Structural analysis of complex ecological economic optimal control problems

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Chapter 4

Stochastic optimal control problems with small noise intensities

In this chapter stochastic optimal control problems with one-dimensional non-convex state dynamics are considered. To solve such a problem at a given noise intensity ε means to find the value function $V(x_0, \varepsilon)$ and the optimal control $u(x_0, \varepsilon)$ for every initial state x_0. The value function $V(x_0, \varepsilon)$ satisfies a second order Hamilton-Jacobi-Bellman equation, which for vanishing noise $\varepsilon = 0$ reduces to a first order differential equation.

Kushner (1967) shows, under suitable assumptions, that the solution of the Hamilton-Jacobi-Bellman equation exists for $\varepsilon > 0$ and is of class C^2. Fleming and Souganidis (1986) obtain an asymptotic expansion for $V(x, \varepsilon)$ in regions of strong regularity, that is for subregions of the state space where the value function of the corresponding unperturbed problem $V(x, 0)$ is smooth. The convergence of $V(x, \varepsilon)$ to $V(x, 0)$ together with its first derivative is shown as $\varepsilon \to 0$. Note that Fleming and Souganidis consider the stochastic control problem over a bounded region with $V(x, \varepsilon)$ satisfying a Dirichlet boundary condition. This chapter considers problems over an unbounded domain with possibly a finite number of interior regularity regions. For each regularity region two conditions are needed to determine the solution of the Hamilton-Jacobi-Bellman equation, which cannot be derived from the Dirichlet conditions of the original problem.

A method is provided of deriving conditions needed to determine the solution of the dynamic programming equation in regularity regions. It is also shown how to obtain an asymptotic series of its solution over the whole state domain.

The structure of the chapter is as follows. In Section 4.1 a stochastic control problem is formulated. Section 4.2 describes the method of approximating its value function when the value function of the corresponding deterministic problem is smooth and there are no irregularity points. In Section 4.3 a method of constructing an asymptotic series for $V(x,\varepsilon)$ is proposed when $V(x,0)$ is not strongly regular everywhere. Section 4.4 concludes.

4.1 Formulation of the problem

Let $x(t)$ and $u(t)$ denote the state of a system and the control applied to the system at time t respectively. It is assumed that $x \in X$ where $X \subset \mathbb{R}$ is an open set and $u \in D$ where $D \subset \mathbb{R}$ is a compact convex set. Let the state evolve according to the following stochastic differential equation

$$dx = f(x,u)dt + \sqrt{2\varepsilon\sigma^2(x)}dw \tag{4.1}$$

where $w(t)$ is a one dimensional Wiener process, ε denotes the noise level, and where f and σ satisfy the following conditions:

1. f satisfies a uniform Lipschitz condition jointly in x and u;
2. f is linear in u: $f(x,u) = A(x) + B(x)u$, where the functions A, B are bounded on X together with their first order derivatives;
3. σ satisfies a uniform Lipschitz condition in x;
4. $\sigma(x) \neq 0 \quad \forall x \in X$.

The reader is referred to Fleming (1971) and Kushner (1967) for a detailed discussion of these conditions.

The state $x(t)$ is assumed to be completely observable at each time t. Thus the control policy may be represented as a function of the state $u = u(x)$. The control $u = u(x)$ is called *admissible* if u takes values in a compact convex set D and is locally Lipschitz.

The benefit functional associated with each u is given by

$$B[x_0, u] = \mathbb{E}^u_{x_0} \int_0^\infty g\left(x, u(x)\right) e^{-\rho t}\, dt \tag{4.2}$$

where the discount rate $\rho > 0$, and where $\mathbb{E}^u_{x_0}$ is the conditional expectation operator given the initial state $x(0) = x_0$ and the control u. The integrand $g(x, u)$ is supposed to satisfy the following conditions:

1. $g(x, u)$ is bounded in any compact set for any admissible u;

2. $g(x, u)$ is locally Lipschitz jointly in x and u;

3. $\exists c < 0$ such that
$$\frac{\partial^2 g(x, u)}{\partial u^2} \leq c \tag{4.3}$$
for all admissible u.

