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*Published in:*  
Journal of Philosophical Logic

[Link to publication](#)

*Citation for published version (APA):*

Bernardi, C., & d' Agostino, G. I. O. V. A. N. N. A. (1996). Translating the Hypergame Paradox: Remarks on the well-founded elements of a relation. *Journal of Philosophical Logic*, 25, 545-557.

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TRANSLATING THE HYPERGAME PARADOX: REMARKS ON  
THE SET OF FOUNDED ELEMENTS OF A RELATION

ABSTRACT. In Zwicker (1987) the hypergame paradox is introduced and studied. In this paper we continue this investigation, comparing the hypergame argument with the diagonal one, in order to find a proof schema. In particular, in Theorems 9 and 10 we discuss the complexity of the set of founded elements in a recursively enumerable relation on the set  $N$  of natural numbers, in the framework of reduction between relations. We also find an application in the theory of diagonalizable algebras and construct an undecidable formula.

1. INTRODUCTION

One feature shared by the majority of logical and set-theoretical paradoxes is the liar's symmetry ' $p$  is true iff  $p$  is not true'; as is well known, the same structure, translated into the diagonal schema, becomes a useful tool to produce proofs.

A few years ago, Zwicker (1987) introduced an asymmetric paradox, the so-called *hypergame paradox*, which also can be used as a schema to produce proofs by contradiction. Consider games between two players  $A$  and  $B$ . We use the word 'game' in two different meanings, to denote both a game in general and any particular competition between  $A$  and  $B$ . Given a game  $G$ , we will say 'a game of  $G$ ', to denote a single competition which proceeds according to the rules of  $G$ . Call a game  $G$  *founded* if every game of  $G$  must terminate after finitely many moves, that is, following the rules, it is impossible for a game of  $G$  to go on for ever, even if a priori there is no fixed bound on the lengths of games of  $G$ .

Now, define the hypergame as follows: player  $A$  chooses a founded game  $G$ , then player  $B$  makes the first move in  $G$  and the game continues according to the rule of  $G$ . (Of course, there are simple winning strategies for  $A$ , but we are not concerned with them – in fact, we only specify legal moves, while the result of the game, who wins and who loses, has no importance.) Does this game have an end? Since  $G$  is founded, the answer is obviously 'yes'. In other words, hypergame is a founded game. As a consequence, in a game of hypergame, player  $A$  can choose the hypergame itself as a founded game, giving  $B$  the right to choose the

founded game. But, if  $B$  decides to be as bizarre as  $A$  and also chooses hypergame, and  $A$  in turn repeats 'let's play hypergame', and so on, we get an unfounded game in which both players move according to the rules of a founded game.

Note that here the argument which leads to the paradox has an asymmetric pattern which consists of two parts: (1) ' $p$  is true' (hypergame is a founded game); (2) 'if  $p$  is true then  $\neg p$  is true' (if hypergame is founded, then it is not founded). In other words, we arrive to a contradiction by showing ' $p$ ' and ' $p \rightarrow \neg p$ ', rather than ' $p \leftrightarrow \neg p$ '. (The hypergame argument involves the classical inference rule 'from  $p \rightarrow \neg p$  deduce  $\neg p$ ' which was called *consequentia mirabilis* in the Renaissance.)

The structure of the hypergame paradox may be adapted and proposed in several contexts. Some earlier statements of the paradox (even if not completely clear) can be found in non-technical papers, as Gardner (1984), Smullyan (1983). A more interesting earlier version is due to Shen-Yuting and dates back to 1953 (Shen-Yuting, 1953): the argument is essentially the same, but it is only concerned with naive set theory. Here the notion of a grounded set is considered: a set  $x$  is *grounded* if there is no sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_0 = x$  and  $x_{n+1} \in x_n$  for each natural number  $n$ . Consider the set  $G$  of all grounded sets. It is easy to see that  $G$  is grounded, because if  $(x_n)$  were a sequence such that  $x_0 = G$  and  $x_{n+1} \in x_n$  for each  $n$ , the sequence  $y_n = x_{n+1}$  would show that  $x_1$  was ungrounded; but this is in contradiction with  $x_1 \in x_0 = G$ . As  $G$  is grounded, we have  $G \in G$ , and the constant sequence  $\dots \in G \in G \in G$  proves that  $G$  is ungrounded, a contradiction.

