Effective Lagrangians and power counting — We start from two-flavor QCD in presence of external sources

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\rm QCD} - \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R \\ &+ \bar{q}_L\gamma^\mu l_\mu q_L + \bar{q}_R\gamma^\mu r_\mu q_R \end{aligned} \tag{1}$$

where $q^T = (u,d)$ and $s(x), p(x), l_\mu(x), r_\mu(x)$ can be written in terms of quark mass, Standard Model gauge fields, and external classical fields $\bar{s}, \bar{p}, \bar{l}_\mu, \bar{r}_\mu$ as follows

$$\chi \equiv B_0(s+ip) = B_0(m_q + \bar{s} + i\bar{p}) \tag{2a}$$
$$l_\mu = -eQ_L^{EM}A_\mu + Q_L^W J_\mu^{\rm lept} + Q_L^{W\dagger} J_\mu^{\rm lept\dagger} + \bar{l}_\mu \tag{2b}$$
$$r_\mu = -eQ_R^{EM}A_\mu + \bar{r}_\mu \ . \tag{2c}$$

Here B_0 is a constant with dimension of mass, m_q is the quark mass matrix, $Q_L^{EM} = Q_R^{EM} = {\rm diag}(q_u, q_d)$ (with $q_u = 2/3, q_d = -1/3$), $Q_L^W = -2\sqrt{2}G_F V_{ud}\,\tau^+$, and $J_\mu^{\rm lept} = \bar{e}_L\gamma_\mu\nu_{eL}$. The Lagrangian in (16) is invariant under local $G = SU(2)_L \times SU(2)_R \times U(1)_V$ transformations

$$q_L \to L(x)e^{\alpha_V(x)}q_L \ , \quad q_R \to R(x)e^{\alpha_V(x)}q_R \ , \tag{3}$$

with $L, R \in SU(2)_{L,R}$, provided $Q_{L,R}^{EM}$ and Q_L^W transform as "spurions" under the chiral group $Q_L^{EM,W} \to LQ_L^{EM,W}L^\dagger$ and $Q_R^{EM} \to RQ_R^{EM}R^\dagger$, and that $\bar{l}_\mu$ and $\bar{r}_\mu$ transform as gauge fields under G. This implies

$$\chi \to R\chi L^\dagger \tag{4a}$$
$$l_\mu \to Ll_\mu L^\dagger + iL\partial_\mu L^\dagger + \partial_\mu\alpha_V \tag{4b}$$
$$r_\mu \to Rr_\mu R^\dagger + iR\partial_\mu R^\dagger + \partial_\mu\alpha_V \ . \tag{4c}$$

Note that the external sources can be decomposed in $SU(2)$ singlet and non-singlet components as follows: $l_\mu = l_\mu^{ns} + l_\mu^s$, $r_\mu = r_\mu^{ns} + r_\mu^s$.

To construct the effective chiral Lagrangians, one introduces the nucleon and pion fields as follows [57, 58],

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \ U = u^2 = e^{i\Pi/(F)}, \ \Pi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \tag{5}$$

and $F \sim F_\pi = 92.4$ MeV. These fields transform under the chiral group as follows

$$u \to LuK^\dagger(u) = K(u)uR^\dagger \tag{6a}$$
$$U \to LUR^\dagger \tag{6b}$$
$$N \to e^{3i\alpha_V}K(u)N \tag{6c}$$

where $K(u)$ is a pion-dependent $SU(2)_V$ transformation.

To construct chiral invariant Lagrangians, it is very useful to use chiral-covariant derivatives

$$D_\mu U \equiv \partial_\mu U - il_\mu U + iUr_\mu \to L(D_\mu U)R^\dagger \tag{7a}$$
$$\nabla_\mu N \equiv \left(\partial_\mu + \Gamma_\mu - i\frac{3(l_\mu^s + r_\mu^s)}{2}\right)N \to K(\nabla_\mu N) \tag{7b}$$
$$\begin{aligned}\Gamma_\mu &= \frac{1}{2}\left[u(\partial_\mu - ir_\mu^{ns})u^\dagger + u^\dagger(\partial_\mu - il_\mu^{ns})u\right] \\ &\to K(u)\Gamma_\mu K(u)^\dagger + K(u)\partial_\mu K(u)^\dagger \ . \end{aligned} \tag{7c}$$

