Structural analysis of complex ecological economic optimal control problems

Kiseleva, T.

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Chapter 5

Regime switching thresholds in stochastic optimal control problems

Recall from Chapter 2 that an indifference point for a deterministic optimal control problem corresponds to an initial state where there are two optimal controls yielding different long run outcomes. This notion of indifference point cannot be extended in a straightforward way to problems with noise as the optimal controls in stochastic problems are always univalued . Therefore a new concept is needed. There has been one attempt in this direction: Dechert and O'Donnell (2006) consider a stochastic control problem in discrete time with a one-dimensional state space and bounded stochastic shocks. They introduce the notion of a stochastic indifference set, which is a stochastic equivalent of a deterministic indifference point, as 'a transient set between two basins of attraction [...] where there is a positive probability that the dynamics will end up in the lower basin of attraction, and a positive probability they will end up in the upper basin of attraction'. The authors analyze numerically the dependence of such sets on the noise level in the context of a stochastic dynamic game. In Bultmann *et al.* (2010) the authors adopt this concept and apply it to a model of illicit drug consumption. They show numerically that these indifference sets expand with the noise variance and that the optimal policy becomes a continuous function of the state for sufficiently high levels of the variance. Unfortunately, the concept of stochastic indifference set is not applicable to models with continuous time and unbounded shocks; for such systems a different notion is needed.

This chapter proposes a concept of stochastic thresholds for one-dimensional continuous time optimal control models, which is invariant with respect to coordinate transformations. The presence of stochastic indifference thresholds implies regime switching behavior of the system. This type of behavior is characterized by persistent state fluctuations around one of the steady state until a large shock forces the system to pass the threshold between the regimes. After that the system fluctuates around another equilibrium until the next large shock hits the system again.

The methodology of obtaining an asymptotic series for the solution of the Hamilton-Jacobi-Bellman equation, developed in Chapter 4, and the concept of stochastic thresholds developed in this chapter are applied to a stochastic model of optimal water pollution control.

5.1 The concept of regime switching thresholds

This section introduces stochastic bifurcations as bifurcations of a certain geometric invariant of the optimally controlled process - the transformation invariant function (see Wagenmakers *et al.* (2005)). Regime switching thresholds are introduced as its local minima.

5.1.1 Processes with constant diffusion

Consider first a stochastic process $x(t)$ with a constant diffusion satisfying the following stochastic differential equation

$$dx = f(x)dt + \sigma dw, \tag{5.1}$$

where w is a Wiener process and σ is constant. It is assumed that given an initial distribution φ_0 of $x(0)$, φ_t exists for all $t \geq 0$. It is well known that $\varphi_t(x)$ satisfies the Fokker-Plank equation[1]

$$\frac{\partial \varphi_t}{\partial t} = -\frac{\partial}{\partial x}\left[f\varphi_t\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2\varphi_t\right]. \tag{5.2}$$

[1]Also known as the Kolmogorov forward equation.

When a stationary solution $\varphi(x)$ of (5.2) exists, it is unique and given as the solution of

$$0 = -\frac{d}{dx}\left[f\varphi\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[\sigma^2\varphi\right]. \tag{5.3}$$

By integrating (5.3) once, the following is obtained

$$-f\varphi + \frac{1}{2}\frac{d}{dx}\left[\sigma^2\varphi\right] = C_1. \tag{5.4}$$

To obtain a probability density, C_1 has to equal 0. Then (5.4) reads as

$$\frac{\sigma^2}{2\varphi}\frac{d\varphi}{dx} = f. \tag{5.5}$$

From (5.5) the following is derived

$$\log\varphi(x) = \frac{2}{\sigma^2}\int_{x_0}^{x} f(\tau)\,d\tau + C_2, \tag{5.6}$$

where the constant C_2 is determined from the normalization condition $\int\varphi = 1$. Thus the probability density function $\varphi(x)$ of the stationary distribution of the process $x(t)$ is given by

$$\varphi(x) = \frac{\exp\left\{\frac{2}{\sigma^2}\int_{x_0}^{x} f(\tau)\,d\tau\right\}}{\int_X \exp\left\{\frac{2}{\sigma^2}\int_{x_0}^{\theta} f(\tau)\,d\tau\right\}\,d\theta}. \tag{5.7}$$

