Kink Dynamics and Quantum Simulation of Supersymmetric Lattice Hamiltonians

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We propose a quantum simulation of a supersymmetric lattice model using atoms trapped in a 1D configuration and interacting through a Rydberg potential. The elementary excitations in the model are kinks or (in a sector with one extra particle) their superpartners—the skinks. The two are connected by supersymmetry and display identical quantum dynamics. We provide an analytical description of the kink (skink) quench dynamics and propose a protocol to prepare and detect these excitations in the quantum simulator. We make a detailed analysis, based on numerical simulation, of the Rydberg atom simulator and show that it accurately tracks the dynamics of the supersymmetric model.

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Introduction.—Models of strongly interacting fermions are key to our understanding of condensed matter systems. At the same time, they are notoriously hard to track, even with sophisticated tools ranging from numerical approaches such as quantum Monte Carlo [1–3] and tensor networks [4,5] to application of gauge-gravity duality [6]. One strategy to make progress is to consider models with special symmetries. A nonstandard but intriguing choice is to consider supersymmetry as an explicit symmetry on the lattice [7–12] or as an emergent symmetry [13–15].

\( \mathcal{N} = 2 \) supersymmetry in a lattice model or a quantum field theory comes with a number of tools, such as the Witten index [16], which facilitate the analysis. Exploiting these tools unveils remarkable features such as extensive ground state degeneracies, a phenomenon dubbed superfrustration [17,18], which can lead to multicriticality [19].

Despite the supersymmetry, many hard questions remain, such as the nature of the quantum phases in higher spatial dimensions. Here a quantum simulator might provide ingenious insights to these questions. We make a step in this direction and propose such a simulator using arrays of neutral atoms trapped in optical potentials and dressed to their Rydberg state. This is motivated by the high versatility of these platforms [20–30] and by the fact that an off-resonant dressing [31–35] naturally implements the constrained dynamics inherent to the supersymmetric lattice model. Specifically, we consider a so-called \( M_1 \) model for spinless fermions on a 1D chain [8]. As a function of a parameter \( \lambda \), this model interpolates between a trivial (\( \lambda = 0 \)) and a quantum critical (\( \lambda = 1 \)) phase, the latter connecting to superconformal field theory [8,36].

The value of the Witten index \( W = 2 \) indicates the existence of two supersymmetric vacua and points at kinks connecting these two vacua as elementary excitations. Furthermore, in a sector with one particle added, the excitations correspond to the superpartners of the kinks, which we call the skinks. We propose a protocol for the (s)kink preparation and solve for their dynamics following a quench (we note a recent study of kink-antikink pair dynamics in a spin chain [37]). We show that it is identical in both cases and accurately reproduced by the quantum simulator. This is a direct consequence and a clear-cut sign of the underlying supersymmetry and we show how it can be revealed by a simple experimental probe based on monitoring single-site particle densities.

The \( M_1 \) model.—An \( \mathcal{N} = 2 \) supersymmetric lattice Hamiltonian for spinless fermions can be defined as

\[
H_Q = \{Q, Q^\dagger\},
\]

where \( Q \) is the nilpotent supercharge, \( Q^2 = 0 \), and the brackets denote the anticommutator. The \( M_1 \) model [8] (on a bipartite graph) arises when \( Q = \sum_i Q_i \), with \( Q_i = (-1)^i \lambda_i c_i c^\dagger_{(i)} \), where \( c_i \) are fermionic annihilation operators, \( \{c_i, c_j\} = \{c^\dagger_i, c^\dagger_j\} = 0 \). \( \{c_i, c^\dagger_j\} = \delta_{ij} \), and \( \lambda_i \in \mathbb{C} \). The \( M_1 \) model constraint, stipulating that fermions are not allowed to occupy nearest-neighbor sites \( \langle ij \rangle \), is implemented via the projector \( P_{(i)} = \prod_{\langle ij \rangle} P_{(j)} \) with \( P_j = 1 - n_j \), \( n_j = c^\dagger_j c_j \). The Hamiltonian \( H_Q \) describes nearest-neighbor hoppings and local interactions; it preserves the number of particles, \( \{H_Q, \sum_i n_i\} = 0 \).

