Measuring more or less: Estimating product period penetrations from incomplete panel data
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6. A variance component model for more than one period

6.1 Introduction

In this chapter we will extend the variance component model that we introduced in chapter 5 so that we can deal with more than one period. For the original model we assumed that the $j$-th interpurchase time of individual $i$ can be written as

$$\log(X_{ij}) = \mu + U_i + E_{ij}$$ (6.1)

where $\mu$ is an overall mean and $U_i$ and $E_{ij}$ are random variables, that are stochastically independent and normally distributed with zero mean and variances $\sigma_u^2$ and $\sigma_e^2$ respectively. The index $j = 1, \ldots, n_i$, the number of purchases of individual $i$, and the index $i = 1, \ldots, N$, the sample size of individuals. The model implies a stationary process that is defined by the parameter set $\mu$, $\sigma_u^2$ and $\sigma_e^2$. Throughout this chapter we will assume that the process is in equilibrium (see Cox, 1962).

It may be the case that the process is controlled by one set of parameter values in one period, but that for different periods different parameter values hold. In that case the process is stationary within each period, but there is a change between the periods. In this sense we may generalise model (6.1) by addition of an index $k$ that denotes the period

$$\log(X_{ijk}) = \mu_k + U_{ik} + E_{ijk}$$ (6.2)

where $k = 1, \ldots, p$, the number of periods, $j = 1, \ldots, n_{ik}$, the number of purchases of individual $i$ in period $k$, and $i = 1, \ldots, N$, the sample size of individuals. Here $\mu_k$ is an overall mean for period $k$, and $U_{ik}$ and $E_{ijk}$ are random variables that are normally distributed with zero mean and variances $\sigma_u^2$ and $\sigma_e^2$ respectively. The random variables $U_{ik}$ and the $E_{ijk}$ are, as in the original model, assumed to be stochastically independent. Also all the error terms $E_{ijk}$ are supposed to be independent. For the variables $U_{ik}$ and $U_{il}$, however, we allow that they are correlated with correlation coefficient $\rho_{kl}$

$$\rho_{kl} = \rho(U_{ik}, U_{il})$$ (6.3)

since an individual that buys a product frequently in one period, is expected to do the same in the next period. The number of parameters of the model depends on the number of periods. Suppose that we observe $p$ periods then there are $3p$ parameters to determine the
marginal distributions and another $\frac{1}{2} p(p-1)$ parameters to determine the correlation coefficients. Thus the total number of parameters is $3p + \frac{1}{2} p(p-1)$.

If we assume that the processes in different periods are defined by different values for the parameters then we have to make extra assumptions about interevent times that do not fall completely within one time period. Figure 6.1 shows a realisation of the purchasing process of a certain individual over four periods. The situation is clear if an event starts and ends within the same period. For example the interevent time $X_2$ is completely within a certain period (period 1), and will be modelled using the parameters of period 1. The situation with the interevent times $X_1$, $X_3$, and $X_4$ is less clear. It is necessary to make some extra assumptions about the distribution of these interevent times.

Figure 6.1 A realisation of a renewal process over four periods.

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>period 1</td>
<td>period 2</td>
<td>period 3</td>
<td>period 4</td>
</tr>
</tbody>
</table>

One option is to assume that an interevent time is distributed according to the model parameters of the period in which it started. Then the observation $X_3$ in Figure 6.1 is modelled according to the parameter values of period 1. The right censored observation $X_4$ is distributed according the parameter values of period 3. For the left censored interevent time $X_1$, we make the extra assumption that the model parameters of the first period also hold in the time previous to the first period. We can give these assumptions a physical interpretation if we think of the classical example of a renewal process where items break down and have to be replaced. Suppose that in different periods items are constructed with parameters of that period. If an item breaks down in period $k$, it will be immediately replaced by an item that was constructed in period $k$, and its renewal time will depend on the parameters of period $k$. We will refer to these assumptions as forward inheritance; a duration inherits the parameters of the period of its start. In a similar way we can define backward inheritance, where a duration inherits the parameter values of the period of its end. An alternative approach that we shall not use in this chapter can be formulated in terms of the hazard rate. We could assume that the changes from period to period are given by a change in hazard function. Such a change can be interpreted as a change in the environment that influences the life expectancy of an item. Unfortunately this approach requires a reformulation of the model in terms of hazard rates, for which the parameter estimation too hard to implement.

In this chapter we will discuss estimation methods for the case where we have two periods. Theoretically it is easy to generalise the model to more than two periods, but its estimation will be more and more difficult, since from the number of parameters is increasing rapidly.
with the number of parameters. In section 6.2 we will discuss a rather heuristic estimation method, where we split the parameter estimation into two stages. In the first stage we estimate the marginal distributions independently, and in the second stage we estimate the dependence structure given these margins. In section 6.3 we will discuss a method where all parameters are estimated simultaneously. In section 6.4 we will discuss how to estimate penetration from the parameter estimates. In section 6.5 we will use the discussed techniques to estimate the model parameters and penetration of certain goods by use of empirical data.

6.2 Parameter estimation based on correlation in a two way table

In this section we will discuss a method to estimate the variance component model that uses two steps. In the first step we estimate the marginal distributions independently, using the methods that were discussed in chapter 5. In the second step we estimate the dependency by use of the two way table of numbers of households that did a certain number of purchases in period 1 and in period 2 (see Table 6.1).

Table 6.1 Two way table showing numbers of households that made \( M_1 \) purchases of pork in \( t_1 \) (ten weeks in the third quarter of 1994) and \( M_2 \) purchases in \( t_2 \) (ten weeks in the fourth quarter of 1994).

