I. GROUND STATE PARTICLE DENSITIES

In this section we recall the expressions for single particle densities for the $M_1$ model at criticality, on an open chain of length $L = 3l$. We used these expressions to produce the grey lines in Fig. 1b. Exploiting the tools of conformal field theory, the model can be mapped to a free boson and the particle densities can be expressed in terms of the correlators of the bosonic vertex operators with a characteristic $Z_3$ pattern [1]. Specifically, they read

$$n_{3j-2} = \frac{1}{3} - \frac{2A}{3} s(x)$$  \hspace{1cm} \text{(S1a)}

$$n_{3j-1} = \frac{1}{3} + \frac{2A}{3} c(x - L'/2)$$ \hspace{1cm} \text{(S1b)}

$$n_{3j} = \frac{1}{3} + \frac{2A}{3} s(x - L')$$ \hspace{1cm} \text{(S1c)}

where $x \in [2, L]$, $L' = L + 3$ and

$$s(x) = \left( \frac{\pi}{2L'} \right)^{\frac{1}{3}} \sin \left( \frac{\pi x}{3L'} \right)$$ \hspace{1cm} \text{(S2a)}

$$c(x) = \left( \frac{\pi}{2L'} \right)^{\frac{1}{3}} \cos \left( \frac{\pi x}{3L'} \right)$$ \hspace{1cm} \text{(S2b)}

$$s(x) = \sin \left( \frac{\pi x}{L} \right)^{\frac{1}{3}}$$ \hspace{1cm} \text{(S2c)}

Here, the parameter $A$ has been determined numerically as $A = 0.77$ [1]. We note that analogous results hold for $L = 3l - 1$ [1].

II. (S)KINK PROFILES AND DESIGN OF THE OBSERVABLES

To motivate the choice of observables, we will first examine the kink and skink profiles. Fig. S1 shows the single site particle densities of kinks (blue data points) and skinks (magenta data points) for $l = 3$ at extreme staggering, $\lambda = 0$, (first two rows) and at criticality, $\lambda = 1$, (third and fourth row) for the leftmost (s)inks $|K_1\rangle$, $|\bar{K}_1\rangle$ (first column), $|K_2\rangle$, $|\bar{K}_2\rangle$ (second column) and the rightmost (s)kinks $|K_4\rangle$, $|\bar{K}_4\rangle$ (third column). We wish to design observables which capture the properties of the overlaps $o(t) = \langle K_{l+1} | K_1(t) \rangle = \langle \bar{K}_{l+1} | \bar{K}_1(t) \rangle = o(t)$ signalling the arrival of the leftmost kink to the right edge. Motivated by the (s)kink profiles, we in particular examine the situation in the last unit cell corresponding to the last three sites of the chain and to staggering 1A1 indicated by the orange boxes in Fig. S1.

For the observable we consider a function of the particle densities on these three sites, $\delta n = \delta n(n_{L-2}, n_{L-1}, n_L)$, and require that $\langle K_i | \delta n | K_j \rangle = \delta_{i,t+1} \delta_{j,t+1}$. We find that these conditions are satisfied by imposing

$$\langle K_1 | \delta n | K_1 \rangle = 0$$ \hspace{1cm} \text{(S3a)}

$$\langle K_{t+1} | \delta n | K_{t+1} \rangle = 1$$ \hspace{1cm} \text{(S3b)}

and similarly for the skinks.

At extreme staggering, the occupation of the last unit cell is 1 and 1/2 for $|K_1\rangle$ and $|K_{l+1}\rangle$ (1 and 3/2 for $|\bar{K}_1\rangle$ and $|\bar{K}_{l+1}\rangle$). We also note, that for kinks the total particle number at the last two sites is different for $|K_1\rangle$ and $|K_{l+1}\rangle$, but the same for the skinks. In contrast to the extreme staggering, at criticality (and generally for $\lambda > 0$) the total
FIG. S1. Single site particle densities of kinks (blue data points) and skinks (magenta data points) for $l = 3$ at extreme staggering $\lambda = 0$ (first two rows) and at criticality $\lambda = 1$ (third and fourth row) for the leftmost (s)inks $|K_1\rangle$, $|\bar{K}_1\rangle$ (first column), $|K_2\rangle$, $|\bar{K}_2\rangle$ (second column) and the rightmost (s)kinks $|K_4\rangle$, $|\bar{K}_4\rangle$ (third column). The gray data points for $\lambda = 1$ indicate the densities evaluated with $|K'_1\rangle$ and $|\bar{K}'_1\rangle$ for kinks and skinks respectively, see text for details.

Particle number in the last unit cell varies and is not equal to integer or half-integer values. This motivates us to introduce the following observables

$$\delta n = \alpha(\lambda, l) \left[ 1 - \beta(\lambda, l) (n_{L-1} + n_{L+1}) \right]$$  \hspace{1cm} (S4a)

$$\delta n_3 = \alpha_3(\lambda, l) \left[ 1 - \beta_3(\lambda, l) (n_{L-2} + n_{L-1} + n_{L+1}) \right]$$  \hspace{1cm} (S4b)

$$\delta \bar{n} = -\bar{\alpha}(\lambda, l) \left[ 1 - \bar{\beta}(\lambda, l) (n_{L-2} + n_{L-1} + n_{L+1}) \right]$$  \hspace{1cm} (S4c)

where the coefficients $\alpha, \beta$ in general depend on $\lambda$ and the chain length $L = 3l + 1$, where the latter is to be expected, similarly to the presence of such scaling in the particle densities of the ground state of chain of length $L = 3l$. At extreme staggering, the value of the coefficients can be read-off directly from Fig. S1 and we obtain

$$\lambda = 0 :$$

$$\left( \alpha, \beta \right) = (1, 1)$$  \hspace{1cm} (S5a)

$$\left( \alpha_3, \beta_3 \right) = (2, 1)$$  \hspace{1cm} (S5b)

$$\left( \bar{\alpha}, \bar{\beta} \right) = (2, 1)$$  \hspace{1cm} (S5c)

which are independent of $l$. For $\lambda = 1$, the values are obtained from the defining property (S3) and their scaling with $l$ is shown in Fig. S2a-f (up to $l = 6$). We use these values in the analysis of the critical dynamics.

To examine the observables, in Fig. S2, we show the (square of the) overlap (gray) together with $\delta n, \delta n_3$ (solid and dashed blue) and $\delta \bar{n}$ (magenta) for a quench from $|K_1\rangle$ and $|\bar{K}_1\rangle$ respectively with the dynamics generated by the supersymmetric Hamiltonian $H_Q$, Eq. (??). We see that the proposed observables nicely capture the overlap function $|\phi|^2$. 


III. STATE PREPARATION

In this section we study state preparation by means of adiabatic following for the two scenarios considered in the main text, an open chain of length $L = 3l$ and staggering $1\Lambda l \ldots 1\Lambda l$ (scenario 1) and an open chain of length $L = 3l + 1$ and staggering $11\Lambda l \ldots 1\Lambda l$ (scenario 2). Motivated by the relative simplicity of the experimental implementation, we focus on the critical case $\lambda = 1$. We comment on the (un)suitability of the adiabatic protocol for critical systems at the end of the section.