We now specialize to 1D and specify real \( \bar{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_L\} \), where \( L \) is the length of the chain, \( \lambda_i \) repeats every three sites in a pattern \( 1 \lambda \) and \( \lambda \geq 0 \). For this choice of staggering, the \( M_1 \) model is known to be integrable [38]. We refer to \( \lambda = 0 \) as “extreme staggering.”

Supersymmetric ground states.—Let us first consider periodic boundary conditions, \( L = 3l \), \( l \in \mathbb{N} \), and \( \bar{\lambda} = (1, 1, \lambda, \ldots, 1, 1, \lambda) \). In this case, there are two supersymmetric ground states with \( E = 0 \), each at 1/3 filling. At
densities have been studied at the critical point (light green diamonds). (d) Scheme of the proposed realization. Atoms in this ground state, perturbatively in \( \lambda \). The corresponding particle densities constitute a \( \mathbb{Z}_3 \) pattern highlighted by the light green, dark green, and black data points. (c) Particle (filled symbols) and energy (empty symbols) densities of \( |K_j\rangle \) for \( \lambda = 0, 0.5, 1 \) (blue circles, orange squares, green diamonds). (d) Scheme of the proposed realization. Atoms in their electronic ground state \( |g\rangle \) tunnel in an optical lattice with spacing \( r_J \) at rate \( J \) subject to dressing to a Rydberg state \( |r\rangle \). Lower graph shows the dressed potential \( W(r) \) for the \( |84S\rangle \) state of \(^4\text{Li}\) with \( \Omega = 2\pi \times 10 \text{ MHz}, \Omega/\Delta = 1/10, C_6 = 645 \text{ GHz } \mu \text{m}^6 \).

Kinks at extreme staggering.—For an open chain of \( L = 3l + 1 \), there are no supersymmetric ground states. Instead, at extreme staggering the lowest energy states with \( l \) particles interpolate between the ground state configurations \( |I\rangle \) and \( |II\rangle \), with an empty site at position \( i = 3j - 2 \), with \( j = 1, \ldots, l + 1 \). We write these “bare kink” states as \( |K_j\rangle = |\Pi_{l+1,j-1}, 0\rangle \), where \( \Pi_{a,b} \) denote the part of the ground state configuration located between sites \( a \) and \( b \). They all have energy \( E = 1 \). The labels \( j = 1(j = l + 1) \) correspond to the leftmost (rightmost) kink, see Fig. 1(a). Acting with the supercharge on the kink increases the number of particles by one creating the kink’s superpartner, the skink, \( |\tilde{K}_j\rangle \equiv Q |K_j\rangle = |\Pi_{l+1,j-1}, 1, 0\rangle \). Consequently, \( Q|\tilde{K}_j\rangle = |K_j\rangle \) such that \( |K_j\rangle \) and \( |\tilde{K}_j\rangle \) form doublets under supersymmetry, see Fig. 1(a) [77]. To characterize the kinks, we introduce a local energy density \( h_i = 1/2 \langle \{Q, Q^\dagger\} + \{Q^\dagger, Q\} \rangle \) such that \( H_Q = \sum_{i=1}^L h_i \).

Figure 1(c) shows the particle density \( n = \langle n_i \rangle \) and energy density \( e = \langle h_i \rangle \) for the leftmost kink \( |K_1\rangle \). The kink is clearly located at the left end of the chain with a corresponding peak in the energy density.

Kinks at general \( \lambda \).—We claim that the notion of one-kink (and multikink) states is well defined also away from extreme staggering, when \( 0 < \lambda \leq 1 \). To illustrate this, we present in the inset of Fig. 2(a) the spectrum of the system for \( l = 4 \). The energies become degenerate for \( \lambda = 0 \), taking odd positive values corresponding to the one-kink, three-kink, etc. states. The unavoidable level crossings, characteristic for integrability, allow us to unambiguously characterize states as multikink states for all \( \lambda \).