Equation (5.5) implies that zeros of the drift function f correspond to critical points of φ, i.e. modes and antimodes of the stationary distribution. The modes of φ indicate the most frequent states of the system. Such a state acts as an attractor: if the system slightly deviates from it, the drift f forces it to come back, as $f = 0$ and $f' < 0$ at such a point. On the contrary, a local minimum of φ, separating two modes, acts as a repeller, as $f = 0$ and $f' > 0$ at such a point. Therefore it seems reasonable to associate the critical points of φ with the stochastic equilibrium states, see Figure 1. However, critical points of the stationary density function φ are not invariant under transformations of the process $x(t)$, which will be shown in the next

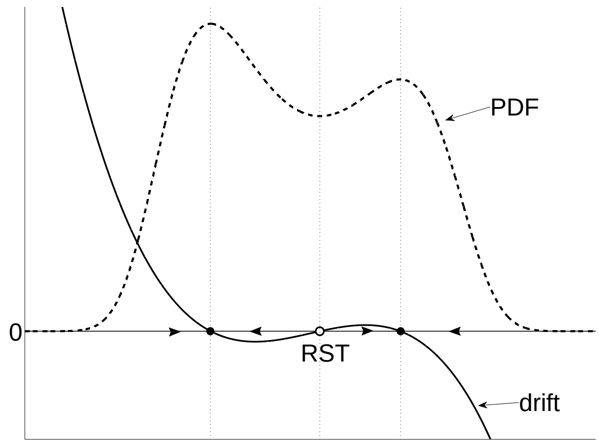


Figure 5.1: Correspondence between the deterministic drift function $f(x)$ and the probability density function $\varphi(x)$ for a process with a constant diffusion σ. Stochastic stable steady states correspond to modes of $\varphi(x)$, which are zeros of $f(x)$ where it changes its sign from positive to negative. Whereas stochastic regime switching thresholds (RST) correspond to antimodes of $\varphi(x)$, which are zeros of $f(x)$ where it changes its sign from negative to positive.

subsection.

5.1.2 Transformation invariant function

Now consider a stochastic process $y(t)$ with non-constant diffusion strength satisfying the following equation

$$dy = g(y)dt + \sqrt{2\varepsilon\sigma^2(y)}dw, \tag{5.8}$$

where the diffusion strength is not necessarily constant. Note that any one-dimensional process with non-constant diffusion strength can be transformed into a process with a constant diffusion by a suitable coordinate transformation. Let $z = r(y)$ be a transformation of the process $y(t)$. Then Itô's differential rule implies

$$\begin{aligned} dz &= r'(r^{-1}(z))dy + \varepsilon\sigma^2(r^{-1}(z))r''(r^{-1}(z))dt \\ &= \left(r'(r^{-1}(z))g(r^{-1}(z)) + \varepsilon\sigma^2(r^{-1}(z))r''(r^{-1}(z))\right)dt + \sqrt{2\varepsilon}r'(r^{-1}(z))\sigma(r^{-1}(z))dw \\ &= \tilde{g}(z)dt + \sqrt{2\varepsilon}\tilde{\sigma}(z)dw. \end{aligned} \tag{5.9}$$

Let $r(y)$ satisfy the following differential equation

$$r'(y) = \frac{A}{\sqrt{2\varepsilon}\sigma(y)}, \tag{5.10}$$

where $A > 0$ is a constant. Then from (5.9) it follows that the process $z(y)$ has a constant diffusion strength which is equal to A.

Let $\psi(y)$ and $\varphi(z)$ denote the stationary solutions of the Fokker-Plank equations corresponding to the process $y(t)$ and $z(t)$ respectively. The following then holds

$$\varphi(z) = \frac{\psi(r^{-1}(z))}{r'(r^{-1}(z))}. \tag{5.11}$$

From (5.11) it follows that the stationary density function ψ is not invariant under coordinate transformation. This implies that $\psi(y)$ and $\varphi(z)$ can have different number of critical points and therefore different number of stochastic stable steady states and regime switching thresholds. This means that the stationary density function is not a proper measure of the equilibrium points of a stochastic process.

The non-invariance under coordinate transformations of stationary density functions has been pointed out by Zeeman (1988). In Wagenmakers *et al.* (2005) the authors suggest to consider *the transformation invariant function* given by

$$I(y) = \sigma(y)\psi(y), \tag{5.12}$$

which is invariant under diffeomorphic transformations of coordinates. To see this, let us combine (5.9) and (5.11) and derive the following

$$\tilde{\sigma}(r(y))\varphi(r(y)) = r'(y)\sigma(y)\frac{\psi(y)}{r'(y)} = \sigma(y)\psi(y). \tag{5.13}$$

Note, that for processes with a constant diffusion the transformation invariant function is proportional to the stationary density function.