We consider a preparation of a ground state of a final Hamiltonian $H_f$ by adiabatic following from $H_i$, such that the time-dependent Hamiltonian can be expressed as

$$H(t) = (1 - F(t))H_i + F(t)H_f.$$  \hfill (S6)

We consider three distinct cases for $H_f$. (i) $H_f = H_Q$ in (scenario 1), (ii) $H_f = H_Q + \mu(n_1 + n_2)$ with $\mu \to \infty$ for kink and (iii) $H_f = H_Q + \mu(-n_1 + n_2 - \bar{\nu}n_3)$ with $\mu, \bar{\nu} \approx 3, 1/2$ for skink preparation in (scenario 2). $H_f$ in (ii) is chosen to ensure no occupation of the first two sites in preparation of the leftmost kink $|K_1\rangle$, a condition well satisfied even at criticality. This is apparent from Fig. S1 ($\lambda = 1$, $|K_1\rangle$ pane, blue data points) where we plot the densities corresponding to the ground state $|K'_1\rangle$ of $H_f$ as gray data points. For skinks, enforcing a particle at first site and zero particle on the second yields a density on the third site which is too small as compared with the ideal skink $|\bar{K}_1\rangle$. To rectify that, we introduce the $H_f$ described in (iii) and optimize the skink fidelity by tuning $\mu, \bar{\nu}$ with optimal values indicated above. The corresponding densities are shown as gray data points in Fig. S1 ($\lambda = 1$, $|\bar{K}_1\rangle$ pane).

The function $F(t)$ in Eq. (S6) has to be chosen such that it satisfies the adiabatic theorem. In particular, $F(0) = 0$, $F(T) = 1$, and it is at least twice differentiable. Here $T$ is the duration of the adiabatic sweep. In order to provide a specific example, we consider

$$F(t) = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi t}{T} \right) \right].$$  \hfill (S7)
While we focus on a critical system, as we are concerned with experiments with a finite number of atoms, the system will remain gapped (we exploit the rigorous results for the finite size scaling of the gap in Sec. [V E]). To this end we consider an approximate but qualitatively sufficient picture that the time duration of the adiabatic sweep $T$ should be much larger than the inverse of the spectral gap $1/\Delta_{sg}$ (see e.g. [2–6] for discussion of adiabaticity conditions).

We define the spectral gap as the energy difference between the lowest and the first excited state of the Hamiltonian $H$. One possibility is to choose $H_i$ such that its ground state is the lowest energy state of $H_Q$ at extreme staggering. For the specific lengths and staggerings considered, the $\lambda = 0$ states are particularly simple as they are given by product states of the form (see Table [1] for a summary of low energy spectral properties of the $M_1$ model at extreme staggering)

$$(i) \quad L = 3l, \ 11l \ldots \rightarrow |\psi_i\rangle = |010010\ldots\rangle$$

$$(ii) \quad L = 3l + 1, \ 11l \ldots \rightarrow |\psi_i\rangle = |001001\ldots\rangle$$

$$(iii) \quad L = 3l + 1, \ 11l \ldots \rightarrow |\psi_i\rangle = |101001\ldots\rangle$$

We thus take the initial Hamiltonian, which enforces the $l$ particles to be trapped at sites $s = \{2, 5, \ldots\}$, $s = \{3, 6, \ldots\}$ for cases $(i)$, $(ii)$ and $l + 1$ particles at sites $s = \{1, 3, 6, \ldots\}$ for $(iii)$, to be of the form

$$H_i = -\sum_i (\alpha i + \mu_0)n_i - \mu \sum_{i \in s} n_i.$$  

The first summand is meant to lift remaining degeneracies and in principle more complicated functions of the position $i$ can be considered. In practice, as we are mainly concerned with preserving the gap between the lowest energy and the first excited state, the details of this function are not essential as far as $F > 0$ for all $i$, which is used in the following. The state is prepared as

$$|\psi(t)\rangle = \mathcal{T} e^{-i \int_0^t dt' H(t')} |\psi_i\rangle,$$

where $H(t)$ is given by ([S0]), $\mathcal{T}$ is the usual time ordering operator and $|\psi_i\rangle$ are given by ([S8]–[S8]). We further denote the prepared state at the end of the adiabatic evolution as $|\psi_{\text{prep}}\rangle \equiv |\psi(T)\rangle$. We then quantify the fidelity of the prepared state as $(i) F = |\langle \psi_{\text{prep}} | \psi_0 \rangle|$, where $|\psi_0\rangle$ is the ground state of $H_Q$ and $(ii) F_j = |\langle \psi_{\text{prep}} | K_j \rangle|$, $(iii) F_j = |\langle \psi_{\text{prep}} | \bar{K}_j \rangle|$. The results for (i) and (ii) are shown in Fig. [S3] see the caption for details (the results of (iii) not shown are similar to (ii)). In summary, the fidelity of preparation $F > 0.92$ ($F > 0.95$ for the boundary kinks $|K_1\rangle, |K_{1+1}\rangle$ and $F > 0.91$ for the boundary skinks $|\bar{K}_1\rangle, |\bar{K}_{1+1}\rangle$) for system sizes we analyzed numerically, i.e. $l \leq 6$.

**Optimization.** Few remarks are in order with respect to the adiabatic procedure considered. A first technical one is that one can optimize the sweep function ([S7]) which adapts the rate of change to the instantaneous gap $\Delta_{sg}(t)$ evolving at a slower rate for smaller gaps, which has the potential to significantly reduce the time required to achieve a desired fidelity for a given system size [4]. This is desirable as the final (and minimal) gap $\Delta_{sg}(t = T)$ is decreasing with system size, so that using the same $\mathcal{F}(t)$ for different system sizes will result in larger times in order to achieve the same target fidelity as is illustrated in Fig. [S3]. The other remark is qualitative and is about the inadequacy of adiabatic protocols for preparing critical states with a vanishing gap in the thermodynamic limit, requiring an infinitely slow sweep akin to the Kibble-Zurek mechanism. While we consider the adiabatic preparation protocol even in this case, we note that other schemes using spatiotemporal quenches have been proposed recently [10]. Whether such a scheme can be implemented in our setup goes beyond the scope of the present work and we leave it for future investigations.

### IV. SADDLE POINT APPROXIMATION

When analyzing the kink dynamics, we have introduced the overlap between the time evolved leftmost kink and the rightmost kink, which, in the thermodynamic limit of large $l$, where $E_k \rightarrow E(\bar{k})$, becomes

$$o(t) = \langle K_{l+1} | K_1(t) \rangle \approx \frac{2}{l+2} \sum_{k=1}^{l+1} \sin \left( \bar{k} \right) \sin \left( \bar{k}(l+1) \right) e^{-iE(\bar{k})t} =: \sum_{k=1}^{l+1} w_k$$

(S11)
FIG. S3. Adiabatic state preparation. (a) Example of the spectrum of the Hamiltonian \[S6\] for case (i), i.e. \( L = 3l \) with \( l = 2 \), as a function of time. The inset shows the spectral gap. (b) Fidelity \(|\langle K'_j | K_j \rangle|\) of the kink states as a function of \( \lambda \) for case (ii), \( L = 3l + 1 \) with \( l = 4 \). The solid (dashed) lines are for the boundary, i.e. rightmost and leftmost, (bulk) kinks respectively. Here, \(|K'_j \rangle\) are the lowest energy states of the Hamiltonian \( H + \mu \sum_{i \in j} n_i \), where the \( \mu \)-term enforces no occupation (for \( \mu \to \infty \)) on the sites of \( j \)-th kink, \( i = (3(j-1), 3(j-1) + 1, 3(j-1) + 2) \) for \( 1 < j < l + 1 \) and \( i = 1, 2 (i = L - 1, L) \) for the leftmost (rightmost) kink \(|K_1 \rangle (|K_{l+1} \rangle)\), see also the main text. The inset shows the finite size scaling of the fidelity of the leftmost kink at criticality for \( l = 2, 3, 4, 5, 6 \). (c) Fidelity of the preparation of the ground state as a function of the adiabatic sweep length \( T \) for \( l = 2, 4 \) (blue, red), i.e. case (i), \( L = 3l, \lambda = 1 \). (d) Fidelity of the preparation of the leftmost kink \(|K_1 \rangle\) as a function of the adiabatic sweep length \( T \) for \( l = 2, 4 \) (blue, red), i.e. case (ii), \( L = 3l + 1, \lambda = 1 \). In (c) and (d) the horizontal axes are in terms of \( T\Delta_{sg} \), where \( \Delta_{sg} = \Delta_{sg}(T) \).

