrm{e}^{\lambda^{u}t}\right),$$

with $\bar{\xi}_1 < 0$ and $\bar{\eta}_1 > 0$ for all μ. Note that it satisfies

$$\begin{aligned}\dot{x}_1 = \frac{\mathrm{d}}{\mathrm{d}t} x(\zeta_1(t)) &= x_\xi \bar{\xi}_1 \mathrm{e}^{\lambda^{uu}t} \lambda^{uu} + x_\eta \bar{\eta}_1 \mathrm{e}^{\lambda^{u}t} \lambda^{u} \\ &= \mathrm{e}^{\lambda^{u}t} \left(\bar{\eta}_1 \lambda^{u} x_\eta + \bar{\xi}_1 \lambda^{uu} \mathrm{e}^{(\lambda^{uu} - \lambda^{u})t} x_\xi\right)\end{aligned}$$

As $\bar{\eta}_1 > 0$, $x_\eta < 0$ and

$$\lim_{t \to -\infty} \mathrm{e}^{(\lambda^{uu} - \lambda^{u})t} x_\xi(\zeta_1(t)) = 0,$$

it follows that there is a constant $T < 0$ such that $\dot{x}_1 < 0$ for all $t < T$ and all μ in a small neighbourhood Γ of 0. If necessary by choosing $\varepsilon > 0$ smaller, it may be assumed that $T = 0$.

By assumption, the point

$$\bar{z}_2 = z(\bar{\xi}_2, \bar{\eta}_2)$$

can be written as

$$x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = \bar{p} = w(\bar{x}) + \alpha. \tag{2.12}$$

Note that for $\alpha = 0$, the point $\bar{p}$ is on W^{uu}, and therefore $\bar{\eta}_2 = 0$. To establish the dependence of η on α, derive first (2.12) with respect to α to obtain

$$x_\xi \left(\bar{\xi}_2\right)_\alpha + x_\eta \left(\bar{\eta}_2\right)_\alpha = 0, \quad p_\xi \left(\bar{\xi}_2\right)_\alpha + p_\eta \left(\bar{\eta}_2\right)_\alpha = 1.$$

Solving for $\left(\bar{\eta}_2\right)_\alpha$ yields

$$\left(\bar{\eta}_2\right)_\alpha = \frac{x_\xi}{\det Dz}$$

from which it follows that

$$\left(\bar{\eta}_2\right)_\alpha > 0.$$

The trajectory $z_2(t)$ through $\bar{z}_2$ has in linearizing coordinates the form

$$\zeta_2(t) = \left(\bar{\xi}_2 \mathrm{e}^{\lambda^{uu}t}, \bar{\eta}_2 \mathrm{e}^{\lambda^u t}\right).$$

Note that $\bar{\xi}_2 > 0$ for all μ, and that

$$\bar{\eta}_2 = \bar{\eta}_2(\alpha(\mu))$$

with $\bar{\eta}_2(0) = 0$. As before

$$\dot{x}_2 = \mathrm{e}^{\lambda^u t}\left(\lambda^u \bar{\eta}_2 x_\eta + \bar{\xi}_2 \lambda^{uu} \mathrm{e}^{(\lambda^{uu}-\lambda^u)t} x_\xi\right).$$

If $\alpha \leq 0$, then $\bar{\eta}_2 < 0$. From $x_\eta < 0$ it then follows that $\dot{x}_2 > 0$ for all t. By continuity, it follows also that $\dot{x}_2(0) > 0$ if $\alpha > 0$ is sufficiently small. Note however that

$$\lim_{t \to -\infty} \dot{x}_2 \mathrm{e}^{-\lambda^u t} = \bar{\xi}_2 \lambda^u x_\eta(0,0) < 0.$$

Consequently, for $\alpha > 0$ there is, by the intermediate value theorem, at least one $t < 0$ such that $x_2(t) < \bar{x}$ and $\dot{x}_2(t) = 0$. Let t_* denote the largest of these t if there are several.

Note that for $\alpha \leq 0$, the continuous curve formed by the union of the trajectories z_1, z_2 and the point e intersects each leaf $\{x = const\}$ exactly once, and defines therefore a continuous function $x \mapsto p^o_\mu(x)$, which is necessarily the optimal costate map.

If $\alpha > 0$, then the trajectory z_2 is tangent to the leaf $L = \{x = x_2(t_*)\}$ at $z_{2*} = z_2(t_*)$, and z_2 cuts all other leaves $\{x = const\}$ transversally for $t^* < t \leq 0$. The leaf L is cut by z_1 at z_{1*}. Since $\dot{x} = \mathcal{H}_p = 0$ at z_{2*} and $\mathcal{H}$ is strictly convex in p, it follows that

$$\mathcal{H}(z_{2*}) < \mathcal{H}(z_{1*}).$$

Since $\xi_1(0) = \delta$, there is $t^* \in (t_*, 0)$ such that $x_2(t^*) = 0$. Again by convexity of $\mathcal{H}$ in p, it follows that

$$\mathcal{H}(z_2(t^*)) > \mathcal{H}(e) = \lim_{t \to -\infty} \mathcal{H}(z_1(t)).$$

Consequently there is $\tilde{t} \in (t_*, t^*)$ such that

$$\mathcal{H}(\tilde{z}_1) = \mathcal{H}(\tilde{z}_2),$$

where $\tilde{z}_i = (\tilde{x}, \tilde{p}_i) = (x_i(\tilde{t}), p_i(\tilde{t}))$, and $\tilde{x}$ is an indifference point by theorem 2.2.1. □

IR$_1$(2) bifurcation

An indifference-repeller singularity of type 2 occurs in certain situations when the dynamics of the repeller is a Jordan node. Specifically, consider the situation that the vector field F on $\mathbb{R}^2$ has an equilibrium $e = (x_e, p_e)$, that its linearization $DF(e)$ has two equal positive eigenvalues $\lambda_1 = \lambda_2 = \lambda > 0$, and such that its proper eigenspace E^{pu} is only one-dimensional. By the Hartman theorem, there is a C^1 curve W^{pu}, the *proper unstable invariant manifold*, which is the image of E^{pu} in general coordinates; trajectories $z(t)$ in W^{pu} are characterized by the requirement that

$$\limsup_{t \to -\infty} \|z(t) - e\| \mathrm{e}^{-\lambda t} < \infty.$$

Definition 2.2.2. *A point $e = (x_e, p_e)$ is a (codimension one) indifference repeller singularity of type 2, notation IR$_1$(2), of an optimization problem with reduced canonical vector field F, if the following conditions hold.*

1. *The point e is an equilibrium of F such that the eigenvalues λ_1, λ_2 of $DF_0(e)$ satisfy $\lambda_1 = \lambda_2 = \frac{\rho}{2}$.*

2. *On some compact interval neighbourhood N of x_e, there is defined a continuous function $p : N \to \mathbb{R}$ such that*
$$p^o(x) = \{p(x)\}.$$
for all $x \in N$, and such that $p_e = p(x_e)$.