Figure 2(a) shows the low-lying part of the spectrum, which includes a band of \( l + 1 \) one-kink eigenstates \( |v_k\rangle \) of energy \( E_k \). We define a localized kink as [78]
\[ |K_j\rangle = \sqrt{\frac{2}{l+2}} \sum_{k=1}^{l+1} \sin(\tilde{k}j)|v_k\rangle, \quad (2) \]

where \( \tilde{k} = \pi k/(l+2) \).

In Fig. 1(c), the orange and green data points show the particle and energy densities in the state \( |K_j\rangle \) obtained numerically using Eq. (2) for \( \lambda = 0.5, 1 \). We see that, even for \( \lambda = 1 \), the kink is well defined, with most of its energy localized at the kink position.

Kink dynamics.—We now proceed with the evaluation of the kink dynamics. We start from the leftmost kink \( |K_j\rangle \) and consider overlap at time \( t \) with the rightmost kink, \( o(t) \equiv \langle K_{j_1}|K_1(t)\rangle \), where \( |K_1(t)\rangle = e^{-i\hat{H}_F t}|K_1\rangle \). It follows from Eq. (2) that

\[ o(t) = \frac{2}{l+2} \sum_{k=1}^{l+1} \sin(\tilde{k}) \sin(\tilde{k}(l+1))e^{-iE_k t}. \quad (3) \]

For simplicity, from now on we focus on the critical case \( \lambda = 1 \). In Fig. 3(a), we show \( |o(t)|^2 \) for \( l = 4 \) (solid blue line). At criticality, the fastest mode propagates with the Fermi velocity \( v_F \), see the discussion after Eq. (4). This results in the onset of the overlap at \( tv_F/l \approx 1 \), with the maximum achieved for a later time, \( tv_F/l \approx 1.75 \).

Kink detection.—To make a connection with experimentally observable quantities, we construct an observable \( \delta n \) that detects the presence of a kink at the right end of the system, by requiring that \( \langle K_j|\delta n|K_j\rangle \approx \delta_{j,j_1}\delta_{j,j_1+1} \). Taking \( \delta n = a(\lambda, l)|1 - \beta(\lambda, l)(n_{l-1} + n_{l+1})| \), we find \( a(0, l) = 1 \) and \( \beta(0, l) = 1 \) for any \( l \), and \( a(1, l) \approx 1.08 \) and \( \beta(1, l) \approx 1.09 \) for \( l = 3, 4 \) [39]. The numerically obtained result for \( \delta n(t) \) is shown as a blue dashed line in Fig. 3(a) and corresponds with good accuracy to \( |o|^2 \).

Kink preparation.—An important question is how the spatially localized kink \( |K_j\rangle \) can be prepared in practice. To this end, we note that the kink site and its nearest neighbors remain approximately empty for all \( \lambda \), cf. Fig. 1(c). We thus consider an adiabatic preparation of a ground state \( |K_j\rangle \) of the final Hamiltonian \( \hat{H}_f = \hat{H}_Q + \mu (n_1 + n_2) \), where \( \mu \to \infty \) ensures the kink condition on the first two sites. The initial Hamiltonian is chosen such that its ground state is a kink at extreme staggering \( \lambda = 0 \) (and similarly for skinks below), cf. Supplemental Material [39]. For \( l = 4 \), we find the fidelities \( F \in [0.95, 1] \), where \( F = \langle |K_j| |K_1\rangle \rangle \), with the highest (lowest) value at extreme staggering (criticality). In Fig. 3(a), we show the numerically evaluated overlap \( |o|^2 = \langle |K_{j_1}| |K_1(t)\rangle \rangle \) and the corresponding observable \( \delta n \) as solid (dashed) red lines. We find that, despite the limited fidelity of the initial state, \( |o|^2 \) and \( \delta n \) agree well with \( |o|^2 \) and \( \delta n \).