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TABLE I. Summary of the low energy properties of the \( M_1 \) model at extreme staggering \( \lambda = 0 \). The three possible staggerings and system sizes are listed in the top row/left column respectively. The three numbers are \((\text{deg}_{\text{lowest}}, E_{\text{lowest}}, \Delta_{sg}) = (\text{degeneracy of the lowest energy manifold, energy of the lowest energy states, spectral gap between the lowest and first excited state})\).

with \( \tilde{k} = \frac{\pi k}{l + 2} \). We note that the form of the overlap is reminiscent of the saddle point approximation for a purely imaginary exponent

\[
o(t) \sim \int dz f(z)e^{ig(z)} \approx e^{i g(z_0) \pm \frac{\pi}{2}} f(z_0) \sqrt{\frac{\pm 2\pi}{g''(z_0)}},
\]

where \( g(z) \in \mathbb{R} \), the saddle point \( z_0 \) satisfies \( g'(z_0) = \partial_z g(z)|_{z=z_0} = 0 \) and the signs are chosen such that the term under the square root is positive. Expanding the second sine term in Eq. \( S11 \) we obtain

\[
\sin \left( \tilde{k} \right) \sin \left( \tilde{k}(l + 1) \right) = \cos (\pi k) \sin \left( \tilde{k} \right)^2 = e^{i(2s-1)\pi k} \sin \left( \tilde{k} \right)^2, \quad s \in \mathbb{Z}.
\]
FIG. S4. Argument of the summands $w_k$ in (S11) for $l = 101, t = 120$ and $s = 1$.

Absorbing the phase factor in the exponential term in (S11), by comparison with (S12) we define
\[ g(k) = (2s - 1)\pi k - E(k)t. \]  
\[ (S14) \]
Differentiating with respect to $k$ and imposing the saddle point condition, we can formally write the $s$-dependent solution for the saddle point as
\[ k_s(t) = (E')^{-1} \left( \frac{(2s - 1)\pi}{t} \right), \quad s = 1, 2, \ldots, \]  
\[ (S15) \]
where the positivity of $s$ follows from the non-negativity of $E'(k)$ in $g'(k) = 0$. We thus get the saddle point which needs to be evaluated at each time $t$. The situation for a specific time is depicted in Fig. S4, where we show the argument of the summands in (S11) for $l = 101$ and $t = 120$. We see a clear rapid oscillatory behaviour resulting in the cancellation of terms from most $k$ but the ones in the vicinity of saddle point, which (here for $s = 1$) is evaluated from (S15) to $k_s = 56$.

Next, we write the condition $g'(k) = 0$ for the saddle point as
\[ \frac{v_{\text{max}}}{l + 2} \hat{v}(k) - (2s - 1) = 0, \]  
\[ (S16) \]
where we have used $\partial_k \hat{k} = \pi/(l + 2)$ and the definition of the group velocity $v(\hat{k}) = \partial_k E(\hat{k})$, which we have written as $v(\hat{k}) = v_{\text{max}} \hat{v}(k)$, so that $0 \leq \hat{v}(k) \leq 1$. It follows from (S16), that with increasing $t$, an increasing number of saddle-points $s = 1, 2, \ldots$ contribute, with the limiting values of $t$ corresponding to the fastest mode, $\hat{v}(k) = 1$. This leads us to the final result, namely Eq. (??) in the main text.

We note that the expressions can be simplified in particular cases, such as at criticality, $\lambda = 1$, where the dispersion takes a simple form $E(\hat{k}) = 2v_F \sin(\hat{k}/2)$. For the first saddle point ($s = 1$), for $x = (l + 2)/(v_F t) < 1$, the overlap Eq. (??) evaluates to
\[ |o(t)| = \frac{2}{\pi} \sin^2(\hat{k}_1) \sqrt{\frac{4\pi}{v_F t \sin(\hat{k}_1)/2}} = \frac{16}{\pi(l + 2)}(1 - x^2)^{3/4}x^{5/2}. \]  
\[ (S17) \]
It follows that the probability $|o(t)|^2$ (green-dashed curve in Fig. ??b) peaks at arrival time
\[ x_{\text{max}} = \sqrt{\frac{5}{8}} \quad \Rightarrow \quad t_{\text{max}} = \sqrt{\frac{8}{5}} \frac{l + 2}{v_F}. \]  
\[ (S18) \]
Similarly, we get the contribution from the second saddle point ($s = 2$, red curve in Fig. ??b) by substituting $x = 3(l + 2)/(v_F t)$ in the above formula.

V. EXPERIMENTAL IMPLEMENTATION

In order to realize the blockade mechanism, we assume the ground state atoms to be excited off-resonantly with detuning $\Delta$ to the Rydberg state $|r\rangle$ as described by the Hamiltonian (??). We assume that the sites can be addressed
individually such that a ground state $|g\rangle$ of an atom at site $i$ is coupled to $|r\rangle$ with Rabi frequency $\Omega_i$, while the detuning $\Delta$ is kept constant for all atoms. When $\Delta \gg \Omega_i$ for all $i$, one can adiabatically eliminate the many-body Rydberg states by means of Brillouin-Wigner perturbation theory carried out to fourth order in $\Omega/\Delta$ \cite{11,12}, which leads to a so-called flat-top potential. Here we present a useful shortcut derivation for two atoms, which, to the order considered, coincides with the results of the systematic adiabatic elimination and which has been also invoked in the analysis of Rydberg atoms in optical lattices \cite{13}.

A. Shortcut derivation of the dressed atomic potential

The Rydberg Hamiltonian of a system of two atoms located at positions $r_i$ and $r_j$ with $r = |r_j - r_i|$ and $i < j$ reads

$$H = \Omega_i \sigma_i^x + \Omega_j \sigma_j^x + \Delta (n_i^r + n_j^r) + V n_i^r n_j^r. \quad (S19)$$

In the basis $\{|gg\rangle, |gr\rangle, |rg\rangle, |rr\rangle\}$ it can be written as

$$H = \begin{pmatrix}
0 & \Omega_j & \Omega_i & 0 \\
\Omega_j & \Delta & 0 & \Omega_i \\
\Omega_i & 0 & \Delta & \Omega_j \\
0 & \Omega_i & \Omega_j & 2\Delta + V_j
\end{pmatrix} \quad (S20)$$

where $V = C_6/r^6$. The Schrödinger equation for the coefficients of the wavefunction is

$$i\dot{c} = Hc, \quad (S21)$$

where

$$c = (c_{gg}, c_{gr}, c_{rg}, c_{rr})^T. \quad (S22)$$

Since $\Omega_{i,j} \ll \Delta$, we eliminate the rapidly oscillating components by setting $\dot{c}_{gr} = \dot{c}_{rg} = \dot{c}_{rr} = 0$ in $(S21)$. Solving for these three components and substituting in the remaining equation for $c_{gg}$, which is of the form $i\dot{c}_{gg} = W'(r)c_{gg}$, yields the effective potential

$$W'(r) = -\frac{2\Omega_i \Omega_j (2\Delta + V(r))}{\Delta(2\Delta + V(r)) - 2\Omega_i \Omega_j}. \quad (S23)$$

Expanding in $\Omega$ and subtracting a global offset $-2\Omega_i \Omega_j/\Delta$ we obtain the effective potential between the two ground state atoms

$$W(r) = \frac{2(\Omega_i \Omega_j)^2 V(r)}{\Delta^3(2\Delta + V(r))} + O(\Omega^6), \quad (S24)$$

with amplitude $W(0) - W(\infty) = 2(\Omega_i \Omega_j)^2/\Delta^3$, which sets the maximum available energy for realizing the blockade. In practice, one needs to accommodate the lattice spacing $r_0$ so that the blockade energy is $W(r_0)$ and the energy scale of the Hamiltonian is $W(2r_0)$. We note that the chemical potentials $\mu_i$ of the ground state atoms are not affected by the dressing. For future convenience let us parametrize the dressed interaction between atoms at site $i$ and $i + n$ by writing the the Eq. $(S24)$ as

$$W_i(nr_0) = \frac{2(\Omega_i \Omega_i n)^2 V(nr_0)}{\Delta^3(2\Delta + V(nr_0))}, \quad (S25)$$

where we have omitted the higher order contributions $O(\Omega^6)$.

