# Reverse time migration-inversion from single-shot data

Stolk, C.C.; De Hoop, M.V.; Op 't Root, T.J.P.M.

**DOI**
10.1190/1.3255475

**Publication date**
2009

**Document Version**
Submitted manuscript

**Published in**
SEG Technical Program Expanded Abstracts

# Reverse time migration-inversion from single-shot data

*Christiaan C. Stolk*, University of Amsterdam, Maarten V. de Hoop, Purdue University and Tim J.P.M. Op't Root, University of Twente*


## SUMMARY

Reverse time migration is well established as an imaging method. In this work we analyze this method from an inverse scattering (that is, true amplitude) point of view. We recall that in the ray-Born approximation the medium is assumed to be the superposition of a smooth velocity model, and a perturbation, containing only high ($\sim$ wavelength scale) wavenumbers. The single reflection data are assumed to be given by the perturbation of the pressure field in the linearization of the acoustic equations. We adapt a wave-equation RTM, based on backpropagation of (common-source) data, such that its resolution only reflects the illumination.


## INTRODUCTION

In reverse time migration (RTM) (Baysal et al., 1983; Biondo, 2006) an image is formed using two fields, the incoming field, or source field $u_{\mathrm{inc}}(\mathbf{x},t)$ and the backpropagated receiver field, denoted by $u_{\mathrm{bp}}(\mathbf{x},t)$. The backpropagated receiver field is obtained by solving the wave equation backwards in time using the data as a source (or boundary condition). An image is obtained by taking the correlation of these two fields, given in the frequency domain by

$$I(\mathbf{x}) = \frac{1}{2\pi}\int \overline{\hat{u}_{\mathrm{inc}}(\mathbf{x},\omega)}\hat{u}_{\mathrm{bp}}(\mathbf{x},\omega)\,d\omega. \tag{1}$$

When multiple sources are present, a summation over sources is done. For a single source the muliplication by $\overline{\hat{u}_{\mathrm{inc}}}$ is replaced by a division by $\hat{u}_{\mathrm{inc}}$ to compensate for amplitude variations in the source field. Other imaging conditions have been proposed in the literature (Chattopadhyay and McMechan, 2008).

Reverse time migration has recently attracted substantial interest. Questions studied include the suppression or attenuation of artifacts (Guitton et al., 2007; Fletcher et al., 2005; Xie and Wu, 2006) and migration weights (in multisource prestack migration) (Plessix and Mulder, 2004).

We follow an approach yielding a seamless integration of RTM, (double-square-root) downward continuation based migration and imaging using generalized Radon transform inversion. Concerning the latter, in the work by Beylkin (1985) a determinant was derived and it was established that, in a ray-Born approximation, Kirchhoff migration can be reformulated as an inversion procedure, in an asymptotic sense, and up to aperture effects; see also (Bleistein et al., 2001), Chapter 4.

We adapt a wave-equation RTM, based on backpropagation of (common-source) data, such that its resolution only reflects the illumination. In ray-Born inversion methods, the medium velocity is written as the sum of a smooth velocity model $v_0$ and a perturbation $\delta v$

$$v(\mathbf{x}) = v_0(\mathbf{x}) + \delta v(\mathbf{x}).$$

(In our analysis the density will be assumed constant) The scattered wave field $u_{\mathrm{scat}}(\mathbf{x},t)$ is approximated by a linearization in $v$ and satisfies hence

$$\left(\frac{1}{v_0^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right)u_{\mathrm{scat}}(\mathbf{x},t) = \frac{2\delta v}{v_0^3}\partial_t^2 u_{\mathrm{inc}}. \tag{2}$$

Our goal is to reconstruct $\delta v(\mathbf{x})$ given $u_{\mathrm{scat}}$ at some (densely sampled) part of the surface, and for some time interval $[0, T_{\max}]$.

In the next section we describe the principles of the inversion procedure. In the subsequent section, we present synthetic examples. We end with some further discussion of the results.