The problem is to find an admissible control $u^*_\varepsilon = u^*(x, \varepsilon)$ that maximizes the benefit functional (4.2) given the state dynamics equation (4.1) and the initial state $x(0) = x_0$. This problem is denoted by $\mathcal{G}(\varepsilon)$.

Introduce the value function

$$V(x_0, \varepsilon) = \sup_u \mathbb{E}^u_{x_0} \int_0^\infty g(x, u) e^{-\rho t}\, dt \tag{4.4}$$

and the current value Hamiltonian

$$H(x, p, u) = g(x, u) + p f(x, u), \tag{4.5}$$

where p is a costate variable.

For each x and $p \in \mathbb{R}$ consider the *maximized current value Hamiltonian*

$$\mathcal{H}(x,p) = \max_u H(x,p,u) = \max_u \left[g(x,u) + pf(x,u)\right] = H(x,p,U(x,p)), \tag{4.6}$$

where

$$U(x,p) = \arg\max_u \left[g(x,u) + pf(x,u)\right]. \tag{4.7}$$

The function $U(x,p)$ is well defined as H is strictly concave in u because of (4.3).

The following theorem of Kushner (1967) gives the existence of an optimal maximizing control.

Theorem 4.1.1. *Let the functions f, σ and g satisfy the conditions above. For every $\varepsilon > 0$ there exists an optimal control $u^*(x,\varepsilon)$ for $\mathcal{G}(\varepsilon)$. The corresponding value function $V(x,\varepsilon)$ has continuous second derivatives w.r.t. x in any compact set and solves the following equation*

$$\varepsilon\sigma^2(x)V_{xx}(x,\varepsilon) + \mathcal{H}(x, V_x(x,\varepsilon)) - \rho V(x,\varepsilon) = 0. \tag{4.8}$$

Equation (4.8) is the Hamilton-Jacobi-Bellman equation of the stochastic optimal control problem $\mathcal{G}(\varepsilon)$. If V solves this equation then the optimal control policy is given by

$$u^*(x,\varepsilon) = U\left(x, V_x(x,\varepsilon)\right). \tag{4.9}$$

According to Theorem 4.1.1 solving $\mathcal{G}(\varepsilon)$ is reduced to solving the second order ordinary differential equation (4.8). The fact that the highest order derivative in (4.8) is multiplied by the perturbation parameter ε makes it a singularly perturbed differential equation: for $\varepsilon = 0$ this term vanishes and the order of the equation changes. This chapter focuses on constructing approximate solutions to such equations using methods of singular perturbations (see for example Holmes (1995) and Verhulst (2005)).

In the case when the solution of (4.8) for $\varepsilon = 0$ is smooth, an approximate solution of (4.8) can be constructed for small $\varepsilon > 0$ as in Fleming and Souganidis (1986). However two conditions are needed to determine the solution. Judd (1998) derives the boundary conditions for such a problem by a Taylor approximation of V in x and ε at the steady state of the deterministic problem. In contrast, this chapter expands $V(x, \varepsilon)$ as a series of ε. The next section shortly describes the method.

4.2 Problems without thresholds

First consider the case that $\mathcal{G}(0)$ has a continuously differentiable solution. That is, there exists a C^1 function $V(x, 0)$ that satisfies the dynamic programming equation of $\mathcal{G}(0)$ for any x

$$\mathcal{H}(x, V_x(x, 0)) - \rho V(x, 0) = 0. \tag{4.10}$$

The so-called *optimal vector field* is then given by

$$\dot{x} = f^o(x) = \frac{\partial \mathcal{H}}{\partial p}(x, V_x(x, 0)). \tag{4.11}$$

It determines the state dynamics under the optimal policy. In this section it is assumed that the optimal vector field f^o of the problem $\mathcal{G}(0)$ has a unique global attractor $x = x^*$.