<table>
<thead>
<tr>
<th>( M_1 )</th>
<th>( M_2 = 0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>65</td>
<td>15</td>
<td>3</td>
<td>7</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>14</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>11</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In order to formulate an estimator for the model parameter for the dependency \( \rho_{12} \) from the observed table we need to study the relationship between the theoretical dependency parameter and the empirical table. Therefore we will derive statistical properties for the observable variable \( M(t) \), the number of events (purchases) in a given time interval \([0,t]\), and for the latent variable \( f \) that denotes the (expected) frequency of action. In section 6.2.1 we will derive these properties for a given individual and in section 6.2.2 for a heterogeneous population. We will study the relation between the empirical and latent variable by discussing the notion of attenuation in section 6.2.3. In section 6.2.4 we will look at the relationship between the latent frequencies of actions in different periods, and finally in section 6.2.5 we will put all ingredients together in order to derive an estimator for the dependency parameter from the two way table.
6.2.1 The number of events in a time interval for a given individual

We are interested in the distribution of \( M(t) \), the number of events in a given time interval \([0, t]\) for a given individual. Although the distribution function is determined by equation (6.1), it is not easy to find an analytical expression for it. From renewal theory (see Cox, 1962, section 3.3) we know that the limiting distribution of \( M(t) \) as \( t \to \infty \) is a normal distribution with asymptotic expectation and variance

\[
E[M(t)] = \frac{t}{v}
\]

and

\[
Var[M(t)] = \frac{\tau^2 t}{v^3}
\]

where \( v \) denotes the expected value of the interevent time \( X \) of the given individual and \( \tau^2 \) denotes its variance. Note that both the asymptotic expectation and variance are linear expressions in \( t \). We will also write \( E[M(t)] = ft \), where \( f = E[M(1)] = 1/v \) is the frequency of the individual renewal process per time unit. We know that for a process that is in equilibrium equation (6.4) is precise, and that for equation (6.5) it is possible to obtain the better approximation (see Cox, 1962, equations 4.1.3 and 4.5.18)

\[
Var[M(t)] = \frac{\tau^2 t}{v^3} + \left( 1 + \frac{\tau^4}{2v^4} - \frac{v_3}{3v^3} \right) + o(1)
\]

where \( v_3 \) is the third central moment of the interevent time \( X \) of the individual, and where \( o(1) \) denotes a function of \( t \) that approaches zero as \( t \) tends to infinity. From equation (6.1) it follows that for an individual for which \( U_j = u \)

\[
\nu = \exp(\mu + \frac{1}{2} \sigma^2 + u), \text{ and}
\]

\[
\tau^2 = \exp(2\mu + 2u + \sigma^2)(\exp(\sigma^2) - 1).
\]

As a consequence we find the following expressions for the expectation and asymptotic variance of \( M(t) \) of an individual for which \( U_j = u \)

\[
E[M(t)] = t \exp(-\mu - \frac{1}{2} \sigma^2 - u)
\]

and

\[
Var[M(t)] = t \exp(-\mu - \frac{1}{2} \sigma^2 - u)(\exp(\sigma^2) - 1)
\]
The frequency of actions $f$ of an individual for which $U_i = u$ is equal to

$$f = E[M(1)] = \exp(-\mu - \frac{1}{2}\sigma_e^2 - u)$$  \hspace{1cm} (6.9)

### 6.2.2 The number of events in a time interval in a heterogeneous situation

In our model the different individuals follow different renewal processes. The heterogeneity in the process is modelled by $U_i$ in equation (6.1). We define $M(t)$ as the number of events during the time interval $[0,t]$ of a randomly drawn individual. Then

$$E[M(t)] = E_U[E[M(t)\mid U]]$$

$$= t \exp(-\mu - \frac{1}{2}\sigma_e^2)E_U[\exp(-U)]$$

$$= t \exp(-\mu - \frac{1}{2}\sigma_e^2 + \frac{1}{2}\sigma_u^2)$$  \hspace{1cm} (6.10)

and if $t$ is large

$$\text{Var}(M(t)) = \text{Var}_U(E[M(t)\mid U]) + E_U[\text{Var}(M(t)\mid U)]$$

$$= t^2 \exp(-2\mu - \sigma_e^2)\text{Var}_U(\exp(-U)) + t \exp(-\mu - \frac{1}{2}\sigma_e^2)(\exp(\sigma_e^2) - 1)E_U[\exp(-U)]$$

$$= t^2 \exp(-2\mu - \sigma_e^2 + \sigma_u^2)(\exp(\sigma_u^2) - 1)$$

$$+ t \exp(-\mu - \frac{1}{2}\sigma_e^2 + \frac{1}{2}\sigma_u^2)(\exp(\sigma_u^2) - 1)$$  \hspace{1cm} (6.11)

Here we used the rough approximation of $\text{Var}(M(t)\mid U)$ that can be improved by use of equation (6.6) instead of (6.5). Note that in the heterogeneous situation the asymptotic expectation is still linear in $t$, but that the asymptotic variance here is quadratic in $t$. Notice further that elimination of the heterogeneity by choosing $\sigma_u^2 = 0$ makes that equations (6.10) and (6.11) become equations (6.4) and (6.5) of the homogeneous case. For the frequency $f$ of the heterogeneous renewal process, we find the following average and variance

$$E[f] = \exp(-\mu - \frac{1}{2}\sigma_e^2 + \frac{1}{2}\sigma_u^2)$$  \hspace{1cm} (6.12)

$$\sigma_f^2 = \text{Var}(f) = \exp(-2\mu - \sigma_e^2 + \sigma_u^2)(\exp(\sigma_u^2) - 1)$$  \hspace{1cm} (6.13)
6.2.3 Attenuation

Attenuation denotes the phenomenon that correlations between observed variables are lower than those between the underlying latent variables because of the measurement error that occurs when variables are observed. In the case of renewal processes a similar thing happens. Relationships between latent variables are attenuated in the relationships between the observed variables because the process is observed through a finite time window. The attenuation in a renewal process is defined as the correlation between \( M(t) \) the observed number of events in time interval \([0, t]\) and \( f \) the expected number of events in time interval \([0, t]\)

\[
c(t) = \rho(M(t), f) \]