B. Implementing the $M_1$ model off criticality

Here we discuss how to implement the $M_1$ model off criticality, $0 \leq \lambda < 1$. It is instructive to write $H_Q$ explicitly. For $l = 2, L = 3l + 1$ and staggering pattern $11\lambda \ldots$ it reads \cite{14}

$$H_Q = \sum_{i=1}^{L-2} \tilde{J}_{i+1} P_{i-1} \left( c_i^\dagger c_{i+1} + \text{H.c.} \right) P_{i+2} + P_2 + \sum_{i=1}^{L-2} \tilde{W}_{i+1} P_i P_{i+2} + P_{L-1}, \quad (S26)$$

$$JH_Q = -\sum_{i=1}^{L-2} \tilde{J}_{i+1} P_{i-1} \left( c_i^\dagger c_{i+1} + \text{H.c.} \right) P_{i+2} + \sum_{i=1}^{L} \mu_i n_i + \sum_{i=1}^{L-2} W_i(2r_0)n_i n_{i+2}. \quad (S27)$$
In the first line, $H_Q$ is expressed in dimensionless units with $\vec{J} = (1, \lambda, 1, \lambda, \lambda)$, $\vec{W} = (1, \lambda^2, 1, 1, \lambda^2)$. In the last line, we have expanded the projectors $P_i = (1 - n_i)$ (except in the kinetic term) and restored the dimensions to make connection with the physical Rydberg Hamiltonian, $\vec{J} = J\vec{J}$, $\vec{W} = J\vec{W}$, where $J = W(2r_0)$. Omitting a constant factor, we have for the chemical potentials $\vec{\mu} = -J(1, 1 + \lambda^2, 2, 1 + \lambda^2, 1 + \lambda^2, 2)$. The situation is summarized in Fig. S5 see the caption for details. We thus have a repeated pattern of period three (starting at the second site for $L = 3l + 1$) of tunneling amplitudes, next-nearest neighbour interaction potentials and chemical potentials, denoted by the black dashed box in Fig. S5. We now comment on the details of implementation related to each of these three types of Hamiltonian contributions.

**Chemical potentials.** The chemical potentials can be realized by a bichromatic optical lattice with the two lattice wave vectors having ratio of 1/3 as depicted in Fig. S5. We note that, due to the boundary conditions, the chemical potentials $\mu_1$, $\mu_L$ on the first and last site get an extra offset which can be realized by for instance additional optical fields.

**Interaction potential.** It is straightforward to show from Eq. (S24), that the off-critical potential pattern $\vec{W}$ can be realized by the pattern of on-site Rabi frequencies $\vec{\Omega} \propto (\lambda, \lambda, 1, \lambda, \lambda, 1)$. In principle, the atoms in the ground and the Rydberg state will experience different polarizability leading to a different AC Stark shift originating both in the driving Rabi field $\Omega$ and the optical lattice potential. Since $\Omega \gg J$, the leading contribution to the AC Stark shift will be from the Rabi frequencies and is proportional to $\Omega^2/\Delta \ll \Delta$ so that it has been neglected in the derivation of Eq. (S24) assuming identical detunings $\Delta$ for all lattice sites.

**Tunneling amplitudes.** Similarly to the chemical and interaction potentials, one needs to tune the tunneling amplitudes, which can be achieved in principle by means of Raman assisted hoppings [15]17, with $J_{i,i+1} \propto \Omega^R_i \Omega^R_{i+1} / \Delta^R$, where $\Omega^R_i$ and $\Delta^R$ label the on-site (single-photon) Raman Rabi frequencies and detunings, respectively, see Fig. S5. These additional laser beams will also contribute to the ground and Rydberg state polarizabilities, however, as $J \ll \Delta$, they will contribute only a subleading correction to the dressed potential as discussed in the previous paragraph.

**FIG. S5.** Scheme of the possible experimental implementation of the $M_1$ model for a generic $\lambda$. The pattern of tunneling amplitudes $J$, next-nearest neighbour interactions $W$ and the corresponding on-site Rydberg laser Rabi frequencies $\Omega$ is given at the bottom in black, red, blue respectively. The necessary periodicity can be achieved with a bi-chromatic optical lattice with ratio 1/3 between the lattice wavelengths. In order to realize the required tunneling amplitudes, one might use Raman assisted hoppings with Raman Rabi frequencies $\Omega^R_i$ and detunings $\Delta^R$ via an intermediate state $|i\rangle$. The dashed rectangle denotes the basic building block which, when repeated, constitutes the whole chain (with the appropriate boundary chemical potentials as indicated).
1. Effect of the interaction tails off-criticality

It is instructive to consider the effect of the dressed interaction beyond next-to-nearest neighbours. Using the pattern of the on-site Rabi frequencies $\Omega_i$ shown in Fig. S5 and the Eq. (S25) we get for the patterns of the next-to and the next-to-next-to nearest neighbours

$$\tilde{\Omega} = \Omega(\lambda, 1, \lambda, 1, \ldots)$$

$$\tilde{W}(2r_0) = (W_1(2r_0), W_2(2r_0), W_3(2r_0), W_4(2r_0), \ldots) \propto \Omega^4(\lambda^2, \lambda^4, \lambda^2, \lambda^2, \ldots)$$

$$\tilde{W}(3r_0) = (W_1(3r_0), W_2(3r_0), W_3(3r_0), \ldots) \propto \Omega^4(\lambda^4, \lambda^4, 1, \ldots)$$  \hspace{1cm} (S27)

It is thus apparent that for sufficiently small $\lambda$, the neglected interactions beyond next-to-nearest neighbours $W(3r_0)$ will become dominant (the “1” terms in $\tilde{W}(3r_0)$) and cannot be neglected anymore. We would like to emphasize that this observation does not affect the discussion of the kink dynamics at the critical point $\lambda = 1$ in the main text, but it clearly limits the exploration of the off-critical regime. To quantify the effect of the long-range interactions we consider system sizes $L = 3l$ featuring a unique ground state, cf. the Table I, and the dressed Hamiltonian (S26) which now includes all the terms beyond $W(2r_0)$

$$JH_Q^{long-range} = JH_Q + \sum_{n=3}^{L-n} \sum_{i=1}^{L-n} W_i(nr_0)n_in_{i+n}$$ \hspace{1cm} (S28)

Let us denote by $|\psi_{long-range}\rangle$, $|\psi_0\rangle$ the ground states of (S28) and (S26), $|\psi_0\rangle$ corresponding to the notation used in Sec. III In Fig. S6a we show the fidelity

$$F = \left| \langle \psi_{long-range} | \psi_0 \rangle \right|$$ \hspace{1cm} (S29)

vs. $\lambda$ for various system sizes (gray lines). It is apparent that the inclusion of the long-range tails limits the exploration of the off-critical regime to $\lambda$ asymptotically approaching 1 in the limit of infinite system sizes. Panes (b,c) of Fig. S6 then show the finite-size scaling of the value of $\lambda$ corresponding to maximum slope (largest gradient) of the fidelity

$$\lambda_{max-slope} = \max_{\lambda} (\partial_\lambda F(\lambda)),$$ \hspace{1cm} (S30)

pane (b), and the error $1 - F$ in the ground state fidelity at criticality, pane (c) (gray lines).