It is also very useful to use combinations of fields that transform homogeneously with $K(u)$:

$$\begin{aligned}u_\mu &= i\left[u(\partial_\mu - ir_\mu)u^\dagger - u^\dagger(\partial_\mu - il_\mu)u\right] \\ &\to K(u)u_\mu K(u)^\dagger \end{aligned} \tag{8a}$$
$$\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u \to K(u)\chi_\pm K(u)^\dagger \tag{8b}$$
$$\mathcal{Q}_L^{EM,W} = u^\dagger Q_L^{EM,W}u \to K(u)\mathcal{Q}_L^{EM,W}K(u)^\dagger \tag{8c}$$
$$\mathcal{Q}_R^{EM} = uQ_R^{EM}u^\dagger \to K(u)\mathcal{Q}_R^{EM}K(u)^\dagger \tag{8d}$$

Finally, in the literature one often finds the combinations of charge building blocks with definite parity

$$\mathcal{Q}_\pm \equiv \frac{1}{2}\left(\mathcal{Q}_L \pm \mathcal{Q}_R\right) \ . \tag{9}$$

The standard χPT power counting assumes that external momenta and meson masses are comparable ($q_{\rm ext} \sim m_\pi$). Including charged lepton masses one assumes $p \sim q_{\rm ext} \sim m_\mu \sim m_\pi \ll \Lambda_\chi \sim 4\pi F_\pi \sim m_N$. Given this, one makes the following assignments:

$$\partial \sim p \ , \quad \chi_\pm \sim B_0 m_q \sim m_\pi^2 \sim p^2 \ . \quad l_\mu, r_\mu \sim p \ , \tag{10}$$

with the latter identification implying $e \sim p$ and $G_F \sim p$ (though we will never go beyond one insertion of G_F and two insertions of the electromagnetic coupling e). The above scalings allow us to assign chiral dimension to each lagrangian vertex in a straightforward way.

The pion Lagrangian has the usual expansion in even chiral powers:

$$\begin{aligned} \mathcal{L}_\pi &= \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + ... && \text{(11a)} \\ \mathcal{L}_\pi^{(2)} &= \mathcal{L}_\pi^{p^2} + \mathcal{L}_\pi^{e^2p^0} \\ &= \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle + e^2 Z_\pi F^4 \langle \mathcal{Q}_L^{EM}\mathcal{Q}_R^{EM}\rangle, && \text{(11b)} \end{aligned}$$

which leads to the identification

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 = 2e^2F_\pi^2 Z_\pi \ . \tag{12}$$

The gauge-kinetic leptonic Lagrangian has chiral dimension $n = 1$:

$$\mathcal{L}_{\rm lept} = \bar e \left(i\partial\!\!\!/ + eA\!\!\!/ - m_e\right) e + \bar\nu i\partial\!\!\!/ \nu \ . \tag{13}$$

To the order we work, we need to include the following purely leptonic counter-term [34]

$$\mathcal{L}_{\rm lept}^{CT} = e^2 X_6 \ \bar e \left(i\partial\!\!\!/ + eA\!\!\!/\right) e \ . \tag{14}$$

The pion-nucleon Lagrangian has both odd and even chiral powers, starting at $n = 1$:

$$\begin{aligned} \mathcal{L}_{\pi N} &= \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + ... && \text{(15a)} \\ \mathcal{L}_{\pi N}^{(1)} &= \mathcal{L}_{\pi N}^{p} = \bar N_v iv\cdot\nabla N_v + g_A \bar N_v S\cdot u N_v && \text{(15b)} \\ \mathcal{L}_{\pi N}^{(2)} &= \mathcal{L}_{\pi N}^{p^2} + \mathcal{L}_{\pi N}^{e^2p^0} && \text{(15c)} \\ \mathcal{L}_{\pi N}^{(3)} &= \mathcal{L}_{\pi N}^{p^3} + \mathcal{L}_{\pi N}^{e^2p} + \mathcal{L}_{\pi N\ell}^{e^2p} && \text{(15d)} \end{aligned}$$

where in the nucleon rest-frame $v^\mu = (1, \mathbf{0})$ and $S^\mu = (0, \boldsymbol{\sigma}/2)$. We have displayed explicitly here only the leading order Lagrangians and we will report below the appropriate higher order terms as needed. All these effective Lagrangian are know in the literature, see for example Ref. [33], except for $\mathcal{L}_{\pi N\ell}^{e^2p}$, which is needed to reabsorb divergences from loops that involve virtual baryons, pions, leptons, and photons. We report here only the terms that play a significant role in our analysis.