It is assumed (see Fleming and Souganidis (1986)) that for small ε the solution $V(x, \varepsilon)$ to $\mathcal{G}(\varepsilon)$ can be expanded as

$$V(x, \varepsilon) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \dots. \tag{4.12}$$

Following Fleming and Souganidis (1986), (4.12) is substituted into (4.8) and terms with the same orders of ε are collected. This yields a series of equations, the first two reading as

$$\mathcal{H}(x, v_0'(x)) - \rho v_0(x) = 0, \tag{4.13}$$

$$\sigma^2(x) v_0''(x) + \mathcal{H}_p(x, v_0'(x)) v_1'(x) - \rho v_1(x) = 0. \tag{4.14}$$

Equation (4.13) is the dynamic programming equation of the unperturbed problem $\mathcal{G}(0)$; it is solved by $v_0(x) = V(x, 0)$. Any subsequent equation is a first order linear differential equation for the function v_k with coefficients depending on v_{k-1}. This allows us to determine the v_k recursively.

As these equations are first-order differential equations, each requires one condition to determine a free parameter in the family of solutions. In order to obtain such a condition for (4.13), the equation is differentiated with respect to x and terms are rearranged as follows

$$v_0''(x) = \frac{\rho v_0'(x) - \mathcal{H}_x(x, v_0'(x))}{\mathcal{H}_p(x, v_0'(x))}. \tag{4.15}$$

Equation (4.11) implies that equilibria of the optimal vector field f^o are zeros of $\mathcal{H}_p(x, v_0'(x))$. From the assumption about existence of the unique global attractor of f^o, it follows that there exists a unique $x = x^*$ such that $\mathcal{H}_p(x^*, v_0'(x^*)) = 0$. If a solution of (4.13) is to be smooth, it is also required that

$$\rho v_0'(x^*) - \mathcal{H}_x(x, v_0'(x^*)) = 0. \tag{4.16}$$

The equations (4.16) and (4.13) yield the value of $v_0(x)$ at $x = x^*$, which gives a condition to determine $v_0(x)$ in (4.13).

Let us rewrite (4.14) as follows

$$v_1'(x) = \frac{\rho v_1(x) - \sigma^2(x) v_0''(x)}{\mathcal{H}_p(x, v_0'(x))}. \tag{4.17}$$

If v_1' is to be smooth, it is required that

$$v_1(x^*) = \frac{\sigma^2(x^*) v_0''(x^*)}{\rho}, \tag{4.18}$$

which yields a condition for equation (4.14) determining $v_1(x)$. Proceeding in the same way $v_2(x), \ldots, v_k(x)$ are obtained iteratively. The $k-$th order approximation to the solution of (4.8) is then given by

$$V_k(x, \varepsilon) = \sum_{j=0}^{k} \varepsilon^j v_j(x). \tag{4.19}$$

4.3 Problems with thresholds

Now consider the situation that the optimal vector field f^o of the deterministic problem has multiple equilibria. As every equilibrium of f^o gives rise to a condition for the equations (4.13), (4.14) etc, for every such equilibrium an approximation of the form (4.19) of the solution to the Hamilton-Jacobi-Bellman equation (4.8) is obtained. These approximations are usually defined only on neighborhoods of the corresponding equilibria. In order to get a global approximation they have to be 'glued' together in such a way that the resulting function is smooth and approximates the value function $V(x, \varepsilon)$ over the whole state space. This can be done using the method of matched asymptotic expansions, which is described next.

The equation (4.13) is a first order differential equation, but its solution $v_0(x)$ has to satisfy more than one condition. This implies that there exist points of non-differentiability of $v_0(x)$; these are indifference points of the deterministic problem (see Chapter 2). They divide the state domain into *outer layers*[1], that is subdomains where (4.8) is approximated by (4.13), where $v_0(x)$ is differentiable and $V(x, \varepsilon)$ converges to $v_0(x)$ uniformly with its derivative as $\varepsilon \to 0$.