5.1.3 Stochastic bifurcations

The transformation invariant function I, introduced above, is a geometrical invariant of the process y. The number of critical points of I is therefore a property of the stochastic process y and not only of its representation in a given coordinate system. The following definitions can then be introduced:

Definition 5.1.1. *Let y be a stochastic process that satisfies (5.8), with transformation invariant function $I(y)$. A local maximum of I is called **a stochastic attractor** of y.*

As mentioned above, for systems with constant diffusion the function I is proportional to the stationary density function. Therefore, for such systems local maxima of I indicate most likely states of the process y.

Definition 5.1.2. *Let y be a stochastic process that satisfies (5.8) with the transformation invariant function $I(y)$. A local minimum of I is called **a regime switching threshold** of the process y. Any interval in the state space bounded by such thresholds is called a regime of the process $y(t)$.*

Each regime of y contains one attractor and is characterized by fluctuations of the process around it. A large shock may change the regime of the process.

Definition 5.1.3. *Let y be a stochastic process that satisfies (5.8) with the transformation invariant function $I(y)$. A change in the number of critical points of $I(y)$ indicates **a stochastic bifurcation** of the process y.*

5.2 The stochastic lake model

In this section the methods developed in Chapter 4 are applied to analyze a stochastic lake model. The deterministic lake model, introduced in Mäler *et al.* (2003), is a prototype for an optimal management problem with conflicting intertemporal interests and non-convex feedbacks.

It has been studied extensively: in Wagener (2003) and Kiseleva and Wagener (2010) a bifurcation analysis of the lake model is performed; Heijnen and Wagener (2008) extended the model by adding an industry whose activities increase the pollution; Kossioris *et al.* (2008) considered a differential lake game on pollution control; Heijdra and Heijnen (2009) studied economic and environmental effects of public abatement in this model; Salerno *et al.* (2007) considered a model of political economy with the underlying lake dynamics. Dechert and O'Donnell (2006) solved a stochastic lake game numerically; however, as mentioned in the introduction, the authors limited themselves to considering discrete bounded shocks.

In the lake model a social planner controls usage of fertilizers by farmers that pollute a lake close to the farmers' fields. The planner's problem is to maximize a social welfare functional which models the conflicting interests of farmers and tourists: farmers indirectly benefit from polluting the lake, and tourists suffer from the polluted lake. If the pollution level at time t is denoted as $x(t)$, and the loading of more pollution due to farming as $u(t)$ then *the stochastic lake problem* is given by

$$\max_u \mathbb{E}_{x_0} \int_0^\infty \left(\log u(t) - cx^2(t)\right) e^{-\rho t}\, dt \tag{5.14}$$

$$dx = \left(u - bx + \frac{x^2}{1+x^2}\right) dt + x\sqrt{2\varepsilon} dw(t), \quad x(0) = x_0. \tag{5.15}$$

The integrand in (5.14) models social utility: the term $\log u$ represents farmers' profits derived from intensity u of the use of fertilizers, whereas the term cx^2 represents tourists' disutility from the pollution in the lake, where the parameter $c > 0$ models the relative costs of pollution. The social planner maximizes the expected social welfare (5.14) subject to the pollution dynamics equation (5.15), where the parameter[2] $b > 0$ is the coefficient that is proportional to the rate of loss of pollution due to sedimentation and where the last term models the biological production

[2]It is assumed that uncertainty enters the lake model via the biological parameter b: rewriting equation (5.15) as

$$\frac{dx}{dt} = u - \left(b - \sqrt{2\varepsilon x^2}\frac{dW(t)}{dt}\right) x + \frac{x^2}{1+x^2}$$

results in the original lake dynamics equation where the parameter b is perturbed by a white noise term.

process.