## THEORY

To explain the basics, we consider a toy problem. The right hand side in (2) can be written in an asymptotic approximation, in the frequency domain as

$$A(\mathbf{x})\omega^{(n+1)/2}e^{i\omega(t-T(\mathbf{x}))}\,\delta v(\mathbf{x})$$

Now assume the wavelength is small, and consider a piece of $\delta v$ that is nonzero only on a region that is several wavelenghts large, but still small. In this region the velocity $v_0(\mathbf{x})$ and the amplitude factor $A(\mathbf{x})$ can be assumed approximately constant, and the wave front can be assumed approximately planar. This is the basis for our simplified problem.

In the simplified problem we consider a field $u$ (which plays the role of $u_{\mathrm{scat}}$) given by the solution of

$$\left(\frac{1}{v_0^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right)u = A\delta(t - \frac{z}{v_0})r(\mathbf{x}). \tag{3}$$

Here the medium velocity is constant, and we considering scattering from a function $r = r(\mathbf{x})$ when the incoming wave field is a vertically downward propagating wave front. (In fact we replace $\frac{2}{v_0^3}\partial_t^2 u_{\mathrm{inc}}$ by a $\delta$-function $A\delta(t - \frac{z}{v_0})$, and we write $r(\mathbf{x})$ instead of $\delta v(\mathbf{x})$.) The steps in arriving at a single source RTM inversion are now as follows:

1. We determine the solution $u$ to (3), given $r$. Then we determine $r$ given $u$. This is the basis for the RTM inversion formula, but it will not yet be in the right form.

2. The next step is to go back from inversion of $u$ from $r$ in (3) to the original inversion problem for $\delta v$ in a variable background. Then this is written in the form of a RTM formula, using a modified imaging condition. It is assumed in this step that the backpropagated field is true amplitude, see the next point.
3. Next we must discuss the details of constructing $u_{\text{bp}}$. The backpropagated field will not contain all waves appearing in the scattered field due to the fact that some scattered waves do not reach the set of receivers. The waves that are measured at the receiver set and are part of the backpropagated field $u_{\text{bp}}$ need to have correct amplitudes, i.e. be equal in amplitude to those in $u_{\text{scat}}$. In our setup we assume that there is a densely sampled receiver surface at $z = 0$, and the medium extends on both sides (after preprocessing, e.g. multiple removal). The backpropagated field is then obtained from an inhomogeneous wave equation with source

$$-2i\omega v_0^{-1}\delta(z)\sqrt{1 + v_0^2\omega^{-2}\nabla_{\mathbf{x}}^2}\hat{u}(\mathbf{x}, \omega). \tag{4}$$

One can show using one-way wave theory that this yields correct amplitudes. We realize that this situation is idealized, and, in practice, we anticipate to achieve this using additional regularization techniques.

### Constant coefficient modeling and inversion

Equation (3) is an instance of the inhomogeneous wave equation

$$\left(\frac{1}{v_0^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right) u = f, \qquad u(t < 0) = 0.$$

The general solution for this inhomogeneous PDE is most easily obtained in the $(t, \boldsymbol{\xi})$ domain, where it is given by

$$\widehat{u}(t, \boldsymbol{\xi}) = \int_0^t \left(e^{iv_0\|\boldsymbol{\xi}\|(t-s)} - e^{-iv_0\|\boldsymbol{\xi}\|(t-s)}\right) \frac{v_0^2\widehat{f}(s, \boldsymbol{\xi})}{2iv_0\|\boldsymbol{\xi}\|}\, ds.$$

We denote by $\boldsymbol{\xi}$ the Fourier variable of $\mathbf{x}$, by $(\xi, \eta, \zeta)$ its coordinates. Application of this formula yields

$$u(t, \mathbf{x}) = \frac{v_0^2 A}{(2\pi)^3} \operatorname{Re} \int_{\mathbb{R}^3} \frac{e^{i\|\boldsymbol{\xi}\|v_0 t + i\mathbf{x}\cdot\boldsymbol{\xi}}}{iv_0\|\boldsymbol{\xi}\|} \widehat{r}(\xi, \eta, \zeta + \|\boldsymbol{\xi}\|)\, d\boldsymbol{\xi} \tag{5}$$

for the solution of (3), assuming $t$ is such that the incoming wave has completely passed the area where $r$ is nonzero.