The next step is to introduce *an inner layer* around each indifference point. Each inner layer contains a point of non-differentiability of $v_0(x)$, therefore the smooth function $V_x(x, \varepsilon)$ needs to change fast in order to approximate the discontinuous function $V_x(x, 0)$. This implies that in an inner layer (4.8) is not well approximated by (4.13) as $V_{xx}(x, \varepsilon)$ will be large and the term εV_{xx} in the stochastic Hamilton-Jacobi-Bellman equation (4.8) will not be negligible. To capture this fact, in each inner layer a suitable local variable is introduced. Writing (4.8) in local variables, a second order equation for the nondeterministic correction term which has a solution with two free parameters is obtained. The missing conditions needed to determine the parameters are obtained by matching the approximate solutions of the Hamilton-Jacobi-Bellman equation (4.8) in the inner and outer layers. This is done in the third step by introducing so called *transition layers* - the intervals where the outer and the inner layers overlap. As a result a collection of approximate solutions to (4.8) in the inner and outer layers is obtained. In the fourth step all

[1] In Fleming and Souganidis (1986) these subdomains are called regions of strong regularity.

approximations are combined in such a way that the resulting function is smooth and solves (4.8) over the whole domain up to a term that tends to 0 as $\varepsilon \to 0$.

Step 1: Outer layer

If f^o has two attracting equilibria, x_L and x_R, then two approximations of $V(x,\varepsilon)$ are obtained using the method described in Section 4.1. Let $V^{L,1}(x,\varepsilon) = v_0^L(x) + \varepsilon v_1^L(x)$ and $V^{R,1}(x,\varepsilon) = v_0^R(x) + \varepsilon v_1^R(x)$ be the first order approximations of $V(x,\varepsilon)$ with conditions derived at $x = x_L$ and $x = x_R$ respectively. The indifference point $x = \hat{x}$ is determined by the condition

$$v_0^L(\hat{x}) = v_0^R(\hat{x}). \tag{4.20}$$

Denote $v_0^L(\hat{x}) = v_0^R(\hat{x}) = \hat{v}$. The *outer* approximation of $V(x,\varepsilon)$ is then given by

$$V^{outer,1}(x,\varepsilon) = \begin{cases} V^{L,1}(x,\varepsilon) & \text{for } x < \hat{x}, \\ V^{R,1}(x,\varepsilon) & \text{for } x \geq \hat{x}. \end{cases} \tag{4.21}$$

However, the function $V^{outer}(x,\varepsilon)$ is generally not of class C^2; it might be even discontinuous at $x = \hat{x}$. It is therefore not a good approximation of the solution to $\mathcal{G}(\varepsilon)$ in a neighborhood of $x = \hat{x}$. Therefore an inner layer is introduced at $x = \hat{x}$ and (4.8) is considered in local variables around $x = \hat{x}$.

Step 2: Inner layer

Introduce the local variable

$$\xi_\alpha = \frac{x - \hat{x}}{\varepsilon^\alpha}, \tag{4.22}$$

where $0 \leq \alpha \leq 1$ is a constant. Rewriting (4.8) in terms of ξ_α and denoting the solution in the inner layer as $V^{inner}(\xi_\alpha,\varepsilon)$ yields

$$\varepsilon^{1-2\alpha}\sigma^2(\hat{x} + \varepsilon^\alpha\xi_\alpha)(V^{inner})'' + \mathcal{H}(\hat{x} + \varepsilon^\alpha\xi_\alpha, \varepsilon^{-\alpha}(V^{inner})') - \rho V^{inner} = 0. \tag{4.23}$$

Define $V^{inner,\alpha}(\xi_\alpha) = \hat{v} + \varepsilon^\alpha W_\alpha(\xi_\alpha)$. It turns out that the appropriate expansion of $V^{inner}(\xi_\alpha, \varepsilon)$ is

$$V^{inner}(\xi_\alpha, \varepsilon) = V^{inner,\alpha}(\xi_\alpha, \varepsilon) + o(\varepsilon^\alpha). \tag{4.24}$$

Then (4.23) reads as follows

$$\varepsilon^{1-\alpha}\sigma^2(\hat{x} + \varepsilon^\alpha \xi_\alpha) W_\alpha'' + \mathcal{H}\left(\hat{x} + \varepsilon^\alpha \xi_\alpha, W_\alpha'\right) - \rho\hat{v} - \rho\varepsilon^\alpha W_\alpha = o(\varepsilon^\alpha). \tag{4.25}$$

In order to determine the value of α the following cases are considered:

1) $\alpha = 0$: if $\varepsilon \to 0$ then (4.25) becomes

$$\mathcal{H}(\hat{x} + \xi_0, W_0') - \rho\left(\hat{v} + W_0\right) = 0; \tag{4.26}$$

