Behavioural models of technological change

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Chapter 2

Optimal diversity in investments with recombinant innovation

2.1 Introduction

When organizations decide on investment in technological innovation, they implicitly or explicitly make choices about diversity of options, strategies or technologies. Such choices should ideally consider the benefits and costs associated with diversity and arrive at an optimal trade-off. One important benefit of diversity relates to the nature of innovation, which often results from combining existing technologies or knowledge bases (Ethiraj and Levinthal, 2004). For instance, the laptop computer combines microelectronics, display technology and a battery; the windmill is a combination of water mill technology and the idea of a sail; the laser is quantum mechanics integrated into an optical device; and the optical fibre used in telecommunication is a laser applied to glass technology. Innovative combinations apply especially to technologies that are relatively close to each other in technology space, such as is common in the bio-pharmaceutical industry and the software industry. Indeed, many multi-product firms choose products in such a way that they can enjoy the spillover effects of learning and innovation.

This chapter is a version of Zeppini and van den Bergh (2008).

Here we propose a theoretical framework for the description of a generic innovative process resulting from the interaction of two existing technologies. The interaction will depend on how these two options match. The ultimate aim of the model is to assess the optimal diversity of technological investments in the context of modular innovation. The main idea is that, in an investment decision where available options may recombine and give birth to an innovative option (technology), some degree of diversity of parent options can lead to higher benefits than specialization. This problem is relevant to both private and public organizations. In addition, the recombinant view of technological innovation can help to explain the diversification pattern of firms and their size distribution, thus contributing to the debate initiated by Penrose (1959).

A motivation for our model is the recent attention for a socio-technological transition to large scale use of renewable energy (Geels, 2002; van den Bergh and Bruinsma, 2008). Diversity here is related to lock-in of an inferior or undesirable technology, such as fossil-fuel-based electricity generation that contributes considerably to global warming. A diversity analysis of energy systems provides insights into the appropriate level of diversity that should be aimed for or maintained in different phases of an energy transition (van den Heuvel and van den Bergh, 2008).

One might distinguish between a bipolar model of recombinant innovation where the two elements being combined are somehow balanced in terms of complexity or importance (e.g., electric and combustion engines in a hybrid car) and a case where an existing complex technology is improved by adding a small, less important or less complex element (e.g. installing a navigation system into a car). The latter is perhaps often seen as an ordinary, gradual innovation, whereas the first case, which we address in the present article, is more associated with major or even radical innovations. Even the restricted set of bipolar recombinant innovations is quite large: the jet engine resulted from combining internal combustion and the turbine concept. In the power generation sector, there are systems that combine different ways of energy transformation, such as photovoltaic collectors using the heat radiation produced by combustion in a gas turbine. Other examples of recombinant innovation are electronic devices like smartphones and ebooks, which integrate pre-existing technologies in a modular way.

Usually in economics and finance, diversity is seen as conflicting with efficiency of specialization. Such efficiency is claimed on the basis of increasing returns to scale arising from fixed costs, learning, network and information externalities, technological complementarities and other self-reinforcement effects. Arthur (1989) studies the dynamics of competing technologies in cases where increasing returns cause path dependence and self-reinforcement, possibly leading to lock-in. This can be seen as a descriptive or positive approach to understanding the dynamics of systems in the presence of positive feedback. Our approach instead is normative, in that it studies the efficiency of the system of different options, considering total net benefits of technologies over time, including innovation-related and scale-related effects of diversity.

A theoretical framework for the study of optimal diversity was proposed by Weitzman (1992) in the context of investment projects for biodiversity protection. The positive role of diversity is recognized in option value and real option theories, which clarify when to keep different options open in the face of irreversible change and uncertain circumstances (Arrow and Fisher, 1974; Dixit and Pindyck, 1994). However, these theories treat diversity as exogenous and do not consider innovation, whereas our model treats diversity as endogenous and contributing to the value of the overall system beyond merely keeping decisions open. Similarly, portfolio theory (Markowitz, 1952; Sharpe, 1964), another classical approach to investment decisions, excludes the possibility of innovation. Moreover, returns to scale are not part of this theory, so that diversification is the usual optimal choice. In the case of technological investments, however, the opposition between returns to scale and recombinant innovation may result in a wider range of optimal solutions depending on the relative strength of each effect, as we will show.

The relevance of our analysis relates to the myopia of economic agents and organizations. Chiu et al. (2008) study empirically the conditions for a positive link between technological diversification and firms' performance. In real-world decision making, short-term interests often prevail, possibly since the advantages of increasing returns are perceived as more clear and certain than the advantages of diversity and recombinant innovation. Fleming (2001) argues that one reason for uncertainty in recombinant innovation is that inventors experiment with unfamiliar technologies and unexploited combinations of technologies. The trade-off between short-term efficiency and long-term benefits from diversity resembles the exploitation versus exploration problem (March, 1991). In fact, recombinant innovation can be regarded as a form of exploration and search. At first sight, diversity benefits as proposed here seem to resemble economics of scope. However, the first notion relates to recombinant innovation, while the second is about synergies in production mainly due to bundled marketing and logistics. Whereas economies of scope are static, diversity benefits are dynamic in nature.

A model of diversity connects not only with the research on modularity but also with the approach of evolutionary economics, as expressed by Nelson and Winter (1982), Dosi et al. (1988), Andersen (1994), Frenken et al. (1999) and Potts (2000), among others. The idea of innovation as recombination dates back to Schumpeter (1934). However, evolutionary economics tends to avoid the notions of optimality and efficiency in terms of maximizing a value function. Our approach can, in fact, be seen as combining diversity-innovation ideas from evolutionary economics with optimality and cost-benefit analysis of neoclassical economics. In an evolutionary approach, one talks of a population of parent options and an offspring to refer to the innovative option. Here we will deal with the smallest population possible: namely, only two parent options, so as to keep the model simple and allow for analytical solutions.

We propose a theoretical model of recombinant innovation with two parent technologies and address the decision problem of optimal diversification of the associated $R\&D$ investment portfolio. The conditions under which diversification or specialization is optimal are studied. The main factors of influence on the optimal allocation of investment are the time horizon and the returns to scale. The model builds upon and generalizes the model by van den Bergh (2008) but differs from it in a number of ways. First, whereas the earlier study was based on numerical analysis, here we derive analytical results, both for model dynamics and optimal investment solutions. Second, in contrast to the earlier study, this analysis addresses heterogeneous returns to scale, as well as non-zero and heterogeneous initial values of parent technologies. All this makes it possible to study asymmetry effects in the investment decision. Third, we consider the effect of the (cumulative) size of parent technologies on the probability of recombinant innovation.

This chapter is organized as follows. Section 2.2 presents the recombinant innovation model, and provides a solution to the dynamics of the recombinant investment. Section 2.3 addresses the problem of optimal diversity in different cases of growing complexity. Section 2.4 concludes and provides suggestions for further research.

2.2 The model

2.2.1 General framework

Consider a system of two investment options that can be combined to give rise to a third. Think of an automotive corporation that is considering the possible benefits of developing a hybrid car. Let I denote investment in the parent options, which in this example are the internal combustion and electrical engines. Investment I_3 is devoted to the third (innovative) option, that is, the development of the hybrid car. The latter is the investment in recombinant innovation, which occurs with probability P_e. The growth rates of parent options are proportional to investments, with shares α and $1-\alpha$. Let O_1 and O_2 represent the values of the cumulative investment in parent options, and O_3 the expected value of the innovative option. Recombinant innovation is a binary event: a new option emerges with probability P_e, and nothing happens with probability $1-P_e$.

Hence, the expected value is simply P_e times the capital invested in the new option. The dynamics of the system can then be described by the set of differential equations:

$$\begin{aligned} \dot{O}_1 &= I_1 = \alpha I, \\ \dot{O}_2 &= I_2 = (1-\alpha)I, \\ \dot{O}_3 &= P_e(O_1, O_2)I_3. \end{aligned} \tag{2.1}$$

The optimization problem that we address is finding an α that maximizes the final total benefits from parent and innovative options. In the hybrid car example, this means to maximize the net benefits from the development of the internal combustion engine, the electrical engine, and their integration.

We assume for parent options a constant allocation of capital I over time $\frac{I_1}{I_2} = \frac{\alpha}{1-\alpha}$, which results in a constant linear growth (accumulation) of O_1 and O_2. The time pattern of the innovative option is non-linear:

$$\begin{aligned} O_1(t) &= O_{10} + I_1 t, \\ O_2(t) &= O_{20} + I_2 t, \\ O_3(t) &= I_3 \int_0^t P_e(\tau) d\tau. \end{aligned} \tag{2.2}$$

We define the probability of emergence of an innovative option P_e as depending positively on the balance $B(O_1, O_2)$ of parent options.[1] Moreover, we assume a positive dependence (with diminishing marginal effect) on the total size of parent options:

$$P_e(O_1, O_2) = eB(O_1, O_2)S(O_1, O_2). \tag{2.3}$$

The size effect is captured by the factor $S(O_1, O_2)$, and will be addressed in some length

[1] An alternative interpretation of P_e is to think of it not as a probability but simply as a matching factor for a recombinant invention that has already occurred. Consequently, O_3 would not be an expected value, while P_e, not being a probability, would not be bounded above and could be larger than 1.

in Section 2.2.3. The coefficient $e \in [0,1]$ can be interpreted as the effectiveness of recombinant $R\&D$, which may change due to learning. In general, as clarified by Stirling (2007), e depends on two other dimensions of diversity: namely, variety (the number of parent options) and disparity (how far apart the options are in the technology space).

Balance expresses how (un)equal the distribution of different options is in a population: the more balanced a system is, the more diversified it is. The idea is that a more balanced investment has a larger probability of recombinant innovation.[2] When one option is zero, we have pure specialization. The balance function must have the following properties: $B(O_1, O_2) \in [0,1]$, $B(O_1 = O_2) = 1$ (maximum diversity or perfect balance) and $\lim_{O_i \to 0} B(O_i, O_j)|_{O_j = const} = 0$ with $i, j = 1, 2$ and $i \neq j$.