(see equation (4.8)). Since the covariance

\[
Cov(M(t), f) = E[M(t)f] - EM(t)Ef
\]

\[
= E_{U} [E[M(t) f | U]] - tE[f]^{2} = E[tf^{2}] - tE[f]^{2}
\]

\[
= t \sigma_{f}^{2}
\]

it follows that

\[
c^{2}(t) = \frac{t \sigma_{f}^{2}}{Var(M(t))} \tag{6.14}
\]

If we approximate \( Var(M(t)) \) by use of equation (6.5), then we obtain the following estimator given the estimates \( \hat{\mu}, \hat{\sigma}_{u}^{2} \) and \( \hat{\sigma}_{r}^{2} \).

\[
\hat{c}^{2}(t) = \frac{t \exp(-\hat{\mu} - 0.5 \hat{\sigma}_{u}^{2} + 0.5 \hat{\sigma}_{r}^{2})(\exp(\hat{\sigma}_{u}^{2}) - 1)}{t \exp(-\hat{\mu} - 0.5 \hat{\sigma}_{u}^{2} + 0.5 \hat{\sigma}_{r}^{2})(\exp(\hat{\sigma}_{u}^{2}) - 1) + (\exp(\hat{\sigma}_{r}^{2}) - 1)} \tag{6.15}
\]

Alternatively we may use the approximation of equation (6.6) that takes the intercept into account or approximate the variance by simulations. Note that if in equation (6.15) \( t = 0 \) then \( \hat{c}(t) = 0 \), and if \( t \to \infty \) then \( \hat{c}(t) \to 1 \). For a heterogeneous renewal process that is deterministic at the individual level, (i.e. \( \sigma_{u}^{2} > 0 \) and \( \sigma_{r}^{2} = 0 \)), we have \( c(t) = 1 \) for all \( t \).
6.2.4 Correlations between frequencies of action in different periods

Model (6.2) implies that the average frequencies of the renewal process in different periods are correlated. For a random individual for which \( U_{ik} = u_k \), the frequency of actions in period \( k \)

\[
    f_{ik} = \exp(-\mu_k - \frac{1}{2} \sigma_k^2) \exp(-u_k)
\]

It follows that

\[
    \rho(f_{ik}, f_{il}) = \rho(\exp(U_{ik}), \exp(U_{il}))
\]

where the vector \( U = [U_k, U_l] \) is bivariate normal distributed with expectation vector zero and covariance matrix

\[
    \Sigma_{kl} = 
    \begin{bmatrix}
        \sigma_{uk}^2 & \rho_{uk} \sigma_{uk} \sigma_{ul} \\
        \rho_{ul} \sigma_{uk} \sigma_{ul} & \sigma_{ul}^2
    \end{bmatrix}
\]

We can write

\[
    \rho(f_{ik}, f_{il}) = \frac{\text{Cov}_U \{E[f_k|U], E[f_l|U]\} + E_U \{\text{Cov}(f_k, f_l|U)\}}{\sigma_{f_k} \sigma_{f_l}}
\]

where \( \sigma_{f_k} \) and \( \sigma_{f_l} \) are given by equation (6.13). Conditional on the vector \( U \) the second term of the numerator is zero. The covariance of the expected frequencies given the vector \( U \) (the first term of the numerator) can be derived from JOHNSON and KOTZ, 1972, equation 34.28.1, leading to

\[
    \rho(f_{ik}, f_{il}) = \frac{\exp(\rho_{uk} \sigma_{uk} \sigma_{ul}) - 1}{\sqrt{(\exp(\sigma_{uk}^2) - 1)(\exp(\sigma_{ul}^2) - 1)}}
\]

(6.16)

6.2.5 An estimator for the dependency parameter given the marginal distributions

At this stage we derived all ingredients to construct an estimator for \( \rho_{kl} \) from the observed two-way table, if we consider the parameter estimates of the marginal distributions to be given. Our estimator is based on the equation
\[
\rho(f_k, f_l) = \frac{\rho(M_k(t_k), M_l(t_l))}{c_k(t_k)c_l(t_l)} 
\]

(6.17)

that relates the correlation of the latent variables to the correlation of the observed variables. The correlation of the latent variables (the left-hand side of equation (6.17)) is related to the model parameter \(\rho_{ul}\) by equation (6.16). For the correlation of the observed variables (the numerator in equation (6.17)) we have a direct estimate \(\hat{\rho}_{\text{2-way}}\), the observed correlation in the two-way table.

Now we are able to formulate a strategy to estimate the full model of equation (6.2) in the case of two periods. We start with estimation of the univariate marginal distributions of period \(k\) and \(l\) separately. Next we estimate the dependency parameter \(\rho_{kl}\) by

\[
\hat{\rho}_{kl} = \frac{1}{\hat{\sigma}_{uk}\hat{\sigma}_{ul}} \ln \left(1 + \hat{\rho}_{\text{2-way}} \sqrt{\frac{(\exp(\hat{\sigma}_{uk}^2) - 1)(\exp(\hat{\sigma}_{ul}^2) - 1)}{\hat{c}_k(t_k)\hat{c}_l(t_l)}}\right) 
\]

(6.18)

where \(\hat{c}_k(t_k)\) and \(\hat{c}_l(t_l)\) are estimates of the attenuation in period \(k\) and \(l\) respectively, given by equation (6.14). We will discuss some results of this estimation approach as we apply these methods to empirical data in section 6.5.

6.3 A method for simultaneous estimation of all parameters

In this section we will describe how we can estimate the variance component model by a simultaneous estimation of all model parameters. We have to assume that the interevent times have the property of either forward inheritance or backward inheritance (see section 6.1). From a theoretical point of view this approach is rather simple, since it is a straightforward extension of the method used to estimate the univariate model. Given the computation problems in univariate case, it is no surprise that estimation in the multivariate case can not be done without some simplifications. Therefore we will reduce the number of parameters. Therefore we will assume homoscedasticity in both variance components.