We next discuss the reconstruction of $r$. The basic idea of imaging is to correlate the source field with the receiver field. Approximating the source field by $A\delta(t - z/v_0)$ this becomes evaluating the receiver field at the arrival time of the incoming wave and multiplication by $A$. So a first guess for the image would be

$$I_0(\mathbf{x}) = Au(z/v_0, \mathbf{x}).$$

This yields an algorithm that is not true amplitude. By some knowledge in advance one defines instead,

$$I(\mathbf{x}) = \frac{2}{v_0^2 A}(\partial_t + v_0\partial_z)u(z/v_0, \mathbf{x}). \tag{6}$$

In several steps one can compute

$$\frac{2}{v_0^2 A}(\partial_t + v_0\partial_z)u(z/v_0, \mathbf{x}) = \frac{2}{(2\pi)^3} \times \operatorname{Re} \int_{\mathbb{R}^3} \left(1 + \frac{\zeta}{\|\boldsymbol{\xi}\|}\right) e^{i\mathbf{x}\cdot(\xi,\eta,\zeta+\|\boldsymbol{\xi}\|)}\, \widehat{r}(\xi, \eta, \zeta + \|\boldsymbol{\xi}\|)\, d\boldsymbol{\xi}. \tag{7}$$

The next step is a change of variables in the integral: $\tilde{\zeta} = \zeta + \|\boldsymbol{\xi}\|$. The image of this transformation is the halfplane $\tilde{\zeta} > 0$, while the Jacobian equals $\frac{\partial\tilde{\zeta}}{\partial\zeta} = 1 + \frac{\zeta}{\|\boldsymbol{\xi}\|}$, a factor already present in (7). Working this out further, we find that in fact $r$ can be reconstructed: $I = r$. (Here we ignore aperture issues. In most acquisition geometries only part of the Fourier component of $r$ is reconstructed, our point is that they are reconstructed with correct amplitude.) So (6) is the basic inversion formula.

### Towards variable coefficients

The next task is to adapt the RTM imaging condition to yield something like (6). The first step is to write (6) in an RTM form We have

$$I(\mathbf{x}) = \int_0^{T_{\max}} \frac{1}{A}\delta(t - \frac{z}{v_0})\, \frac{2}{v_0^2}(\partial_t + v_0\partial_z)u_{\text{bp}}\, dt$$

We can convert this to the frequency domain by Parseval's theorem (Fourier convention $\hat{f}(\omega) = \int e^{-i\omega t} f(t)\, dt$). This theorem says for real functions $A(t), B(t)$ that

$$\int A(t)B(t)\, dt = \frac{1}{2\pi} \int \overline{\hat{A}(\omega)}\hat{B}(\omega)\, d\omega.$$

It follows that

$$I(\mathbf{x}) = \frac{1}{2\pi} \int \frac{1}{Ae^{-i\omega\frac{z}{v_0}}}\, \frac{2}{v_0^2}(i\omega + v_0\partial_z)\hat{u_{\text{bp}}}(\omega, \mathbf{x})\, d\omega.$$

Using heuristics, we now explain how a modified imaging condition for RTM is obtained:

**Step 1**: In $Ae^{-i\omega\frac{z}{v_0}}$ we recognize the Fourier transform of the field $A\delta(t - \frac{z}{v_0})$ premultiplying $r$ in (3). However, the Born approximation reads

$$\left(\frac{1}{v_0^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right) u = \partial_t^2 u_{\text{inc}} \frac{2\delta v(\mathbf{x})}{v_0^3}. \tag{8}$$

We aim to reconstruct $\delta v(\mathbf{x})$, which is premultiplied by $-\frac{2\omega^2}{v_0^3}\hat{u}_{\text{inc}}(\omega, \mathbf{x})$. We therefore make the replacement

$$Ae^{-i\omega\frac{z}{v_0}} \quad \longrightarrow \quad -\frac{2\omega^2}{v_0^3}\hat{u}_{\text{inc}}(\omega, \mathbf{x})$$

The formula for the image becomes

$$I(\mathbf{x}) = \frac{1}{2\pi} \int -\frac{v_0}{\omega^2\hat{u}_{\text{inc}}}\, (i\omega + v_0\partial_z)\hat{u_{\text{bp}}}\, d\omega$$