3. *Let W^{uu} denote the strong unstable manifold of F at e, parameterized as the graph of $w : N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of N. For all $x \in \partial N$, we have*

that

$$\nu(x)\Big(w(x) - p(x)\Big) > 0. \tag{2.13}$$

This singularity also gives rise to an indifference repeller bifurcation, as in the previous case, but through a different mechanism. See Figure 2.2: at bifurcation, the equilibrium of the reduced canonical vector field is a Jordan node. When the eigenvalues move off the real axis, it turns into a focus. This precludes the possibility of an optimal repeller. When the eigenvalues remain on the real axis but separate, two independent eigenspaces E^{uu} and E^u are generated. Condition (2.13) then implies the existence of an optimal repeller.

Theorem 2.2.5. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_1$(2) singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$. Let $D(\mu)$ and $T(\mu)$ denote the trace and the determinant of $DF_\mu(e_\mu)$.*

2. *The function $\alpha : \Gamma \to \mathbb{R}$, defined by*

$$\alpha(\mu) = \frac{T(\mu)^2}{4} - D(\mu)$$

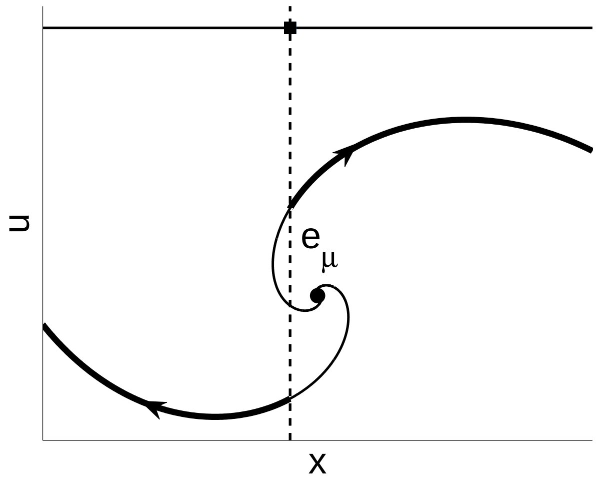
   and for which $\alpha(0) = 0$, satisfies

$$D\alpha(0) \neq 0.$$

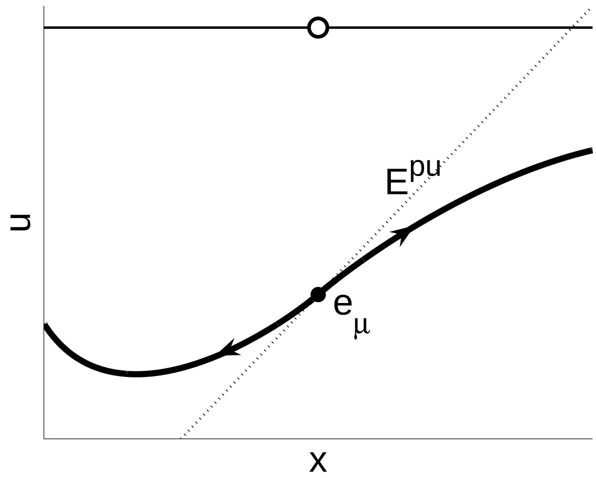
3. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*

$$p^o(x) = \{p(x, \mu)\}$$

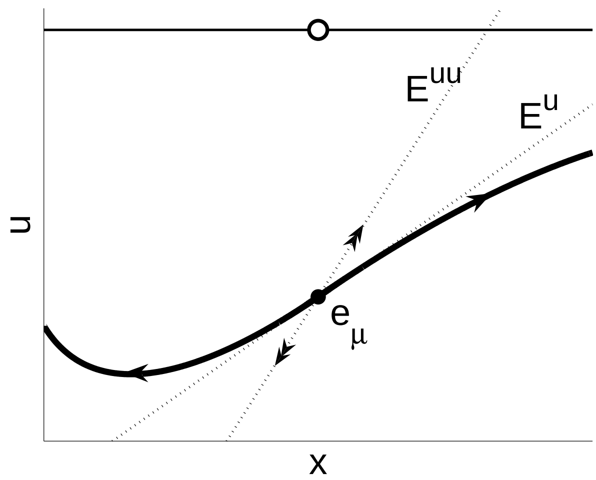
   for all $x \in \partial N$ and all $\mu \in \Gamma$.



(a) $\alpha(\mu) < 0$



(b) $\alpha(\mu) = 0$



(c) $\alpha(\mu) > 0$

Figure 2.2: Before, at and after the type 2 indifference-repeller bifurcation point.

Then the optimal vector field f^o restricted to N is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \begin{cases} -1 & x < 0, \\ \{-1, 1\} & x = 0, \\ 1 & x > 0. \end{cases}$$

whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = x.$$

Proof. There is a linear map C_0 such that

$$C_0^{-1} DF_0(e) C_0 = \begin{pmatrix} \frac{\rho}{2} & 1 \\ 0 & \frac{\rho}{2} \end{pmatrix}.$$

Arnol'd's matrix unfolding theorem (Arnold (1988)) then implies that there is a family of maps $C(\alpha)$, smoothly depending on α, such that $C(0) = C_0$ and such that

$$A_\alpha = C(\alpha)^{-1} DF_\mu(e_\mu) C(\alpha) = \begin{pmatrix} \frac{\rho}{2} & 1 \\ \alpha & \frac{\rho}{2} \end{pmatrix},$$

where $\alpha = \alpha(\mu)$. The eigenvalues of $DF_\mu(e_\mu)$ and consequently also those of A_α take the form

$$\lambda^u = \frac{\rho}{2} - \sqrt{\alpha}, \quad \lambda^{uu} = \frac{\rho}{2} + \sqrt{\alpha};$$

the corresponding eigenvectors of A_α take the form

$$v^u = (1, -\sqrt{\alpha}), \quad v^{uu} = (1, \sqrt{\alpha}).$$

Note that for $\alpha > 0$, these eigenvectors have the same ordering as the corresponding eigenvectors of $DF_\mu(e)$; cf. lemma 2.2.2. It follows that the matrix $C(\alpha)$ is necessarily orientation preserving for $\alpha > 0$ and, by continuity, for all other values of α.