The one-loop diagrams with virtual nucleons, pions, and photons generate divergences which are absorbed by counterterms in the $\mathcal{L}_{\pi N}^{e^2p}$ Lagrangian. When constructing the baryon electromagnetic Lagrangian, it has been common practice in the literature [31–33] to use charge spurions corresponding to the nucleon charge matrix $\bar Q = diag(1, 0)$. Now $\bar Q$ differs from the quark charge matrix only in its $SU(2)$ singlet component: therefore the two objects have the same transformation properties under the chiral group and this procedure is justified. In what follows we indicate all the chiral building blocks built from the nucleon charge matrix with a bar. A minimal version of $\mathcal{L}_{\pi N}^{e^2p}$ was constructed in Ref. [33]

$$\mathcal{L}_{\pi N}^{e^2p} = e^2 \sum_{i=1,12} g_i \ \bar N_v \, O_i^{e^2p} \, N_v \ , \tag{16}$$

Only four operators contribute to neutron decay at tree level,

$$O_1^{e^2p} = \langle \bar{\mathcal{Q}}_+^2 - \bar{\mathcal{Q}}_-^2 \rangle \, S\cdot u \tag{17a}$$
$$O_2^{e^2p} = \langle \bar{\mathcal{Q}}_+ \rangle^2 \, S\cdot u \tag{17b}$$
$$O_9^{e^2p} = \frac{i}{2}[\bar{\mathcal{Q}}_+, v\cdot c^+] + \text{h.c.} \tag{17c}$$
$$O_{11}^{e^2p} = \frac{i}{2}[\bar{\mathcal{Q}}_+, S\cdot c^-] \ , \tag{17d}$$

with

$$c_\mu^\pm = -\frac{i}{2}\left(u[l_\mu,\bar{Q}]u^\dagger \pm u^\dagger[r_\mu,\bar{Q}]u\right) \ . \tag{18}$$

As standard practice in χPT, the divergences are subtracted as follows [28]:

$$\begin{aligned} g_i &= \eta_i \ \lambda(\mu) + g_i^r(\mu) \ , \\ \lambda(\mu) &= \frac{\mu^{d-4}}{(4\pi)^2}\left(\frac{1}{d-4} - \frac{1}{2}\left(-\gamma + \log 4\pi + 1\right)\right) . \end{aligned} \tag{19}$$

We use the same subtraction scheme for all LECs. The coefficients η_i can be found in Table 5 of Ref. [33]. We checked that the g_i couplings absorb correctly the divergences of diagrams without virtual leptons, thus providing a consistency check on our calculation.

The one-loop diagrams with virtual nucleons, pions, photons, and charged leptons generate divergences which are absorbed by counterterms in the new $\mathcal{L}_{\pi N\ell}^{e^2p}$ Lagrangian. These are the analogue of the operators introduced in the meson sector in Ref. [34], that contribute to (semi)leptonic meson decays to $O(e^2p^2)$. We find five structures, of which only the first three contribute to neutron decay at tree level

$$\mathcal{L}_{\pi N\ell}^{e^2p} = e^2 \sum_{i=1,5} \tilde{X}_i \, \tilde{O}_i \ , \tag{20}$$

where

$$\tilde{O}_1 = \bar{e}\gamma_\alpha\nu_L \ \bar{N}_v v^\alpha \mathcal{Q}_L^W N_v \tag{21a}$$
$$\tilde{O}_2 = \bar{e}\gamma_\alpha\nu_L \ \bar{N}_v v^\alpha [\mathcal{Q}_L^W, \bar{\mathcal{Q}}_R^{EM}] N_v \tag{21b}$$
$$\tilde{O}_3 = \bar{e}\gamma_\alpha\nu_L \ \bar{N}_v S^\alpha [\mathcal{Q}_L^W, \bar{\mathcal{Q}}_R^{EM}] N_v \tag{21c}$$
$$\tilde{O}_4 = \bar{e}\gamma_\alpha\nu_L \ \bar{N}_v v^\alpha \langle \mathcal{Q}_L^W \bar{\mathcal{Q}}_R^{EM}\rangle N_v \tag{21d}$$
$$\tilde{O}_5 = \bar{e}\gamma_\alpha\nu_L \ \bar{N}_v S^\alpha \langle \mathcal{Q}_L^W \bar{\mathcal{Q}}_R^{EM}\rangle N_v \ . \tag{21e}$$

The couplings $\tilde{X}_i$ are dimensionless (note that $\mathcal{Q}_W$ carries dimension via the G_F factor).