**Step 2**: The derivative $v_0\partial_z$ comes from the fact that we have a vertically downward propagating wave. In general it should be replaced by

$$v_0\partial_z \quad \longrightarrow \quad v_0^2(\nabla T)\cdot\nabla$$

where $T = T(\mathbf{x})$ is the traveltime function of the incoming wave. The vector $\nabla T$ is not directly available. However, we have

$$\hat{u}_{\mathrm{inc}}(\omega,\mathbf{x}) \approx A(\omega,\mathbf{x})e^{-i\omega T(\mathbf{x})},$$

and therefore

$$\nabla T \approx \frac{\nabla \hat{u}_{\mathrm{inc}}}{-i\omega\hat{u}_{\mathrm{inc}}}.$$

Combining all this, we arrive at the reverse time migration/inversion formula,

$$I(\mathbf{x}) = \frac{1}{2\pi}\int\left(-\frac{iv_0}{\omega}\frac{\hat{u}_{\mathrm{bp}}}{\hat{u}_{\mathrm{inc}}} - \frac{iv_0^3}{\omega^3}\frac{\nabla_{\mathbf{x}}\hat{u}_{\mathrm{inc}}\cdot\nabla_{\mathbf{x}}\hat{u}_{\mathrm{bp}}}{\hat{u}_{\mathrm{inc}}^2}\right)d\omega \tag{9}$$

We note that the imaging formula includes a division by the source field. Source field caustics will therefore lead to singularities. Indeed this condition of absence of source field caustics is well known, see (Nolan and Symes, 1997).

Elsewhere, we will present a full analysis of this procedure and, in particular, estimate the associated errors. Part of these errors can be suppressed with techniques that correct for limited illumination, for example, using wave packets.

## EXAMPLES

We have done computations with several models, of which some results are shown here. The general setup of the example was as follows. First a model was chosen, consisting of a background medium $v_0$, a medium perturbation $\delta v$, a domain of interest and a computational domain. The latter was larger than the domain of interest and included absorbing boundaries. Data was generated by solving the wave equation with velocity $v_0+\delta v$, and a Ricker wavelet source at position $x_{\mathrm{s}} = (0,0)$ using finite differences. The direct wave was eliminated. The backpropagated field was computed, the source field was computed and finally the imaging condition (9) was applied to obtain an approximate reconstruction of $\delta v$.

With regard to the choice of $\delta v$, it is well known that the inversion yields only a band-limited reconstruction. To be able to compare the original and reconstructed reflectivity we used bandlimited functions for $\delta v$.

Our first example concerned a gradient type medium $v_0 = 2.0+0.001z$, with $v_0$ in km/s and $z$ in meters. Our model region was the square with $x$ and $z$ between 0 and 2000 meters. The purpose was to show a successful reconstruction of velocity perturbations at different positions and with different orientations in the model. We therefore chose for $\delta v$ a linear combination of three wave packets at different locations, with central wave vector well within in the inversion aperture. We included one with large dip, as one of the interesting abilities of RTM is imaging of large dips. The results of the above procedure are shown in Figures 1 and 2. The reconstruction of the phase is excellent. However, the reconstructed amplitude is around 8-10 % smaller than the original amplitude. Possible explanations for this are inaccuracies related to the several approximations involved: The linearization, the geometrical optics approximation (e.g. smiles) and the discretization.

Our second example concerns a bandlimited continuous reflector. For a continuous reflector one might expect less loss in amplitude when compared to the localized velocity perturbations. One of the strengths of RTM and wave equation migration in general is that multipathing is easily incorporated, where in our case of single source RTM, multipathing is only allowed between the reflector and the receiver point. To see this in an example we included in our background model a low velocity lens at $(800, 1200)$ m. The background medium including some rays, as well as some data are plotted in Figure 3. The velocity perturbation was located at $z = 1600$ m. The results of this example are given in Figure 4. The reconstruction of the phase is again excellent. The amplitude varies somewhat depending on location, being about 0-10 % to low. The smooth tapering which was applied has diminished smiles and amplitude variations, but not fully eliminated them. The multipathing leads to singularities in the inverse of the source field $\hat{u}_{\mathrm{inc}}^{-1}$, around $(x,z) = (1900, 1000)$ m, which leads to the two artifacts that can be seen there.