Define the value function of the stochastic lake problem

$$V(x_0,\varepsilon) = \max_u \mathbb{E}_{x_0} \int_0^\infty \left(\log u(t) - cx^2(t)\right) e^{-\rho t}\, dt. \tag{5.16}$$

It is easy to check that the functions in the model satisfy the conditions of Theorem 4.1.1 if the control u takes values in a compact interval[3] $[m, M]$. Then Theorem 4.1.1 implies that the value function $V(x,\varepsilon)$ solves (4.8), which here takes the form

$$\varepsilon x^2 V_{xx} + \max_u \left[\log u - cx^2 + V_x \left(u - bx + \frac{x^2}{1+x^2}\right)\right] - \rho V = 0. \tag{5.17}$$

Solving the optimization problem in (5.17) the optimal policy is obtained

$$u^*(x,\varepsilon) = -\frac{1}{V_x(x,\varepsilon)}. \tag{5.18}$$

Then the value function $V(x,\varepsilon)$ of the stochastic lake problem solves

$$\varepsilon x^2 V_{xx}(x,\varepsilon) - \log(-V_x) - cx^2 - 1 + V_x \left(-bx + \frac{x^2}{1+x^2}\right) - \rho V(x,\varepsilon) = 0. \tag{5.19}$$

Now, the method developed in Chapter 4 to approximate the solution of (5.19) can be applied. For that the parameter values (b, c, ρ) have to be fixed. Let us first consider a case with a unique optimum of the corresponding deterministic lake problem: $b = 0.65,\ c = 0.7,\ \rho = 0.03$. The corresponding stochastic optimal policy and the long run distribution of the state are shown in Fig. 2. As the noise level ε in the model is small, the stochastic optimal policy is almost indistinguishable from the deterministic one. The long run distribution is unimodal.

To consider a case when the deterministic lake problem exhibits an indifference point, the parameters are set at $b = 0.65,\ c = 0.5,\ \rho = 0.03$. As shown in Fig. 5.4(a) the deterministic optimal policy is discontinuous in this case: a part of the optimal control is formed by a part of

[3] u is bounded from above by M, where $M >> 1$ is an upper limit of the amount fertilizers that land can bear. And u is bounded from below by m, where $0 < m << 1$, to assure sustainability of the agricultural sector.

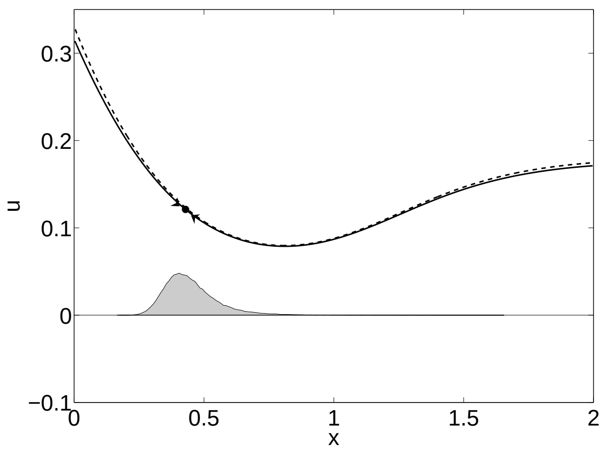


Figure 5.2: The graph of the optimal policy function for the stochastic lake problem (dotted) and the corresponding deterministic lake problem (solid). The corresponding long run distribution is shown at the bottom of the figure. It is obtained by running the process (5.15) with $u = u^*(x, \varepsilon)$ for 10^5 periods after 10^2 transient periods. The model parameters are $b = 0.65$, $c = 0.7\,\rho = 0.03$, $\varepsilon = 0.001$.

the stable manifold of one equilibrium, another part by the stable manifold of another equilibrium. The discontinuity is located at the deterministic indifference point, which separates the basins of attraction with respect to the deterministic optimal vector field f^o of the two equilibria. Starting at that point the controller is indifferent between steering the system to either of these equilibria.

Let us apply the method described in Section 4.3 to solve the corresponding stochastic lake problem. In Fig. 5.3 the deterministic and stochastic value functions are shown. At the indifference point the deterministic value function has a kink, whereas the stochastic value function is smooth and everywhere differentiable.

It is clear from Figure 5.3(a) that the value function of the stochastic problem is located above the value function of the deterministic problem. It suggests that stochasticity increases the expected social welfare, at least for small noise levels.

The corresponding optimal policy and the long run state distribution are shown in Fig. 5.4(a). The discontinuity of the optimal policy at the indifference point in the deterministic model is smoothed out by uncertainty in the stochastic model and the optimal policy $u^*(x, \varepsilon)$ is a continuous differentiable function, see Fig. 5.4(a).