Define $\bar{x}_i$, $\bar{p}_i$ and $\bar{z}_i$ as in (2.10) and (2.11).

When $\alpha < 0$, the eigenvalues are complex, and the trajectories z_1 and z_2 emanating from $\bar{z}_1$ and $\bar{z}_2$ respectively spiral towards e as $t \to -\infty$. Let t_* be the largest $t \leq$ such that $\dot{x}_2(t) = 0$. Then necessarily

$$x_* = x_2(t_*) < x_e.$$

The trajectory z_2, restricted to $[t_*, 0]$, can be parameterized as the graph of a continuous function $p_2 : [x_*, \bar{x}_2] \to \mathbb{R}$. In the same way, if $t^* < 0$ is the largest t such that $\dot{x}_1(t) = 0$, then z_1 restricted to $[t^*, 0]$ can be parameterized as the graph of the function $p_1 : [\bar{x}_1, x^*] \to \mathbb{R}$, where

$$x^* = x_1(t^*) > x_e.$$

Moreover, as $\mathcal{H}$ is strictly convex and $\mathcal{H}_p(x_*, p_2(x_*)) = 0$, it follows that

$$\mathcal{H}(x_*, p_2(x_*)) < \mathcal{H}(x_*, p_1(x_*));$$

likewise

$$\mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)).$$

By continuity, there is a point $\tilde{x} \in [x_*, x^*]$ such that

$$\mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})).$$

By Theorem 2.2.1, this is an indifference point.

Point 2 of Definition 2.2.2 implies that for $\mu = 0$, the sets $p^o(x)$ contain a single element $p(x)$ for all $x \in N$. Necessarily, the graph of p is formed by two trajectories of F_0 as well as the equilibrium point e. These trajectories intersect the lines $\{x = \mathit{const}\}$ transversally and they are tangent to E^{pu} at x_e; put differently, the graph of p is tangent to the proper unstable manifold W^{pu} at e. By point 3 of the definition, $p(x) > w(x)$ if $\bar{x}_1 < x < x_e$ and $p(x) < w(x)$ if $x_e < x < \bar{x}_2$.

For $\alpha(\mu) > 0$, there is a family W_μ^{uu} of strong unstable manifolds, depending continuously on μ, and parameterized as the graph of a family of C^1 functions w_μ around x_e. In particular, if N and Γ sufficiently small

$$\nu(x)\,(w_\mu(x) - p(x,\mu)) > 0$$

for all $x \in \partial N$ and $\mu \in \Gamma$. By continuity, the backward trajectories through $\bar{z}_1$ and $\bar{z}_2$ intersect all lines $\{x = const\}$ transversally and are tangent to the weak unstable direction E^u at x_e. But this implies that they form, together with the equilibrium e, the graph of a C^1 function p_μ that is defined on N, and for which

$$p_\mu^o(x) = \{p_\mu(x)\}.$$

□

IA$_1$ bifurcation

Definition 2.2.3. *A point $e = (x_e, p_e)$ is a (codimension one) indifference attractor singularity, notation IA$_1$, of an optimization problem with reduced canonical vector field F, if the following conditions hold.*

1. *The point e is an equilibrium of F such that the eigenvalues λ^s, λ^u of $DF(e)$ satisfy $\lambda^s < 0 < \lambda^u$.*

2. *On some compact interval neighbourhood N of x_e, there is defined a continuous function $p : N \to \mathbb{R}$ such that*

$$p^o(x) = \{p(x)\}.$$

for all $x \in N$, and such that $p_e = p(x_e)$.

3. *Let W^s and W^u denote respectively the stable and the unstable manifold of F at e, parameterized as the graph of functions $w^s, w^u : N \to \mathbb{R}$. If $\partial N = \{\bar{x}_1, \bar{x}_2\}$, then*

$$p(\bar{x}_1) = w^u(\bar{x}_1), \quad p(\bar{x}_2) = w^s(\bar{x}_2). \tag{2.14}$$

Note that this definition does not require the points $\bar{x}_1$ and $\bar{x}_2$ to be ordered in a certain way.

Theorem 2.2.6. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IA$_1$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu(e_\mu)$ satisfy $\lambda_\mu^s < 0 < \lambda_\mu^u$. Let the stable and the unstable manifolds W_μ^s and W_μ^u of e_μ be parameterized as graph $p = w^s(x, \mu)$ and $p = w^u(x, \mu)$ of differentiable functions $w^s, w^u : N \times \Gamma \to \mathbb{R}$.*

2. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*

$$p^o(x) = \{p(x, \mu)\}$$

for all $x \in \partial N$ and all $\mu \in \Gamma$.

3. *The function*

$$\alpha(\mu) = \nu(x_1)\big(p(x_1, \mu) - w(x_1, \mu)\big),$$

for which $\alpha(0) = 0$ by (2.14)*, is defined on Γ and satisfies*

$$D\alpha(0) \neq 0.$$

4. *For all $\mu \in \Gamma$ such that $\alpha(\mu) < 0$, the equality*

$$p(x_2, \mu) = w_\mu^s(x_2)$$

holds.