The optimization problem of the investment decision is addressed by considering the joint benefits of parents and innovative options. In order to model the trade-off between diversity and scale advantages of specialization, we introduce a returns to scale parameter s_i for each technology $i = 1, 2, 3$. This acts on the cumulative investment in each option, capturing not only economies of scale but also learning over time. For instance, there is a s_1 for the investment in internal combustion engine, a s_2 for the electrical engine, and a s_3 for the hybrid car. The overall benefits from investment can be expressed as:

$$V(\alpha; t) = O_1(\alpha; t)^{s_1} + O_2(\alpha; t)^{s_2} + O_3(\alpha; t)^{s_3}, \tag{2.4}$$

where t is the time horizon of the investment. To find the optimal α, an explicit solution to $O_3(\alpha; t)$ is required, i.e. we need to compute the integral in the third equation of (2.2).

[2]This idea is consistent with both codified and tacit knowledge. In the first case, recombination will most likely occur through engineers that are specialised in different technologies exchanging or combining tacit knowledge about these. More balance will then mean more engineers in either technological area and therefore more opportunities to cooperate or exchange information. In the case of codified knowledge, a single individual will be able to combine knowledge about separate technologies. More balance may then go along with better accessibility and quality of codified information in either technological area, which in turn will enhance opportunities for successful recombination by a single researcher. Of course, codified knowledge is flexible in that it also allows recombinant innovation to follow the route of cooperation among individuals with different technological expertise (see van den Bergh (2008)).

2.2.2 The effect of balance

A balance function is defined in the positive octant of an n-dimensional space. A functional specification of the balance of two options x and y should have the following properties:

1. it is symmetric in its arguments $B(x, y) = B(y, x)$;

2. the maximum value is attained on the diagonal $B(x, x) \geq B(x, y)\ \forall x, y \geq 0$;

3. the minimum value (lowest balance) is attained when one of the two options is zero: $B(x, 0) = B(0, x) = 0 < B(x, y)\ \forall y > 0$;

4. it is homogeneous of degree zero: $B(\lambda x, \lambda y) = B(x, y)$.

The latter means that the balance of two quantities can be expressed as a function of their ratio $b = O_1/O_2$ (simply put $\lambda = 1/x$). The functional specification of the balance that we adopt is the "Gini" measure (Fig. 2.1):

$$B(O_1, O_2) = 1 - \frac{(O_1 - O_2)^2}{(O_1 + O_2)^2} = 4\frac{O_1 O_2}{(O_1 + O_2)^2}. \tag{2.5}$$

This specification is a rather obvious way of expressing the symmetry of a system, and it is a standard measure of concentration in industrial organization studies.[3] Expressed as a function of the ratio, the above specification reads $B(b) = 4\frac{b}{(1+b)^2}$.

Suppose that the total size of the population of parent options has a negligible effect on the probability of emergence, and set the size factor to the value $S(O_1, O_2) = 1$ in Eq. (2.3), so that the probability of emergence only depends on the balance $B(O_1, O_2)$

[3]Notice the differentiability in $O_1 = O_2$. Other specifications are possible, for instance $B(O_1, O_2) = 1 - \frac{|O_1 - O_2|}{O_1 + O_2}$ and $B(O_1, O_2) = \frac{min\{O_1, O_2\}}{max\{O_1, O_2\}}$ (see also Stirling (2007)). A detailed analysis of the latter specification is available upon request. The case $O_1 = O_2 = 0$ is excluded by all these specifications. This is a rather degenerate and irrelevant case, however, as we are only interested in systems with at least one option ($\exists\ i = 1, 2 \mid O_i > 0$). Otherwise, we can always define $B(0, 0) = lim_{O_1, O_2 \to 0} B(O_1, O_2) = 1$.

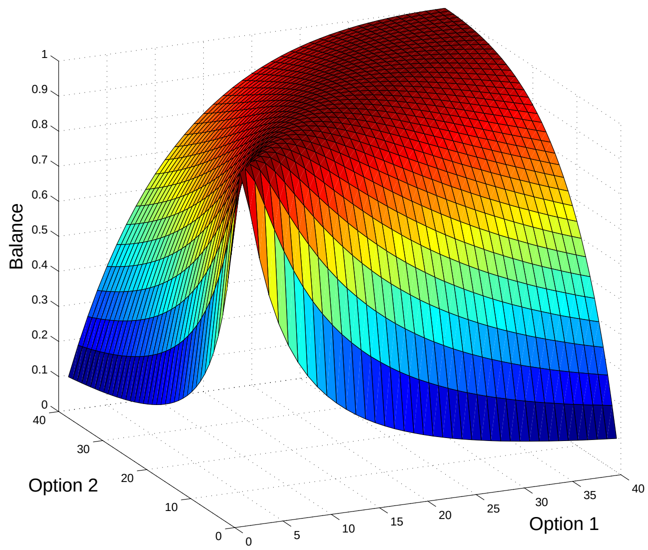

Figure 2.1: Graph of the diversity function with two parent options.

through the proportionality factor e. The value of the innovative option at time t is then:

$$O_3(t) = 4I_3 \int_0^t P_e(\tau)d\tau = eI_3 \int_0^t \frac{O_1(\tau)O_2(\tau)}{\big(O_1(\tau) + O_2(\tau)\big)^2} d\tau. \tag{2.6}$$

If the initial value of parent options is zero ($O_{10} = O_{20} = 0$), the balance is constant and equal to $4\alpha(1-\alpha)$. In this case, the innovative option grows linearly in time.

If we allow for positive initial values O_{10}, O_{20}, we obtain the following function of time:

$$B(t) = 4\frac{(O_{10} + \alpha It)(O_{20} + (1-\alpha)It)}{(O_0 + It)^2}, \tag{2.7}$$

where $O_0 = O_{10} + O_{20}$ is the total initial size. Notice that $\lim_{t\to\infty} B(t) = 4\alpha(1-\alpha)$, and $B \simeq 4\alpha(1-\alpha)$ as soon as $t \gg O_{i0}/(\alpha I)$, $i = 1, 2$. We can then state the following:

Proposition 2.2.1. *In the long-run the balance converges to the constant value* $B(\alpha) = 4\alpha(1-\alpha)$, *which is independent of the initial values of the parent options.*

The dynamics of the balance in the transitory phase ($t \sim O_{i0}/(\alpha I)$) depends on initial conditions and on the investment share α, and can be understood easily by looking at options trajectories in (O_1, O_2) space. From the first two equations of (2.2) we have:

$$O_2 = O_{20} - \frac{1-\alpha}{\alpha} O_{10} + \frac{1-\alpha}{\alpha} O_1.$$

The starting point ($t = 0$) of each trajectory is determined by the initial values (O_{10}, O_{20}). The slope is the ratio of investment shares. For our recombinant innovation system we identified seven major cases, which are reported in Fig. 2.2. This figure must be read as follows: the more a trajectory gets close to the line $O_1 = O_2$, the more balanced is the investment, and the larger the probability of recombinant innovation (for a detailed analysis of each of these cases, see Zeppini and van den Bergh (2008)). In principle,

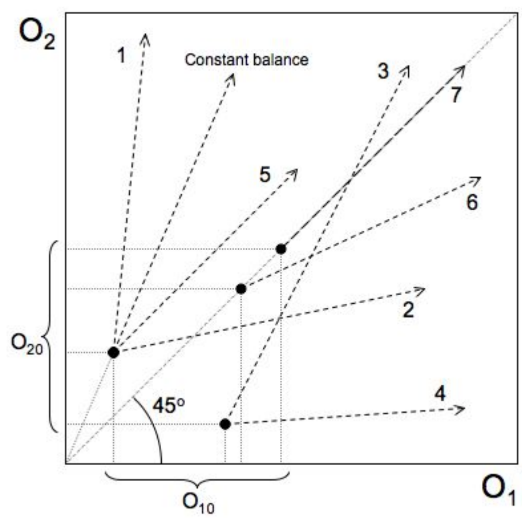

Figure 2.2: Trajectories of the two parent options in (O_1, O_2) space. Trajectory "1" has $O_{10} < O_{20}$ and $\alpha < 1/2$; trajectory "2" has $O_{10} < O_{20}$ and $\alpha > 1/2$; trajectory "3" has $O_{10} > O_{20}$ and $\alpha < 1/2$; trajectory "4" has $O_{10} > O_{20}$ and $\alpha > 1/2$; trajectory "5" has $O_{10} \neq O_{20}$ and $\alpha = 1/2$; trajectory "6" has $O_{10} = O_{20}$ and $\alpha < 1/2$; trajectory "7" has $O_{10} = O_{20}$ and $\alpha = 1/2$. For "Constant balance" the slope is equal to the ratio O_{20}/O_{10}.

the optimal condition for recombinant innovation is when the balance is constant and maximal (Case 7). In general, for constant balance the following condition applies:

Proposition 2.2.2. *The balance is constant and equal to $B(\alpha) = 4\alpha(1-\alpha)$ iff*

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1-\alpha}. \tag{2.8}$$

For a proof of this proposition see Appendix 2.A. This configuration falls into Cases 1, 4 and 7 of Fig. 2.2. As a function of time, the balance may have a critical point t^* where it reaches its maximum value.[4] Fig. 2.3 shows two examples of monotonic and non-monotonic dynamics. Here we have set $I = 4$, with initial values $O_{10} = 1$ and $O_{20} = 2$.

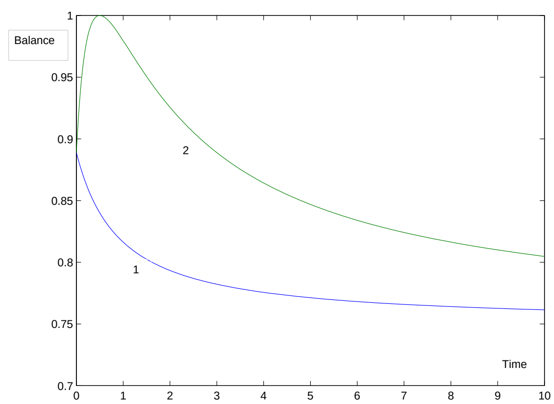

Figure 2.3: Two examples for the balance as a time function ($I = 4$, $O_{10} = 1$, $O_{20} = 2$). Case 1: $\alpha = 1/4$. Case 2: $\alpha = 3/4$.