A not-too-different structure can be found in other known paradoxes, for instance in Girard's one in  $\lambda$ -calculus. In any case, asymmetric paradoxes belonging to the hypergame family can be readily constructed. Let us see an example in topology. Call a topological space *Noetherian* if it contains no infinite strictly descending chain of closed sets (with respect to the relation of inclusion). Given a family  $(X_i)_{i \in I}$  of Noetherian spaces, we introduce a topology on the disjoint union  $X$  of the spaces  $X_i$ : the open sets are the empty set and the subsets  $A$  of  $X$  such that  $A \cap X_i$  is open in  $X_i$  for each  $i \in I$ , and moreover  $A \cap X_i = X_i$  for almost all  $i$ . It is easy to show that  $X$  is in turn a Noetherian space. Consider now a family  $(X_i)_{i \in I}$  of spaces that contains (a homeomorphic copy of) every Noetherian space, and let  $X$  be the space obtained as before. By hypothesis, there is an index  $i_0$  such that  $X_{i_0}$  is homeomorphic to  $X$  by means of a function  $f: X \rightarrow X_{i_0}$ . We have  $f(X) = X_{i_0} \subset X$ , where the inclusion is strict and  $X_{i_0}$  is closed in  $X$ . It follows that  $f^{n+1}(X) \subset f^n(X)$  for each natural  $n$ , a contradiction, because each  $f^n(X)$  is closed.

In natural language, the most popular version of a symmetric paradox is obtained by considering a barber that shaves every man that does not shave himself; but, if one consider a barber that shaves every man that does not shave himself, *except* for the barber himself, the paradox is avoided. Van Benthem (1978) showed that a similar construction is not enough to avoid Russell's paradox in naive set theory: if we assume that a set  $d$  exists such that, for every set  $n$  different from  $d$ ,  $(n \in d \leftrightarrow n \notin n)$ , we again arrive at a contradiction, just considering the set  $d - \{d\}$  if  $d \in d$ , and the set  $d \cup \{d\}$  otherwise. The same observation applies to the Shen-Yuting paradox: if we assume the existence of a set  $i$  such that for every set  $n$  different from  $i$   $(n \in i \leftrightarrow n$  is grounded), we again arrive at a contradiction. Indeed, starting from  $i$ , we can construct a set  $h$  that contains exactly all grounded sets, regardless of whether  $n = i$  or not; it is enough to define  $h = i \cup \{i\}$  if  $i$  is grounded or  $h = i - \{i\}$  if  $i$  is not grounded. In both cases, we get a contradiction following the same argument as in Shen-Yuting's paradox.

In this paper we investigate the hypergame paradox, comparing it with diagonal paradoxes, and showing that also the hypergame paradox can be translated into a proof schema. Our aim, first of all, is to give insight into the situation; to this end, some related questions are discussed and some applications are provided.

## 2. THE HYPERGAME SCHEMA

The core of the diagonal method consists of a binary relation  $R$  on a set  $X$ , and the observation that there is no element  $d$  of  $X$  such that  $xRd$  iff not  $xRx$  (otherwise, by replacing  $x$  by  $d$  we would have  $dRd$  iff not  $dRd$ , a symmetric contradiction). This proof schema, adapted to various situations by choosing appropriately a set  $X$  and a relation  $R$ , is used in different contexts of mathematical logic, such as recursion theory, proof theory, and set theory.

Let us come to the task of extracting a proof schema from the hypergame paradox. Again, the core of the method is a binary relation  $R$  on a set  $X$ ; we say that an element  $x \in X$  is *founded* if there is no sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_0 = x$  and  $x_{n+1}Rx_n$  for each natural number  $n$ . The *hypergame method* consists of the following observation: there is no element  $i$  in  $X$  such that  $xRi$  iff  $x$  is founded (w.r.t.  $R$ ). If there were such an  $i$ , it would be a founded element: every sequence  $(x_n)$  contradicting the foundedness of  $i$  is such that  $x_1$  is founded, while the sequence  $y_n = x_{n+1}$  shows that  $x_1$  is not founded. But if  $i$  is founded, then  $iRi$  and  $i$  is not founded, a contradiction.

Let us compare the diagonal method with the hypergame method. First, remark that, while negation has a crucial rôle in the diagonal method, in the sentence ‘ $xRi$  iff  $x$  is founded’ negation does not appear (at least if ‘infinite’ is regarded as negation of ‘finite’, and not conversely).

**EXAMPLE 1.** We can prove the uncountability of the set of real numbers both in the usual way, that fits the schema of a diagonal proof, and using a hypergame argument. In both cases  $X = N$ ; if  $(a_n)_{n \in N}$  is a sequence that contains all real numbers, for each  $n$  fix a decimal representation of  $a_n$ , and call  $a_{n,m}$  the  $m$ th digit of  $a_n$ . Then define the binary relation  $R$  on  $N$  as  $mRn$  iff  $a_{n,m} = 1$ . Up to this point, the diagonal and the hypergame proofs follow the same path. Now, the classical diagonal proof refers to the number  $z = 0, b_1 b_2 \dots$ , where  $b_n = 1$  if  $a_{n,n} \neq 1$ , and  $b_n = 2$  otherwise, and gets a contradiction.