2) $0 < \alpha < 1$: if $\varepsilon \to 0$ then (4.25) becomes

$$\mathcal{H}(\hat{x}, W_\alpha') - \rho\hat{v} = 0; \tag{4.27}$$

3) $\alpha = 1$: if $\varepsilon \to 0$ then (4.25) becomes

$$\sigma^2(\hat{x}) W_1'' + \mathcal{H}(\hat{x}, W_1') - \rho\hat{v} = 0. \tag{4.28}$$

Note that for $\alpha = 0$ the leading terms of the inner expansion $V^{inner,0}(\xi_0, \varepsilon) = \hat{v} + W_0(\xi_0)$ and in the outer expansion

$$V^{outer,0}(x, \varepsilon) = \begin{cases} v_0^L(x) & \text{for } x < \hat{x}, \\ v_0^R(x) & \text{for } x \geq \hat{x}, \end{cases}$$

coincide, compare (4.13) and (4.26). Note also that for $\alpha = 1$ the inner expansion $V^{inner,1}(\xi_1, \varepsilon)$ approximates the solution $V(x, \varepsilon)$ in the inner layer, as $\alpha = 1$ is the distinguished limit for

(4.25). Therefore for $0 < \alpha < 1$ the expansion $V^{inner,\alpha}(\xi_\alpha, \varepsilon)$ approximates the solution in the *transition layers*[2]- two regions (left and right) where the outer layers overlap with the inner layer. This implies that the leading terms in the two approximations must coincide if both are rewritten in $\xi_\alpha-$coordinates and the limit $\varepsilon \to 0$ is taken. This gives us two conditions that are needed to determine the solution of (4.28): one condition is derived from matching the inner and the outer expansions at the left transition layer, another one from matching them at the right transition layer. The process of matching is done in the next step.

Step 3: Matching

Outer and inner solutions are matched by writing the two solutions in the transition layers' coordinates and equating their values. If the outer solutions are rewritten in the transition layers coordinates

$$\xi_\alpha = \frac{x - \hat{x}}{\varepsilon^\alpha} = \varepsilon^{-\alpha}\xi_0 = \varepsilon^{1-\alpha}\xi_1 \tag{4.29}$$

then from (4.21) the following is obtained

$$V^{outer}(\xi_0) = V^{outer}(\varepsilon^\alpha \xi_\alpha) = \begin{cases} v_0^L(\hat{x} + \varepsilon^\alpha \xi_\alpha) + \varepsilon v_1^L(\hat{x} + \varepsilon^\alpha \xi_\alpha) & \text{for } \xi_\alpha < 0 \\ v_0^R(\hat{x} + \varepsilon^\alpha \xi_\alpha) + \varepsilon v_1^R(\hat{x} + \varepsilon^\alpha \xi_\alpha) & \text{for } \xi_\alpha > 0 \end{cases}$$

$$= \begin{cases} \hat{v} + (v_0^L)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^L(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha < 0 \\ \hat{v} + (v_0^R)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^R(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha > 0 \end{cases} \tag{4.30}$$

Rewriting the inner solution in the transition layer coordinates yields, using (4.24) and (4.29)

$$V^{inner,\alpha}(\xi_1) = V^{inner,\alpha}(\varepsilon^{\alpha-1}\xi_\alpha) = \hat{v} + \varepsilon W_1(\varepsilon^{\alpha-1}\xi_\alpha) + o(\varepsilon). \tag{4.31}$$

[2]This follows from the *overlap hypothesis* (see Verhulst (2005)), which assumes that if there are two neighboring expansions, then there exists a common subdomain where both expansions are valid. Equation (4.26) is obtained from (4.27) for $\alpha \to 0$, and equation (4.28) is obtained from (4.27) for $\alpha \to 1$.