In Example 2 we have $\alpha/(1-\alpha) = 3$: there is a time $t^* = 1/2$ when the balance is equal to 1 (a perfectly similar pattern would obtain in Case 3). In Example 1 the balance is decreasing, with $\alpha/(1-\alpha) = 1/4$. In general, $B(t)$ is decreasing when $\frac{\alpha}{1-\alpha} < \frac{O_{10}}{O_{20}} < 1$, and increasing when $\frac{\alpha}{1-\alpha} > \frac{O_{10}}{O_{20}} > 1$, while a non-monotonic behaviour is obtained for $\frac{\alpha}{1-\alpha} < 1 < \frac{O_{10}}{O_{20}}$ or $\frac{O_{10}}{O_{20}} < 1 < \frac{\alpha}{1-\alpha}$.

We now proceed to the integration of balance, giving the value of the innovative option at time t. We assume $e = 1$. Eq. (2.6) becomes:

$$O_3(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha I\tau)(O_{20} + (1-\alpha)I\tau)}{(O_0 + I\tau)^2} d\tau. \tag{2.9}$$

[4]The critical time value is $t^* = (O_{20} - O_{10})/(2\alpha - 1)I$.

The detailed solution of this integral is in Appendix 2.B. The final result is the following:

$$O_3(t) = \frac{4I_3}{I}\left[\alpha(1-\alpha)It + \big(O_{10}-\alpha O_0\big)^2\left(\frac{1}{O_0+It} - \frac{1}{O_0}\right) + \big(O_{10}-\alpha O_0\big)(1-2\alpha)\ln\frac{O_0+It}{O_0}\right]. \tag{2.10}$$

If condition (2.8) holds, $O_{10} = \alpha O_0$ and the expression of the innovative option reduces to $O_3(t) = 4I_3\alpha(1-\alpha)t$, which is the same as with zero initial values. This linear expression of $O_3(t)$ is also valid in the early stages of innovation: namely, when $It \ll O_0$. In the long run, however, the logarithmic term can not be neglected and the value of innovation is approximately given by:

$$O_3(t) \simeq 4\frac{I_3}{I}\left[\big(O_{10}-\alpha O_0\big)(1-2\alpha)\ln\frac{It}{O_0} + \alpha(1-\alpha)It\right]. \tag{2.11}$$

The coefficient of the logarithmic term determines whether the time pattern of the innovative option is concave (positive sign) or convex (when the sign is negative). This feature has economic relevance, in that it reflects the marginal effect of extending the time horizon of the investment. A concave pattern results when $\alpha < 1/2$ and $\alpha < O_{10}/O_0$ or $\alpha > 1/2$ and $\alpha > O_{10}/O_0$. These are exactly the conditions of Cases 3 ($\alpha < 1/2$ and $O_{10} > O_{20}$) and 2 ($\alpha > 1/2$ and $O_{10} < O_{20}$) of Fig. 2.2, when the balance has a critical point t^*. The convex time pattern occurs when the balance does not have a critical point. For example, take $O_0 = 3$, $O_{10} = 1$, $O_{20} = 2$, $\alpha = 2/3$. Since $O_{10}/O_{20} = 1/2 < \alpha/(1-\alpha) = 2$, we have that option 3 follows a concave time pattern, $O_3(t) = \frac{4}{3}\left[2t + \ln(1+t) - \frac{t}{1+t}\right]$.

2.2.3 Introducing a size effect

The size factor $S(O_1, O_2)$ in expression (2.3) is meant to capture the positive effect that a larger cumulative size has on the probability of emergence, i.e. a kind of economies of scale effect in the innovation process. Such a factor is designed to have the following properties: first, it is increasing in the size of each parent option with marginally diminishing effects. Second, it must be bounded, to guarantee that the probability P_e is in the interval $[0, 1]$. In addition, it should not overlap with the balance factor, which means that only the total sum of the sizes of options matters and not their distribution. These properties capture increased learning subject ultimately to saturation. We adopt a Weibull cumulative distribution specification:

$$S(O_1, O_2) = 1 - exp[-\sigma(O_1 + O_2)]. \tag{2.12}$$

Here $\partial S/\partial O_i = \partial S/\partial O = \sigma/exp(\sigma O)$, with $O = \sum_i O_i$. The parameter σ captures the sensitivity of P_e to the size when the balance is kept constant.[5] After including the size factor, the probability of emergence (2.3) is expressed as follows:

$$P_e(O_1, O_2) = 4e\frac{O_1 O_2}{(O_1 + O_2)^2}\{1 - exp[-\sigma(O_1 + O_2)]\}. \tag{2.13}$$

This expression depends on the sum and the difference of O_1 and O_2 (see Eq. 2.5). As a function of time, the probability of emergence reads

$$P_e(t) = 4e\frac{(O_{10} + \alpha It)(O_{20} + (1 - \alpha)It)}{(O_0 + It)^2}\{1 - exp[-\sigma(O_0 + It)]\}. \tag{2.14}$$

We might think of the event of innovation as occurring suddenly at a time t_E, and write $P_e(t) = Prob(t_E < t)$. Note how the effect of size on P_e does not depend on whether it comes from "old" value O_0 or from "new" investment It. This is not true for the balance.[6]

The size factor $S(t)$ describes a saturation effect of the probability of emergence P_e. After a sufficiently long time ($It \gg O_0$), the effect of cumulative size on P_e vanishes, since $lim_{t\to\infty} S(t) = 1$ and $lim_{t\to\infty} P_e(t) = 4\alpha(1 - \alpha)$, from Eq. (2.7). In cases other than the symmetric one ($\alpha = 1/2$), the balance is suboptimal ($B < 1$), and $P_e(t) < 1$ $\forall t$. This is summarized in the following proposition:

[5] One could allow for heterogeneous effects with the specification $1 - exp(-\sigma_1 O_1 - \sigma_2 O_2)$. This can address two different technologies operating in different sectors with different sensitivities σ_1 and σ_2.

[6] Formally, $S(t)$ is invariant to a time shift $t \to t^*$, such that $O_0 + It = O_0^* + It^*$, while $B(t)$ is not.

Proposition 2.2.3. *When a marginal diminishing size effect is introduced in the probability of emergence, innovation occurs almost surely iff the balance is constant, and the investment is maximally diversified ($\alpha = 1/2,\ B = 1$).*

We now integrate the third equation of the model (2.1) with a full specification of the probability of emergence, taking into account the balance and the size effect together. Before doing this, it is useful to write down the general expression of the probability of emergence as a function of time:

$$P_e(t) = 4\frac{(O_{10} + \alpha It)(O_{20} + (1-\alpha)It)}{(O_0 + It)^2}\big\{1 - exp[-\sigma(O_0 + It)]\big\}. \tag{2.15}$$

We will now proceed in steps in order to better understand the effect of size in the model. First assume the balance is constant, i.e. condition (2.8) holds and $B = 4\alpha(1-\alpha)$. The probability P_e then becomes $P_e(t) = eB\big\{1 - exp[-\sigma(O_0 + It)]\big\}$, and we obtain the following time pattern for the innovative option value:

$$O_3(t) = eI_3B\bigg\{t + \frac{exp(-\sigma O_0)}{\sigma I}\big[exp(-\sigma It) - 1\big]\bigg\}. \tag{2.16}$$

The first term of this expression is what we have without the size factor. The second term comes from the size effect. Here $\dot{O}_3(t) > 0$ and $\ddot{O}_3(t) > 0\ \forall t \geq 0$.[7] This means the innovative option has a convex time pattern. Such a behaviour accounts for a transitory phase in which the innovation 'warms up' before becoming effective. This is a stylised fact of innovation processes (Fig. 2.4).

The time pattern of $O_3(t)$ tends to the asymptote $eI_3B\big[t - exp(-\sigma O_0)/\sigma I\big]$: after a sufficiently long time, the innovative option attains linear growth. An indication of the characteristic time interval of the transitory phase is given by the intercept $\hat{t} = \frac{exp(-\sigma O_0)}{\sigma I}$. Interestingly, this characteristic time depends neither on the recombinant innovation ef-

[7]The first derivative is $\dot{O}_3(t) = I_3P_e(t)$, the second derivative is $\ddot{O}_3(t) = I_3eB\sigma Iexp[-\sigma(O_0 + It)]$.

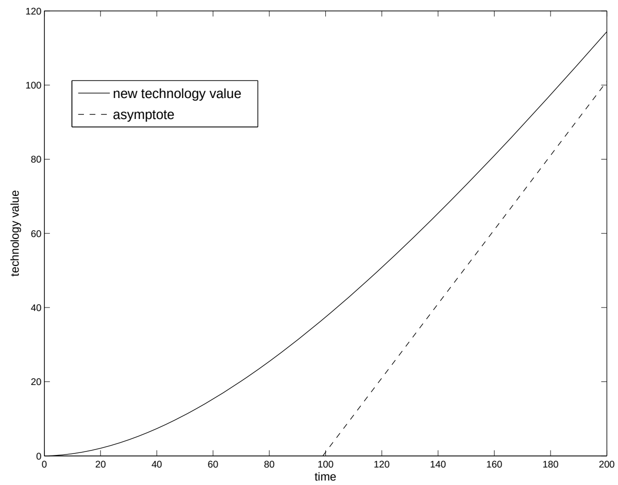

Figure 2.4: Value of the innovative option as a function of t, for the case of constant balance. Here $\alpha = 1/2$, $e = 1$, $I_3 = 1$, $I = 4$, $\sigma = 1/400$, and $O_{10} = O_{20} = 2$. Then $O_3(t) = t + 100e^{-0.01t}(e^{-0.01t} - 1)$ and the asymptote is $t - 100e^{-0.01}$.

fectiveness e nor on the investment I_3. The higher the sensitivity σ or the initial value O_0 or the investment rate I, the shorter the transitory phase and the faster the innovative option gets to linear growth.