If we prefer a hypergame proof, instead, we define the number  $t = 0, b_1 b_2 \dots$  as the real number such that  $b_n = 1$  iff  $n$  is founded w.r.t.  $R$ ,  $b_i = 2$  otherwise. Let  $i$  be such that  $a_i = t$ . We conclude that  $nRi$  iff  $a_{i,n} = 1$ , iff  $b_n = 1$  iff  $n$  is founded, reaching in this way a contradiction.

**EXAMPLE 2.** A classical application of the diagonal method is Cantor’s proof of the non-existence of a surjective function from  $X$  to the power set of  $X$ . Let us compare diagonal and hypergame methods in this context. Suppose that there is such a function  $h$ ; then define a relation  $R$  on  $X$  as  $xRy$  iff  $x \in h(y)$ . When using a diagonal argument, we consider an element  $d$  such that  $h(d) = \{x \in X / \text{not } xRx\}$ ; if we choose the hypergame method, the set of non-reflexive elements of the relation  $R$  is replaced by the set of founded elements of  $R$ . If  $i$  is such that  $h(i) = \{n/n \text{ is founded w.r.t. } R\}$ , then  $mRi$  iff  $m \in h(i)$  iff  $m$  is founded, and we get a contradiction.

**EXAMPLE 3.** Here we work in recursion theory. Let  $xEy$  iff  $x \in W_y$  for natural numbers  $x$  and  $y$ , where  $(W_x)_{x \in N}$  is the standard enumeration of recursively enumerable sets. The diagonal method on  $E$  gives us a proof that the set  $\bar{K} = \{x \in N / x \notin W_x\}$  is not r.e. By using a hypergame argument, we show that the set  $F = \{n/n \text{ is } E\text{-founded}\}$  is not r.e. Indeed, if this were not the case, there would be an  $i$  such that  $F = W_i$ , and again we would have  $nEi$  iff  $n \in W_i$  iff  $n$  is founded, a contradiction.

For any binary relation  $R$  on a set  $X$ , the set  $F_R = \{n/n \text{ is founded}\}$  is a subset of the set  $\bar{D} = \{n / \text{not } nRn\}$ , even if these sets can differ

from each other. In fact,  $F_R$  and  $\overline{D}$  present some analogies, but do not share all properties. Let us compare, in particular, the sets  $F_E$  and  $\overline{K}$  of Example 3. They are both productive in a very natural way: if  $W_x \subseteq F_E$ , then  $x$  is founded, and therefore cannot belong to  $W_x$ . However,  $F_E$  is much more complicated than  $\overline{K}$ : in fact,  $\overline{K}$  is co-r.e. and  $\Pi_1^0$ -complete, while we will show that  $F_E$  is  $\Pi_1^1$ -complete.

In general, we may appreciate the difference between  $F_R$  and  $\overline{D}$  by considering the operator  $dom: P(X) \rightarrow P(X)$  that maps the set  $A$  to  $dom A = \{x \in X / \{y / yRx\} \subseteq A\}$  (this definition is from recursion theory – see for instance (Rogers, 1967)). It can be shown that  $F_R$  is the least fixed point of the operator  $dom$ . Moreover, if  $I = \{A \in P(X) / dom A \subseteq A \text{ and } A \cap D = \emptyset\}$ , the set  $F_R$  is the least element in  $I$ , with regard to the inclusion relation, while  $\overline{D}$  is the greatest element. Note that the hypergame argument works for every  $A$  in  $I$ : there exists no  $i$  in  $X$  such that  $xRi$  iff  $x \in A$ . Indeed, if  $\{x/xRi\} \subseteq A$ , then  $i \in dom A \subseteq A$  but  $i \notin \{x/xRi\}$  because  $A \cap D = \emptyset$ . Adapted to the relation  $E$  of Example 3, this argument proves that every set in  $I$  is productive, with the identity map as a productive function. Actually, in recursion theory it is not hard to prove more: every set  $A$  containing all indices of the empty set and no index of  $N$  is productive\*.

For instance, call a number  $x$  *finite* if there exists an  $n$  that bounds the length of any descending  $E$ -chain starting from  $x$ : the set of finite numbers is productive, but does not belong to the family  $I$  (that means that the hypergame argument does not work for this set).

We end this section by fitting into the hypergame schema a paradox and a proof already known in set theory.

As usual, consider an ordinal number  $\alpha$  as the set of all the ordinal numbers that are less than  $\alpha$ . Now, for every ordinal number  $\alpha$  we can construct a founded game as follows: player  $A$  chooses an ordinal  $\beta$  in  $\alpha$ , that is, a  $\beta$  less than  $\alpha$ , then  $B$  chooses a  $\gamma$  in  $\beta$  and so on (as before, we are not concerned in winning strategies). For every  $\alpha$  the game is founded because the ordinal number  $\alpha$  is a well-ordered set. If only these games are considered, the hypergame is started by choosing an element of the ‘set’  $\Omega$  of all ordinal numbers. In some sense, we can think of  $\Omega$  as the set of all well-ordered sets. But since  $\Omega$  is in turn a well-ordered set (we are in the realm of naive-set theory), we can write  $\Omega \in \Omega$ , and we conclude that  $\Omega$  is not well ordered. From this point of view, the hypergame paradox is closely related to Burali-Forti’s paradox (cf. (Burali-Forti, 1897)).