In the following section we discuss how to significantly suppress the unwanted effect of the tails of the interaction using doubly-dressed Rydberg potential. This not only allows to extend the region of high fidelity ground states to smaller $\lambda$ and larger system sizes but also achieves tenfold improvement in the $W(2r_0)/W(3r_0)$ ratio over the single dressing scheme.

C. Improving the scheme using double dressing

The idea behind the improvement is to suppress the long-range tail $W_i(nr_0)$, $n > 2$, of the dressed potential (S25) by a second potential with asymptotically the same behaviour in the long separation limit, but with opposite sign. Specifically, from (S25) we have

$$W_i(nr_0)|_{n \rightarrow \infty} = \frac{(\Omega_i\Omega_{i+n})^2}{\Delta^4} V(nr_0).$$ \hspace{1cm} (S31)

With a slight abuse of notation, let us denote the quantities corresponding to the second potential with a prime (not to be confused with (S23)). We thus require

$$W_i(nr_0)|_{n \rightarrow \infty} = - W_i'(nr_0)|_{n \rightarrow \infty} \Rightarrow \Omega' = \Omega \left| \frac{\Delta'}{\Delta} \right| \left| \frac{C_6}{C'_6} \right|^4,$$ \hspace{1cm} (S32)

where $\Omega^{(i)}$ is the amplitude of the pattern of the on-site Rabi frequencies, $\tilde{\Omega}^{(i)} = \Omega^{(i)}(\lambda, 1, \ldots)$, and we recall that the bare Rydberg interaction reads $V^{(i)}(nr_0) = C_6^{(i)}/(nr_0)^6$ and $C'_6 < 0$. The Eq. (S32) thus represents a condition for the amplitude of the second dressing Rabi frequency with other parameters given. We also note that in order
to avoid the resonance $\Delta' + V'(r) = 0$, which leads to the vanishing denominator in (S25) and can be exploited for instance to realize a spin model featuring both attractive and repulsive interaction [18], we take $\Delta' < 0$.

The considered construction is allowed by an appropriate choice of the additional Rydberg state which features attractive rather than repulsive interaction. Specifically, in addition to the $|nS\rangle$ state of $^7$Li which has $C_6 > 0$ we choose the $|nD_{3/2}\rangle$ state with $C_6 < 0$. An example of $W$, $W'$ and $W + W'$ is shown in Fig. S7 as solid red, blue and black lines respectively. Here we have assumed $\lambda = 1$, i.e. $\Omega_i = \Omega \forall i$ for the sake of simple illustration and considered the Rydberg level $|74D_{5/2}\rangle$ with $C_6' = -6.005 \text{ GHz} \cdot \mu\text{m}^6$ [19, 20] in addition to the $|84S\rangle$ one. Clearly, the details of the resulting potential depend on the precise choice of the parameters in the now extended parameter space spanned by $\Omega, \Delta, C_6, \Delta', C_6'$ (with $\Omega'$ given by (S32)). The detailed exploration of this parameter space goes beyond the scope of the present work, however we note that the resulting choice is typically a compromise between maximizing the energy scale $W_{\text{tot}}(2r_0)$ and maximizing the ratios $W_{\text{tot}}(r_0)/W_{\text{tot}}(2r_0)$ and $W_{\text{tot}}(2r_0)/W_{\text{tot}}(3r_0)$ which characterize the blockade and the suppression of the long-range tails respectively. Here $W_{\text{tot}} = W + W'$ is the total dressed potential. To illustrate this, the inset of Fig. S7 shows the detail of the potential for the next-to and next-to-next-to nearest neighbours for $|84D_{5/2}\rangle$ with $C_6' = -24.200 \text{ GHz} \cdot \mu\text{m}^6$ [19, 20] (dashed line) in addition to the potential stemming from the $|74D_{5/2}\rangle$ state (solid line). We get for these two situations

$$
|84S\rangle, |74D_{5/2}\rangle : (W_{\text{tot}}(2r_0) = J, W_{\text{tot}}(r_0)/W_{\text{tot}}(2r_0), W_{\text{tot}}(2r_0)/W_{\text{tot}}(3r_0)) \approx (1 \text{ kHz}, 74, 94)
$$

$$
|84S\rangle, |84D_{5/2}\rangle : (W_{\text{tot}}(2r_0) = J, W_{\text{tot}}(r_0)/W_{\text{tot}}(2r_0), W_{\text{tot}}(2r_0)/W_{\text{tot}}(3r_0)) \approx (2.4 \text{ kHz}, 35, 56).
$$

which illustrate the point discussed, i.e. the increase of the effective Hamiltonian energy scale $W_{\text{tot}}(2r_0)$ at the expense of reducing the quality of the blockade and the suppression of the interaction tails. In particular the values $(W_{\text{tot}}(r_0)/W_{\text{tot}}(2r_0), W_{\text{tot}}(2r_0)/W_{\text{tot}}(3r_0))$ should be contrasted with the values 21 and 11 respectively of the single dressing scheme. We thus see that both of these ratios can be enhanced by a factor 5-10 with the double dressing scheme.

Two minor comments are in order. Firstly, as $\Omega'$ is constrained by the Eq. (S32) one has to check whether it still satisfies $\Omega' \ll |\Delta'|$ required for the perturbative description to hold. Since we have chosen $|\Delta'| \approx |\Delta|$ and we have $|C_6'| > |C_6|$, it follows from (S32) that it is indeed the case (specifically, for the chosen value $\Delta' = -500 \text{ MHz}$ we get $|\Omega'|/|\Delta'| \approx 1/17.5$ with $\Omega' \approx 2\pi \times 4.5 \text{ MHz}$ for $|74D_{5/2}\rangle$ and $|\Omega'/\Delta'| \approx 1/24.5$ with $\Omega' \approx 2\pi \times 3.2 \text{ MHz}$ for $|84D_{5/2}\rangle$).

Secondly, one should also verify whether the separation between adjacent Rydberg levels is larger than the considered detunings so that one still selectively addresses the target Rydberg state. For the range $n \sim 70 - 80$ of the principal quantum numbers, the typical separation between adjacent $|nS\rangle - |(n + 1)S\rangle$ and $|nD\rangle - |(n + 1)D\rangle$ states is of the order of 10 GHz which is well above the values of the considered detunings $|\Delta|, |\Delta'| \sim 0.5 \text{ GHz}$.

Finally, let us go back to the discussion of Sec. V B 1 about the effect of the long-range tails of the interactions.
as quantified by the ground state fidelity of the supersymmetric Hamiltonian $\text{S}26$. The fidelity, $\lambda_{\text{max}}$ and slope $1 - F$ at criticality corresponding to the doubly-dressed potential are shown in red in Fig. $\text{S}6$. We see that the double dressing scheme significantly improves over the single dressing one reducing $\lambda_{\text{max}}$ and slope by a factor of $\gtrsim 3$ and the error at criticality by two orders of magnitude for the same system size.