Skinks.—Supersymmetry guarantees that the one-skink energies [the lower dashed magenta lines in Fig. 2(a)] in the sector with \( l + 1 \) particles are identical to the one-kink energies \( E_k \). As a consequence, the quench dynamics for the skinks is again given by Eq. (3). For the detection of \( |K_{j_l+1}\rangle \), we propose \( \delta n = -\alpha(\lambda, l)|1 - \tilde{\beta}(\lambda, l)l(2n_{l-1} + n_{l+1} + n_{l+1})| \), with \( \alpha(0, l) = 2 \) and \( \tilde{\beta}(0, l) = 1 \), and \( \alpha(1, l) \approx 1.46 \) and \( \tilde{\beta}(1, l) \approx 0.98 \) for \( l = 3, 4 \). For the preparation, we find that the ground state \( |K_1\rangle \) of \( \hat{H}_f = \hat{H}_Q + 3(-n_1 + n_2 - 0.5n_3) \) corresponds well to \( |K_j\rangle \) [39]. The \( l = 4 \) fidelities are \( F \in [0.93, 1] \) with \( F = \langle |K_j| |K_1\rangle \rangle \).

Kink (skink) dynamics at large \( l \).—Surprisingly, the kink arrival amplitude Eq. (3) is analytically tractable, for general \( \lambda \), in the large-\( l \) limit. A key element for this is the continuum limit \( E(\tilde{k}) \) of the kink dispersion relation. Exploiting a relation between the \( M_1 \) model and the XYZ spin-1/2 chain [79], we have found [78]

\[ E(\tilde{k}) = (3s + \lambda^2)^{3/2}/2 \sqrt{2\sqrt{\lambda^2 + s} + \lambda^2}, \quad (4) \]

where \( s = \sqrt{8 + \lambda^2} \). In Fig. 2(b), we show the dispersion for \( \lambda = 0.1, 0.5, 1 \) and \( s = 0 \). We denote by \( v_{\text{max}}(\lambda) \) the maximum dispersion.

FIG. 3. (a) Time evolution of \( |o(t)|^2 \), Eq. (3) (solid lines) and \( \delta n \) (dashed lines) for a quench from the exact (blue) and reduced fidelity (red) kink state \( |K_j\rangle \) for \( l = 4 \). The gray line corresponds to \( |o(t)|^2 \) evaluated with Eq. (4) for the eigenenergies. (b) Numerical evaluation [gray, Eq. (3)] and saddle point approximation [green dashed, red, Eq. (5)] of the overlap \( |o(t)|^2 \) for \( l = 101 \). The red (green) lines correspond to considering the first two (only the second) saddle points. The inset shows the onset of oscillations around \( tv_F/l = 3 \). (c) Quench dynamics for ten-site chain \( (l = 3) \), with initial states \( |K_j\rangle \) (blue) and \( |K_j\rangle \) (magenta). Solid lines show dynamics under the full Hamiltonian \( \hat{H}_\text{Ry} \), whereas the gray curve is for a truncation of \( H_\text{Ry} \), neglecting interactions beyond next-nearest neighbors. Dashed lines show dynamics under \( \hat{H}_Q \). The red line shows the average population in the Rydberg state \( \bar{n}_\text{R} = 1/l \sum_n n_r \), whereas the green line tracks the nearest-neighbor occupation of ground state atoms, \( \bar{n}_{n_{l+1}} = 1/l \sum_f n_r n_{r+1} \). Parameters as in Fig. 1(d).
value of the group velocity \( v(\mathbf{k}) = \partial_t E(\mathbf{k}) \). At criticality, 
\( v_{\text{max}}(\lambda = 1) = v_F = 3\sqrt{3}/4 \), with \( v_F \) as the Fermi velocity. This gives real space velocity (since kinks hop three sites at a time) \( 3v_F = 9\sqrt{3}/4 \), in agreement with \[36\].