To compute the neutron decay amplitude to $O(G_F\alpha\epsilon_\chi)$ we must consider one-loop diagrams with insertions of $\mathcal{L}_{\pi N}^{p^2}$, for which (in the notation of Ref. [59]) we use

$$\begin{aligned} \mathcal{L}_{\pi N}^{p^2} = \bar{N}\Bigg[& \frac{1}{2m_N}\left((v\cdot\mathcal{D})^2 - \mathcal{D}^2\right) - i\frac{g_A}{2m_N}\{S\cdot\mathcal{D}, v\cdot u\} + c_1 \mathrm{Tr}(\chi_+) + \left(c_2 - \frac{g_A^2}{8m_N}\right)(v\cdot u)^2 \\ & + c_3 u\cdot u + \left(c_4 + \frac{1}{4m_N}\right)[S^\mu, S^\nu]\, u_\mu u_\nu + c_5\tilde{\chi}_+ - \frac{i}{4m_N}[S^\mu, S^\nu]\left((1+\kappa_1) f_{\mu\nu}^+ + \frac{1}{2}(\kappa_0 - \kappa_1)\mathrm{Tr}\left(f_{\mu\nu}^+\right)\right)\Bigg] N. \end{aligned} \tag{22}$$

Given these Lagrangians, Weinberg's power counting argument [35–37] implies that connected diagrams scale as $\mathcal{A} \sim p^\nu$ with

$$\nu = 2L + 1 + \sum_{n=2,4,\ldots} (n-2) N_n^M + \sum_{m=1,2,\ldots} (m-1) N_m^F \tag{23}$$

where L is the number of loops and N_n^M (N_m^F) is the number of mesonic (fermionic) vertices with chiral dimension n (m). In deriving this formula, pion propagators are counted as p^{-2} and baryon / lepton propagators are counted as p^{-1}.

Using this power counting one sees that the amplitude for neutron decay can be organized as follows

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \mathcal{A}^{(4)} + ... && \text{(24a)}\\
\mathcal{A}^{(1)} &= \mathcal{A}^{G_F p^0} && \text{(24b)}\\
\mathcal{A}^{(2)} &= \mathcal{A}^{G_F p} && \text{(24c)}\\
\mathcal{A}^{(3)} &= \mathcal{A}^{G_F p^2} + \mathcal{A}^{e^2 G_F p^0} && \text{(24d)}\\
\mathcal{A}^{(4)} &= \mathcal{A}^{G_F p^3} + \mathcal{A}^{e^2 G_F p} && \text{(24e)}\\
&... &&
\end{aligned}$$

We are interested in computing the leading and next-to-leading electromagnatic corrections to the neutron decay, which appear at chiral order $n = 3$ and $n = 4$, respectively. Using Eq. (38) one can easily identify the tree-level and one-loop diagrams that contribute to a given order, up to $\mathcal{A}^{(4)}$:

- The amplitude $\mathcal{A}^{(1)}$ is given by a tree-level diagram with insertion of the weak vertices from $\mathcal{L}_{\pi N}^{(1)}$.

- The amplitude $\mathcal{A}^{(2)}$ is obtained by tree-level graphs with one insertion of $\mathcal{L}_{\pi N}^{(2)}$ and any number of insertions from $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi}^{(2)}$. It contributes to neutron decay at order $G_F \epsilon_{\rm recoil}$.

- The amplitude $\mathcal{A}^{(3)}$ is given by tree-level graphs with one insertion of $\mathcal{L}_{\pi N}^{(3)}$ and any number of insertions from $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi}^{(2)}$; and by one-loop diagrams with vertices from $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi}^{(2)}$. In Fig. 1 (upper panel) we show all one-loop topologies contributing up to $O(e^2 G_F p^0)$, These involve virtual pions, virtual photons, and virtual charged leptons.

- The amplitude $\mathcal{A}^{(4)}$ is given by tree-level graphs with one insertion of $\mathcal{L}_{\pi N}^{(4)}$ and any number of insertions from $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi}^{(2)}$; and by one-loop diagrams with one vertex from $\mathcal{L}_{\pi N}^{(2)}$ and any number of vertices from $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi}^{(2)}$. Note that tree level graphs with insertion of $\mathcal{L}_{\pi N}^{(e^2 p^2)}$ do not contribute. In Fig. 1 (lower panel) we show representative one-loop diagrams contributing up to $O(e^2 G_F p)$, These involve virtual pions, virtual photons, and virtual charged leptons.