An example of a proof that follows a schema not too different from the hypergame paradox occurs in (axiomatic) set theory when proving, without the axiom of choice, that there are uncountably many ordinal numbers. One proves that the collection  $\Gamma(\omega)$  of all ordinals which are embeddable in  $\omega$  is in turn an ordinal number, called the Hartogs number of  $\omega$  (this is the affirmative part of the hypergame schema); then that if  $\Gamma(\omega)$  were countable one would have  $\Gamma(\omega) \in \Gamma(\omega)$ , a contradiction (this is the part ' $p \rightarrow \neg p$ ' of the hypergame).

### 3. THE COMPLEXITY OF THE SET OF FOUNDED ELEMENTS

Let us restrict ourselves to relations on natural numbers. If a relation  $R$  is r.e., the set  $\bar{D}$  of non-reflexive elements of  $R$  is co-r.e., while the set  $F_R$  is  $\Pi_1^1$ . In fact,  $F_R$  is  $\Pi_1^1$  also if  $R$  is an arithmetical relation: let  $A(x)$  be a formula that defines the relation  $R$  in the set  $N$ , and let

$$V = \{u / \text{there is a number } n \text{ such that } u = \langle x_1, \dots, x_n \rangle \text{ and } x_1 R x_2 \dots R x_n\}$$

where  $\langle -, \dots, - \rangle$  denotes a code for finite sequences.  $V$  is still an arithmetical set, and  $x \in F_R$  iff  $\forall f (f(0) = x \rightarrow \exists n \langle f(n), \dots, f(0) \rangle \notin V)$ .

At the end of this section we will find a global property of r.e. relations that implies the  $\Pi_1^1$ -completeness of the set of founded elements of  $R$ . For the moment, let us consider the case where the relation is the membership relation of Example 3. In this case the set  $F_E$  of founded elements is recursively isomorphic to the domain  $O$  of Kleene's ordinal notation. This fact is not surprising: an effective version of Burali-Forti construction moves from the 'set'  $\Omega$  of all ordinal numbers to the set  $O$ . At the same time,  $F_E$  is related to the set  $T$  of indices of founded recursive trees, and it is known that this set is  $\Pi_1^1$ -complete.

*Remark 4.* A direct proof of the  $\Pi_1^1$ -completeness of  $F_E$  may be obtained as follows. Let  $A$  be a  $\Pi_1^1$  set and let  $Q$  be a recursive relation such that

$$x \in A \text{ iff } \forall f \exists y Q(x, \langle f(0), f(1), \dots, f(y) \rangle)$$

(we are applying the normal form for  $\Pi_1^1$  sets, cf. for example (Odifreddi, 1989)).

Let  $g$  be a recursive function such that

$$W_{g(x,u)} = \{g(x,t) / u \text{ is a sequence number, } t = u * \langle k \rangle, k \in N, \neg Q(x,t)\}.$$

If we define  $h$  as  $h(x) = g(x, \langle \emptyset \rangle)$ , one can prove that

$$h(x) \notin F_E \text{ iff there exists a function } f \text{ with} \\ \neg Q(x, \langle f(0), f(1), \dots, f(n) \rangle),$$

for each  $n \in N$ . It is easy to show that the function  $h$  reduces the set  $A$  to the set  $F_E$  and this proves that  $F_E$  is  $\Pi_1^1$ -complete.

Obviously, the same argument applies to the case in which a relation  $E'$  is defined referring to an acceptable system of indices for partial recursive functions  $(\psi_x)_{x \in N}$  as  $x E' y$  iff  $x \in \text{domain} \psi_y$ . In fact, it is enough that the system satisfies the *parametrization* property: there is a recursive function  $f$  such that  $\phi_x \sim \psi_{f x}$ , for each number  $x$ . If this is the case, let  $t(x, y)$  be a recursive injective function such that  $W_{t(z,x)} = f \phi_z W_x$ ; by the  $s_m^n$ -Theorem in the standard system of indices, there exists a recursive function  $h(z)$  such that  $\phi_{h z} x = t(z, x)$ . Let  $a$  be a number that verifies  $\phi_{h a} \sim \phi_a$ . We have:

$$f \phi_a(W_y) = W_{t(a,y)} = W_{\phi_{h a}(y)} = W_{\phi_a(y)} = \text{domain} \psi_{f \phi_a(y)}.$$

It follows that  $x E y$  implies  $f \phi_a(x) E' f \phi_a(y)$ , and that if  $u E' f \phi_a(y)$ , there is a number  $x$  such that  $x E y$  and  $f \phi_a(x) = u$ . Thus we have  $x \in F_E$  iff  $f \phi_a(x) \in F_{E'}$ , and  $F_{E'}$  is  $\Pi_1^1$ -complete.