Relaxing the assumption of constant balance, we have to solve the following integral:

$$O_3^\sigma(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha I\tau)(O_{20} + (1-\alpha)I\tau)}{(O_0 + I\tau)^2} \big\{1 - exp[-\sigma(O_0 + I\tau)]\big\} d\tau.$$

We call this solution $O_3^\sigma(t)$ to differentiate it from the solution without size effect. Appendix 2.B contains the detailed derivation. The result is:

$$\begin{aligned} O_3^\sigma(t) \quad &= \quad BI_3t + B\frac{exp(-\sigma O_0)}{\sigma I}\big[exp(-\sigma It) - 1\big] - \frac{4I_3}{I}\sigma E^2 \ln\frac{O_0 + It}{O_0} + \\ &- \quad \frac{4I_3}{I}E^2\Big[\frac{1}{O_0}\big[1 - exp(-\sigma O_0)\big] - \frac{1}{O_0 + It}\big[1 - exp[-\sigma(O_0 + It)]\big]\Big] + \\ &- \quad \frac{4I_3}{I}E\Big[\sigma + E(H - F)\Big]\Big[\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0 + It)\big)^k}{k \cdot k!} - \sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k \cdot k!}\Big], \end{aligned} \tag{2.17}$$

where $B = 4\alpha(1-\alpha)$ is the value of the balance when it does not depend on time;

$E = O_{10}(1-\alpha) - \alpha O_{20}$; $F = \alpha$; $G = -E$; and $H = (1-\alpha)$. When the balance is constant, we have $O_{10}(1-\alpha) = O_{20}\alpha$, and the expression of $O_3(t)$ only contains the first two terms since $E = G = 0$. When the balance is not constant, the time pattern of the third option contains a logarithmic term, a negative exponential divided by a linear function, and two infinite sums, one constant and the other dependent on time. As argued in Appendix 2.B, the two sums converge to negative exponentials. This means that the infinite sum which depends on time goes to zero for $It \gg O_0$. In the long run, the time pattern of O_3^{σ} is given by the following expression:

$$O_3^{\sigma}(t) \simeq 4\alpha(1-\alpha)I_3 t - 4\frac{I_3}{I}\sigma[O_{10}(1-\alpha) - O_{20}\alpha]^2 \ln\frac{It}{O_0}. \tag{2.18}$$

Without the size effect, we have (see equation (2.11)):

$$O_3(t) \simeq 4\alpha(1-\alpha)I_3 t + 4\frac{I_3}{I}[O_{10}(1-\alpha) - O_{20}\alpha](1-2\alpha) \ln\frac{It}{O_0}.$$

When a size factor is present, the logarithmic term adds negatively to the value of the innovative option, producing the expected convex time pattern which reveals the diminishing marginal contribution of parent technologies. Without the size effect, the logarithmic term can be either positive or negative. This shows how a marginally diminishing size effect is important in reproducing the typical threshold effect of recombinant innovations. The contribution of the logarithmic term depends to a great extent on the value of the sensitivity σ, which should be assessed empirically for each context.

2.3 Optimization of diversity

2.3.1 A simple case

We now address the problem of optimal diversity $\max_{\alpha\in[0,1]} V(\alpha;t)$, where the objective function is given by (2.4). In general the solution depends on the time horizon. Here we consider different cases, starting from the simplest one, where parent options have zero initial value, there is no size effect, and returns to scale are the same for the three technologies. Later we relax these assumptions.

Assume zero initial value for parent options, then $O_1(t) = \alpha I t$, and $O_2(t) = (1-\alpha)It$ and the balance is constant. Assume, moreover, no size effect ($S = 1$). Also the innovative option grows linearly with time:

$$O_3(t) = 4eI_3\alpha(1-\alpha)t. \tag{2.19}$$

Assume, finally, that returns to scale are the same for all three technological options, $s_1 = s_2 = s_3 \equiv s$. The maximization problem of optimal diversity then becomes:

$$\max_{\alpha\in[0,1]} V(\alpha;t) = t^s I^s \big[\alpha^s + (1-\alpha)^s + C^s\alpha^s(1-\alpha)^s\big], \tag{2.20}$$

where $C = \frac{4eI_3}{I}$. This factor weights the contribution of recombinant innovation to total benefits. This contribution is larger for a larger effectiveness e. It is useful to normalize the benefits function to its value in the case of specialization $V(\alpha = 0; t) = V(\alpha = 1; t) = I^s t^s$:

$$\tilde{V}(\alpha) \equiv \frac{V(\alpha;t)}{I^s t^s} = \alpha^s + (1-\alpha)^s + C^s\alpha^s(1-\alpha)^s. \tag{2.21}$$

The function $\tilde{V}(\alpha)$ reaches a maximum for $\alpha = 1/2$ (maximum diversity), or for either $\alpha = 0$ or $\alpha = 1$ (specialization). Fig. 2.5 reports the benefits curve (2.21) in the case of increasing returns to scale ($s = 1.2$) for four different values of the factor e. Either specialization or diversity can be optimal, depending on factor $C = 4eI_3/I$. If the effectiveness e is insufficiently large, for instance, returns to scale may be too large for diversity to be the optimal choice. This result is in accordance with Dasgupta and Maskin (1987): in an uncertain environment parallelism of investments should not be considered as waste, unless increasing returns outweigh the benefits from diversification.

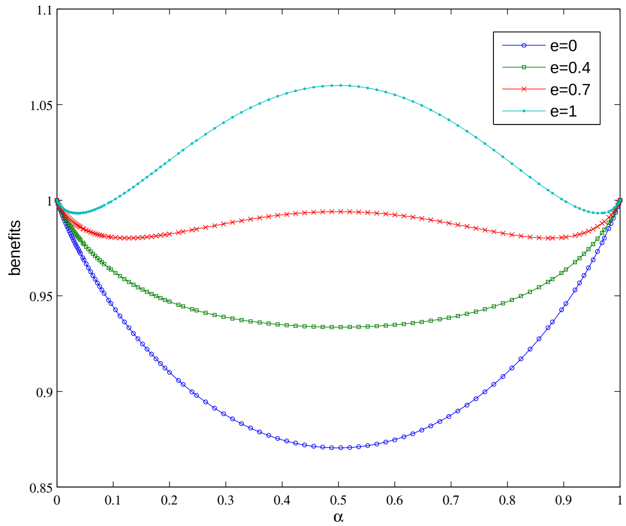

Figure 2.5: Final benefits $\tilde{V}$ as a function of the investment share α under increasing returns to scale ($s = 1.2$) for different values of the innovation effectiveness $e = 0$, 0.4, 0.7, 1 (here $I = 4I_3$).

This theoretical result goes beyond the usual message from portfolio theory, according to which diversification is good. In Fig. 2.6 there are four examples with different values of returns to scale for a given value of the factor C.

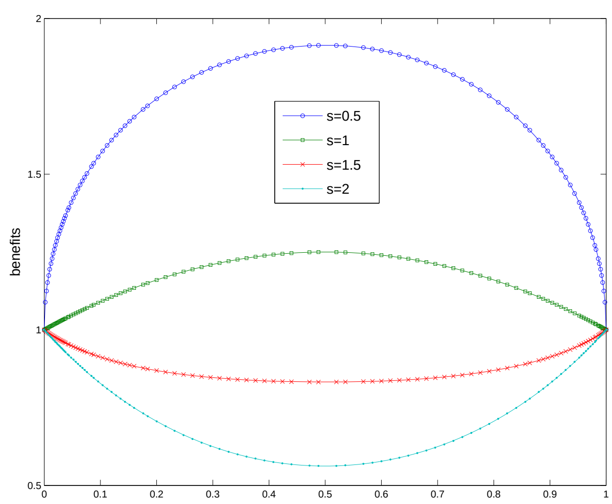

Figure 2.6: Final benefits $\tilde{V}$ as a function of the investment share α for different values of returns to scale, with $C = 1$ (for instance $e = 1/4$, $I = I_3$).

For a systematic analysis, different cases need to be distinguished. There is a threshold value $\overline{e}$ of effectiveness, such that for $e < \overline{e}$ the optimal decision is specialization, while for $e > \overline{e}$ diversity is optimal. Conversely, given the effectiveness of recombinant innovation e, one can derive the turning point $\overline{s}$ of returns to scale at which maximal diversity ($\alpha = 1/2$) becomes optimal. This is given by the threshold level $\overline{s}$ that solves the equation:

$$\tilde{V}\big(\alpha = 1/2\big) = \frac{1}{2^{\overline{s}}}\left[2 + \left(\frac{C}{2}\right)^{\overline{s}}\right] = 1. \tag{2.22}$$

If $C = 0$ (for instance with $e = 0$), we have $\overline{s} = 0$. If $C = 1$ (for instance, with $I = 4I_3$ and $e = 1$), we find $\overline{s} \simeq 1.2715$. There is no closed form solution $\overline{s}$ as a function of other parameters, but we can instead solve for C. For $s > 1$ this solution is:

$$\overline{C} = 2(2^s - 2)^{1/s}. \tag{2.23}$$

Since $C = 4eI_3/I$, Eq. (2.23) links the ratio of investments and the effectiveness e to the returns to scale: as soon as $C = 4eI_3/I > \overline{C}$ diversity is the optimal solution. Furthermore, since $\overline{C}(s)$ is increasing, concave and converging to four, there is a saturation effect: as returns to scale get larger, less and less investment is needed in the new technology to make diversified investment the best choice.[8] In the limit of infinite returns to scale, the threshold value of I_3/I approaches $1/e$. This leads to:

Proposition 2.3.1. *For any given values of the effectiveness e and returns to scale s, benefits from diversity are larger than benefits from specialization iff $I_3/I > 1/e$.*

This means that in principle a diversified investment can always be rendered the optimal choice of the allocation problem, if one has enough resources I_3 to assign to the recombinant innovation, no matter how small the recombination effectiveness e, and no matter how large the returns to scale s.

[8] We have $\frac{d}{ds}2(2^s-2)^{1/s} = (2^s-2)^{1/s}\left[\frac{2^s \ln 2}{s(2^s-2)} - \frac{\ln(2^s-2)}{s^2}\right]$. The first term is $\frac{2^s \ln 2}{2^s-2} \geq \frac{(2^s-2)\ln 2}{2^s-2} = \ln 2$, while $\frac{\ln(2^s-2)}{s}$ is increasing and converges to $\ln 2$ from below. This means that $\frac{d}{ds}\overline{C}(s) \geq 0 \; \forall s > 1$.