Under the assumption of forward inheritance we can write down the distribution of an observed vector of interevent times. If we assume that we are dealing with an ordinary renewal process, then the logarithmic transformed interevent times are multivariate normal with known expectation vector and covariance structure. Suppose that we observe one individual during two periods with \(m_1\) observations in the first and \(m_2\) observations in the second period, then the logarithmic transformed interevent times are multivariate normal with an \(m_1 + m_2\)-dimensional expectation vector consisting of \(m_1\) times \(\mu_1\) and \(m_2\) times \(\mu_2\). The covariance matrix has the form
The likelihood function of an observed vector of interevent times can be written as a product of conditional probabilities (see equation (5.2)). The distribution of (logarithmic transformed) interevent times given the past, are normal distributions with known conditional expectations $\xi_i$ and variances $\tau_i^2$ (see equations (5.26) and (5.27)). Computation of $\xi_i$ and $\tau_i^2$ involves the inversion of (top left) submatrices of $\Sigma$. The inversion can be done numerically, but this would require large computation times to find ML-estimates. The estimation procedure would speed up drastically if we would have analytical expressions at our disposal.

For the case with two periods we can derive analytic expressions for $\xi_i$ and $\tau_i^2$. We distinguish two situations. If the history consists of observations of the first period only then a top left submatrix is of the simple form (6.19), and we are in the situation of chapter 5 where we use equations (5.4) and (5.5) for $\xi_i$ and $\tau_i^2$ respectively. If the history contains both observations from the first and the second period, then we have to invert a matrix of the form (6.19), where $\Sigma_{11}$ is a matrix of dimension $m_1$ and $\Sigma_{22}$ is a square matrix of dimension $i - m_1 - 1$. In that case

$$\xi_i = \mu_2 + \gamma_1 m_1 (\bar{x}_1 - \mu_1) + \gamma_2 (i - m_1 - 1)(\bar{x}_2 - \mu_2)$$  \hspace{1cm} (6.20)$$

and

$$\tau_i^2 = \sigma_u^2 + \sigma_e^2 - \gamma_1 \rho \sigma_u^2 - \gamma_2 \sigma_e^2$$  \hspace{1cm} (6.21)$$

where $\bar{x}_1$ and $\bar{x}_2$ are averages of the first $m_1$ and the last $i - m_1 - 1$ log-transformed interevent times respectively, and where $\gamma_1$ and $\gamma_2$ are functions of the model parameters $\sigma_u^2$, $\sigma_e^2$ and $\rho$ (see proposition 1 in section 6.6).
So far we ignored left censoring and thereby implicitly assumed that the process is an ordinary renewal process. We can deal with left censoring by use of a distribution that approximates the forward recurrence time (see section 5.3.2). In proposition 2 in section 6.7 it is shown, that if we use approximation (B) then the conditional expectation becomes

\[ \mathbb{E} = \beta_i + m_i \left( \bar{x}_i - \mu_i \right) + \gamma_i (i - m_i - 1)(\bar{x}_i - \mu_i) - \gamma_i, \quad i = 1, \ldots, A \]  

(6.22)

and \( \tau_i \) is given by equation (6.21), the same as for the ordinary renewal process. The use of approximation (A) for the forward recurrence time puts us for computational problems, because the top left element of the covariance matrix (6.19) is now equal to \( \sigma_e^2 + \sigma_{e_i}^2 + c_i^{(a)} \).

For the multivariate case where we deal with \( p \) \((p > 2)\) periods, we have a matrix that consists of \( p^2 \) submatrices \( \Sigma_{ij} \). Then the inversion has to be done numerically, and we may expect large computation times.

### 6.4 Estimating product period penetrations for the combined period.

The product period penetration is the probability that a household buys at least one item of a certain product in the given period. If we assume forward inheritance then the \( t \)-period penetration is

\[ P(X_t \leq t) = F_{N(\mu, \sigma^2_{\mu}, \sigma^2_e)}(\log(t)) \]  

(6.23)

(see equation 5.16). An estimator for the penetration is obtained by substitution of the parameter estimators for \( \mu, \sigma^2_{\mu}, \) and \( \sigma^2_e \). Note that this quantity is determined by the parameter estimates of the first period only. Suppose we use the estimation method where we first estimate the marginal distribution independently and afterwards estimate the dependency parameter from the two-way table (see section 6.2). Then with respect to penetration estimation, it is sufficient to estimate the parameters of the first margin only! The consumer behaviour in the second period and dependencies of behaviour between the periods are of no consequence. Also if we estimate the full model (see section 6.3), then the estimator of the penetration does depend on parameter estimates of the first period only, and is independent of the behaviour in the last period. Under the assumption of backward inheritance the situation is similar. In that case the penetration would entirely depend on the parameters of the last period.

### 6.5 Results for empirical data

In this section we will apply both methods to the empirical data, that consist of purchases of the products pork, beef and veal in the third and in the fourth quarter of 1994. We used the
data of 524 households that reported 10 weeks in the third quarter and 10 week in the fourth quarter of 1994. In section 6.5.1 we will discuss the application of the estimation method where the marginal distributions and the dependency structure are estimated separately (see section 6.2). In section 6.5.2 we will discuss the application of the method where the full model is estimated with the constraint that $\sigma^2_{uk} = \sigma^2_{ul} = \sigma^2_{u}$ and $\sigma^2_{dk} = \sigma^2_{dl} = \sigma^2_{d}$ (see section 6.3).