## DISCUSSION

We discussed a RTM procedure that results in images that have only been affected by the illumination. The numerical examples show that amplitudes are reconstructed quite well. A further question of interest is of course the multi-source case. This is complicated by the fact that for each image point and each source point the aperture in the $\boldsymbol{\xi}$ domain is different. The summation over sources in combination with the different apertures leads to complicated correction factors. We refer to (Ten Kroode et al., 1998) for theory on this topic.

## ACKNOWLEDGEMENTS

CCS and TJPMOtR thank the Netherlands Organisation for Scientific Research for support under grant no. 639.032.509. MVdH thanks the members of the Geo-Mathematical Imaging Group for financial support.

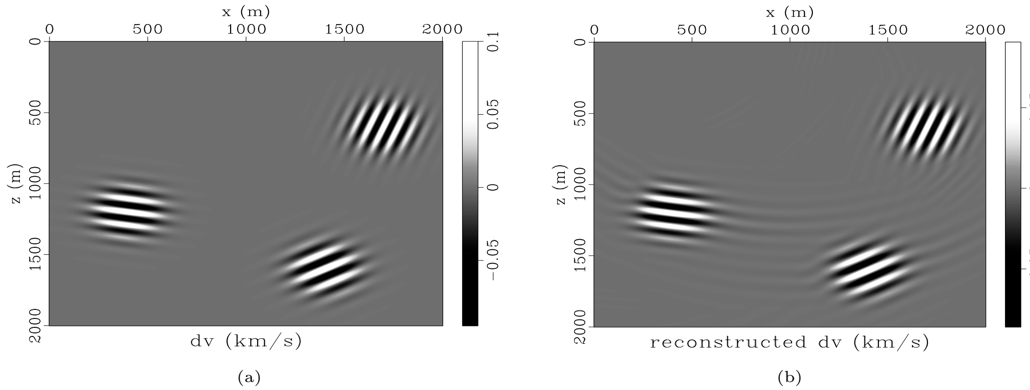


Figure 1: Example 1: Velocity perturbation and reconstructed velocity perturbation for our first example. The background medium is a gradient $v = 2.0 + 0.001z$, with $z$ in meters and $v$ in km/s.

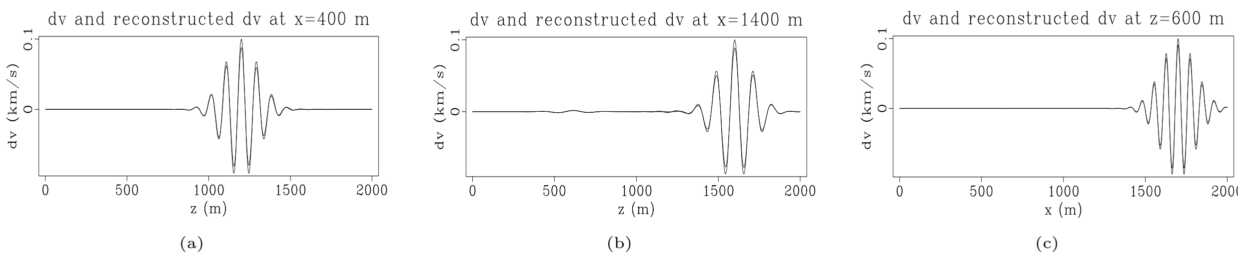


Figure 2: Example 1: Comparison of some traces from Figure 1 at $x = 400$ m, $x = 1400$ m and $z = 600$ m. The amplitudes of the reconstructed signal are about 8-10 % too low.