Then the optimal vector field f^o restricted to N is for $\alpha(\mu) < 0$ topologically conjugate to

$$Y(x) = \begin{cases} \{-x\}, & x < 1, \\ \{-1, 1\}, & x = 1, \\ \{1\}, & x > 1, \end{cases}$$

whereas for $\alpha(\mu) > 0$ it is conjugate to

$$Y(x) = \{1\}$$

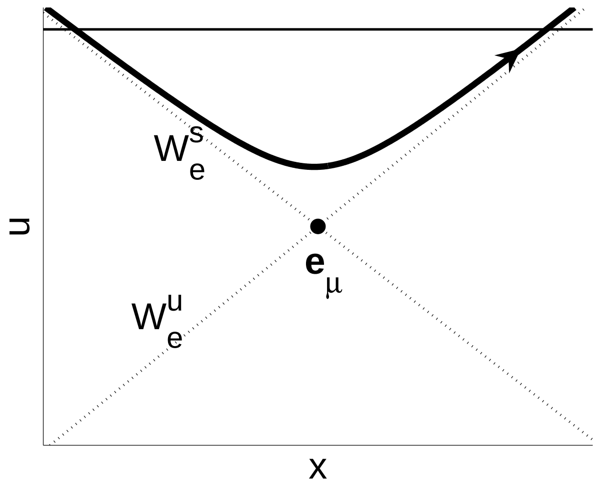
The theorem is illustrated in Figure 2.3. As for the $\mathrm{IR}_1(1)$ bifurcation, at the bifurcation the relative position of the optimal trajectories and the 'most unstable' invariant manifold changes.

If $\alpha(\mu) < 0$, the backward extension of the optimal trajectory through the point $\bar{z}_1 = (\bar{x}_1, p(\bar{x}_1, \mu))$ has a vertical tangent at a certain point. Past this point, the trajectory cannot be optimal, even locally. It follows that x_e is locally optimal. For $\alpha(\mu) > 0$, the trajectory through $\bar{z}_1$ intersects the line $x = x_e$. Theorem 2.2.1 then implies that the constant trajectory e cannot be optimal at all in this case.

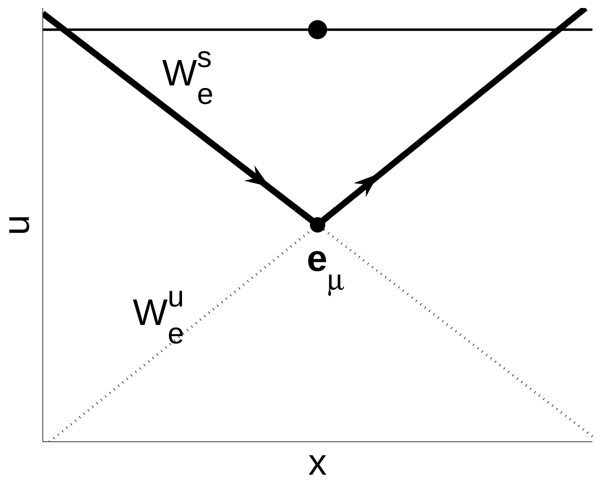
In many applications, the optimal trajectory through $\bar{z}_1$ is on the stable manifold of another equilibrium e'. For $\alpha(\mu) = 0$, we have also that $\bar{z}_1$ is in the unstable manifold of e, and the trajectory of F through $\bar{z}_1$ then forms a *heteroclinic connection* between e and e'. In this form, the indifference-attractor bifurcation was investigated in Wagener (2003). The present formulation in terms of the optimal costate rule is more general as it captures, for instance, also the situation that the optimal trajectory through $\bar{z}_1$ tends to infinity as $t \to \infty$ (cf. Hinloopen *et al.* (2010)).

Proof. Restricted to a neighbourhood of the saddle, in linearizing coordinates the vector field F_μ takes the form

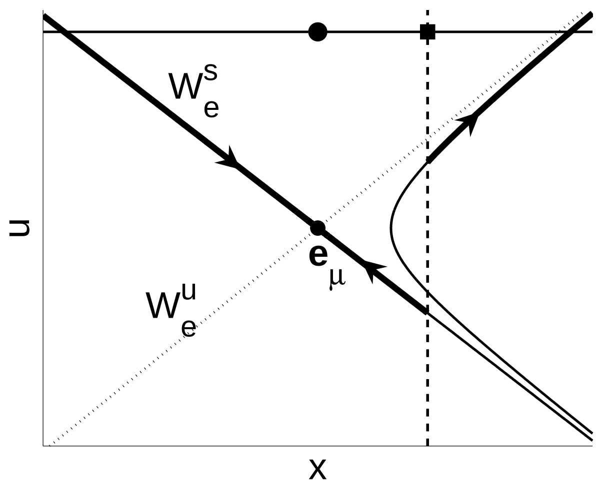
$$\dot{\zeta} = \begin{pmatrix} \lambda^u & 0 \\ 0 & \lambda^s \end{pmatrix} \zeta.$$



(a) $\alpha(\mu) < 0$



(b) $\alpha(\mu) = 0$



(c) $\alpha(\mu) > 0$

Figure 2.3: Before, at and after the indifference-attractor bifurcation point.

The coordinates are chosen such that the coordinate transformation is orientation preserving; moreover, the direction of the axes is chosen such that

$$x_\xi > 0, \quad x_\eta < 0.$$

Note that the unstable and stable manifolds are in these coordinates equal to the horizontal and vertical coordinate axes respectively.

As in the proof of theorem 2.2.4, set $\bar{x}_i$, $\bar{p}_i$ and $\bar{z}_i$ as in (2.10) and (2.11).

Assume that $\bar{x}$ of point 2 of Definition 2.2.3 satisfies $\bar{x} = \bar{x}_2$; the opposite situation can be handled analogously. If $\bar{\xi}_2$ and $\bar{\eta}_2$ are defined as

$$x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = w(\bar{x}) + \alpha,$$

then it follows as in the proof of theorem 2.2.4 that

$$(\bar{\eta}_2)_\alpha > 0$$

and $\bar{\eta}_2 = 0$ if $\alpha = 0$.

The trajectory $z_2(t) = (x_2(t), p_2(t))$ through $\bar{z}_2$ has in linearizing coordinates the form

$$\zeta_2(t) = (\xi_1(t), \eta_1(t)) = \left(\bar{\xi}_2 \mathrm{e}^{\lambda^u t}, \bar{\eta}_2 \mathrm{e}^{\lambda^s t}\right).$$

It follows that

$$\dot{x}_2 = \mathrm{e}^{\lambda^s t}\left(\lambda^s \bar{\eta}_2 x_\eta + \lambda^u \bar{\xi}_2 x_\xi \mathrm{e}^{(\lambda^u - \lambda^s)t}\right). \tag{2.15}$$

If $\alpha(\mu) > 0$, then $\bar{\eta}_2 > 0$; as both $\lambda^s x_\eta > 0$ and $\lambda^u \bar{\xi}_2 x_\xi > 0$, it follows from (2.15) that $\dot{x}_2 > 0$ for all t. That is, the trajectory z_2 intersects each line $x = const$ exactly once, and therefore defines a C^1 function $x \mapsto p(x, \mu)$, which then necessarily satisfies

$$p^o_\mu(x) = \{p(x, \mu)\}$$

for all $x \in N$.