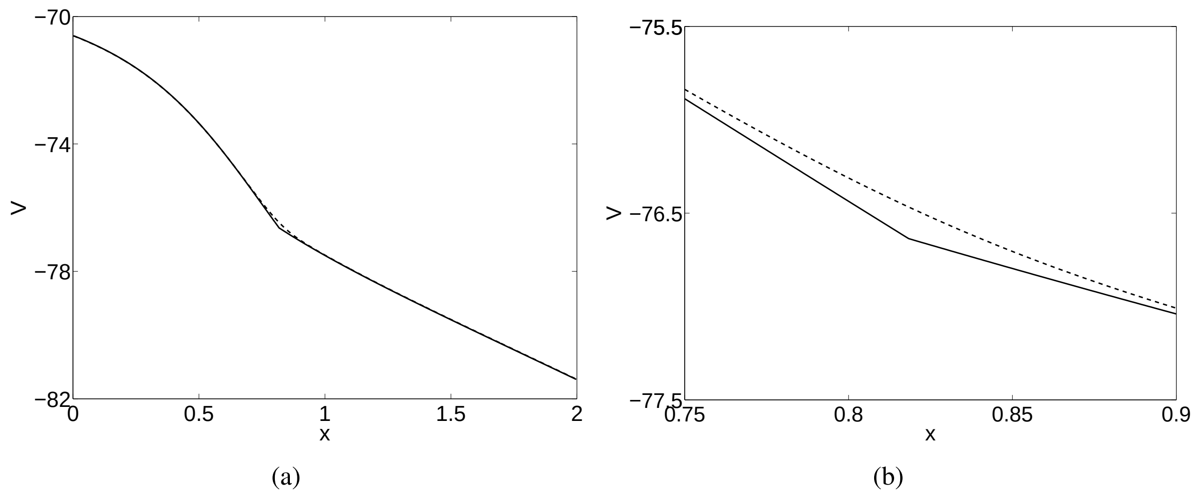


Figure 5.3: Figure 5.3(a) shows the deterministic (solid line) and the stochastic (dotted line) value functions for the stochastic lake model. The deterministic value function has a kink at the indifference point, whereas the stochastic value function is smooth. Figure 5.3(b) shows a blow up in a neighborhood of the deterministic indifference point. The model parameters $b = 0.65$, $c = 0.5\,\rho = 0.03$, $\varepsilon = 0.003$.

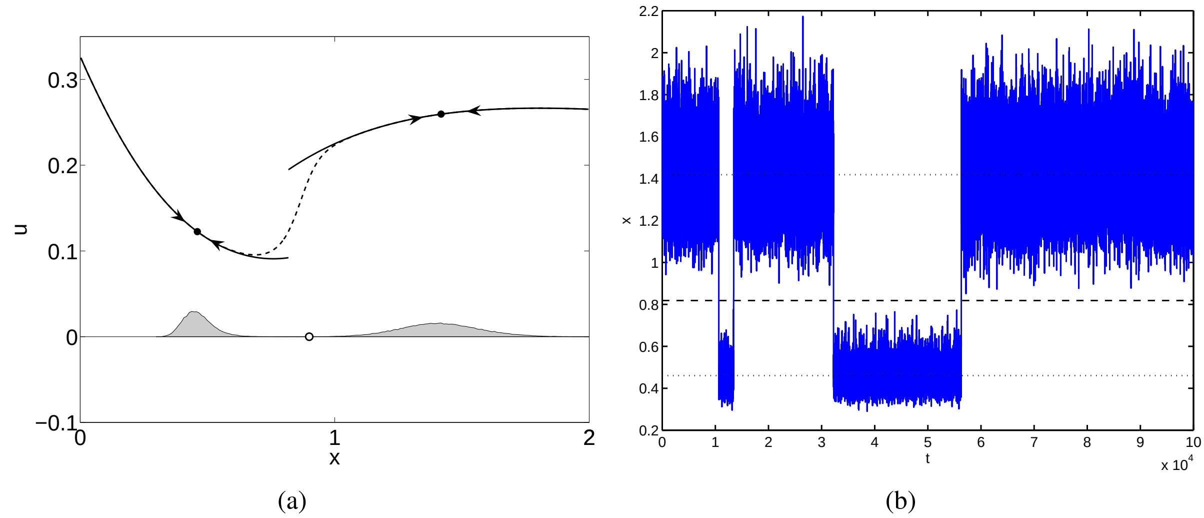


Figure 5.4: Figure 5.4(a) shows the graphs of the deterministic optimal policy (solid discontinuous line), the stochastic optimal policy (dotted line) and the long run distribution of the state. The stochastic regime switching threshold is marked as an empty circle. Figure 5.4(b) shows the corresponding times series. It is obtained by running the process (5.15) with $u = u^*(x, \varepsilon)$ for 10^4 periods after 10^2 transient periods. Figure 5.4(b) shows that the considered stochastic model exhibits regime switching phenomenon: the system fluctuates around one of the steady states until it is pushed away to another steady state due to a large shock. The basins of fluctuations are separated by the stochastic regime switching threshold. The model parameters are $b = 0.65$, $c = 0.5$, $\rho = 0.03$, $\varepsilon = 0.003$.