Assume the ratio of investments I_3/I is given. For $s = 1$ (constant returns to scale), we have $\tilde{V}(1/2)_{s=1} = 1 + C/4 \geq 1$, since $C \geq 0$. If a positive level of investment I_3 is devoted to the innovative technology, the following statement holds true:

Proposition 2.3.2. *The threshold $\overline{s}$, below which a diversified system is the optimal choice, has the property that $\overline{s} \geq 1$; and $\overline{s} > 1$ iff $e > 0$.*

Corollary 2.3.1. *For all decreasing or constant returns ($s \leq 1$), a maximum value of final benefits is realized for the allocation $\alpha = 1/2$, i.e. for maximum diversity.*

This is true for any value of C, that is for any set of values of e, I and I_3.[9] In other words, in all cases of decreasing returns to scale up to constant returns it is better to split equally the investment between the two parent options. Notice that diversity is also optimal in absence of recombinant innovation, when returns to scale are low enough.

The case of increasing returns to scale is the one that better represents technological innovation, because of fixed costs and learning. In this regime we have a tradeoff between scale advantages and benefits from diversity. This is the case studied numerically by van den Bergh (2008). In general, we have the following result, which completes Proposition 2.3.2:

Corollary 2.3.2. *Diversity ($\alpha = 1/2$) can also be optimal with increasing returns to scale, which happens when $1 < s < \overline{s}$.*

Our analytical model shows how and when diversification of investments can be harmful. This result should be compared with the usual message arising from the *R&D* portfolio literature, where generally diversification of investments is encouraged to secure firm success (much in line with the financial portfolio literature). Such a message can be wrong, depending on the relevant returns to scale and probability of innovation.

[9]Consider the function $f(s) \equiv \left(2 + (C/2)^s\right)/2^s$. The statement is true if $f(s) \geq 1\ \forall s \in [0,1]$. Since $f'(s) < 0\ \forall s \geq 0$, $f(s)$ is a decreasing function for fixed C. For fixed s, f is an increasing function of C. When $C = 0$ $f(1) = 1$ and $f(s) \geq 1\ \forall s \in [0,1]$. When $C > 0$ $f(1)|_{C>0} > f(1)|_{C=0} = 1$ and $f(s)|_{C>0} > f(s)|_{C=0} = 1\ \forall s \in [0,1]$. This proves Corollary 2.3.1.

As Fig. 2.5 shows, there can be either one or three stationary points for the benefits curve $\tilde{V}(\alpha)$. The first-order necessary condition for maximization of final benefits is:

$$\frac{\partial \tilde{V}}{\partial \alpha} = s\alpha^{s-1} - s(1-\alpha)^{s-1} + C^s s\big[\alpha(1-\alpha)\big]^{s-1}(1-2\alpha) = 0. \tag{2.24}$$

The symmetric solution $\alpha = 1/2$ always exists. Depending on returns to scale s, two other solutions are present, $\alpha_1(s)$ and $\alpha_2(s)$. They are symmetric with respect to $\alpha = 1/2$ (i.e. $\alpha_1 + \alpha_2 = 1$), and if they exist they are always associated with a minimum level of benefits, while $\alpha = 1/2$ may be either a minimum or a maximum. The transition from $\alpha = 1/2$ as a minimum to $\alpha = 1/2$ as a maximum occurs together with the appearance of these two solutions of Eq. (2.24). For a given value of C there is a level of returns to scale $\hat{s}$ at which $\alpha = 1/2$ is neither a maximum or a minimum. The threshold value is given by a tangency requirement $\frac{\partial^2 \tilde{V}}{\partial \alpha^2}\Big|_{\alpha=1/2} = 0$, which turns into the following condition:

$$\hat{s} = \left(\frac{C}{2}\right)^{\hat{s}} + 1. \tag{2.25}$$

The threshold value $\hat{s}$ is a fixed point of the function $f(s) = \left(\frac{C}{2}\right)^s + 1$. With $C = 1$ (for instance, with $I = 4I_3$ and $e = 1$) we have $\hat{s} \simeq 1.3833$. Note that $\hat{s} > 1$ since $C \geq 0$. Then we have the following proposition:

Proposition 2.3.3. *A necessary condition for only one stationary point ($\alpha = 1/2$ a local and global minimum) is increasing returns to scale. With decreasing returns there are always three stationary points.*

Conversely, given a value s of returns to scale, one can compute the transition value in terms of the other factors, with $\hat{C} = 2(s-1)^{1/s}$. For $C > \hat{C}$, there are three stationary points. Note how $\hat{C} > 0$ only with increasing returns to scale.

We can compare the transition value $\hat{s}$ with the value $\overline{s}$: three different regions can be identified in the returns to scale domain, as shown in Fig. 2.7.

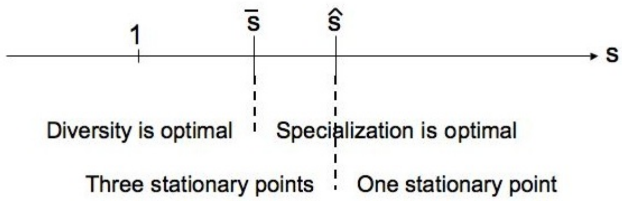

Figure 2.7: With a positive effectiveness of recombinant innovation e, we have $\hat{s} > \overline{s} > 1$.

Proposition 2.3.4. *In general, $\hat{s} \geq \overline{s} \geq 1$, and $\hat{s} = \overline{s} = 1$ for $e = 0$ (no recombination).*

Fig. 2.8 shows $\tilde{V}(\alpha)$ and its derivative for two different values of s.[10] In the first case ($s = 1.5$), the only stationary point $\alpha = 1/2$ is a global minimum of final benefits. Global maxima are the corner solutions $\alpha = 0$ and $\alpha = 1$. In the second case ($s = 1.2$), there are three stationary points: $\alpha = 1/2$ is a global maximum, while the two symmetric stationary points, α_1 and α_2, are global minima.

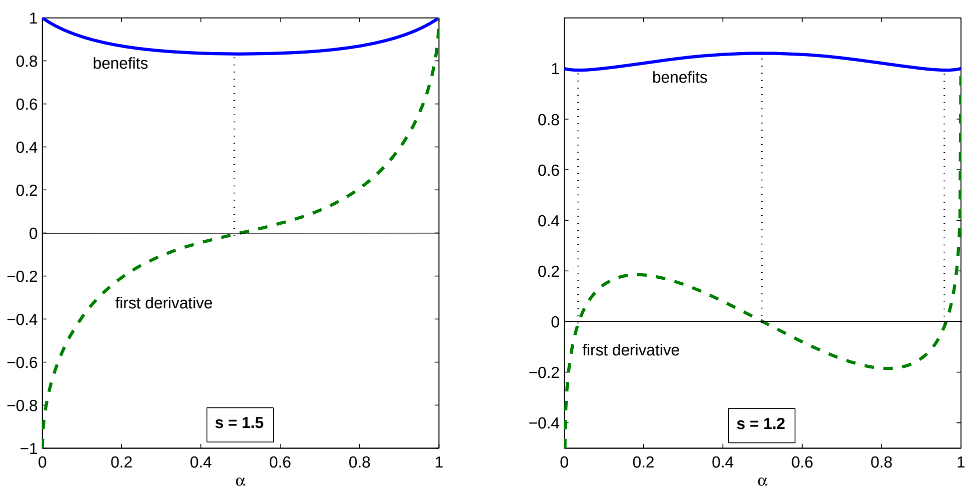

Figure 2.8: Normalized benefits $\tilde{V}(\alpha)$ and derivative $\tilde{V}'(\alpha)$, for two values of returns to scale s.

2.3.2 Optimization with a size effect and zero initial values

In this subsection, we study the effect of size in the problem of optimal diversification, still assuming zero initial values for the parent options and equal returns to scale $s_1 = s_2 = s_3$.

[10]Fig. 2.8 reports $\tilde{V}'(\alpha)/s = \alpha^{s-1} - (1-\alpha)^{s-1} + \left[\alpha(1-\alpha)\right]^{s-1}(1-2\alpha)$.

Without initial values, the balance is constant, but P_e depends on time because of the size effect. The expression of the innovative option is given by Eq. (2.16). Substituting this into the objective function of the maximization problem (2.4), we obtain:

$$V(\alpha, t) = (\alpha I t)^s + \big((1-\alpha) I t\big)^s + \big[4 e I_3 \alpha (1-\alpha)\big]^s \big[t + g(t)\big]^s, \tag{2.26}$$

where $g(t) = (e^{-\sigma I t} - 1)/\sigma I$. The normalized version (divide by $I^s t^s$) reads:

$$\tilde{V}(\alpha, t) = \alpha^s + (1-\alpha)^s + C^s m(t)^s \alpha^s (1-\alpha)^s, \tag{2.27}$$

where the constant factor is again $C = 4eI_3/I$. Now, a time dependent factor shows up, $m(t) = 1 + \frac{exp(-\sigma I t)-1}{\sigma I t}$, so that $m'(t) > 0$, $\lim_{t\to 0} m(t) = 0$ and $\lim_{t\to\infty} m(t) = 1$. The factor $m(t)$ monotonically modulates the contribution of innovative recombination to final benefits, being very small in the early stages and converging to 1 as $\sigma I t \gg 1$. In the long-run, the model converges to the simplest case analysed before.

One can incorporate $m(t)$ into C, defining a function $C(t) = Cm(t)$. Final benefits with size effect (equation (2.27)) are formally the same as before (equation (2.21)): the only difference is that constant C now depends on time. Consequently, the solution (optimal diversity) depends on the time horizon t.[11] Nevertheless, since the system remains symmetric, the optimal solution will be either $\alpha = 0, 1$ or $\alpha = 1/2$. This is better understood by looking at Fig. 2.5: given I, I_3 and e, as time flows, the factor $C(t)$ increases and the benefits curve goes from the lower curve $e = 0$ (representing $C = 0$) to the upper curve $e = 1$ (which stands for $C = 1$).