Consider now the case that $\alpha(\mu) < 0$. By equation (2.15), if $\alpha(\mu)$ and hence $\bar{\eta}_2$ is sufficiently close to 0, then $\dot{x}_2(0) > 0$. Let $T_\mu < 0$ be such that $\eta_2(T_\mu) = -\bar{\xi}_2$. Then

$$T_\mu = \frac{1}{\lambda^s} \log \frac{\bar{\xi}_2}{(-\bar{\eta}_2)}$$

and equation (2.15) yields

$$\begin{aligned}\dot{x}_2(T_\mu)\mathrm{e}^{-\lambda^s T_\mu} &= \lambda^s \bar{\eta}_2 x_\eta + \lambda^u \bar{\xi}_2 x_\xi \left(\frac{-\bar{\eta}_2}{\bar{\xi}_2}\right)^{1+\lambda^u/|\lambda^s|} \\ &= \lambda^s \bar{\eta}_2 x_\eta + o(\bar{\eta}_2).\end{aligned}$$

This is negative if $\bar{\eta}_2$, and hence $\alpha(\mu)$, is sufficiently close to 0. For such values of α, there exists $t < 0$ such that $\dot{x}_2(t) = 0$. Let t_* denote the largest value of t with this property, and introduce

$$x_* = x_2(t_*).$$

For $x_* \leq x \leq \bar{x}_2$, the trajectory $(x_2(t), p_2(t))$ parameterizes the graph of a function $p_2(x)$. As $\dot{x} = \mathcal{H}_p$ along trajectories, note that

$$\mathcal{H}_p(x_*, p_2(x_*)) = 0.$$

Let $p_1 : N \to \mathbb{R}$ be such that its graph parameterizes the stable manifold W^s of s. Strict convexity of $\mathcal{H}$ implies the inequality

$$\mathcal{H}(x_*, p_2(x_*)) < \mathcal{H}(x_*, p_1(x_*)). \tag{2.16}$$

Define functions V_1 on N and V_2 on $[x_*, \bar{x}_2]$ by

$$V_1(x) = \frac{\mathcal{H}(x, p_1(x))}{\rho}$$

and

$$V_2(x) = \frac{\mathcal{H}(x, p_1(x))}{\rho}.$$

Then

$$V_2(x_*) < V_1(x_*).$$

To establish the opposite inequality for some $x^* \in [x_*, \bar{x}_2]$, consider the situation for $\alpha(\mu) = 0$, when $\bar{z}_2 \in W^u$. Then V_2 is defined for all $x_s < x < \bar{x}_2$. Moreover,

$$\lim_{x \downarrow x_s} V_2(x) = V_1(x).$$

Note that since $V_i'(x) = p_i(x)$ and

$$p_2(x) > p_1(x)$$

for all $x_s < x < \bar{x}_2$, it follows that

$$V_2(x) - V_1(x) = \int_{x_s}^{x} (p_2(\sigma) - p_1(\sigma))\, \mathrm{d}\sigma > 0$$

for all $x > x_s$. This implies in particular that

$$\mathcal{H}(x, p_2(x)) > \mathcal{H}(x, p_1(x))$$

for all $x > x_s$, if $\alpha(\mu) = 0$.

Fix $x^* \in (x_*, \bar{x}_2)$. Then for $\alpha(\mu) < 0$ sufficiently close to 0, by continuity

$$\mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)). \tag{2.17}$$

As a consequence of (2.16) and (2.17), there is $\tilde{x} \in (x_*, x^*)$ such that

$$\mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})).$$

By theorem 2.2.1, it follows that $\tilde{x}$ is an indifference point. $\square$

The saddle-node bifurcation

The saddle-node bifurcation of dynamical systems has a natural counterpart as a bifurcation of optimal vector fields.

Recall that a family of vector fields $f_\mu : \mathbb{R}^m \to \mathbb{R}^m$ can be viewed as a single vector field $g : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ by writing

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = g(x, \mu) = \begin{pmatrix} f_\mu(x) \\ 0 \end{pmatrix}.$$

Consider the situation that for $\mu = 0$ the point $\bar{z}$ is an equilibrium of f_0, and that $Df_0(\bar{z})$ has a single eigenvalue 0. Then $Dg(\bar{z}, 0)$ has two eigenvalues zero and an associated two-dimensional eigenspace E^c. The center manifold theorem applied to g implies that there is a differentiable invariant manifold W^c of g that is tangent to E^c at $(\bar{z}, 0)$. The manifold W^c can be viewed as a parameterized family of invariant manifolds W^c_μ, which are defined for μ taking values in a full neighbourhood of $\mu = 0$. Note that the center manifolds need not be unique.

Definition 2.2.4. *A point $e = (x_e, p_e)$ is a (codimension one) saddle-node singularity, notation SN$_1$, of an optimization problems with reduced canonical vector field F, if the following conditions hold.*

1. *The point e is an equilibrium of F such that the eigenvalues λ_1, λ_2 of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.*

2. *There is a compact interval N of X containing x_e and a function $p : N \to \mathbb{R}$ such that*

$$p^o(x) = \{p(x)\}$$

for all $x \in N$, and such that the graph of p is a center manifold W^c of F at e.

3. *The restriction*

$$F^c(x) = F_1(x_e + x, p(x_e + x)).$$

of F to W^c satisfies

$$F^c(0) = 0, \quad (F^c)'(0) = 0, \tag{2.18}$$

and

$$(F^c)''(0) \neq 0. \tag{2.19}$$

Theorem 2.2.7. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ a SN$_1$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

1. *There is a function $p : N \times \Gamma \to \mathbb{R}$ such that*

$$p^o_\mu(x) = \{p(x, \mu)\}$$

for all $(x, \mu) \in N \times \Gamma$.