6.5.1 Estimation based on correlation in a two way table

To estimate the marginal distributions we used the methods that were discussed in chapter 5. We had to decide on how we wanted to deal with left censoring and on which imputation method we wanted to use for the purchase times. We used the lognormal approximation (A) distribution for the forward recurrence time (see section 5.3), and imputed purchase times by the uniform$^a$ imputation method (see section 5.4 and 5.5). The results of these univariate estimations are shown in Table 6.2. The parameters $\mu_1$, $\sigma_{u1}^2$ and $\sigma_{d1}^2$ refer to the third quarter and the parameters $\mu_2$, $\sigma_{u2}^2$ and $\sigma_{d2}^2$ refer to the fourth quarter of 1994. Note that the estimates of the third quarter in Table 6.2 differ slightly from those in chapter 5 (see Table 5.6). This is a consequence of the fact that the estimates here are based on a subsample of 524 households of the total sample of 1625 households. We also see a slight difference between the values of the parameter sets for the two periods. The last column of Table 6.2 shows direct estimates of $p_{2-way}$, the correlation in the two-way tables. The correlation is somewhat lower for veal than for the products pork and beef. We expect that this is a result of a higher attenuation due to the fact that 10 weeks is a relative short period for the product veal.

Table 6.2 Parameter estimates for marginal distributions of the full variance component model and direct estimates of correlations in the two way table for the products pork, beef and veal.

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1$</th>
<th>$\sigma_{u1}^2$</th>
<th>$\sigma_{d1}^2$</th>
<th>$\mu_2$</th>
<th>$\sigma_{u2}^2$</th>
<th>$\sigma_{d2}^2$</th>
<th>$p_{2-way}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pork</td>
<td>0.465</td>
<td>0.595</td>
<td>1.038</td>
<td>0.030</td>
<td>0.587</td>
<td>1.011</td>
<td>0.793</td>
</tr>
<tr>
<td>beef</td>
<td>0.994</td>
<td>0.891</td>
<td>0.925</td>
<td>0.836</td>
<td>0.841</td>
<td>0.960</td>
<td>0.836</td>
</tr>
<tr>
<td>veal</td>
<td>5.810</td>
<td>4.672</td>
<td>1.274</td>
<td>4.997</td>
<td>2.207</td>
<td>1.001</td>
<td>0.650</td>
</tr>
</tbody>
</table>

As a next step in the estimation procedure of the parameter $\rho_{12}$, we need to estimate the attenuation in each period (see equation (6.18)) by use of equation (6.14). We have three alternatives to estimate the variance $\text{Var}(M(t))$ given a set of parameters. The first two alternatives are based on the asymptotic approximations. Alternative 1 is to use the linear approximation of equation (6.5) and leads to the estimator of equation (6.15). Alternative 2 is to use the approximation of equation (6.6) that uses both the linear part and the intercept. As a third alternative we used simulations (sample size 50,000) to approximate the variance. Table 6.3 shows results for all three alternatives. For the product veal we found that alternatives 2 and 3 lead to estimates for $\text{Var}(M(t))$ that are lower than those of $t^2\sigma^2_f$. 

103
Therefore we fixed the estimate for the attenuation at its upper bound 1. Estimates for the model parameter $\rho_{12}$ are obtained from equation (6.18). Note that in almost all cases we find estimates larger than 1. The independent estimation of the marginals and the dependency structure does not give us good estimates. We hope that a simultaneous estimation of the dependency structure with the margins will prove to be better.

Table 6.3. Estimates for the attenuations and for the dependency parameter $\rho_{12}$ the products pork, beef and veal.

<table>
<thead>
<tr>
<th></th>
<th>asymptotic approximation</th>
<th></th>
<th>asymptotic approximation</th>
<th></th>
<th>approximation based on simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'linear'</td>
<td></td>
<td>'linear plus intercept'</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_1(t_1)$</td>
<td>$c_2(t_2)$</td>
<td>$\rho_{12}$</td>
<td>$c_1(t_1)$</td>
<td>$c_2(t_2)$</td>
</tr>
<tr>
<td>pork</td>
<td>0.832</td>
<td>0.856</td>
<td>1.084</td>
<td>0.886</td>
<td>0.892</td>
</tr>
<tr>
<td>beef</td>
<td>0.880</td>
<td>0.877</td>
<td>1.054</td>
<td>0.922</td>
<td>0.919</td>
</tr>
<tr>
<td>veal</td>
<td>0.933</td>
<td>0.606</td>
<td>1.104</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

6.5.2 Simultaneous estimation of all parameters

In this section we will discuss the results of simultaneous estimation of all model parameters for the purchasing processes of the products pork, beef and veal in the third and in the fourth quarter of 1994. We used the constraint that $\sigma_{u1}^2 = \sigma_{e1}^2$ and $\sigma_{u2}^2 = \sigma_{e2}^2$. This seems a reasonable assumption given the values in Table 6.2 and keeping the standard errors in mind that we saw in chapter 5 (see Table 5.6). The constraint leaves us with five parameters $\mu_1$, $\sigma_{u1}^2$, $\sigma_{e1}^2$, $\mu_2$ and $\rho$.

Table 6.4 Estimates for the parameter and the penetration under the forward inheritance assumption.

<table>
<thead>
<tr>
<th></th>
<th>$\mu_0$</th>
<th>$\sigma_{u1}^2$</th>
<th>$\sigma_{e1}^2$</th>
<th>$\mu_2$</th>
<th>$\rho$</th>
<th>penetration</th>
</tr>
</thead>
<tbody>
<tr>
<td>pork</td>
<td>0.520</td>
<td>0.858</td>
<td>0.962</td>
<td>0.392</td>
<td>0.964</td>
<td>0.961</td>
</tr>
<tr>
<td>beef</td>
<td>0.967</td>
<td>1.119</td>
<td>0.935</td>
<td>0.909</td>
<td>0.999</td>
<td>0.921</td>
</tr>
<tr>
<td>veal</td>
<td>5.783</td>
<td>4.751</td>
<td>1.300</td>
<td>5.601</td>
<td>1.000</td>
<td>0.142</td>
</tr>
</tbody>
</table>

Table 6.4 shows results under the assumption of forward inheritance where we used approximation (B) for the forward recurrence times. A comparison with the values of the parameter estimates in Table 6.2 shows us that the values are different, but that there is no clear pattern. We found that the Newton Raphson algorithm was insensitive for starting values: the same estimates were obtained if different starting values for the parameter estimates were used. We computed penetrations by use of equation (6.23) only for the products pork and beef where we obtained parameter estimates inside their domain. For the product veal we found a parameter estimate for $\rho$ outside its domain.
If we change the assumption of forward inheritance into backward inheritance we find parameter estimates that are rather different (see Table 6.5). The estimator of the penetration however is not as sensitive to this change of assumption.