The corresponding long run distribution is bimodal. The stochastic regime switching point acts as a repeller separating two regimes: ‘clean lake’ and ‘turbid lake’, see Fig. 5.4(b).

5.2.1 Bifurcation analysis with respect to the noise level

Figure 5.5(a) shows a bifurcation diagram of the stochastic lake problem with respect to the noise level parameter ε for $b = 0.65,\ c = 0.49,\ \rho = 0.03$. For small noise levels the long run distribution of the state is bimodal with the stochastic regime switching point separating two modes. When the noise level increases the threshold collides with the highest mode and they both disappear via a saddle-node bifurcation. As the noise level increases further the long run distribution remains unimodal. The same scenario is illustrated in Figure 5.5(b) for $c = 0.52$. Larger costs of pollution yield more restrictive policies, and as a result uniqueness of the ‘clean lake’ regime for lower values of ε.

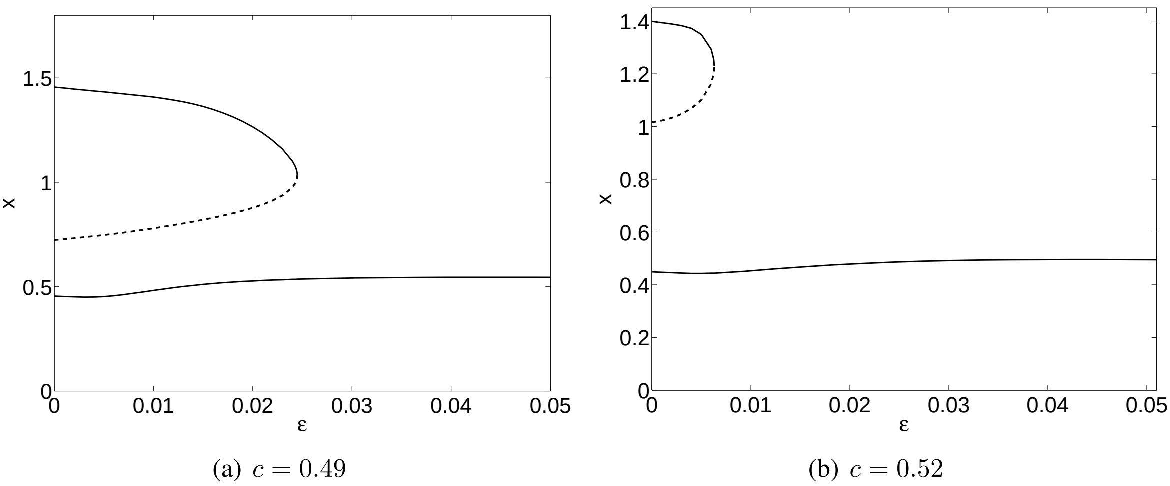


Figure 5.5: Bifurcation diagrams of the long run distribution in the stochastic lake model with respect to the noise level ε. Solid lines indicate modes of the transformation invariant function $\sigma(x)\varphi(x)$, the dashed lines indicates the antimode of $\sigma(x)\varphi(x)$ - the stochastic regime switching threshold. For small level of noise the long run distribution is bimodal. As the noise parameter ε increases the eutrophic steady state and the stochastic regime switching point merge in a saddle-node bifurcation and disappear, the long run distribution becomes unimodal. The model parameters are $b = 0.65,\ \rho = 0.03$.