2. *For $\mu \in \Gamma$, the graphs of $x \mapsto p(x, \mu)$ form a family of center manifolds W^c_μ of F at e.*

3. *If F^c_μ is*

$$F^c_\mu(x) = (F_\mu)_1 (x_e + x, p(x_e + x, \mu))$$

then the function

$$\alpha(\mu) = F^c_\mu(0)$$

satisfies

$$D\alpha(0) \neq 0.$$

Then the optimal vector field f^o_μ restricted to N is for $\mu \in \Gamma$ topologically conjugate to

$$Y_\mu(x) = \{\alpha(\mu) - \sigma x^2\}$$

where $\sigma \in \{-1, 1\}$ is given as

$$\sigma = \operatorname{sgn}(F_0^c)''(0).$$

Proof. This is a direct consequence from the usual saddle-node bifurcation theorem.

□

2.2.3 Codimension two bifurcations

Most codimension two situations are straightforward extensions of the corresponding codimension one bifurcations. The results in this subsection will in most cases be stated more briefly and less formally. An exception is made for the indifference-saddle-node bifurcation.

A model case: the IR$_2$(1,1) bifurcation

Definition 2.2.5. *A point $e = (x_e, p_e)$ is a (codimension two) indifference repeller singularity of type (1,1), notation IR$_2$(1,1), of an optimization problem with reduced canonical vector field F, if all conditions of definition 2.2.1 hold, but with* (2.8) *and* (2.9) *replaced by the condition that*

$$p(x) = w(x)$$

for all $x \in \partial N$.

Theorem 2.2.8. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_2$(1,1) singularity. Let all the conditions of Theorem 2.2.4 hold, excepting point 3, which is replaced by the following.*

The function

$$\alpha(\mu) = \big(\nu(x_1)\big(p(\bar{x}_1, \mu) - w(\bar{x}_1, \mu)\big), \nu(x_2)\big(p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu)\big)\big).$$

for which $\alpha(0) = (0, 0)$, is defined on Γ and satisfies

$$\operatorname{ran} D\alpha(0) = 2.$$

Then the optimal vector field f^o restricted to N is topologically conjugate to

$$Y(x) = x$$

if $\alpha_1(\mu) \leq 0$ and $\alpha_2(\mu) \leq 0$, whereas it is conjugate to

$$Y(x) = \begin{cases} -1 & x < 0, \\ \{-1, 1\} & x = 0, \\ 1 & x > 0. \end{cases}$$

if $\alpha_1(\mu) > 0$ or $\alpha_2(\mu) > 0$. In particular, the curves $\alpha_1(\mu) = 0$, $\alpha_2(\mu) < 0$ and $\alpha_2(\mu) = 0$, $\alpha_1(\mu) < 0$ are codimension one indifference-repeller bifurcation curves.

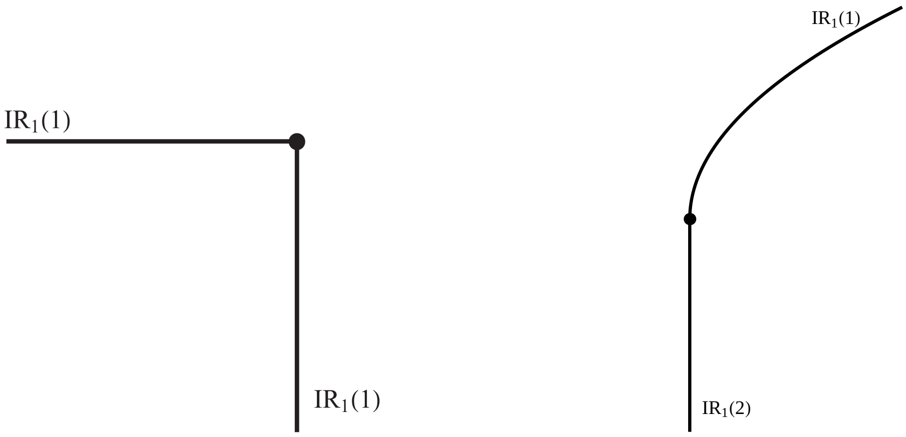
The proof is a simple modification of the proof of the codimension one case and is therefore omitted.

Other indifference-repeller and indifference-attractor bifurcations

Looking at the definition of the $\mathrm{IR}_1(2)$ bifurcation, it is clear that bifurcations of higher codimension are obtained when condition (2.13) is violated at a boundary point. If this happens at one of the boundary points, a codimension two situation is obtained where an $\mathrm{IR}_1(1)$ and an $\mathrm{IR}_1(2)$ curve meet in a $\mathrm{IR}_2(1{,}2)$ point. If it happens at both boundary points, a codimension three situation arises, denoted IR_3, where two $\mathrm{IR}_1(1)$ and a $\mathrm{IR}_1(2)$ surface meet. In order to avoid unnecessary repetitions, the exact definitions for these bifurcations are not formulated; they can all be modeled on Definition 2.2.5 and Theorem 2.2.8. Their bifurcation diagrams are given in Figures 2.4(b) and 2.5.

Likewise, a codimension two bifurcation is obtained if condition 2.14 is replaced by

$$p(x_1) = w^u(x_1), \quad p(x_2) = w^u(x_2). \tag{2.20}$$



(a) $IR_2(1,1)$ bifurcation diagram (b) $IR_2(1,2)$ bifurcation diagram

Figure 2.4: Indifference-repeller bifurcations of co-dimension two.

This a two-sided or double indifference attractor bifurcation, denoted DIA_2. Its bifurcation diagram is given in Figure 2.6.

Degenerate saddle-node bifurcations

The degenerate saddle-node bifurcations like the cusp (SN_2), the swallowtail (SN_3) etc. can be treated entirely analogously to the saddle-node itself.

The indifference-saddle-node bifurcation

The indifference-attractor and indifference-repeller bifurcations correspond to global bifurcations involving hyperbolic equilibria of the reduced canonical vector field; in contrast, the saddle-node bifurcation corresponds to a local bifurcation. The final bifurcation to be considered is the indifference-saddle-node bifurcation, which corresponds to a global bifurcation involving a nonhyperbolic equilibrium.