The first-order condition for optimal diversity in this dynamic setting is as follows:

$$s\alpha^{s-1} - s(1-\alpha)^{s-1} + C(t)^s \big[\alpha(1-\alpha)\big]^{s-1} (1-2\alpha) = 0. \tag{2.28}$$

[11] It is important to note that we only deal with one-period investment decisions and do not engage in dynamic optimization. The optimal investment share may depend on time, in the sense that it may be different for a different investment time horizon.

The analysis of the shape of the benefits curve can be done as before by simply substituting the constant C with the function $C(t)$. The transition value $\hat{s}$, where $\alpha = 1/2$ is neither a minimum nor a maximum of benefits, is now time dependent and given by:

$$\hat{s}(t) = \left(\frac{C(t)}{2}\right)^{\hat{s}(t)} + 1. \tag{2.29}$$

It is also interesting to think in terms of a transition time $\hat{t}$: for a given value of returns to scale s, this is the threshold value of the time horizon above which one finds three stationary points. Such value is obtained implicitly from the following condition:

$$C(\hat{t}) = 2(s-1)^{1/s}. \tag{2.30}$$

Formally the threshold analysis of optimal diversity is also the same as before: we define the returns to scale $\overline{s}(t)$ as the level where, for a given time horizon t, the benefits with $\alpha = 1/2$ are the same as the benefits from specialization ($\alpha = 0, 1$):

$$\tilde{V}\big(\alpha = 1/2\big) = \frac{1}{2^{\overline{s}(t)}}\left[2 + \left(\frac{C(t)}{2}\right)^{\overline{s}(t)}\right] = 1. \tag{2.31}$$

Proposition 2.3.5. *For a given time horizon t, diversity ($\alpha = 1/2$) is optimal iff $s < \overline{s}(t)$.*

How does $\overline{s}(t)$ behave? The larger t is, the larger $\overline{s}(t)$. The intuition behind this is as follows. $C(t)$ is increasing, which means that time works in favour of recombinant innovation. As time goes by, the region of returns to scale where diversity is optimal enlarges, and $\overline{s}(t)$ converges to the value $\overline{s}$ of the simplest case (see Fig. 2.9). Diversity may never become the optimal choice if returns to scale are too high ($\overline{s} < s$). But, if investment I_3 is large enough, diversity will always become optimal. This is consistent with Proposition 2.3.1: given returns to scale s, if one has infinite disposal of investment I_3, threshold $\overline{s}$ can always be made such that $\overline{s} > s$, so that, at some time t, one will see $\overline{s}(t) > s$.

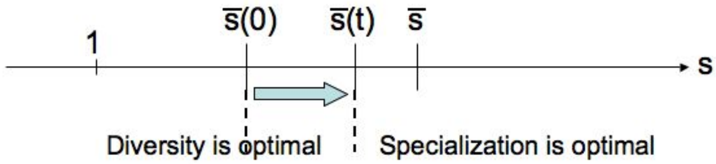

Figure 2.9: As time goes, the region of returns to scale values where diversity is optimal becomes larger.

In the time domain, one can define a threshold time horizon $\overline{t}$, such that for $t < \overline{t}$ specialization is optimal, while for $t \geq \overline{t}$ diversity is the best choice:

$$C(\overline{t}) = 2\big(2^s - 2\big)^{1/s}. \tag{2.32}$$

The function $C(t)$ is increasing: the inverse $C^{-1}(\cdot)$ is increasing as well, and a unique $\overline{t}$ exists. The right-hand side of (2.32) is increasing[12] in s. The following holds true, then:

Proposition 2.3.6. *For higher returns to scale s, the threshold time horizon $\overline{t}$ is larger, and it takes a longer time for diversity ($\alpha = 1/2$) to become the optimal decision.*

Concluding, the size factor introduces a dynamic scale effect into the system. The optimal solution may change through time, but symmetry is unaffected, and it can only switch from $\alpha = 0, 1$ to $\alpha = 1/2$ (not vice versa). This happens if and only if the effectiveness of recombination e is sufficiently large (see corollary 2.3.2 in Section 2.3.1).

Finally, in the long-run the size effect vanishes, with $\lim_{t\to\infty} S(t) = 1$. If one considers a time horizon long enough the size factor can be discarded in the probability of emergence of recombinant innovation. Beyond the transitory phase, the optimal diversity is approximated by the solution of the case without the size effect.

[12] We have $\frac{d}{ds}2^{s+1}\big(2^{s-1} - 1\big) = 2^{s+1}\ln 2(2^s - 1) > 0$ since $s > 0$.

2.3.3 The effect of non-zero initial values on the optimal investment decision

We now consider a first type of symmetry-breaking in the $R\&D$ investment portfolio, allowing for non-zero initial value of parent options in the optimization of final benefits. Initial values address in particular the case of a main (core) technology recombining with a smaller (and therefore possibly a younger) one. To focus on this, we assume no size effect ($\sigma = \infty$) and homogeneous returns to scale ($s_1 = s_2 = s_3$). Equation (2.10) shows the value of the innovative option in this case:

$$O_3(t) = C\Big[f(\alpha, t) + \alpha(1-\alpha)It\Big], \tag{2.33}$$

where $C = 4eI_3/I$ and the nonlinear time-dependent factor is:

$$f(\alpha, t) = \big(O_{10} - \alpha O_0\big)^2\Big(\frac{1}{O_0 + It} - \frac{1}{O_0}\Big) + \big(O_{10} - \alpha O_0\big)(1-2\alpha)\ln\frac{O_0 + It}{O_0}.$$

This is the sum of two terms: one is hyperbolic and converges to a negative value. The other is logarithmic and monotonically increasing or decreasing, depending on the factor $\big(O_{10} - \alpha O_0\big)(1-2\alpha)$. The objective function for maximization is:

$$V(\alpha, t) = \big(O_{10} + \alpha It\big)^s + \big(O_{20} + (1-\alpha)It\big)^s + C^s\Big[f(\alpha, t) + \alpha(1-\alpha)It\Big]^s, \tag{2.34}$$

and normalized benefits are:[13]

$$\tilde{V}(\alpha, t) = \left(\frac{O_{10}}{It} + \alpha\right)^s + \left(\frac{O_{20}}{It} + 1 - \alpha\right)^s + C^s\left[\frac{f(\alpha, t)}{It} + \alpha(1-\alpha)\right]^s. \tag{2.35}$$

[13] Normalizing this function to $(It)^s$ is less meaningful now, since $(It)^s$ no longer represents the value of benefits with specialization. Nevertheless, this normalization leaves us with a dimensionless function and enables us to compare the results with other versions of the model.

The first-order necessary condition for a maximum reads:

$$\left(\frac{O_{10}}{It}+\alpha\right)^{s-1}-\left(\frac{O_{20}}{It}+1-\alpha\right)^{s-1}+ \tag{2.36}$$
$$+\quad C^s\left[\frac{f(\alpha,t)}{It}+\alpha(1-\alpha)\right]^{s-1}\left(\frac{1}{It}\frac{\partial f(\alpha,t)}{\partial\alpha}+1-2\alpha\right)=0.$$

The solution to this equation is rather complicated. Note that $\alpha = 1/2$ is not a solution in general.[14] Optimal diversity is represented by a function of time $\alpha^*(t)$. In Fig. 2.10 we report an example where initially one option is much larger than the other, i.e. where $O_{10} = 10O_{20}$. Diversification happens to be the optimal choice at all times (maximum benefits are normalized to the value in the long-run). For a very short time horizon, one should invest more in the larger option (nearly 60 per cent). As the time horizon becomes more distant, the optimal solution approaches a perfectly diversified portfolio ($\alpha = 50$ per cent). There is an "overshooting" effect: the optimal solution becomes larger than 1/2,

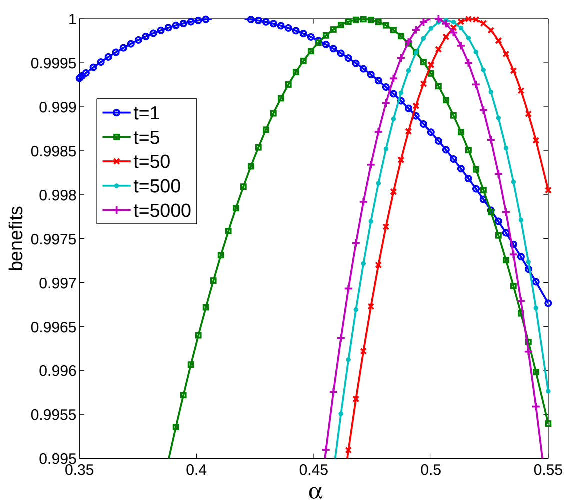

Figure 2.10: Final benefits with positive initial values and no size effect for five different time horizons. Here $O_{10} = 1$, $O_{20} = 10$, $s = 1.2$, $e = 1$ and $I = 4I_3 = 1$. Time horizons t are in units of $1/I$.

reaches a top and then goes back to the symmetric allocation. In the long-run ($t \gg O_0/I$), symmetry is restored. The effect of initial values has then dissipated, and one is back in

[14]The symmetric allocation is still a solution in the particular case of equal initial values $O_{10} = O_{20}$.

the case of Section 2.3.1.