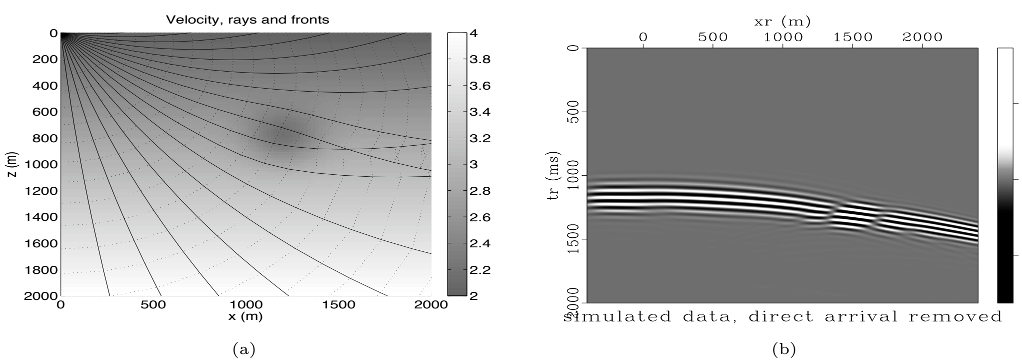


Figure 3: Example 2: (a) A velocity model with some rays; (b) Simulated data, with direct arrival removed.

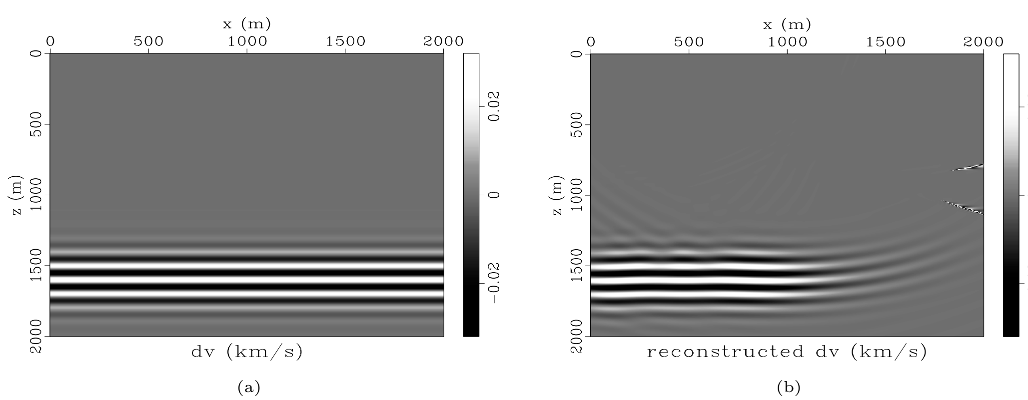


Figure 4: Example 2: (a) Velocity perturbation; (b) Reconstruction of the velocity perturbation.

## REFERENCES


Baysal, E., D. D. Kosloff, and J. W. C. Sherwood, 1983, Reverse time migration: Geophysics, **48**, 1514–1524.

Beylkin, G., 1985, Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform: J. Math. Phys., **26**, 99–108.

Biondo, B. L., 2006, 3D seismic imaging: Society of Exploration Geophysicists.

Bleistein, N., J. K. Cohen, and J. W. Stockwell, Jr., 2001, Mathematics of multidimensional seismic imaging, migration, and inversion: Springer.

Chattopadhyay, S. and G. A. McMechan, 2008, Imaging conditions for prestack reverse-time migration: Geophysics, **73**, S81–S89.

Fletcher, R. F., P. Fowler, P. Kitchenside, and U. Albertin, 2005, Suppressing artifacts in prestack reverse time migration: 75th Ann. Internat. Mtg., Expanded Abstracts, 2049–2052, Society of Exploration Geophysicists.

Guitton, A., B. Kaelin, and B. Biondi, 2007, Least-square attenuation of reverse-time migration artifacts: Geophysics, **72**, S19–S23.

Nolan, C. J. and W. W. Symes, 1997, Global solution of a linearized inverse problem for the wave equation: Communications in Partial Differential Equations, **22**, 919–952.

Plessix, R. E. and W. A. Mulder, 2004, Frequency-domain finite-difference amplitude-preserving migration: Geophys. J. Int., **157**, 975–987.

Ten Kroode, A., D. Smit, and A. Verdel, 1998, A microlocal analysis of migration: Wave Motion, **28**, 149–172.

Xie, X. and R. Wu, 2006, A depth migration method based on the full-wave reverse-time calculation and local one-way propagation: 76th Ann. Internat. Mtg., Expanded Abstracts, 2333–2336, Society of Exploration Geophysicists.