Definition 2.2.6. *A point $e = (x_e, p_e)$ is a (codimension two) indifference-saddle-node singularity, notation ISN_2, of an optimization problem with reduced canonical vector field F, if the following conditions hold.*

1. *The point e is an equilibrium of F, such that the eigenvalues λ_1, λ_2 of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.*

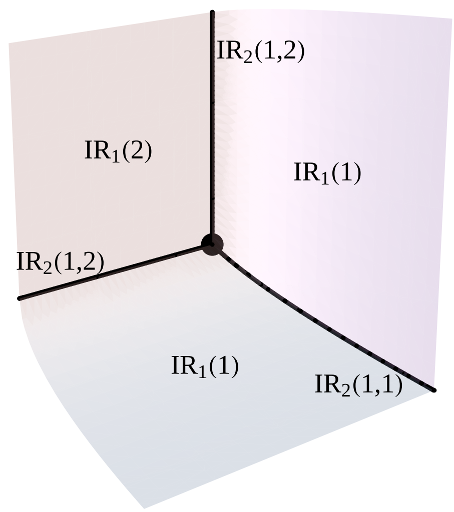


Figure 2.5: IR$_3$ bifurcation diagram.

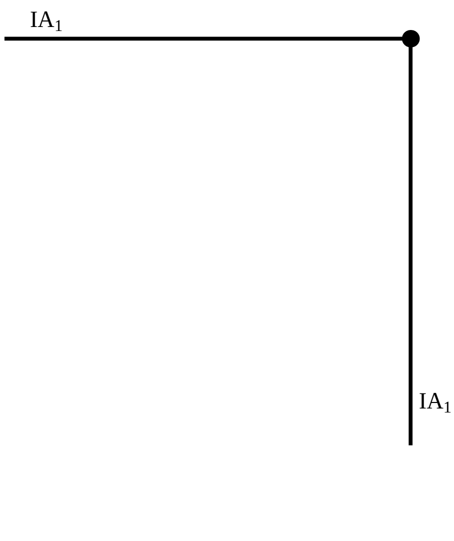


Figure 2.6: DIA$_2$ bifurcation diagram.

2. *On some compact interval neighbourhood N of x_e, there is defined a continuous function $p : N \to \mathbb{R}$ such that*

$$p^o(x) = \{p(x)\}$$

for all $x \in N$, and such that $p_e = p(x_e)$.

3. *Let W^u denote the unstable manifold of F at e, parameterized as the graph of a function $w^u : N \to \mathbb{R}$. There is a unique $\bar{x} \in \partial N$ such that*

$$p(\bar{x}) = w^u(\bar{x}). \tag{2.21}$$

4. *There is a center manifold W^c of F at e, parameterized as the graph of $w^c : N \to \mathbb{R}$, such that for $x \in \partial N$ and $x \neq \bar{x}$, we have that*

$$p(x) = w^c(x). \tag{2.22}$$

5. *The restriction*

$$F^c(x) = F_1(x_e + x, w^c(x_e + x))$$

of F to W^c satisfies

$$F^c(0) = 0, \quad (F^c)'(0) = 0,$$

and

$$(F^c)''(0) \neq 0.$$

Theorem 2.2.9. *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an ISN$_2$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

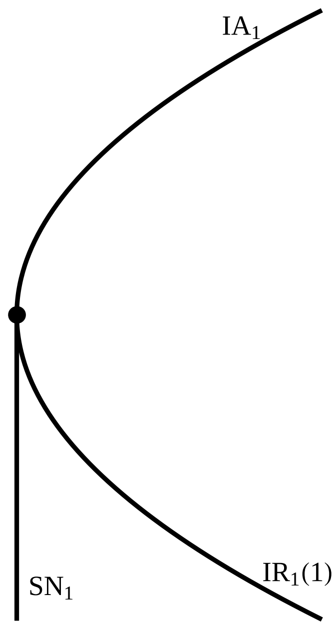


Figure 2.7: ISN$_2$ bifurcation diagram.

1. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in the second argument, such that*

$$p^o_\mu = \{p(x, \mu)\}$$

for all $(x, \mu) \in \partial N \times \Gamma$, and such that

$$\alpha_2(\mu) = p(\bar{x}, \mu) - p(\bar{x}, 0)$$

satisfies

$$D\alpha_2(0) \neq 0.$$

2. *There is a family of center manifolds W^c_μ, parameterized as the graphs of functions $x \mapsto w^c(x, \mu)$, such that $p(x, 0) = w^c(x, \mu)$ if $x \in \partial N \backslash \{\bar{x}\}$.*

3. Let F^c_μ be the restriction

$$F^c_\mu(x) = (F_\mu)_1 \left(x_e + x, w^c(x_e + x, \mu)\right)$$

of F to W^c. Then the function

$$\alpha_1(\mu) = F^c_\mu(0)$$

satisfies

$$D\alpha_1(0) \neq 0.$$

4. Let $\alpha(\mu) = (\alpha_1(\mu), \alpha_2(\mu))$. Then $\operatorname{ran} D\alpha(0) = 2$.

Then there is a differentiable functions $C(\alpha_2)$ such that $C(0) = C'(0) = 0$ and $C''(0) \neq 0$, and such that the problem has an indifference-attractor bifurcation if

$$\alpha_1 = C(\alpha_2), \qquad \alpha_2 > 0,$$

an indifference-repeller bifurcation if

$$\alpha_1 = C(\alpha_2), \qquad \alpha_2 < 0,$$

and a saddle-node bifurcation curve if

$$\alpha_1 = 0, \qquad \alpha_2 < 0.$$

Proof. Assume without loss of generality that $(F^c)''(0) > 0$.

The system is first put, by an orientation preserving transformation, in coordinates $\zeta = (\xi, \eta)$ such that the center manifold W^c_μ corresponds to $\eta = 0$ for all μ close to $\mu = 0$, and the unstable manifold W^u corresponds to $\xi = 0$ at $\mu = 0$. In these coordinates, the system, augmented by the parameter equation $\dot{\mu} = 0$, takes the form

$$\dot{\xi} = \alpha_1(\mu) + f_0(\xi, \mu) + \eta f_1(\zeta, \mu), \tag{2.23}$$

$$\dot{\eta} = \rho\eta \qquad\qquad\qquad + \eta g_1(\zeta, \mu), \tag{2.24}$$

$$\dot{\mu} = 0 \tag{2.25}$$

where by assumption $f_0(\xi, \mu) = c(\mu)\xi^2 + O(\xi^3)$ with $c(0) > 0$, and where $D\alpha_1(0) \neq 0$. These conditions imply that a saddle-node bifurcation occurs at $(\xi, \eta) = (0, 0)$ if $\alpha_1(\mu) = 0$, generating a family of hyperbolic saddle and one of hyperbolic unstable equilibria of F.