**FIG. S7.** The doubly dressed potential $W_{\text{tot}} = W + W'$ (solid black line) composed of the repulsive $W$ (solid red line) and attractive $W'$ (solid blue line) as a function of the distance $r$. The red circles indicate the potential at the positions of the atoms spaced by $r_0 = 2.5 \mu m$. Parameters used for $W$ are that of Fig. $\text{S}7$, i.e. $\Omega = 2\pi \times 10$ MHz, $\Omega/\Delta = 1/10$, $C_6 = 645$ GHz $\cdot \mu m^6$ corresponding to the state $|84\text{S}\rangle$ of $^6\text{Li}$. Parameters used for $W'$ are $\Delta' = -500$ MHz, $C_6' = -6005$ GHz $\cdot \mu m^6$ for $|74\text{D}\rangle$ with $\Omega' = 2\pi \times 4.5$ MHz given by the Eq. $\text{S}32$. The inset shows, in addition to $W_{\text{tot}}$ of the main plot (solid black line), $W_{\text{tot}}$ corresponding to taking $|84\text{D}\rangle$ instead of $|74\text{D}\rangle$ for the second Rydberg state with parameters $C_6' = -24200$ GHz $\cdot \mu m^6$ yielding $\Omega' = 2\pi \times 3.2$ MHz (dashed black line).

### D. Arbitrary number of dressing potentials

In this section we generalize the above considerations and address the following question: Provided an arbitrary number of dressing potentials $W$ of the form (S24) is available, is it possible to generate a total potential $W_{\text{tot}} = \sum_i W_i$ which matches the target potential implementing the supersymmetric model, i.e. satisfying (we again focus on the critical case $\lambda = 1$ for simplicity)

$$\frac{W_{\text{tot}}(r_0)}{W_{\text{tot}}(2r_0)} \to \infty \quad \text{(perfect blockade)} \quad (\text{S34a})$$

$$\frac{W_{\text{tot}}(2r_0)}{W_{\text{tot}}(nr_0)} \to \infty \quad \text{for } n > 2 \quad \text{(no interaction tails)} \quad (\text{S34b})$$

with $W(2r_0)$ the relevant energy scale? To this end we first write the $j$-th potential (S24) in a customary form as

$$W(r, \rho_j) = A_j \left(\frac{1}{\rho_j r^6} + \frac{1}{\rho_j r^6} \right) \quad (\text{S35a})$$

$$A_j = 2 \frac{\Omega^{(j)}}{\Delta^{(j)^3}} \quad (\text{S35b})$$

$$\rho_j = \left(\frac{2\Delta^{(j)}}{C_6^{(j)}}\right)^{\frac{1}{2}} \quad (\text{S35c})$$

where $A_j$ is the potential amplitude and $\rho_j$ the characteristic inverse radius. The functions $W(r, \rho_j)$ constitute a set on which we wish to decompose our target potential $W_{\text{target}}$. If we allow for an infinite such set with smoothly varying
A and \( \rho \), we can formulate this requirement as

\[
W_{\text{target}}(r) = \int_{0}^{\infty} d\rho W(r, \rho) A(\rho).
\]  

(S36)

This is nothing but the inhomogeneous Fredholm equation of the first kind with kernel \( W(r, \rho) \) and an unknown function \( A(\rho) \). In order to proceed, we need to specify the target potential \( W_{\text{target}} \), which has to be chosen to satisfy the conditions \( (S34) \). Motivated by the fact that at long distances the dressed potentials decay as \( r^{-6} \), we consider a function

\[
\frac{W_{\text{target}}(nr_{0})}{W_{\text{target}}(2r_{0})} = \begin{cases} 
    s & \text{for } n = 1 \\
    1 & \text{for } n = 2 \\
    \frac{1}{s} & \text{for } n = 3 \\
    \frac{1}{s} \left( \frac{1}{n} \right)^{6} & \text{for } n > 3,
\end{cases}
\]  

(S37)

where \( s \) is the suppression factor such that \( W_{\text{target}}(r_{0})/W_{\text{target}}(2r_{0}) = W_{\text{target}}(2r_{0})/W_{\text{target}}(3r_{0}) = s \) and \( W_{\text{target}}(nr_{0})/W_{\text{target}}((n + 1)r_{0}) = (n + 1)^{6}/n^{6} \). One obtains the ideal supersymmetric potential in the limit \( s \to \infty \).

Next, we need to determine the potential amplitudes \( A(\rho) \). To proceed, we consider a discrete set of separations \( \{r_{i}\} \) and inverse characteristic radii \( \{\rho_{j}\} \) such that we get from the Fredholm equation \( (S36) \)

\[
W_{\text{target}}(r_{i}) = \sum_{j} W(r_{i}, \rho_{j}) A(\rho_{j}).
\]  

(S38)

The equation \( (S38) \) is thus a matrix equation for the amplitudes \( A \) which can be inverted using pseudoinverse \( W^{+} \) as

\[
A = W^{+} W_{\text{target}}.
\]  

(S39)

Here we consider the pseudoinverse since we are dealing in general with a rectangular rather than a square matrix \( W(r_{i}, \rho_{j}) \). In fact, even when dealing with a square matrix, it might not be invertible (and in general is not). Since we use a finite set \( \{r_{i}\} \) in \( (S38) \), the resulting total potential

\[
W_{\text{tot}}(r) = \sum_{j} W(r, \rho_{j}) A(\rho_{j})
\]  

(S40)

will differ from \( W_{\text{target}} \) for a general \( r \not\in \{r_{i}\} \).

It would be desirable to investigate the mathematical properties of \( (S36), (S38) \) in detail. For the purpose of demonstrating that such approach can yield arbitrary suppression \( s \), yielding the ideal potential satisfying Eqs. \( (S34) \), we consider specific examples illustrated in Fig. \ref{fig:example}. We refer the reader to the caption for all relevant details.

The results presented in the Fig. \ref{fig:results} indicate that arbitrary suppression \( s \), and thus the ideal supersymmetric potential corresponding to the next-to-nearest neighbour interaction \( (S34) \) can be achieved in principle, provided a sufficient number of suitable dressing potentials is available. To assess an experimental feasibility of such approach would however require a detailed study of available van der Waals couplings provided by the Rydberg levels and realistic Rabi frequencies and detunings. More involved dressing schemes would also increase the decoherence rates as we discuss in the next section. While such a detailed study of experimental feasibility goes beyond the scope of the present work, the possibility of engineering step-like potentials through multiple dressings remains an interesting opening which it might be interesting to address in the future.

### E. Coherent evolution

In practice, any experiment is prone to detrimental effects, such as decoherence due to off-resonant scattering from the Rydberg state. On the one hand, one can reduce the scattering, the rate of which \( \propto (\Omega/\Delta)^{2} \), by reducing \( \Omega/\Delta \). On the other hand, to limit the influence of such scattering, it is desirable to reduce the time of the experiment, by increasing the energy scale of \( H_{\text{Ry}} \). Since \( J = W(2r_{0}) \propto \Omega^{4}/\Delta^{3} \), it can be achieved by increasing \( \Omega \) and reducing \( \Delta \) while still in the perturbative regime \( \Omega/\Delta \ll 1 \). For this reason, to weigh between these two effects, we choose as a figure of merit the achievable system size, for which \( |K_{1}\rangle \) traverse the chain coherently in time \( t = LT_{c} \), where \( T_{c} = 1/(3v_{F}W(2r_{0})) \) is the characteristic propagation time between lattice sites. To obtain \( L_{\text{max}} \), \( t \) should be equated
FIG. S8. The total potential $W_{\text{tot}}$ (blue and yellow solid lines in (a-c)), Eq. (S40), obtained from the discrete version of the Fredholm equation (S38) for suppression factors $s = 10^3$ (a,c) and $s = 10^5$ (b). The red circles indicate the set $\{r_i\}$ of atomic distances used in (S38). In (a,b) [(c)] we use 8 [200] equally spaced inverse radii $\rho_j$ from the interval $\rho_j r_0 \in [0.01, 2]$ [$\rho_j r_0 \in [0.01, 3]$]. Since we use the logarithmic plot, the blue (yellow) colour indicate the positive (negative) part of $W_{\text{tot}}$. One can see from (a-c) that the total potential matches the specified points $\{r_i\}$ but deviates from $W_{\text{target}}$, Eq. (S37) (solid gray line), for $r \notin \{r_i\}$. In particular, one has to verify that the condition (S34b) is satisfied for large $r$. In (a), (b) we get $W_{\text{tot}}(2r_0)/W_{\text{tot}}(25r_0) \approx -5200$ and $6 \cdot 10^6$ respectively, where the reference point $n = 25$ is indicated by the green circle. Furthermore, in (d) we show the form of the amplitude function $A(\rho_j)$ for the parameters used in (c).