If a size effect is also present, the value of the innovative option is given by Eq. (2.17). Maintaining the assumption of homogeneous returns to scale, the value of final benefits from the overall investment is as follows:

$$\begin{aligned} V(\alpha,t) &= \big(O_{10}+\alpha It\big)^s + \big(O_{20}+(1-\alpha)It\big)^s + \\ &+ C^s\Big[B(\alpha)It + B(\alpha)\frac{exp(-\sigma O_0)}{\sigma}\big[exp(-\sigma It)-1\big] + h(\alpha,t)\Big]^s, \end{aligned} \tag{2.37}$$

where $h(\alpha,t)$ collects all terms in the expression of O_3 but the first two. Note that it is not possible to separate this expression into two factors depending separately on t and α as we managed to do in Section 2.3.2 (Eq. 2.26). The contribution of innovation consists of three terms. The first is the linear one, which already appears when all the simplifying assumptions hold. The second is a direct effect of the size factor. The third is due to the presence of non-zero initial values of parent options. This expression combines the effects that we have been analysing separately so far. If we normalize this expression by dividing it by $I^s t^s$ we obtain:

$$\tilde{V}(\alpha,t) = \left(\frac{O_{10}}{It}+\alpha\right)^s + \left(\frac{O_{20}}{It}+1-\alpha\right)^s + C^s\left[B(\alpha)n(t) + \frac{h(\alpha,t)}{It}\right]^s, \tag{2.38}$$

where $n(t) = 1 + exp(-\sigma O_0)/(\sigma It)[exp(-\sigma It)-1]$. This time factor can be expressed in terms of the factor $m(t)$ that we introduced in Section 2.3.2: $n(t) = exp(-\sigma O_0)m(t) + 1 - exp(-\sigma O_0)$, $n(0) \simeq 1 - exp(-\sigma O_0)$, $n'(t) = exp(-\sigma O_0)m'(t) > 0$ and $\lim_{t\to\infty} n(t) = 1$. The smaller the sum of initial values (O_0), the closer $n(t)$ is to $m(t)$. With no initial values $n(t)|_{O_0=0} = m(t)$. The effect of $n(t)$ is symmetric: the benefits curve rises from lower values where the contribution of innovation is negligible to higher values where diversity may be the optimal choice eventually. The other terms of equation (2.38) are similar to the case without the size effect (Eq. 2.35), with the factor $h(\alpha,t)$ doing a job similar to $f(\alpha;t)$.

In the long-run ($It \gg O_0$), the initial values become negligible and the size factor converges to 1. In other words, if the time horizon is long enough, this more general case reduces to the case analysed in Section 2.3.1.

2.3.4 Heterogeneous returns to scale

The previous model version with homogeneous returns to scale is restrictive. A more general version of the model would allow for heterogeneity of returns to scale. Assume zero initial values of capital stock and no size effect. The balance of the parent options is constant in this condition, $B(\alpha) = 4\alpha(1-\alpha)$, and all three investment options grow linearly in time. Consider the symmetry-breaking of different returns to scale for each technology. The final benefits of investment are:

$$V(\alpha; t) = (\alpha It)^{s_1} + [(1-\alpha)It]^{s_2} + [4\alpha(1-\alpha)eI_3t]^{s_3}. \tag{2.39}$$

The analytical expression of the optimal investment share $\alpha^*(t)$ is difficult to compute. Three examples of $V(\alpha; t)$ with three different choices of s_1, s_2, s_3 for a given time horizon t are shown in Fig. 2.11. Nevertheless, some general insights can be easily derived. The

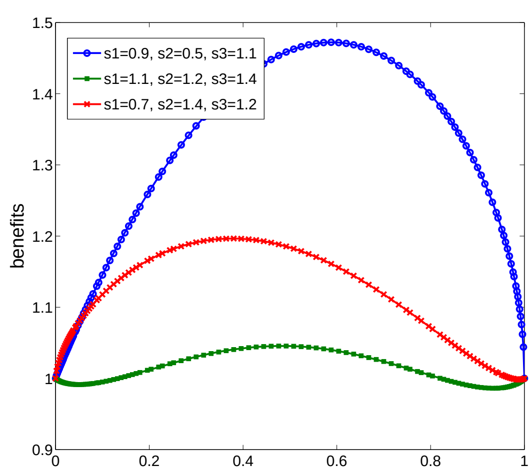

Figure 2.11: Final benefits for three different choices of returns to scale values (here $I = 4eI_3$ and $It = 1$).

main intuition is that for $s_2 > s_1$, one should invest more in the second option ($\alpha^* > 1/2$), and vice versa for $s_2 < s_1$. If we think of the time horizon as a variable, final benefits (2.39) are a polynomial with three different powers of t: the term with higher power eventually overcomes the other two and this dictates the optimal choice α^* in the long-run:

- $s_1 > s_2, s_3 \Rightarrow \alpha^*_{t\to\infty} = 1$;
- $s_2 > s_1, s_3 \Rightarrow \alpha^*_{t\to\infty} = 0$;
- $s_3 > s_1, s_2 \Rightarrow \alpha^*_{t\to\infty} = 1/2$.

For a finite time horizon, an investment share other than these three values is the optimal choice. If $s_3 > s_1, s_2$ but e and I_3 are relatively low, there is a range of time horizons for which it is still better to opt for specialization. On the other hand, if $s_3 < s_1, s_2$ but e and I_3 are relatively high, for some short time horizon a diversified investment is better.

The most general case of our model entails relaxing all assumptions made in this section simultaneously. We have analysed the role of three factors separately: the size effect; the initial value of parent options; and the heterogeneous returns to scale. There is a big difference in the long-run between the first two and the last one. The effect of initial values and the effect of cumulative size vanish in the long-run, making the investment problem in the limit the same as if these factors were not present. Heterogeneous returns to scale do exactly the opposite: their effect gets larger as time goes on. The optimal choice in the long-run is not converging to the simplest case when returns to scale are different, but to one of the limit values 0,1, and 1/2 (see above).

2.4 Conclusion

This study proposes a model of an investment allocation problem where the decision maker faces a trade-off between scale advantages and recombinant innovation. The first calls for specialization, while the second benefits from a diversified portfolio.

The initial part of the analysis consists of deriving a solution for the model dynamics. A condition for constant probability of recombinant innovation (probability of emergence) is that the ratio of the investment shares equals the ratio of the initial values of the parent options. When this is not the case, the probability of emergence changes over time and may be increasing, decreasing or non-monotonic, depending on the relative value of these two ratios. In all cases, it converges to the same constant value. The time pattern of the innovative option in the long-run only contains a linear and a logarithmic term, which is either convex or concave, depending on initial values and investment shares.

In order to account for a diminishing marginal effect of parent options in recombinant innovation, a size factor is included in the innovation probability. In the long-run, the value of recombinant innovation reduces again to a linear plus logarithmic term. But, in this case, there can only be a convex time pattern. This shape reflects the typical threshold effect of recombinant innovations.

The second part of the analysis is devoted to the optimal allocation of investment between the two technological options, which boils down to finding an optimal trade-off between the benefits of recombinant innovation and the benefits associated with returns to scale. We derive conditions for optimal diversification under different regimes of returns to scale. A perfectly symmetric portfolio ($\alpha = 1/2$) may be either a local maximum or a local minimum of final benefits, depending on returns to scale. When $\alpha = 1/2$ is a local maximum, two other stationary points are present. We define two threshold values of returns to scale: the first one is the value where the system makes a transition from one to three stationary points of final benefits. The second threshold is the returns to scale level below which diversity is a global maximum of final benefits.

The presence of a size factor in the probability of emergence makes the returns to scale threshold time-dependent. This suggests a threshold analysis in the time domain: for a given level of returns to scale, when the investment time horizon is beyond a critical value, the best choice becomes diversity. This threshold time is larger, the higher are returns to scale. Introducing initial values of parent options breaks the symmetry of the portfolio. The share $\alpha = 1/2$ is no longer a solution to the maximization problem. Only in the long-run is symmetry restored, that is, approximated through convergence, and $\alpha = 1/2$ will be optimal eventually, if increasing returns are not too high.

Finally, we study the effect of heterogeneous returns to scale of the different technologies involved. This constitutes another symmetry-breaking of the investment portfolio and causes optimal diversity to depart from $\alpha = 1/2$ when diversification is preferred to specialization. One important result is that, in the long-run, the option with the highest returns to scale overcomes the others. Furthermore, this dictates the allocation decision when the time horizon is distant enough.

One final methodological remark is in order. When returns to scale are homogeneous, the long-run limit is well approximated by the simplest case that we have analysed: namely, where no initial values of parent options are considered, and no size factor enters the probability of emergence. With heterogeneous returns to scale, however, the case of reference is different: the effect of initial values and total size vanishes, but the effect of returns to scale grows.

Several directions for future research can be identified. Investment in the innovative option can be endogenized, i.e. made part of the allocation decision. Extending the number of parent options allows for an examination of the role of technological distance, as well as for assessing the marginal effect of new options (e.g. diminishing returns) and the optimal number of options. Finally, the value of parent options can be modelled as a stochastic process, which suggests an analogy between the innovative option and a financial derivative: parent options would then play the role of underlying assets.

Appendix

2.A Condition for constant balance

Here we give a proof of the necessary and sufficient conditions of constant balance for the "Gini" specification.

In order to prove necessity, we differentiate the expression $B\big(O_1(t), O_2(t)\big)$ with respect to time, and see under which conditions the derivative is equal to zero. Using the chain rule we have:

$$\frac{dB}{dt} = \frac{\partial B}{\partial O_1}\frac{dO_1}{dt} + \frac{\partial B}{\partial O_2}\frac{dO_2}{dt}, \tag{2.40}$$

where

$$\frac{\partial B}{\partial O_i} = \frac{O_j(O_j - O_i)}{(O_i + O_j)^3} \qquad i, j = 1, 2 \qquad i \neq j.$$

Time derivatives are given by the specifications of the model (2.1). If we now substitute the time flow of each option value, $O_1(t) = O_{10} + \alpha I t$ and $O_2(t) = O_{20} + (1-\alpha)It$, the time derivative of balance becomes:

$$\frac{dB}{dt} = \frac{O_{10} - O_{20} + (2\alpha - 1)It}{(O_{10} + O_{20} + It)^3}\Big[(O_{10} + \alpha It)(1-\alpha)I - (O_{20} + (1-\alpha)It)\alpha I\Big]. \tag{2.41}$$

Setting this derivative to zero, we obtain:

$$(O_{10} + \alpha It)(1-\alpha) = (O_{20} + (1-\alpha)It)(\alpha I).$$

This equation must hold true for any value of t. For instance, taking $t = 1/I$, we have:

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1-\alpha},$$

which is condition (2.8).