Going back to the general case of a binary relation  $R$ , we note that the arithmetical complexity of the set of founded elements does not reflect the complexity of the relation itself. Indeed,  $R$  may be arbitrarily complex but have  $F_R = \emptyset$  (it suffices to add a minimum reflexive element to a relation of the needed complexity), while the following relation  $R$  is recursive and the set of founded elements is  $\Pi_1^1$ -complete.

**EXAMPLE 5.** Let  $A$  be a  $\Pi_1^1$ -complete set, and let  $Q$  be a recursive relation such that

$$x \in A \text{ iff } \forall f \exists y Q(x, \langle f(0), f(1), \dots, f(y) \rangle).$$

Let  $R$  be

$$\langle x, a_0, \dots, a_n \rangle R \langle x, a_0, \dots, a_{n-1} \rangle \text{ iff } \neg Q(x, \langle a_0, \dots, a_n \rangle).$$

We have

$$x \in A \text{ iff } \langle x \rangle \text{ is founded w.r.t. the relation } R.$$

Thus  $A$  reduces to  $F_R$  via the recursive function  $h(x) = \langle x \rangle$ .

Let us now look for a property of relations that implies the  $\Pi_1^1$ -completeness of  $F_R$ .



DEFINITION 6 (cf. for instance (Bernardi and Sorbi, 1983)). A relation  $S$  is said to be *reducible* to a relation  $R$  if there exists a recursive function  $f$  such that  $xSy$  iff  $f(x)Rf(y)$ . We say that the function  $f$  *reduces*  $S$  to  $R$ .

An r.e. relation  $R$  is said to be *m-complete* (respectively, 1-complete) if, for each r.e. relation  $S$ , there exists a recursive function (respectively, injective)  $f$  that reduces  $S$  to  $R$ .

LEMMA 7. *An m-complete relation is 1-complete.*

*Proof.* Let  $R$  be  $m$ -complete, and let  $S$  be an arbitrary r.e. relation. Embed  $N$  in  $N \times \{0, 1\}$ , by sending  $x$  to  $(x, 0)$ ; define  $S'$  on  $N \times N$  as:  $(x, 0)S'(y, 0)$  iff  $xSy$ , and  $(x, 0)S'(x, 1)$ , for each  $x, y \in N$ . By codifying  $N \times \{0, 1\}$  in  $N$ , we may look at  $S'$  as a relation on  $N$ . If  $f'$  is the recursive function that reduces  $S'$  to  $R$ , the function  $f(x) = f'(x, 0)$  must be injective and reduces  $S$  to  $R$ .  $\square$

Since  $m$ -completeness and 1-completeness coincide, we will use only the term completeness.

DEFINITION 8. An r.e. relation  $S$  is said to be *strongly reducible* to an r.e. relation  $R$  if there is a recursive function  $f$  such that  $f$  reduces  $S$  to  $R$ , and moreover  $zRf(x)$  implies  $z \in \text{Im}f$ . An r.e. relation  $R$  is said to be *strongly complete* if for each r.e. relation  $S$  there exists a recursive function  $f$  that strongly reduces  $S$  to  $R$  (as before, we may require that the function  $f$  is injective without affecting the generality of the definition).

THEOREM 9. *If  $R$  is strongly complete, then the set of founded elements of  $R$  is  $\Pi_1^1$ -complete.*

*Proof.* If  $f$  is a recursive function that strongly reduces the relation  $E$  to  $R$ , then  $F_E$  is reducible, via  $f$ , to the set  $F_R$  of founded elements of  $R$ . Since  $F_E$  is  $\Pi_1^1$ -complete, we conclude that  $F_R$  is  $\Pi_1^1$ -complete too.  $\square$

THEOREM 10. *The relation  $E = \{(x, y)/x \in W_y\}$  is strongly complete.*

*Proof.* Given an r.e. relation  $S$ , let  $\psi(x, y)$  be a recursive function such that  $W_{\psi(x, y)} = \{\phi_x(t)/tSy\}$ . By the  $s_m^n$ -Theorem we may assume that  $\psi$  is injective. By the Recursion Theorem, there exists a number  $n$  such that the partial recursive functions  $\psi(n, -)$  and  $\phi_n(-)$  are equal. Note that  $\phi_n$  is in turn total and injective, since so is  $\psi$ ; thus  $W_{\phi_n(y)} = \{\phi_n(t)/tSy\}$ , and the function  $\phi_n$  strongly reduces the relation  $S$  to  $E$ . Since this holds for each r.e. relation  $S$ , we conclude that  $E$  is strongly complete.  $\square$

EXAMPLE 11. There exists a complete relation that is not strongly complete.