with the effective decay time $\tau = 1/\gamma$, where $\gamma \approx L/3\gamma_0(\Omega/\Delta)^2$ is the sum of $\approx L/3$ individual atomic far-detuned decay rates [21]. Combining these expressions we get

$$L_{\text{max}} = 3 \sqrt{\frac{\tau_0 v_F W(2r_0)}{(\Delta)^2}} = 3\Omega \sqrt{\frac{2\tau_0 v_F}{\Delta \left(1 + \frac{2\Delta}{\Omega}\right)}}.$$  

(S41)

Fig. S9 shows $L_{\text{max}}$ vs. $\Delta/\Omega$ for the state $|84S\rangle$ of $^6\text{Li}$ and parameters used in the main text, $C_6 = 645 \text{GHz} \cdot \mu\text{m}^6$, $r_0 = 2.5 \mu\text{m}$. It is apparent from the Figure that system sizes of the order of 100 sites might be achievable for realistic parameters. While this is encouraging, it is known that the dressing schemes are sensitive to detrimental effects, such as line broadening [22] leading to avalanche dephasing [23, 24]. It has been suggested, however, that these effects can be mitigated by cooling to reduce the black-body radiation or quenching the contaminant states [23].

Next, in analogy to the derivation of Eq. (S41) we can derive the scaling of the system size taking into account both adiabatic preparation, cf. Sec. III, and the subsequent time evolution. We note that the estimation Eq. (S41) holds for any $\lambda$, in which case $v_F$ should be replaced by $v_{\text{max}}(\lambda)$.

To proceed, we quantify the preparation time as a multiple $\kappa$ of the inverse spectral gap,

$$T = \frac{\kappa}{\Delta_{\text{sg}}}.$$  

(S42)

Working at criticality, we use the formula for finite size scaling of the gap [1]

$$\Delta_{\text{sg}} \approx \frac{2\pi E_{\text{SCFT}} 3v_F W(2r_0)}{L},$$  

(S43)
where \( E_{SCFT} = 1/3 \) (2/3) for \( L = 3l + 1 \) \((L = 3l)\) respectively. We write the dispersive decay rates of \( l \approx L/3 \) atoms as

\[
\Gamma = \frac{L}{3} \frac{1}{\tau_0} \left( \frac{\Omega}{\Delta} \right)^2 \bar{\rho}, \quad \gamma = \frac{L}{3} \frac{1}{\tau_0} \left( \frac{\Omega}{\Delta} \right)^2,
\]

\( \text{(S44)} \)

where \( \bar{\rho} \) stands for the average probability of being in the Rydberg state during the adiabatic sweep and depends on the form of the sweep. For the function \( \text{(S7)} \) it is taken to be

\[
\left( \frac{\Omega}{\Delta} \right)^2 \bar{\rho} = \frac{1}{T} \int_0^T dt \left( \frac{\Omega(t)}{\Delta} \right)^2 = \left( \frac{\Omega}{\Delta} \right)^2 \frac{1}{T} \int_0^T dt \sqrt{\mathcal{F}(t)} = \left( \frac{\Omega}{\Delta} \right)^2 \frac{2}{\pi} \rightarrow \bar{\rho} = \frac{2}{\pi},
\]

\( \text{(S45)} \)

where the ramping of the Hamiltonian is implemented by tuning the Rabi frequency so that \( \mathcal{F}(t) \propto \Omega(t)^4 \) and correspondingly for the tunnelings \( J(t) \). Setting

\[
\Gamma T + \gamma t = 1
\]

\( \text{(S46)} \)

and recalling that \( t = LT_c = L/[3v_F W(2r_0)] \), we can express \( L \) after substituting \( \text{(S42-S44)} \) to \( \text{(S46)} \) as

\[
L_{\text{max}} = 3\Omega \frac{\tau_0 2v_F}{\Delta \left( 1 + \frac{2\Delta}{\pi E_{SCFT} \kappa \bar{\rho} + 1} \right)}.
\]

\( \text{(S47)} \)

We note that setting \( \kappa = 0 \) corresponds to ignoring the preparation stage and we recover the formula \( \text{(S41)} \). Similarly, ignoring the time evolution, as in the case of the preparation of the ground states for chains with \( L = 3l \), amounts to neglecting the “+1” term in the second denominator.

**FIG. S9.** Maximum chain length \( L_{\text{max}} \) for a coherent evolution using Rydberg state \|84S\rangle of \(^6\text{Li}\) for temperatures \( T = (0, 30, 273) \) K (blue, green, red) corresponding to \( \tau_0 = (8.6, 2.8, 0.42) \) ms \[19, 20\].

**F. Energy scales and quality of the approximations**

*Estimates of energy scales.* It is apparent from the previous sections that engineering the supersymmetric Hamiltonian is a tradeoff between a number of requirements. On one hand, one wishes to maximize the energy scales, namely the dressing Rabi frequency \( \Omega \), which determines the interaction Eq. \( \text{(S24)} \) to overcome the decoherence, cf. Eq. \( \text{(S47)} \). Tuning the Hamiltonian to the supersymmetric point \( J = W(2r_0) \), it also maximizes the tunneling rate, cf. Eq. \( \text{(S26)} \). Same effects result from reducing \( \Delta \).

On the other hand, one wishes to minimize \( \Omega/\Delta \) to be well in the adiabatic approximation regime, where the flat-top interaction potential \( \text{(S24)} \) remains valid, the population in the Rydberg state is negligible and where the Rydberg Hamiltonian, Eq. \( \text{(??)} \) of the main text, approaches the supersymmetric one.
Choosing \( \Omega = 2 \)

The ratio \( k_\tau \) is instructive to consider specific example. Focusing on fermionic \( \Omega \) increasing \( \Omega \), the \( r \) line in Fig. \( ?? \) dynamics generated by the Rydberg Hamiltonian with interactions truncated beyond next-to-nearest neighbour (gray line in Fig. \( ?? \)) this occurs in the limit \( V(\mu)/E_r \gg 1 \), where \( E_r = (\hbar k^2)/(2m) \) is the recoil energy, \( m \) the atom mass and \( k = 2\pi/\lambda \) the lattice wavevector. In the tight-binding regime, the tunneling rate is given by \( J/E_r = \frac{4}{\sqrt{\pi}} \left( \frac{V_0}{E_r} \right)^{\frac{3}{2}} \exp \left( -2\sqrt{\frac{V_0}{E_r}} \right) \) (S48) and the band width \( \Delta E = 4J \). Furthermore, taking a harmonic approximation for the minima of the potential \( V(x) \), one can identify the corresponding harmonic oscillator frequency separating the ground and the first excited state \( \omega = \sqrt{2V_0k^2/m} \). This indicates the relevant energy scale for the temperature of the atoms, namely \( k_\eta T < \omega \) in order to avoid thermal excitations of the higher bands of the lattice. Put together we thus require \( V_0 > E_r \) and \( \Delta E, k_\eta T < \omega \).