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1$</th>
<th>$\sigma^2_a$</th>
<th>$\sigma^2_v$</th>
<th>$\mu_2$</th>
<th>$p$</th>
<th>penetration</th>
</tr>
</thead>
<tbody>
<tr>
<td>pork</td>
<td>0.438</td>
<td>0.954</td>
<td>0.943</td>
<td>0.570</td>
<td>0.974</td>
<td>0.964</td>
</tr>
<tr>
<td>beef</td>
<td>0.904</td>
<td>1.159</td>
<td>0.902</td>
<td>1.049</td>
<td>0.983</td>
<td>0.929</td>
</tr>
<tr>
<td>veal</td>
<td>3.996</td>
<td>2.081</td>
<td>2.286</td>
<td>4.811</td>
<td>1.220</td>
<td>0.141</td>
</tr>
</tbody>
</table>

6.6 Discussion and conclusions

In this chapter we studied the possibilities of applying the variance component model to the situation where we have more than one period. Theoretically it is easy to generalise the model, but we need extra assumptions to deal with spells that cross periods. We introduced the notions of forward and backward inheritance in order to deal with this problem. Next we described two methods to derive estimates for the model. The first method is a rather ad hoc method that starts with an independent estimation of the marginal distributions and where the dependence structure is estimated by a generalised moment estimator based on the observed 2-way table is derived. The second method is the maximum likelihood estimation method. For the case where we have two periods, we were able to derive an analytical form of the likelihood function under the restriction that $\sigma^2_{u1} = \sigma^2_{u2}$ and $\sigma^2_{v1} = \sigma^2_{v2}$.

We applied the two methods to empirical data in order to obtain parameter estimates for the purchasing processes of the products pork, beef and veal in the third and in the fourth quarter of 1994. We found that the first method gave us estimates for the dependency parameter outside its domain. For the second option we obtained reasonable estimates for the parameters, although we found different estimates under different assumptions of inheritance. With respect to the estimation of the penetration the method does not guarantee consistency in the sense that the penetration of the combined period is by definition higher than those in the individual periods. This is because the penetration under forward (or backward) inheritance is completely determined by the parameters of the first (or last) period, and totally independent from the parameters of the other period. As a way out we may consider to use a weighted average between the estimates based on forward and backward inheritance. This may solve the problem of inconsistent estimates for the penetration, although a strategy of determining the weights is an object of future research.
6.7 Appendix

In this appendix we will derive the conditional distribution of the logarithmic transformed interevent time (or spell) \( \log(X_i) \) given the set of previous spells \( X_1, X_2, \ldots, X_{i-1} \) for the variance component model in the case that we have two periods. Suppose that the spells \( X_{ij} \) of a renewal process follow the model

\[
\log(X_{ij}) = \mu_k + U_{ik} + E_{ijk}
\]  

(6.24)

where \( U_{ik} \) and \( E_{ijk} \) are stochastic random variables that are normal distributed with zero expectation and variances \( \sigma^2_{uk} \) and \( \sigma^2_{ijk} \) respectively. The index \( i \) refers to an individual, the index \( k \) to a period, and the index \( j \) to an event of individual \( i \) in period \( k \). We assume that within a period the random variables \( U_{ik} \) and \( E_{ijk} \) are independent, and the same for the random variables \( E_{ijk} \) and \( E_{ijl} \). For the random variable \( U_{ik} \) and \( U_{il} \), however, we assume that they may be correlated with a coefficient \( \rho_{ul} \). If we have two periods then the model has seven parameters: \( \mu_1, \sigma^2_{u1}, \sigma^2_{e1}, \mu_2, \sigma^2_{u2}, \sigma^2_{e2} \) and \( \rho \). The first three parameters describe the process in the first period, the second three describe the process in the second period, and \( \rho \) describes the dependence between the two processes. We will use reduce the number of parameters to five by the restrictions that \( \sigma^2_{u1} = \sigma^2_{e2} \) and \( \sigma^2_{e1} = \sigma^2_{e2} \). For spells that cross periods we will use the assumption of forward inheritance (see section 6.1). We are interested in the conditional distribution of \( \log(X_i) \) given the set of previous spells \( X_1, X_2, \ldots, X_{i-1} \). The case where the previous observations are all from the same period (period 1) was discussed in chapter 5. For the case where the \( i-1 \) previous observations exist of \( m \) observations in period 1 and \( n \) observations in period 2, we have the following proposition

**Proposition 1.** We observe spells of a renewal process in two periods, and assume that these spells follow model (6.24) with the restriction that \( \sigma^2_{u1} = \sigma^2_{u2} \) and \( \sigma^2_{e1} = \sigma^2_{e2} \). We further assume forward inheritance for spells that cross periods (see section 6.1). If the process is an ordinary renewal process then \( \log(X_i) \) given previous spells follows a normal distribution

\[
\log(X_i) \mid \log(X_1) \ldots \log(X_{i-1}) \sim N(\xi_i, \tau^2_i)
\]  

(6.25)

with expectation

\[
\xi_i = \mu_2 + m\gamma_1(\bar{x}_1 - \mu_1) + n\gamma_2(\bar{x}_2 - \mu_2)
\]  

(6.26)
and variance

\[ \tau_i^2 = \sigma_u^2 + \sigma_e^2 - \rho \sigma_u^2 \gamma_1 - \sigma_u^2 \gamma_2 \]  