Let  $R$  be the relation

$$R = (S + 1) \cup \{(0, n) / n \in N\},$$

where  $S$  is a complete relation, for example  $E$ , and  $S + 1$  is  $\{(x + 1, y + 1) / \langle x, y \rangle \in S\}$ . Since  $0Rn$  for each  $n \in N$ , the set of founded elements in  $R$  is empty, and the relation  $R$  cannot be strongly complete. On the other hand, if a recursive injective function  $f$  reduces an r.e. relation to  $S$ , the recursive injective function  $f'(x) = f(x) + 1$  reduces the same relation to  $R$ .

*Remark 12.* Theorem 10 allows us to embed any r.e. relation  $R$  into  $E$ . We can get a more precise statement: if  $R$  is an r.e. relation on  $N$ , then there is a transitive element  $z$  (that is, an element  $z$  such that  $xEz$  and  $yEx$  implies  $yEz$ ) such that  $R$  is recursively isomorphic to  $E|_{W_z \times W_z}$ ; moreover,  $z \in F_E$  iff  $R$  is founded. The proof follows along the same line as in Theorem 10.

Finally, we note that Myhill's Theorem does not hold for strong reduction between relations; there are relations which are strongly complete but not isomorphic. Indeed, we define a relation  $K_0$  as  $\langle x, z \rangle K_0 \langle y, z \rangle$  iff  $\langle x, y \rangle \in W_z$ ; it is readily seen that  $K_0$  is strongly complete. On the other hand,  $E$  and  $K_0$  are not recursively isomorphic: if  $z$  is an index for the empty set,  $\{y / \langle x, z \rangle K_0 y\} = \emptyset$ , while for each  $x \in N$  the set  $\{y / xEy\}$  is not empty.

#### 4. AN APPLICATION TO DIAGONALIZABLE ALGEBRAS

We shall see how the strong completeness of a relation  $R$  may be used to construct a generic Magari algebra. Recall that a *Magari algebra* (formerly called diagonalizable algebra) is a pair  $(D, \square)$ , where  $D$  is a Boolean algebra with operations  $0, 1, \wedge, \neg, \vee$ , and  $\square$  is a map from  $D$  to  $D$  satisfying the following identities:

- (1)  $\square 1 = 1$ ;
- (2)  $\square(a \wedge b) = \square a \wedge \square b$ ;
- (3)  $\square(\square a \rightarrow a) = \square a$  (where, as usual,  $a \rightarrow b$  is  $\neg a \vee b$ ).

The main example of a Magari algebra is the Lindenbaum algebra  $D_T$  of an r.e. theory  $T$  extending Peano Arithmetic  $PA$ , where the

$\square$  is defined as  $\square[p] = [Theor_T(\bar{p})]$  (here  $[p]$  denotes the equivalence class of the sentence  $p$  with respect to provable equivalence in  $T$ , while  $Theor_T(x)$  is the standard provability predicate of  $T$ ). It has been proved that this algebra is generic in the class of Magari algebras, in the sense that every identity true in  $D_T$  holds in all Magari algebras (cf. (Smorynski, 1982) for a general introduction to Magari algebras).

Given any binary relation  $R$  on a non empty set  $X$ , we may endow the power set  $P(F_R)$  with a structure of Magari algebra by defining the  $\square$  operator on  $P(F_R)$  as the operator  $dom$  of the transitive closure of  $R$ :

$$dom_{\bar{R}}A = \{x/\{y/\exists x_1 \cdots x_n \text{ with } yRx_nR \cdots Rx_1Rx\} \subseteq A\}.$$

The structure  $(F_E, dom_{\bar{E}})$  is generic in the class of Magari algebras. More generally:

**THEOREM 13.** *If  $R$  is strongly complete, then the Magari algebra  $(P(F_R), dom_{\bar{R}})$  is generic in the class of Magari algebras.*

*Proof.* For the sake of brevity, we only sketch the proof. It is known that the class of finite Magari algebras is generic. Moreover, one can obtain any finite Magari algebra from a finite set  $X$  and from an irreflexive and transitive relation  $S$  on  $X$ , by considering  $P(X)$  endowed with the operator  $dom_S$  as  $\square$ . Thus to conclude that the algebra  $(P(F_R), dom_{\bar{R}})$  is generic, we only have to show that, for each finite subset  $X$  of natural numbers and for each irreflexive and transitive relation  $S$  on  $X$ , there is an injective function  $f$  from  $X$  to  $F_R$ , such that:

- (1)  $xSy$  implies  $f(x)\bar{R}f(y)$ ;
- (2)  $z\bar{R}f(y)$  implies the existence of an  $x \in X$  such that  $xSy$  and  $z = f(x)$ . Indeed, if this is the case, the function  $f^*: P(F_R) \rightarrow P(X)$  defined as  $f^*(A) = f^{-1}(A)$  is a surjective homomorphism of Magari algebras.