Figure 5.6 shows average time spent in one regime for different values of the noise parameter ε. It exponentially declines with the noise level. Of course, as ε increases large shocks

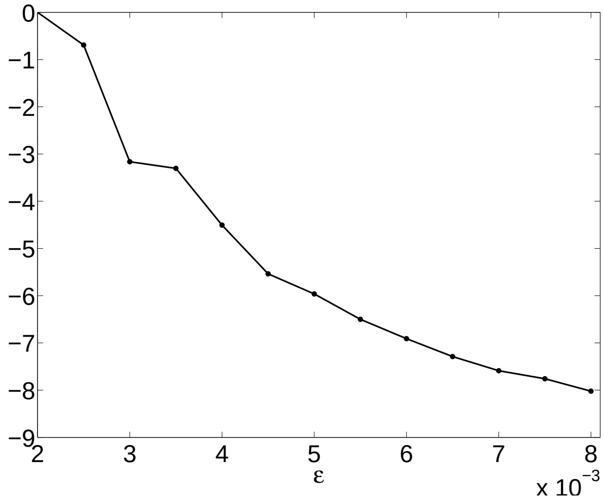


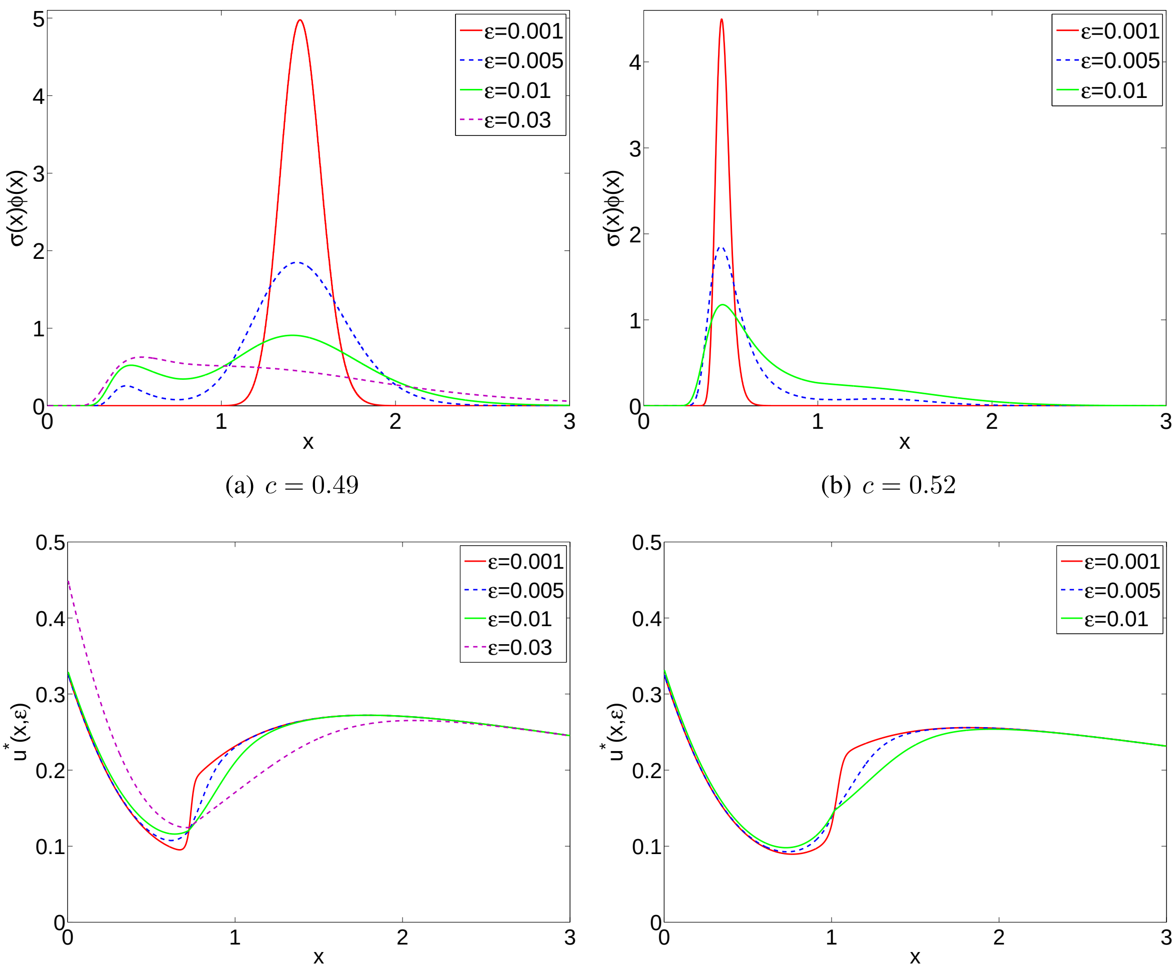
Figure 5.6: Average time (in log) spent by the lake system in a regime before switching to another regime for different noise levels. The model parameters are $b = 0.65$, $c = 0.5$, $\rho = 0.03$.

become more likely, pushing the state process out of one basin of fluctuation to the other.