The counterterm contributions to the amplitude at $O(e^2 G_F p^0)$ are captured by the combinations $\hat{C}_{V/A}$ (see Eq. (6)) as follows:

$$\begin{aligned}
\hat{C}_A &= 8\pi^2 \left[-\frac{X_6}{2} + \frac{1}{g_A^{(0)}} \left[\tilde{X}_3 + \left(g_1 + g_2 + \frac{g_{11}}{2} \right) \right] \right] \\
\hat{C}_V &= 8\pi^2 \left[-\frac{X_6}{2} + 2 \left(\tilde{X}_1 - \tilde{X}_2 \right) + g_9 \right] . \qquad (25)
\end{aligned}$$

Neutron decay rate — We now present the neutron differential decay rate up-to-and-including $\mathcal{O}(G_F \epsilon_{\rm recoil}$, $\mathcal{O}(G_F \alpha)$, $\mathcal{O}(G_F \alpha \epsilon_\chi)$, and $\mathcal{O}(G_F \alpha \epsilon_\not{\pi})$ corrections. We follow Refs. [9, 25, 26] and write

$$\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} = \frac{(G_F V_{ud})^2}{(2\pi)^5} (1+3\lambda^2) w(E_e) D(E_e, \vec{p}_e, \vec{p}_\nu, \vec{\sigma}_n) , \qquad (26)$$

where $\vec{\sigma}_n$ denotes the neutron polarization and $\lambda = g_A/g_V$. The spectrum of the electron is described by

$$w = |\vec{p}_e| E_e (E_0 - E_e)^2 F(E_e) \left(1 + \frac{\alpha}{2\pi} \delta_\alpha^{(1)}(E_e) \right) , \qquad (27)$$

where $E_0 = (m_n^2 - m_p^2 + m_e^2)/(2m_n)$ is the maximal electron energy, and $F(E_e)$ is the Fermi function for an electron in the field of the final-state proton. The radiative correction $\delta_\alpha^{(1)}$ is discussed below. The function D can be parametrized[1] as

$$\begin{aligned}
D &= 1 + c_0 + c_1 \frac{E_e}{m_N} + \frac{m_e}{E_e} \bar{b} + \bar{a} \frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu} + \bar{A} \frac{\vec{\sigma} \cdot \vec{p}_e}{E_e} \\
&\quad + \bar{B} \frac{\vec{\sigma} \cdot \vec{p}_\nu}{E_\nu} + \bar{C}_{aa} \left(\frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu} \right)^2 \\
&\quad + \bar{C}_{aA} \frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu} \frac{\vec{\sigma} \cdot \vec{p}_e}{E_e} + \bar{C}_{aB} \frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu} \frac{\vec{\sigma} \cdot \vec{p}_\nu}{E_\nu} . \qquad (28)
\end{aligned}$$

The various coefficients can be further decomposed through [9, 26]

$$\begin{aligned}
\bar{a} &= \left(a_{\rm LO} + c_0^{(a)} + c_1^{(a)} \frac{E_e}{m_N} \right) \left(1 + \frac{\alpha}{2\pi} \delta_\alpha^{(2)}(E_e) \right) , \\
\bar{A} &= \left(A_{\rm LO} + c_0^{(A)} + c_1^{(A)} \frac{E_e}{m_N} \right) \left(1 + \frac{\alpha}{2\pi} \delta_\alpha^{(2)}(E_e) \right) , \\
\bar{B} &= B_{\rm LO} + c_0^{(B)} + c_1^{(B)} \frac{E_e}{m_N} + \frac{m_e}{E_e} b_\nu , \\
\bar{C}_{aa} &= c_1^{(aa)} \frac{E_e}{m_N} , \\
\bar{C}_{aA} &= c_1^{(aA)} \frac{E_e}{m_N} , \\
\bar{C}_{aB} &= c_0^{(aB)} + c_1^{(aB)} \frac{E_e}{m_N} . \qquad (29)
\end{aligned}$$

[1] A possible correction to the time-reversal-odd D coefficient only enters at $\mathcal{O}(G_F \alpha \epsilon_{\rm recoil})$ [60].