This is also a sufficient condition for constant balance as can be seen by direct substitution:

$$
\begin{aligned}
B(t) &= 4\frac{(O_{10}+\alpha It)(O_{20}+(1-\alpha)It)}{(O_{10}+O_{20}+It)^2} = 4\frac{(O_{10}+\alpha It)(O_{10}\frac{1-\alpha}{\alpha}+(1-\alpha)It)}{(O_{10}+O_{10}\frac{1-\alpha}{\alpha}+It)^2} \\
&= 4\frac{(1+\frac{\alpha}{O_{10}}It)(\frac{1-\alpha}{\alpha}+\frac{1-\alpha}{O_{10}}It)}{(1+\frac{1-\alpha}{\alpha}+\frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\frac{(1+\frac{\alpha}{O_{10}}It)^2}{(\frac{1}{\alpha}+\frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\alpha^2 = 4\alpha(1-\alpha).
\end{aligned}
$$

2.B General model solution

Here we report the steps of the integration of the probability of emergence as defined in (2.15), that is, the integration of the third equation of the model (2.1) leading to the time value of the third option O_3. This computation contains the solution without size effect as a particular case. In what follows, we set $I_3 = 1$ for investment in the innovative option:[15]

$$
O_3(t) = \int_0^t 4\frac{(O_{10}+\alpha I\tau)(O_{20}+(1-\alpha)I\tau)}{(O_0+I\tau)^2}\left(1-e^{-\sigma(O_0+I\tau)}\right)d\tau. \tag{2.42}
$$

We substitute $\tau = (x-O_0)/I$ and obtain:

$$
O_3 = \frac{4}{I}\int_{O_0}^{O_0+It} \frac{(E+Fx)(G+Hx)}{x^2}\left(1-e^{-\sigma x}\right)dx, \tag{2.43}
$$

where $E = O_{10}(1-\alpha) - \alpha O_{20}$, $F = \alpha$, $G = -E$ and $H = (1-\alpha)$. The expression above is the difference of two integrals (for ease of notation, we consider indefinite integrals for the moment). The first one is:

$$
\begin{aligned}
\int \frac{(E+Fx)(G+Hx)}{x^2}dx &= EG\int\frac{dx}{x^2} + (EH+FG)\int\frac{dx}{x} + FH\int dx \\
&= -\frac{EG}{x} + (EH+FG)lnx + FHx.
\end{aligned}
$$

[15]We assume $e = 1$ for the effectiveness of recombination. Here e denotes the function $exp(\cdot)$

And for the second integral, we have:

$$\begin{aligned}\int \frac{(E+Fx)(G+Hx)}{x^2}e^{-\sigma x}dx &= EG\int \frac{e^{-\sigma x}}{x^2}dx + (EH+FG)\int \frac{e^{-\sigma x}}{x}dx + \\ &+ FH\int e^{-\sigma x}dx = \\ &= -\frac{FH}{\sigma}e^{-\sigma x} - EG\frac{e^{-\sigma x}}{x} + \\ &+ [EH+FG-\sigma EG]\Big[lnx + \sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k\cdot k!}\Big].\end{aligned}$$

When substituting the latter two results into equation (2.43) we obtain:

$$\begin{aligned}\int \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx &= -\frac{EG}{x} + FHx + FH\frac{e^{-\sigma x}}{\sigma} + \\ &+ EG\frac{e^{-\sigma x}}{x} + \sigma EGlnx + \\ &+ [\sigma EG - (EH+FG)]\sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k\cdot k!}.\end{aligned}$$

It is instructive to look first at the case of constant balance. The necessary and sufficient condition can be written as $O_{10}(1-\alpha) = O_{20}\alpha$. Then $EG = 0$, $EH + FG = 0$, and $FH = \alpha(1-\alpha)$, and the integral above simplifies to:

$$\int \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx\Big|_{B=const} = \alpha(1-\alpha)\Big(x + \frac{e^{-\sigma x}}{\sigma}\Big). \tag{2.44}$$

The solution for the value of the third option as a function of time is then:

$$O_3(t) = \frac{4}{I}\alpha(1-\alpha)\Big(x+\frac{e^{-\sigma x}}{\sigma}\Big)\Big|_{x=O_0}^{x=O_0+It} = Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\Big(e^{-\sigma It}-1\Big), \tag{2.45}$$

where $B = 4\alpha(1-\alpha)$. It is useful to check the units of the solution just obtained. The first term Bt is *time* (balance is dimensionless). The second term is *time* again, since σ is *capital*$^{-1}$, while I is capital per unit of time. Not surprisingly O_3 has a *time* dimension, after we have set $I_3 = 1$.

Relaxing the condition of constant balance, we have the following general result for the value of the innovative option at time t:

$$\begin{aligned} O_3(t) &= \frac{4}{I}\int_{x=O_0}^{x=O_0+It} \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx = \\ &= Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\big(e^{-\sigma It}-1\big) + \frac{4}{I}\sigma EG\log\frac{O_0+It}{O_0} + \\ &+ \frac{4}{I}EG\Big[\frac{1}{O_0}\big(1-e^{-\sigma O_0}\big) - \frac{1}{O_0+It}\big(1-e^{-\sigma(O_0+It)}\big)\Big] + \\ &+ \frac{4}{I}\Big[\sigma EG-(EH+FG)\Big]\Bigg[\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} - \sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!}\Bigg]. \end{aligned} \tag{2.46}$$

The first two terms are what we have with constant balance. In the short run ($It \ll O_0$), we have $O_3(t) \simeq Bt$. A bit more complex is the analysis of the long-run behaviour ($t \gg O_0/I$). The second term vanishes. In the logarithmic term, the value of the new investment It overcomes the initial option value O_0. The fifth term vanishes even faster than the second term, because of the presence of t in the denominator. Finally, the infinite sum containing t goes to zero at least exponentially: this can be seen by noting that, for even values of k, we have ($O_0 + It = y$):

$$\frac{(-y)^k}{2^k\cdot k!} < \frac{(-y)^k}{k\cdot k!} < \frac{(-y)^k}{k!}.$$

For odd values of k, the inequalities are reversed. This means that our series is bounded between the functions $-1+e^{-(O_0+It)}$ and $-1+e^{-(O_0+It)/2}$, implying that it goes to zero at least exponentially:

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} &= -\sigma(O_0+It) + \frac{\sigma^2(O_0+It)^2}{2\cdot 2} - \frac{\sigma^3(O_0+It)^3}{3\cdot 3!} + \frac{\sigma^4(O_0+It)^4}{4\cdot 4!} - \dots \\ &< -\sigma(O_0+It) + \frac{\sigma^2(O_0+It)^2}{2} - \frac{\sigma^3(O_0+It)^3}{3!} + \frac{\sigma^4(O_0+It)^4}{4!} - \dots \\ &= -1+e^{-\sigma(O_0+It)} \le 0, \end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} &= -\sigma(O_0+It)+\frac{\sigma^2(O_0+It)^2}{2\cdot 2}-\frac{\sigma^3(O_0+It)^3}{3\cdot 3!}+\frac{\sigma^4(O_0+It)^4}{4\cdot 4!}-\dots\\
&> -\frac{\sigma(O_0+It)}{2}-\frac{\sigma(O_0+It)}{2}+\frac{\sigma^2(O_0+It)^2}{2^2\cdot 2!}-\frac{\sigma^3(O_0+It)^3}{2^3\cdot 3!}+\dots\\
&= -1-\frac{\sigma(O_0+It)}{2}+e^{-\frac{\sigma(O_0+It)}{2}}.
\end{aligned}
$$

Alternatively, one can think that, for $k \gg 1$, we have $k\cdot k! \simeq ke^{k\log k-k} \simeq k!$. This means that the infinite sums in the expression of $O_3(t)$ do not differ too much from negative exponential functions. In particular, the one depending on t goes to zero as time is sufficiently long ($It \gg O_0$). Consequently, we are left with the following long-run functional behaviour:

$$
\begin{aligned}
O_3(t) &\simeq B\Big(t-\frac{e^{-\sigma O_0}}{\sigma I}\Big)+\frac{4}{I}\sigma EG\log\frac{It}{O_0}+\\
&+ \frac{4}{I}EG\Big[\frac{1}{O_0}\big(1-e^{-\sigma O_0}\big)\Big]-\frac{4}{I}\Big[\sigma EG-(EH+FG)\Big]D(\sigma,O_0).
\end{aligned}
\tag{2.47}
$$

The factor $D(\sigma,O_0)=\sum_{k=1}^{\infty}\frac{(-\sigma O_0)^k}{k\cdot k!}$ only depends on parameters σ and O_0; as we noticed for the series dependent on t, we can say that such a quantity is bounded between e^{-O_0} and $e^{-O_0/2}$. In particular, it can easily be seen that $C(\sigma,O_0)$ is finite:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!} &= -\sigma O_0+\frac{\sigma^2O_0^2}{2\cdot 2}-\frac{\sigma^3O_0^3}{3\cdot 3!}+\frac{\sigma^4O_0^4}{4\cdot 4!}-\dots\\
&< -\sigma O_0+\frac{\sigma^2O_0^2}{2}-\frac{\sigma^3O_0^3}{3!}+\frac{\sigma^4O_0^4}{4!}-\dots\\
&= -1+e^{-\sigma O_0}\le 0,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!} &= -\sigma O_0+\frac{\sigma^2O_0^2}{2\cdot 2}-\frac{\sigma^3O_0^3}{3\cdot 3!}+\frac{\sigma^4O_0^4}{4\cdot 4!}-\dots\\
&> -\frac{\sigma O_0}{2}-\frac{\sigma O_0}{2}+\frac{\sigma^2O_0^2}{2^2\cdot 2!}-\frac{\sigma^3O_0^3}{2^3\cdot 3!}+\frac{\sigma^4O_0^4}{2^4\cdot 4!}-\dots\\
&= -1-\frac{\sigma O_0}{2}+e^{\frac{-\sigma O_0}{2}}.
\end{aligned}
$$

Obviously, the expression in (2.47) must be positive. The third and fourth terms are constant, and, since we consider the long-run behaviour of the system, it does not really matter whether they are positive or negative. Actually, the third term is negative, while the fourth can be either negative or positive, depending on σ, the investment share α, and the initial values O_{10} and O_{20}. The second term is negative, since $G = -E$. But, in the long-run, the linear function overcomes the logarithmic one. Then we can be sure that what we obtain for $O_3(t)$ in the long-run is a positive quantity.