Being finite, the relation  $S$  is recursive. We may then apply Definition 8 to obtain an injective recursive function  $h$  such that  $xSy$  iff  $h(x)Rh(y)$  and  $\{t/tRh(y)\} \subseteq Imh$ . This implies  $xSy$  iff  $h(x)\bar{R}h(y)$ , and, by considering the restriction of  $h$  to  $X$ , we obtain points (1) and (2) above.  $\square$

### 5. UNDECIDABLE FORMULAS BASED ON THE HYPERGAME

In this section we use the hypergame method to build undecidable formulas in  $ZF$  and in  $PA$ . In the proofs, we will refer to the standard models

of these theories that we denote respectively by  $U$  and  $N$ , assuming in both cases the soundness hypothesis.

We identify natural numbers with formulas of  $ZF$  in one free variable, and we define  $R \subseteq N \times N$  by

$$\beta R \alpha \text{ iff } ZF \vdash \alpha(\bar{\beta})$$

(where  $\bar{\beta}$  represents the natural number  $\beta$  in  $ZF$ ).

The relation  $R$  is r.e. and can be represented in the universe  $U$  by means of a formula  $R(x, y)$ :

$$\beta R \alpha \text{ iff } U \models R(\bar{\alpha}, \bar{\beta}).$$

Following the diagonal method, let  $\bar{D} = \{\alpha / ZF \not\vdash \alpha(\bar{\alpha})\}$ . The formula  $\delta(x) = \neg R(x, x)$  represents  $\bar{D}$  in  $U$ , and  $ZF \not\vdash \delta(\bar{\delta})$  iff  $U \models \delta(\bar{\delta})$ . It follows that the formula  $\delta(\bar{\delta})$  is undecidable: if  $ZF \vdash \delta(\bar{\delta})$  then  $U \models \delta(\bar{\delta})$  and  $ZF \not\vdash \delta(\bar{\delta})$ ; if  $ZF \vdash \neg \delta(\bar{\delta})$ , then  $U \models \neg \delta(\bar{\delta})$ , and  $ZF \vdash \delta(\bar{\delta})$ . In other words, assuming the completeness of  $ZF$ , we have a symmetric contradiction of type  $p \leftrightarrow \neg p$ .

Now with the hypergame: the set  $F_R$  of founded elements of the relation  $R$  is representable in  $U$  by the formula

$$\begin{aligned} \gamma(x) = & \forall f(f: \omega \rightarrow \omega \wedge f(0) = x) \\ & \rightarrow \exists n[n \in \omega \wedge \neg R(f(n+1), f(n))]. \end{aligned}$$

**THEOREM 14.** *The formula  $\gamma(\bar{\gamma})$  is undecidable in  $ZF$ .*

*Proof.* If  $\alpha R \gamma$ , then  $ZF \vdash \gamma(\bar{\alpha})$  and  $U \models \gamma(\bar{\alpha})$ . From the definition of  $\gamma$  it follows that  $\alpha$  is a founded formula with respect to  $R$ . We have  $\{\alpha / \alpha R \gamma\} \subseteq F_R$  and so  $\gamma$  is founded. Thus  $U \models \gamma(\bar{\gamma})$  and  $ZF \not\vdash \neg \gamma(\bar{\gamma})$ . On the other hand, if  $ZF \vdash \gamma(\bar{\gamma})$ , then  $\gamma R \gamma$ , contradicting the foundedness of  $\gamma$ . We conclude that the formula  $\gamma(\bar{\gamma})$  is undecidable in  $ZF$ .  $\square$

*Remark 15.* A similar construction is not allowed in  $PA$ . Identify formulas with one free variable with the corresponding Gödel numbers, and denote with  $\bar{\beta}$  the numeral of (the Gödel number of)  $\beta$ . Now define the relation  $R$  as follows:  $\beta R \alpha$  iff  $PA \vdash \alpha(\bar{\beta})$ . The relations  $R$  and  $E$  are recursively isomorphic: if  $V_\alpha = \{\beta / PA \vdash \alpha(\bar{\beta})\}$ , then  $(V_\alpha)$  is an acceptable numbering and, by a Theorem of Blum (see (Odifreddi, 1989), Th.II.5.8), there exists a recursive permutation  $h$  such that  $x \in W_y$  iff  $hx \in V_{hy}$ . In particular, it follows that the set of founded elements with respect to  $R = \{(\beta, \alpha) / PA \vdash \alpha(\bar{\beta})\}$  is not representable in  $\omega$ , being  $\Pi_1^1$ -complete.

A not completely satisfactory way out is to require a kind of uniformity on descending sequences, in such a way to obtain an arithmetical set.

More precisely, we define the set of *uniformly founded* formulas  $UF$ :

$$\begin{aligned} \alpha \in UF \text{ iff there is no formula } \phi(x, y) \text{ such that} \\ PA \vdash \forall y \phi(0, y) \leftrightarrow \alpha(y) \text{ and, for each natural } k, \\ PA \vdash \phi(k, \phi(k + 1, y)) \end{aligned}$$

(where the formula  $\phi$  has at most two free variables, and we omitted signs for numerals). In other words, in the definition of founded elements we consider only the sequences  $(\alpha_k)_{k \in \mathbb{N}}$  of kind  $\alpha_k(x) = \phi(k, x)$ , for a suitable formula  $\phi$ .