The LO coefficients are well known and given by

$$\begin{aligned} a_{\rm LO} &= \frac{1-\lambda^2}{1+3\lambda^2}\,, \\ A_{\rm LO} &= \frac{2\lambda-2\lambda^2}{1+3\lambda^2}\,, \\ B_{\rm LO} &= \frac{2\lambda+2\lambda^2}{1+3\lambda^2}\,. \end{aligned} \tag{30}$$

We write the remaining coefficients in terms of $\bar{\mu}_V = \mu_p - \mu_n - \frac{\alpha Z_\pi}{2\pi}\frac{g_A^2 m_N \pi}{m_\pi}$ and $c_T = \frac{\alpha}{2\pi}\frac{g_A m_N \pi}{3 m_\pi}$

$$\begin{aligned} \bar{b} &= -\frac{m_e}{m_N}\frac{1+2\lambda(\bar{\mu}_V+3c_T)+\lambda^2}{1+3\lambda^2}\,, \\ c_0 &= -\frac{E_0}{m_N}\frac{2\lambda(\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_1 &= \frac{3+4\lambda\bar{\mu}_V+9\lambda^2}{1+3\lambda^2}\,, \\ c_0^{(a)} &= \frac{E_0}{m_N}\frac{2\lambda(\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_1^{(a)} &= -\frac{4\lambda(3\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_0^{(A)} &= \frac{E_0}{m_N}\frac{(\lambda-1)(\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_1^{(A)} &= \frac{\lambda(7-5\lambda)+\bar{\mu}_V(1-3\lambda)}{1+3\lambda^2}\,, \\ c_0^{(B)} &= -\frac{E_0}{m_N}\frac{2\lambda(\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_1^{(B)} &= \frac{\lambda(7+5\lambda)+\bar{\mu}_V(1+3\lambda)}{1+3\lambda^2}\,, \\ b_\nu &= -\frac{m_e}{m_N}\frac{(1+\lambda)(\lambda+\bar{\mu}_V)+2c_T(1+2\lambda)}{1+3\lambda^2}\,, \\ c_1^{(aa)} &= -\frac{3(1-\lambda^2)}{1+3\lambda^2}\,, \\ c_1^{(aA)} &= \frac{(\lambda-1)(5\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_0^{(aB)} &= \frac{E_0}{m_N}\frac{(1+\lambda)(\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,, \\ c_1^{(aB)} &= -\frac{(1+\lambda)(7\lambda+\bar{\mu}_V)}{1+3\lambda^2}\,. \end{aligned} \tag{31}$$

The explicit expressions for the radiative corrections $\delta_\alpha^{(1)}$ and $\delta_\alpha^{(2)}$ are given by

$$\begin{aligned} \delta_\alpha^{(1)} &= 2\hat{C}_V + 3\log\frac{\mu}{m_e} + \frac{1}{2} \\ &\quad + \frac{1+\beta^2}{\beta}\log\frac{1+\beta}{1-\beta} + \frac{1}{12\beta}\left(\frac{\bar{E}}{E_e}\right)^2\log\frac{1+\beta}{1-\beta} \\ &\quad + 4\left[\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right]\left[\log\frac{2\bar{E}}{m_e}-\frac{3}{2}+\frac{\bar{E}}{3E_e}\right] \\ &\quad + \frac{1}{\beta}\left[-4\,{\rm Li}_2\left(\frac{2\beta}{1+\beta}\right)-\log^2\left(\frac{1+\beta}{1-\beta}\right)\right]\,, \end{aligned} \tag{32}$$

$$\begin{aligned} \delta_\alpha^{(2)} &= \frac{1-\beta^2}{\beta}\log\frac{1+\beta}{1-\beta} \\ &\quad + \frac{4(1-\beta^2)}{3\beta^2}\frac{\bar{E}}{E_e}\left[\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right] \\ &\quad + \frac{1}{6\beta^2}\frac{\bar{E}^2}{E_e^2}\left[\frac{1-\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right]\,, \end{aligned} \tag{33}$$

where $\beta = |\vec{p}_e|/E_e$ and $\bar{E} = E_0 - E_e$. Our expression for $\delta_\alpha^{(1)}$ coincides with the combination of $\delta_\alpha^{(1)} + e_V^R(\mu)$ in Ref. [25] upon identifying $2\hat{C}_V$ with the combination of counterterms $e_V - (e_1+e_2)/2$] in Ref. [25]. Finally, expressing $\delta_\alpha^{(1)}$ in terms of the Sirlin function $g(E_e, E_0)$ [17], we find

$$\delta_\alpha^{(1)} = 2\hat{C}_V + \frac{5}{4} + 3\log\frac{\mu}{m_p} + g(E_e, E_0)\,. \tag{34}$$