Figure 5.7(a) shows a collection of transformation invariant functions for different values of ε and $c = 0.49$. For small ε the functions are bimodal (see Figure 5.5(a)), the most of the mass is concentrated around the high steady state, so the lake is polluted most of the time. As the noise in the system increases the mass is shifted to the regime of the 'clean lake'. For large noise the long run distribution is unimodal with a peak at the 'clean' equilibrium. In the face of uncertainty the planner acts to avoid serious or irreversible potential harm to the environment. Higher uncertainties in the system yield more restrictive pollution policy when the pollution level is high, see Figure 5.7(c)-5.7(d).

For $c = 0.52$, see Figure 5.7(b), the transformation invariant function is bimodal for small values of ε (see Figure 5.5(b)), and the mass is concentrated around the 'clean' steady state for all values of the noise parameter ε. It shows that high relative costs of pollution c force the decision maker to keep the lake in the 'clean' regime.

Figure 5.8 shows the critical values of c for different values of the noise parameter ε that correspond to the invariant distributions such that the mass concentrated around either of the regimes is equal to $1/2$. For values of c below that curve the lake is turbid most of the time,



(a) $c = 0.49$ (b) $c = 0.52$

(c) $c = 0.49$ (d) $c = 0.52$

Figure 5.7: Transformation invariant functions and optimal control policies for different values of ε and c. The model parameters are $b = 0.65,\ \rho = 0.03$.

and for values of c above it the lake is clean most of the time. As can be seen from Figure 5.8 (right), the larger the noise in the system the lower the critical value of c.

5.3 Concluding remarks

This chapter introduces a concept of a stochastic regime switching threshold. From the definition it follows that such a point separates two basins of fluctuations, or two regimes of the system. In one regime the system fluctuates around one of the steady states, until it is pushed away by a large shock to another basin, and the regime changes.

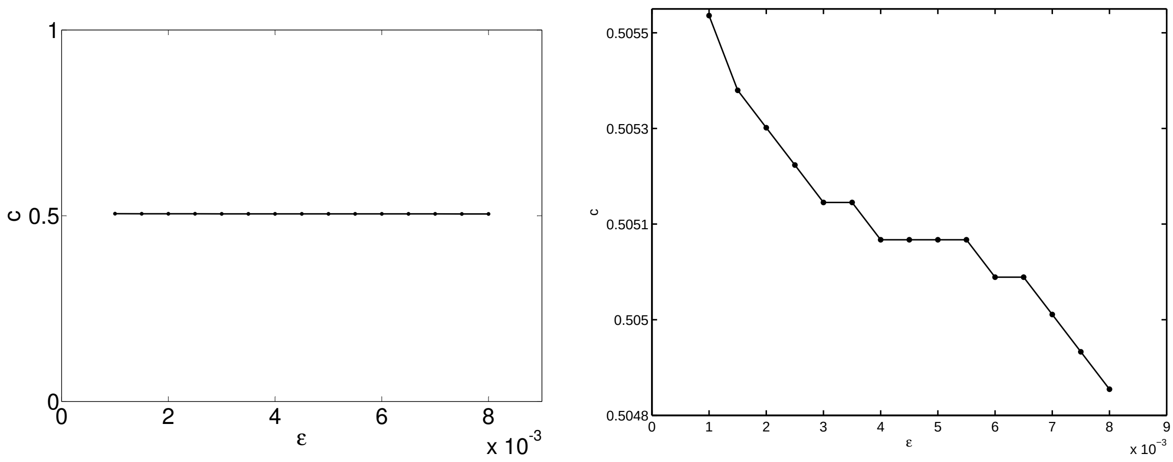


Figure 5.8: The bifurcation curve in $(\varepsilon, c)-$parameter space corresponding to the case when the mass concentrated around each of the regimes is equal to 1/2. For the parameter values below this curve the process $\{x_t\}$ spends more time in the 'turbid lake' regime, and vice versa. The model parameters are $b = 0.65,\ \rho = 0.03$.

This concept as well as the solution method developed in Chapter 4 are applied to perform a bifurcation analysis of the stochastic lake model. Approximations of the solution for different values of the parameters are obtained, including the case with the indifference point. The optimal policy function is computed, which is smooth for any values of the model parameters. Stochastic regime switching thresholds are computed for different noise levels, and it is shown that as the noise increases the threshold point collides with one of the steady states and disappears via a saddle-node bifurcation.