The above parameters fix the tunneling rate \( J \approx 4 \text{kHz} \), which implies \( V_0 \approx 5.5 \times 10^{2} \text{kHz} \) (for \( \alpha = 2r_0 = 5 \mu \text{m} \) of the optical lattice light) and \( \omega \approx 2\pi \times 6 \times 10^{2} \text{ kHz} \) corresponding to \( h\omega/k_\eta \approx 0.3 \mu \text{K} \). To this end we note that temperatures of order \( O(10 \text{ nK}) \) have been achieved when cooling fermions in optical lattices \( [27,29] \). Clearly, the value \( V_0/E_r \approx 5.5 \) puts in question the appropriateness of the tight-binding approximation. One way to ensure its applicability is to increase \( V_0/E_r \), reducing however the tunneling rate significantly. For instance for \( V_0/E_r = 40 \) we get \( J = 6 \times 10^{2} \text{ Hz} \). To remedy this, one could implement the Raman assisted hopping as suggested in Fig. \( ?? \) in the context of the off-critical implementation of the \( M_1 \) model, leading to \( J \propto \Omega R^2/\Delta_R \).

Quality of the approximations. In the proposed quantum simulation of the supersymmetric lattice Hamiltonian, we have made two crucial approximations, namely we have adiabatically eliminated the Rydberg states, cf. Sec. \( \text{V}A \) and neglected the long range tails of the dressed interaction beyond next-to-nearest neighbours. The former approximation effectively neglects a finite population in the Rydberg state which results in photon scattering with associated energy scale \( \gamma \sim \gamma_0(\Omega/\Delta)^2 \), cf. Sec. \( \text{V}E \) while the energy scale of the latter is \( W(3r_0) \). These two scales then set a bound on the times where the approximations remain valid. Importantly, one also requires coherent evolution up to times \( tJ/l \sim O(1) \), where \( J = W(2r_0) \), to see the effects of the (s)kink propagation.

Here we specifically focus on evaluating the effect of neglecting the tails of the interaction versus neglecting the scattering. We further parametrize \( W(nr_0) = w_n \Omega^4/\Delta^3 \) and ask what is the effect of both for a given time \( tJ \). We get for the scattering and neglecting the tails respectively

\[
\frac{t\gamma}{tJ} \propto \frac{1}{w_2} \frac{\Delta \gamma_0}{\Omega} \frac{tW(3r_0)}{tJ} = \frac{W(3r_0)}{W(2r_0)}. \tag{S49}
\]

The ratio \( W(3r_0)/W(2r_0) \ll 1 \) by construction. In comparison, \( \Delta \gamma_0/(w_2 \Omega^2) \approx 290 \) for the parameters used in the main text, i.e. \( \Omega = 2\pi \times 10 \text{ MHz} \), \( \Omega/\Delta = 1/10 \), \( \Omega_0 = 645 \text{ GHz} \cdot \mu \text{m}^6 \), \( r_0 = 2.5 \mu \text{m} \) and taking \( \tau_0 = 1/\gamma_0 = 8.6 \text{ ms} \) corresponding to zero temperature lifetime of \( 84 \text{S} \) state of \( ^6 \text{Li} \). It is thus clear that the off-resonant scattering is the dominant limiting factor for the coherent evolution. Furthermore, this trend will become even more pronounced in the double dressing scheme discussed in Sec. \( \text{V}C \) which further reduces the ratio \( W(3r_0)/W(2r_0) \) and increases the scattering due to the off-resonant coupling to the additional Rydberg state. This is indeed compatible with the time dynamics generated by the Rydberg Hamiltonian with interactions truncated beyond next-to-nearest neighbour (gray line in Fig. \( ??c \)). In this case, the only discrepancy with respect to the dynamics generated by the supersymmetric Hamiltonian (blue dashed line in Fig. \( ??c \)) comes from the leakage to the Rydberg state. While this increases with increasing \( \Omega/\Delta \), the overall optimization is dictated by the requirements in Eq. \( \text{S49} \), namely it is favorable to increase \( \Omega/\Delta \) while maximizing \( \Omega \), cf. Sec. \( \text{V}E \).
Symmetry considerations. Let us recap the effects of the finite energy scales from the point of view of the (super)symmetry of the target Hamiltonian. As discussed in the previous paragraph, for the relevant parameter regime, the imperfect blockade $W(r_0)/W(2r_0) < \infty$ dominates over the effect of the interaction tails $W(nr_0), n > 2$, cf. Eq. (S28), and both represent a weak breaking of the supersymmetry.

In order to quantify the effect of the tails in the limit of large system sizes one could treat the effects of the $W(nr_0)$ terms, $n > 2$, perturbatively (as suggested in a similar context in [31]) or using large-scale numerical simulations for real time evolution such as the DMRG [32] or neural networks [30] based methods. Similarly, one could perform a systematic study of the imperfect blockade by replacing the projectors $P_i$ in the definition of the supercharges $Q_i$, cf. [S26], by $P_i \rightarrow P_i + \eta n_i$, where $\eta = O(1/W(r_0))$ [34] and simultaneous deformation of $H_Q$ from the supersymmetric point with a potential $\sum_i W(r_0)n_in_{i+1}$ to account for the nearest-neighbour interaction. However, such detailed analysis is beyond the scope of the present work.

Importantly, we note that $H_Q$, Eq. (S26), is recovered from the parent Rydberg Hamiltonian (??) in the limit of $W(r_0)/W(2r_0), W(2r_0)/W(nr_0) \rightarrow \infty$, $\forall n > 2$. This is reminiscent of a situation encountered in the early proposals [37, 38] of the cold atom based quantum simulators of lattice gauge theories (LGT) [we thank the anonymous Referee for bringing up this issue]. There, the local Hamiltonian symmetry corresponds to the Gauss law which was recovered only asymptotically in the limit of infinite interaction strength of the parent Hubbard-like model from which the effective Hamiltonian corresponding to the LGT was derived. The engineering of the Gauss law has then been improved in [39] by invoking the Higgs field, cf. also [40, 41] for recent developments.

In this context, it would be interesting to explore alternative avenues for engineering the supersymmetric lattice Hamiltonians in order to recover the exact supersymmetry, for instance through the mapping to spin Hamiltonians as discussed in the Outlook of the main text.

[7] This is in contrast to the gap $E_{\text{gap}}$ used in the main text in scenario (ii), which was the energy of the lowest excited state, i.e. its distance from the zero energy corresponding to the true ground state of the supersymmetric Hamiltonian on a chain with periodic boundaries. In the thermodynamic limit the lowest excited states become however equidistant as a consequence of the conformal symmetry and $E_{\text{gap}}$ coincides with $\Delta_{\text{sgq}}$.
[8] We numerically implement the time evolution using Crank-Nicholson discrete time-step evolution given by $|\psi(t+\Delta t)| = (1+iH(t)\Delta t/2)^{-1}(1-iH(t)\Delta t/2)|\psi(t)|$.
[14] Here it becomes apparent the site-dependent factor $(-1)^i$ in the definition of the supercharge $Q = \sum_i (-1)^i \lambda_i c_i^\dagger P_i$ - this is required, for $\lambda_i$ real, for the kinetic term to be negative. Another consequence of this choice is the appearance of triplets in the ground state |l⟩. Conversely, the omission of the $(-1)^i$ factor leads to positive kinetic term and singlets instead of triplets in |l⟩.