$UF$  is an arithmetical set; thus there exists a formula  $\gamma(x)$  such that

$$\alpha \in UF \text{ iff } N \models \gamma(\bar{\alpha}).$$

**THEOREM 16.** *The formula  $\gamma(\bar{\gamma})$  is undecidable in  $PA$ .*

*Proof.* We only sketch the proof. First one proves easily that the formula  $\gamma(x)$  is uniformly founded; thus  $\gamma(\bar{\gamma})$  is a true formula, and since we supposed  $PA$  to be sound, the formula  $\neg\gamma(\bar{\gamma})$  is not provable in  $PA$ . From  $\gamma \in UF$  it follows  $PA \not\vdash \gamma(\bar{\gamma})$ : if  $PA \vdash \gamma(\bar{\gamma})$ , the formula  $\phi(x, y) = \gamma(x)$  would prove that  $\gamma \notin UF$ .  $\square$

In spite of the use of uniformity in the last proof, we followed the hypergame method in a genuine way. The schema of the proof, indeed, is not symmetric: assuming the completeness of  $PA$ , the implication  $PA \vdash \gamma(\bar{\gamma}) \Rightarrow PA \vdash \neg\gamma(\bar{\gamma})$  is still a part of the proof, but we do not have  $PA \vdash \neg\gamma(\bar{\gamma}) \Rightarrow PA \vdash \gamma(\bar{\gamma})$ . In other words,  $PA \vdash \neg\gamma(\bar{\gamma})$  has to be excluded not because it implies  $PA \vdash \gamma(\bar{\gamma})$ , but because from it we deduce that  $\gamma(x)$  is a non uniformly-founded formula, contradicting the first part of the proof.

## 6. A FINAL REMARK

Referring to the concept of *ungrounded sentence*, suggested by Kripke (1975), we can try to avoid paradoxes in natural languages in the following way. Given a sentence  $a$ , consider all the sentences  $b$  whose truth is mentioned in  $a$ , then the sentences  $c$  whose truth is mentioned in some sentence  $b$ , and so on; call the initial sentence  $a$  'grounded' if this process terminates. Now, 'we can assign a truth value only to grounded sentences'. And, of course, since this last statement refers only to grounded sentences, it is in turn grounded, ... is it?

## ACKNOWLEDGEMENT

We are grateful to Raymond Smullyan for drawing our attention to the hypergame paradox.

## NOTES

\* Indeed,  $A$  is contraproductive with respect to a total recursive function  $g$ . To see this, for every  $n$  define the set  $W_{fx}$  to be the empty set if  $x \notin W_n$ , or  $N$  if  $x \in W_n$ . Choose an  $a$  such that  $W_a = W_{fa}$  and put  $gn = a$ . Now, suppose  $A \subseteq W_n$ . From  $gn \notin W_n$  we deduce  $W_{gn} = \emptyset$  and  $gn \in A \subseteq W_n$ . Therefore  $gn \in W_n$ ; we conclude  $W_{gn} = N$  and  $gn \notin A$ .

## REFERENCES

- Bernardi, C. and Sorbi, A.: 1983, Classifying positive equivalence relations. *J. Symbolic Logic* **48**, 529–539.
- Burali-Forti, C.: 1897, Una questione sui numeri transfiniti. *Rend. Circ. Mat. Palermo* **11**, 154–164.
- Kripke, S.: 1975, Outline of a Theory of Truth. *J. Philosophy* **72**(19), 690–716.
- Gardner, M.: 1984, *Puzzles from Other Worlds: Fantastical Brainteasers from Isaac Asimov's Science Fiction Magazine*. New York, Random House, pp. 60–61, 163.
- Odifreddi, P.: 1989, *Classical Recursion Theory*. Amsterdam, North-Holland.
- Rogers, H. Jr.: 1967, *Theory of Recursive Functions and Effective Computability*. New York, McGraw-Hill.
- Shen-Yuting: 1953, Paradox of the class of all grounded classes. *J. Symbolic Logic* **18**, 114.
- Smorynski, C.: 1982, Fixed point algebras. *Bull. Amer. Math. Soc. (N.S.)* **6**, 324–337.
- Smullyan, R.: 1983, *5000 B.C. and Other Philosophical Fantasies*. New York, St. Martin's Press, pp. 40–41.
- Van Benthem, J.F.A.K.: 1978, Four paradoxes. *J. Philos. Logic* **7**, 49–72.
- Zwicker, W.S.: 1987, Playing Games with games: the Hypergame Paradox. *Amer. Math. Monthly* **94**, 507–514.

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