In Fig. 3(a), the gray line shows the overlap Eq. (3) evaluated with the energies \( E(\mathbf{k}) \) instead of \( E_k \) (blue line). The difference is a consequence of finite \( l \), cf. the green diamonds vs green solid line in Fig. 2(b).

Using the dispersion \( E(\mathbf{k}) \), we can evaluate the large-\( l \) limit of Eq. (3) in a saddle point approximation \[39\], giving

\[
\alpha(t) \approx \frac{2}{l + 2} \sum_{s=1}^{\infty} \theta \left( \frac{v_{\text{max}}l}{2} - (2s - 1) \right) \sin(\mathbf{k}_s)^2 \\
\times e^{i(2s-1)\mathbf{k}_s + E(\mathbf{k}_s)t} + i(5\pi/4) \sqrt{\frac{2\pi}{-E''(\mathbf{k}_s)t}}.
\]

where \( \theta \) is the Heaviside step function, \( k_s = E^{-1}[(2s - 1)\pi/l] \), \( E' = \partial_k E(\mathbf{k}) \), \( E'' = \partial_k^2 E(\mathbf{k}) \), and \( s \) labels the saddle point corresponding to the arrival times \( t = (2s - 1)l/(2s - 1)l/\nu_{\text{max}} \approx (2s - 1)l/\nu_{\text{max}}, s \in \mathbb{N} \), of the kink front (maximum velocity mode). At criticality, where \( E(\mathbf{k}) = 2v_F \sin(\mathbf{k}/2) \), the saddle point expression takes a simple closed form \[39\].

In Fig. 3(b), we show an example of the dynamics for \( l = 101 \) evaluated using Eq. (3) (gray line) together with the prediction of Eq. (5) (green dashed line). We see a close to perfect agreement, with the inset showing the details around \( tv_F/l = 3 \), where the second saddle point \( s = 2 \) starts to generate the characteristic modulation of the overlap due to the interference of the kink front propagating at \( v_F \) incident on the right edge (after it has undergone one round-trip) and the kink tail. We note the frequency chirp of the modulation due to the nontrivial time dependence of \( \mathbf{k}_s \).

Here we do not show the observable \( \delta n(t) \), as for large \( l \) the Hamiltonian cannot be diagonalized exactly.

Experimental implementation.—We now discuss how \( H_Q \) can be engineered using Rydberg dressed atoms \[40,41\]; see Ref. \[80\] for an experimental realization of fermions hopping in an optical lattice and interacting through a Rydberg dressed potential.

We consider effectively two-level atoms with the ground and Rydberg states \( |g \rangle \) and \( |r \rangle \), where the ground state atoms experience an optical lattice potential and the atoms in a Rydberg state experience a repulsive van der Waals interaction described by

\[
H_{Ry} = -J \sum_{i=1}^{L-1} (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + \sum_{i=1}^{L} \mu_i n_i \\
+ \sum_{i=1}^{L} \Omega \sigma_i^z + \Delta n_i^z + \sum_{i>j=1}^{L-1} V_{ij} n_i^z n_j^z.
\]

Here, \( J > 0 \) is the hopping amplitude, \( \sigma^z = |r \rangle \langle g| + |g \rangle \langle r| \), \( n^z = |r \rangle \langle r| + |g \rangle \langle g| \), and \( V_{ij} = C_6/(r_{ij}^6) \) with \( C_6 \) as the van der Waals coefficient and \( r_{ij} \) as the lattice spacing. We consider a regime of large detuning \( \Omega/\Delta \ll 1 \), where the ground state atoms interact, up to order \( \Omega^4 \), through an effective flat top potential \( W(r = r_0|i-j|) = 2\Omega^2 V_{ij}/\Delta^2 (V_{ij} + 2\Delta) \), cf. Fig. 1(d). To obtain the supersymmetric \( H_Q \), the interaction and chemical potentials \( W \) and \( \mu \) and the hopping \( J \) need to be tuned as follows.