(6.27)

where \( \bar{x}_1 \) is the average over the first \( m \) \( \log(X_1) \)'s, \( \bar{x}_2 \) is the average over the last \( n \) \( \log(X_j) \)'s, and

\[ \gamma_1 = \rho \sigma_u^2 \beta (1 - n \sigma_u^2 \alpha) \]

\[ \gamma_2 = -m \rho^2 \sigma_u^2 \alpha \beta + \sigma_u^2 \alpha (1 + n m \rho^2 \alpha \beta) \]

\[ \alpha = \frac{1}{\sigma_e^2 + n \sigma_u^2} \]

\[ \beta = \frac{\sigma_u^2 + n \sigma_u^2}{(\sigma_e^2 + m \sigma_u^2)(\sigma_e^2 + n \sigma_u^2) - mn \rho^2 \sigma_u^4} \]

**Proof.** Under the assumptions of proposition 1 the vector of observed log-transformed spells is multivariate normal distributed. Suppose that the vector contains \( m \) spells of the first period and \( n \) of the second, then we have that the \( m + n \)-vector \( \log(X) \) is normal distributed with expectation vector \( \mu \) and covariance matrix \( \Sigma \).

\[ \log(X) \sim N(\mu, \Sigma) \]

The expectation vector \( \mu \) consists of two parts containing \( m \) times \( \mu_1 \) and \( n \) times \( \mu_2 \)

\[ \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix} \]  

(6.28)  

Likewise the covariance matrix \( \Sigma \) consists of four parts

\[ \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \]


\[
\begin{bmatrix}
\sigma_u^2 + \sigma_e^2 & \sigma_u^2 & \cdots & \sigma_u^2 & \rho \sigma_u^2 & \cdots & \rho \sigma_u^2 \\
\sigma_u^2 & \sigma_u^2 + \sigma_e^2 & \rho \sigma_u^2 & \cdots & \rho \sigma_u^2 & \cdots & \rho \sigma_u^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\rho \sigma_u^2 & \rho \sigma_u^2 & \cdots & \sigma_u^2 + \sigma_e^2 & \rho \sigma_u^2 & \cdots & \rho \sigma_u^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\rho \sigma_u^2 & \rho \sigma_u^2 & \cdots & \rho \sigma_u^2 & \sigma_u^2 & \cdots & \sigma_u^2 + \sigma_e^2 \\
\end{bmatrix}
\]

(6.29)

From multivariate theory we know that

\[
\log(X_i)|\log(X_j),...\log(X_{i-1})
\]

is again normal distributed with expectation and variance given by equations (5.26) and (5.27) respectively. If \( i \leq m + 1 \) then we can obtain the conditional expectation and variance in the way we did in chapter 5. If, however, \( i > m + 1 \) then we need to invert top left submatrices of \( \Sigma \) that are of the same form as \( \Sigma \) where \( \Sigma_{11} \) is of dimension \( m \) and \( \Sigma_{22} \) is of dimension \( i - m - 1 \).

We will use basic matrix algebra (see equation (5.28) through (5.31)) to obtain the inverse of a matrix that has the form of the covariance matrix. Define a matrix \( M \) of dimension \((m+n)\times(m+n)\) as

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} a & c & d & \cdots & d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & a & d & \cdots & d \\
d & \cdots & d & a & c \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
d & \cdots & d & c & a \\
\end{bmatrix}
\]

We will use equation (5.29) in order to derive \( M^{-1} \), and need to find an expression for \((M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1}\). We start with finding the inverse of the \( n \times n \)-matrix \( M_{22} \) from (5.32)

\[
M_{22}^{-1} = \begin{bmatrix} p & q & q \\
q & p & q \\
q & q & p \\
\end{bmatrix}
\]
where
\[ p = \frac{a + (n-2)c}{(a-c)(a+(n-1)c)} \]
and
\[ q = \frac{-c}{(a-c)(a+(n-1)c)}. \]

We find
\[
(M_{11} - M_{12}M_{22}^{-1}M_{21}) = \begin{bmatrix}
    a' & c' & \cdots & c' \\
    c' & a' & & \\
    \vdots & & \ddots & \\
    c' & c' & a'
\end{bmatrix}
\]

where
\[ a' = a - \frac{nd^2}{a + (n-1)c} \]
and
\[ c' = c - \frac{nd^2}{a + (n-1)c}. \]

The matrix \(M_{11} - M_{12}M_{22}^{-1}M_{21}\) is an \(m \times m\) matrix and by use of (5.32) we find its inverse
\[
(M_{11} - M_{12}M_{22}^{-1}M_{21})^{-1} = \begin{bmatrix}
p' & q' & \cdots & q'
q' & p' & & \\
\vdots & & \ddots & \\
q' & q' & \cdots & p'
\end{bmatrix}
\]

where
\[ p' = \frac{a' + (m-2)c'}{(a' - c')(a' + (m-1)c')} \quad \text{(6.30)} \]
and
\[ q' = \frac{-c'}{(a' - c')(a' + (m-1)c')} \quad \text{(6.31)} \]

Now we can use equation (5.29) and put all ingredients together to obtain the inverse of the full matrix \(M\). We find
where $p'$ and $q'$ are given by equations (6.30) and (6.31) respectively, and where

\[ r = -d\alpha\beta, \]
\[ s = p - dm\alpha, \]
\[ t = q - dm\alpha, \]
\[ \alpha = p + (n-1)q, \]
\[ \beta = p' + (m-1)q'. \]