The saddle equilibria have associated to them unique unstable invariant manifolds W^u_μ; the unstable equilibria have associated to them strongly unstable manifolds W^{uu}_μ, which are also unique. An indifference-attractor bifurcation occurs if $(x, p(x, \mu)) \in W^u_\mu$; an indifference-repeller bifurcation occurs if $(x, p(x, \mu)) \in W^{uu}_\mu$. The main thing to prove is that the manifolds W^u_μ and W^{uu}_μ can be parameterized as graphs of differentiable functions

$$x \mapsto w^u(x, \mu), \quad x \mapsto w^{uu}(x, \mu).$$

This is not automatic, for the function w^u and w^{uu} will not be differentiable as functions of μ, having necessarily at $\mu = 0$ a singularity of the order $\sqrt{\mu}$.

In the following, it will however be shown that the closure of the invariant set

$$W = \bigcup_\mu W^u_\mu \cup W^{uu}_\mu$$

forms a differentiable manifold. From figure 2.8, it seems likely that W can be described as the level set

$$W \; : \quad \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu),$$

where w is a function yet to be determined. The condition that W is invariant under the flow of (2.23)–(2.25) leads to a first order partial differential equation for the function w; this equa-

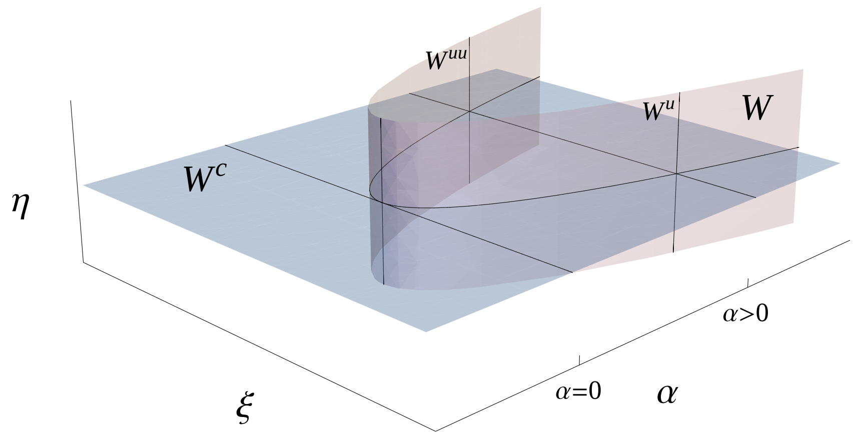


Figure 2.8: The manifold W

tion is singular for $\eta = 0$.

To solve this equation using the method of characteristics, introduce $w = w(t)$ as an independent variable by setting

$$\eta w = \alpha_1 + f_0(\xi, \mu).$$

Deriving with respect to time and using equations (2.23)–(2.25) yields

$$\eta \dot{w} = -\dot{\eta} w - \frac{\partial f_0}{\partial \xi}\dot{\xi} = -w(\rho + g_1)\eta + \frac{\partial f_0}{\partial \xi}(w + f_1)\eta.$$

Dividing out η formally, an equation for $\dot{w}$ is obtained. Together with equations (2.23)–(2.25), the following system is obtained:

$$\dot{\xi} = \eta w + \eta f_1, \qquad \dot{w} = -\rho w - w g_1 - \frac{\partial f_0}{\partial \xi}(w + f_1),$$
$$\dot{\eta} = \rho\eta + \eta g_1, \qquad \dot{\mu} = 0.$$

Linearizing the new system at $(\xi, \eta, w, \mu) = (0, 0, 0)$ yields

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{w} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \rho & & \\ & & -\rho & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ w \\ \mu \end{pmatrix}$$

Again invoking the center manifold theorem, we find that there is an invariant center-unstable manifold W^{cu} that is tangent to the center-unstable eigenspace $E^{cu} = \{w = 0\}$. Let this manifold be parameterized, in a neighbourhood of the origin, as

$$W^{cu} \; : \; w = w^{cu}(\xi, \eta, \mu).$$

Then w^{cu} is the function we have been looking for.

A final note on W: as for $\mu = 0$ the unstable manifold W^u is tangent to $\alpha_1 = 0$ at $\xi = 0$, the function w in

$$W \; : \quad \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu), \tag{2.26}$$

has to satisfy $w = \xi^2 \tilde{w}$.

Indifference-attractor or indifference repeller bifurcations occur if $(\bar{x}, p(\bar{x}, \mu)) \in W$. The equations

$$x = \bar{x}, \quad p = p(\bar{x}, 0) + \alpha_2$$

take in (ξ, η)-coordinates the form

$$\xi = c_1\alpha_2 + O(\varepsilon^2 + \alpha_2^2), \quad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Note however that if $\mu = 0$, then W is given by $\xi = 0$. Moreover, by assumption $p(\bar{x}, 0) \in W$; therefore the equations actually read as

$$\xi = \alpha_2 \left(c_1 + O(\varepsilon + \alpha_2)\right), \quad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Substitution in equation (2.26) yields the indifference-attractor and indifference-repeller bifurcation curves

$$\alpha_1 = c(\mu)c_1^2\alpha_2^2 + O(\alpha_2^3).$$

Taken together with the saddle-node curve

$$\alpha_1 = 0,$$

this yields the bifurcation diagram. Finally, note that if $p(\bar{x}, \mu) > w^{uu}(\bar{x})$, the saddle node bifurcation does not correspond to a bifurcation of the optimal vector field. $\square$

DISN$_3$ bifurcation

It is possible that a 'double' ISN singularity, denoted DISN$_3$, occurs if conditions (2.21) and (2.22) of definition 2.2.6 are replaced by the condition that

$$p(x) = w^u(x)$$

for all $x \in \partial N$. This is clearly a codimension three situation.