Admissible statistics from a latent variable perspective

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Appendix A

A Bisymmetrical Structure for Vending Machine Bias

In order to show that the bias in Lord's vending machines is a quantitative property that allows for an interval representation, we use the representational measurement theorems describing bisymmetric structures by Krantz et al. (1971, p. 294), which were developed for mean structures (means of pairs of numbers). We use a slightly adapted version, which we denote as an Abelian bisymmetric structure, for purposes of simplification[1].

An Abelian idempotent bisymmetrical structure $\langle A, \succcurlyeq, \circ \rangle$, where A is a nonempty set of objects, $\succcurlyeq$ is a binary relation on A and $\circ$ is a binary operation from $A \times A$ into A, exists iff, for all $a, b, b^{-}, b_{-}, c, d \in A$, the five following axioms hold:

(1)	weak order	$\langle A, \succcurlyeq \rangle$is a weak order.
(2)	commutativity	$a \circ b \sim b \circ a$(where $\sim$ denotes equality).
(3)	idempotency	$a \circ a \sim a$.
(4)	monotonicity	$a \succcurlyeq b$ iff $a \circ c \succcurlyeq b \circ c$ iff $c \circ a \succcurlyeq c \circ b$.
(5)	bisymmetry	$(a \circ b) \circ (c \circ d) \sim (a \circ c) \circ (b \circ d)$.
(6)	restricted solvability	if $b^{-} \circ c \succcurlyeq a \succcurlyeq b_{-} \circ c$ then there exists $b' \in A$ such that $b' \circ c \sim a$.
(7)	Archimedean	every strictly bounded standard sequence is finite where $a_i \in A, i \in N$ is a standard sequence iff there exist $p, q \in A$ such that $p \succ q$ and, for all $i, i+1 \in N$, $a_i \circ p \sim a_{i+1} \circ q$.

[1] We would like to thank professor Duncan Luce for his suggestion to simplify the bisymmetric structure in this way.

The objects in our empirical relational structure are the football number producing machines; relations between these objects are represented in a numerical relational structure by the set of positive real numbers $R+$. We are interested in the bias towards low numbers in these machines, introduced by replacing high numbers with low numbers. This method of introducing bias by replacement is assumed here because its formalization is the most straightforward. We ascertain the bias in a machine by drawing a number from the machine with replacement an infinite number of times and taking the mean of these draws. The empirical relational structure further consists of the relation of weak ordering of the machines according to their means, denoted by the symbol $\succcurlyeq$. In the numerical relational structure this relation is represented by the numerical relation symbol $\geq$. The empirical relational system also includes an empirical concatenation operation, denoted by the symbol $\circ$. Concatenation of two machines a and b is achieved by alternately drawing a number from each machine with replacement an infinite number of times. The concatenation of bias in these machines is the mean of these draws, which equals $.5\mu a + .5\mu b$, where μa and μb represent the population means of machines a and b. Trivially, this matches the result of taking the mean of the means of the two machines, represented in the numerical relational structure by $\frac{(a+b)}{2}$, where a and b represent the means of two machines. Concatenating machines a and b into $a \circ b$ and then concatenating c is achieved by drawing alternately from $(a \circ b)$ and c. The resulting bias equals $.25\mu a + .25\mu b + .5\mu c$. In the numerical relational structure this corresponds to first taking the mean of the means of a and b ($\frac{(a+b)}{2}$), then taking the mean of the resulting mean and the mean of c: $\frac{((\frac{(a+b)}{2})+c)}{2}$. It is important to note that this method of concatenation ensures that the size of a machine is irrelevant, which automatically entails that the operation is not associative. The concatenation $(a \circ b) \circ c$ does not result in the same level of bias as the concatenation of $a \circ (b \circ c)$. In each individual concatenation the two machines (or concatenations of machines) always contribute equally.

The axioms of weak order, commutativity, idempotency, monotonicity and bisymmetry hold. The first requirement is met: the machines can be weakly ordered according to their means, where a lower values obviously indicates more bias; the ordering is transitive, connected and reflexive. Our structure can be called commutative when the order in which two machines are added does not matter; this is clearly the case. A structure is idempotent when the concatenation of two objects, equal in terms of the relevant property, is equal to either of the original objects. This holds for our vending machines: adding two machines that are equally biased results in a machine with the same amount of bias as the original two. The requirement of monotonicity holds also; if one machine has a lower mean than another, then concatenating them with the same machine will not affect their ordering.

The requirement of bisymmetry is somewhat more complicated to explain. Bisymmetry means that the order in which concatenations are primarily and secondarily concatenated is irrelevant. Concatenating a and b into ab, c and d into cd, and the resulting concatenations ab and cd into $abcd$ is equivalent to concatenating a and c into ac, b and d into bd and ac and bd into $acbd$. In the primary concatenations, the same number of machines are concatenated, namely two. The elements a, b, c and d all contribute equally in the primary and the secondary concatenation. Therefore the combination of elements one chooses to concatenate first is arbitrary. Thus, the requirement of bisymmetry is also met.

Axiom (6) and (7) are structural, or existential axioms. Axiom (6), restricted solvability, ensures that a bounded set of objects is sufficiently dense. When two objects are concatenated, a third can always be found that is equal to the concatenation of the first two. Normally, such an axiom could never be shown to hold in a real setting with a finite set of objects. Fortunately, in a thought experiment we can imagine an infinite number of machines, limited only by the structure of the machines and the tampering method we assumed. We only have to know whether we could construct a machine to be equal to any concatenation of two other machines. With a hundred quadrillion numbers per vending machine, that Lord providentially provided, we could at least approximate any concatenation extremely well. Axiom (7), the Archimedean axiom, entails that no element can be infinitely small or large. This axiom obviously holds in our case, where a and b are always positive rational numbers[2].

[2]If the reader is uncomfortable with these structural axioms: Krantz et al. (1971, p. 297) also provide a representation and uniqueness theorem for the finite, equally spaced case. For such a case the axioms of restricted solvability and the Archimedean axiom can be dropped. Adding the assumption of equal spacing and the assumption that the operation is idempotent and commutative results in the bisymmetry axiom being dropped as well. This version of the bisymmetric structure however, seems less intuitive and therefore the more readily interpretable theorems for the general case were used here.

With all the requirements met, we may assume that the structure of bias in the machine is additive. Now we want to know how different instantiations of numerical representations relate to each other, i.e. what level of measurement can be achieved. Krantz et al. (1971, p. 295) have proven that when a bisymmetric structure exists, representations of this structure are related linearly. The uniqueness theorem for bisymmetrical structures that states this fact, has again been altered to fit our Abelian idempotent bisymmetrical structure:

Given the Abelian idempotent bisymmetric structure $\langle A, \succsim, \circ \rangle$, there exist real numbers $\mu = \nu = \frac{1}{2}$ and a real-valued function ϕ on A, such that for $a, b \in A$:

I. $a \succsim b$ iff $\phi(a) \geq \phi(b)$.

II. $\phi(a \circ b) = \mu\phi(a) + \nu\phi(b)$.

III. If μ', ν', ϕ' is another representation fulfilling I. and II., then there exist constants $\alpha > 0$ and β such that:
$\phi' = \alpha\phi + \beta$,
$\mu' = \mu, \nu' = \nu$.

Krantz et al. (1971, p. 295) show that the fact that μ and ν equal $\frac{1}{2}$ follows directly from the fact that our concatenation operation is idempotent and commutative. The function that translates one numerical representation of the bisymmetric structure into another is linear ($\phi' = \alpha\phi + \beta$). This means that vending machine bias can be represented on the interval level by any linear transformation of the mean of the football numbers.