Now we return to the computation of the conditional expectation and variance in the case that $i > m + 1$. If we define $n = i - m - 1$ then we have from equation (5.26)

\[ \xi_i = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\log(X) - \mu) \]

where $\Sigma_{21}$ is a vector that consists of $m$ times $\rho\sigma_u^2$ and $n$ times $\sigma_u^2$, where $\Sigma_{11}$ is a square matrix of dimension $m + n$ and of the form (6.19), and where $\mu$ is a vector that consists of $m$ times $\mu_1$ and $n$ times $\mu_2$. If we define $a = \sigma_u^2 + \sigma_v^2$, $c = \sigma_u^2$ and $d = \rho\sigma_u^2$ then $\Sigma_{11} = \mathbf{M}$, and

\[ \xi_i = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\log(X) - \mu) \]

\[ = \mu_2 + \begin{bmatrix} d & \cdots & d & c & \cdots & c \\
\end{bmatrix} \begin{bmatrix} p' & q' & r & \cdots & r \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
q' & p' & r & \cdots & r \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
r & \cdots & r & s & t \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
r & \cdots & r & t & s \\
\end{bmatrix} \begin{bmatrix} \log(X_1) - \mu_1 \\
\vdots \\
\log(X_m) - \mu_1 \\
\vdots \\
\log(X_{m+n}) - \mu_1 \\
\end{bmatrix} \]

where
\[ = \mu_2 + m(d(p' + (m-1)q') + ncr)(\bar{x}_1 - \mu_1) + n(mdr + c(s + (n-1)t))(\bar{x}_2 - \mu_2) \]

where \( \bar{x}_1 \) is the average over the first \( m \) \( \log(X_j) \)'s, \( \bar{x}_2 \) is the average over the last \( n \) \( \log(X_j) \)'s.

\[ \gamma_1 = d(p' + (m-1)q') + ncr \]

\[ = \rho \sigma_v^2 \beta (1 - n \sigma_u^2 \alpha) \]

and

\[ \gamma_2 = mdr + c(s + (n-1)t) \]

\[ = -m \rho^2 \sigma_v^2 \alpha \beta + \sigma_u^2 \alpha (1 + n m \rho^2 \alpha \beta) \]

where, as before,

\[ \alpha = p + (n-1)q \]

\[ = \frac{1}{\sigma_v^2 + n \sigma_u^2} \]

and

\[ \beta = p' + (m-1)q' \]

\[ = \frac{\sigma_v^2 + n \sigma_u^2}{(\sigma_v^2 + m \sigma_u^2)(\sigma_v^2 + n \sigma_u^2) - mn \rho^2 \sigma_u^4} \]

thus proving the conditional expectation in equation (6.26). For the conditional variance we have

\[ \tau^2_i = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \]

\[ = \sigma_u^2 + \sigma_v^2 - \begin{bmatrix} d & \cdots & c \end{bmatrix} \begin{bmatrix} \begin{array}{cccc} p' & q' & r & \cdots & r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & \cdots & r & \cdots & r \end{array} \end{bmatrix} \begin{bmatrix} d \\ \vdots \\ r \end{bmatrix} \]

\[ = \sigma_u^2 + \sigma_v^2 - \begin{bmatrix} d & \cdots & d & c & \cdots & c \end{bmatrix} \begin{bmatrix} \begin{array}{cccc} q' & r & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ r & \cdots & r & \cdots & r \end{array} \end{bmatrix} \begin{bmatrix} q' \\ \vdots \\ r \end{bmatrix} \]

\[ = \sigma_u^2 + \sigma_v^2 - \begin{bmatrix} d & \cdots & d & c & \cdots & c \end{bmatrix} \begin{bmatrix} \begin{array}{cccc} r & \cdots & r & s & \cdots & t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r & \cdots & r & s & \cdots & t \end{array} \end{bmatrix} \begin{bmatrix} c \\ \vdots \\ s \end{bmatrix} \]
\[ = \sigma_u^2 + \sigma_e^2 - m(d(p' + (m-1)q' + ncr) - nc(mdr + c(s + (n-1)r)). \]

\[ = \sigma_u^2 + \sigma_e^2 - m\rho \sigma_u^2 \gamma_1 - n\sigma_u^2 \gamma_2. \]

with \( \gamma_1 \) and \( \gamma_2 \) as in equations (6.32) and (6.33). This proofs equation (6.27), and completes the proof of proposition 1.

Proposition 2. We observe spells of a renewal process in two periods, and assume that these spells follow model (6.24) with the restriction that \( \sigma_{u1}^2 = \sigma_{u2}^2 \) and \( \sigma_{e1}^2 = \sigma_{e2}^2 \). We further assume forward inheritance for spells that cross periods. If the process is in equilibrium and we use approximation (B) for the forward recurrence time then

\[ \log(X_1)|\log(X_1)...\log(X_{i-1}) \sim N(\xi_i, \tau_i^2) \]

(6.34)

follows a normal distribution with expectation

\[ \xi_i = \mu_2 + \gamma_1m(\bar{x}_1 - \mu_1) + \gamma_2n(\bar{x}_2 - \mu_2) - \gamma_1'(\sigma_e^2 + c^{(B)}) \]

(6.35)

and variance

\[ \tau_i^2 = \sigma_u^2 + \sigma_e^2 - \rho \sigma_u^2 \gamma_1 - \sigma_u^2 \gamma_2 \]

(6.36)

Proof. Proposition 2 can be proved in exactly the same way as proposition 1. We need to replace the distribution of \( \log(X_1) \) - the normal \( N(\mu_1, \sigma_u^2 + \sigma_e^2) \) distribution - by the approximation (B) distribution of the forward recurrence time

\[ N(\mu_1 + \sigma_u^2 + c^{(B)}, \sigma_u^2 + \sigma_e^2) \]

(see equation (5.14)). This effects the first element in the expectation vector of the multivariate normal distribution in equation (6.28). The covariance matrix in equation (6.19) remains unchanged, since the variance and the covariances do not change. The use the same covariance matrices as in proposition 1 makes that we find the same conditional variance \( \tau_i^2 \). With respect to the conditional expectation we find a correction of

\[ -\gamma_1'(\sigma_e^2 + c^{(B)}) \]

leading to equation (6.22). Finally we remark that the use of approximation (A) instead of approximation (B) changes the entire covariance structure (see equation (5.13)). This would make the inversion of top left submatrices, and thereby the derivation of \( \xi_i \) and \( \tau_i^2 \), a lot more complicated.