The values of the inner and the outer expansions in the transition layers have to be matched, therefore it is required that

$$\hat{v} + \varepsilon W_1(\varepsilon^{\alpha-1}\xi_\alpha) = \begin{cases} \hat{v} + (v_0^L)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^L(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha < 0 \\ \hat{v} + (v_0^R)'(\hat{x})\varepsilon^\alpha \xi_\alpha + \varepsilon v_1^R(\hat{x}) + o(\varepsilon) & \text{for } \xi_\alpha > 0, \end{cases} \tag{4.32}$$

which implies for fixed x two asymptotic conditions for W_1

$$\lim_{\varepsilon\to 0} \left[W_1(\varepsilon^{\alpha-1}\xi_\alpha) - (v_0^L)'(\hat{x})\varepsilon^{\alpha-1}\xi_\alpha - v_1^L(\hat{x})\right] = 0, \text{ for } \xi_\alpha < 0 \tag{4.33}$$

$$\lim_{\varepsilon\to 0} \left[W_1(\varepsilon^{\alpha-1}\xi_\alpha) - (v_0^R)'(\hat{x})\varepsilon^{\alpha-1}\xi_\alpha - v_1^R(\hat{x})\right] = 0, \text{ for } \xi_\alpha > 0. \tag{4.34}$$

Write $w = W_1'$. Equation (4.28) takes the form

$$\sigma^2(\hat{x})w' + \mathcal{H}(\hat{x}, w) - \rho\hat{v} = 0. \tag{4.35}$$

If $w(\tau, C)$ designates the general solution of (4.35), where $w(0, C) = C$, then (4.33), (4.34) can be rewritten as follows

$$W_1(0) - \int_{-\infty}^{0} \left\{w(\tau, C) - (v_0^L)'(\hat{x})\right\} d\tau = v_1^L(\hat{x}), \tag{4.36}$$

$$W_1(0) + \int_{0}^{+\infty} \left\{w(\tau, C) - (v_0^R)'(\hat{x})\right\} d\tau = v_1^R(\hat{x}). \tag{4.37}$$

From equalities (4.36), (4.37) the conditions $W_1(0)$ and $W_1'(0) = C$ needed to determine the solution of (4.28) are obtained.

Step 4: Composite expansion

The description of the solution consists of two parts V^{outer} and V^{inner}, which now have to be combined to form a *composite expansion*. This is done by adding the expansions and subtracting common parts. Thus the first order approximation of the solution to (4.8) is given by

$$V_1(x,\varepsilon) = \begin{cases} v_0^L(x) + \hat{v} + \varepsilon W_1\left(\dfrac{x-\hat{x}}{\varepsilon}\right) - \hat{v} - \left(v_0^L\right)'(\hat{x})(x-\hat{x}), & \text{for } x < \hat{x} \\ v_0^R(x) + \hat{v} + \varepsilon W_1\left(\dfrac{x-\hat{x}}{\varepsilon}\right) - \hat{v} - \left(v_0^R\right)'(\hat{x})(x-\hat{x}), & \text{for } x \geq \hat{x} \end{cases}$$

$$= \begin{cases} v_0^L(x) + \varepsilon W_1\left(\dfrac{x-\hat{x}}{\varepsilon}\right) - \left(v_0^L\right)'(\hat{x})(x-\hat{x}), & \text{for } x < \hat{x} \\ v_0^R(x) + \varepsilon W_1\left(\dfrac{x-\hat{x}}{\varepsilon}\right) - \left(v_0^R\right)'(\hat{x})(x-\hat{x}), & \text{for } x \geq \hat{x} \end{cases} \tag{4.38}$$

The function $V_1(x,\varepsilon)$ given by (4.38) is continuous and smooth at $x = \hat{x}$ as the following is true

$$\lim_{x\uparrow\hat{x}} V_1(x,\varepsilon) = \lim_{x\downarrow\hat{x}} V_1(x,\varepsilon) = \hat{v}, \tag{4.39}$$

$$\lim_{x\uparrow\hat{x}} (V_1)_x(x,\varepsilon) = \lim_{x\downarrow\hat{x}} (V_1)_x(x,\varepsilon) = W_1'(0). \tag{4.40}$$

4.4 Concluding remarks

In this chapter a solution approximation method for stochastic optimal control problems with one-dimensional state space and infinite time horizon has been developed. This method is general and can be applied to problems with small noise intensities. The algorithm of constructing a solution approximation involves solving a number of ordinary differential equations and integral equalities, which can be implemented efficiently numerically. In Chapter 5 this method is applied to a stochastic lake model.