For simplicity, we refer the discussion of general \( \lambda \) to the Supplemental Material \[39\] and focus on \( \lambda = 1 \). In this case, the chemical potential terms in \( H_Q \) become site-independent up to the boundary terms originating from \( P_L P_3 \) and \( P_{L-3} P_L \), which can be accounted for by setting \( \mu_1 = \mu_L = J \).

Next, the \( M_1 \) model Hamiltonian forbids nearest-neighbor occupation and the potential terms are of the form \( P_i P_{i+1} \), with no interactions beyond lattice distance 2. For this to be captured by the flat top potential, we need \( W(r_0)/W(2r_0) \gg 1 \) and \( W(2r_0)/W(3r_0) \gg 1 \) with the maximum achieved in the limit \( r_0 \rightarrow \infty \). However, to counteract experimental imperfections \[39\], one should reduce the duration of the simulation by maximizing the relevant energy scale, here \( W(2r_0) \), which happens for \( r_0 \rightarrow 0 \), and one has to set \( J = W(2r_0) \). This corresponds to the optimal approximation of \( H_Q \) using single dressing. Written explicitly,

\[
JH_Q = -J \sum_{i=1}^{L-2} (c_i^\dagger c_{i+1} + \text{H.c.}) P_{i+2} \\
+ J(n_1 + n_L) + J \sum_{i=1}^{L-2} n_i n_{i+2}
\]

(up to global energy offset), see Supplemental Material \[39\] for details. Importantly, we also show in \[39\] that \( H_Q \) can be reached, in principle, with an arbitrary number of dressings with already a tenfold increase in \( W(r_0)/W(2r_0) \) and \( W(3r_0)/W(2r_0) \) for a double dressing with realistic parameters.

As a specific example, we consider the fermionic \(^4\)Li dressed with the \( |8S4\rangle \) state with \( C_6 = 645 \text{ GHz} \mu \text{m}^6 \) \[42,43\] and lattice spacing \( r_0 = 2.5 \mu \text{m} \). The resulting dressed potential is shown in Fig. 1(d). We get \( W(2r_0) = J \approx 4 \text{ kHz} \), which for the optical lattice laser wavelength \( \lambda = 2r_0 = 5 \mu \text{m} \) corresponds to lattice depth \( \approx 5.5 E_r \), with \( E_r \) being the recoil energy \[44,81\].

Figure 3(c) shows the quantum simulation of \( H_Q \), where we compare the dynamics generated by the Rydberg Hamiltonian \( H_Q \) with that of \( H_Q \) quenching from \( |K'_1 \rangle \) and \( |K'_1 \rangle \), see caption for details. We draw two main conclusions. First, the quantum simulator accurately tracks the dynamics set by the model Hamiltonian \( H_Q \) and, second, the dynamics in the \( l \)-particle sector (blue lines).
is highly similar to that in the $l + 1$ particle sector (magenta lines). The latter observation is direct evidence of the supersymmetry of $H_Q$.

**Outlook.**—We have proposed a realization of a supersymmetric lattice Hamiltonian $H_Q$ based on atoms interacting through a Rydberg dressed potential $[82,83]$. Our results constitute a stepping stone to quantum simulations of supersymmetric lattice models in higher dimensions $[18,84–87]$, which can require $n$-body, rather than two-body, interactions. In this context, it would be interesting to consider a scheme relying on coupling the Rydberg atoms with phonons $[88]$ or to use cold molecules with permanent or electric-field-induced dipole moments, avoiding the need for off-resonant dressing $[45–48]$. Another interesting avenue is to exploit the mapping of the supersymmetric lattice Hamiltonians to spins $[9,19,78,79]$, which would allow for simulations with platforms such as superconducting devices with $n$-body interactions $[89,90]$.

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