Admissible statistics from a latent variable perspective

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Appendix B

The Axioms of ACM Applied to the Guttman and Rasch Models

In ACM (Luce & Tukey, 1964) the empirical relational structure is made up of two disjoint sets of objects, their Cartesian product and a weak order relation. In our case the two sets of objects are items ($I = \{a, b, c, \ldots\}$) and persons ($P = \{x, y, z, \ldots\}$). The Cartesian product is the pairing of each item a with each person x, resulting in the probability (a, x) that the person will endorse the item. The empirical relation between these objects is a weak ordering on the latent property of interest denoted by $\succcurlyeq$. The numerical relational structure that can represent this relation between items, persons and probabilities, consists of the reals, and the numerical relation $\geq$.

The first axiom states that the objects in each of the two sets have to form a weak order. This is the case if and only if, when we let any two persons x and y (from P) answer the same item a (from I), and the resulting probabilities stand in relation $(x, a) \succcurlyeq (y, a)$, then the persons also stand in relation $x \succcurlyeq y$. The relation $\succcurlyeq$ should be transitive and connected. The same should hold when we present any two items a and b to the same person x. This axiom of weak order is met by the probabilities associated with data that perfectly fit the Rasch model. A numerical example is provided in Table 4.1b. In the Rasch model the probability of answering an item correctly is a strictly increasing monotonic function of the difference between ability and difficulty. A higher (or equal) probability of correctly answering an item can only be achieved if the ability is higher (or equal), since all other parameter values remain unchanged. Also, connectedness and transitivity hold; any item can be paired with any person (connectedness) and if $(x, a) \succcurlyeq (y, b)$ and $(y, b) \succcurlyeq (z, c)$, then $(x, a) \succcurlyeq (z, c)$ (transitivity).

Whether the axiom of weak order holds for the probabilities associated with a perfectly fitting Guttman model is less straightforward. An example of Guttman data is given in Table 4.1a. If person y has a probability of 1 to answer item b correctly and person z has a probability of 0, then according to the axiom person y must have a higher or equal ability than z. Let us say for example that person y is smarter. But what if, for item c (a more difficult item), person y has the same probability of 0 as person z to answer this item correctly? In this case $(y, c) \succcurlyeq (z, c)$ and therefore $y \succcurlyeq z$ holds, but so does $(z, c) \succcurlyeq (y, c)$, and therefore $z \succcurlyeq y$. This seems counter-intuitive to say the least. It is strange to have to accept that while we know person y is smarter, this is not reflected in the equivalent scores of 0 on the harder item for both persons. In the Rasch model, if the probabilities show a specific order or equivalence for two persons on an item, then the persons themselves will show this same specific order or equivalence. Mathematically however, the reasoning in the Guttman case is sound, given the nature of the relation $\succcurlyeq$. This weak order relation entails that the 'greater than' and equivalence relation cannot be unraveled. The relation is not required to be antisymmetric (if $a \succcurlyeq b$ and $b \succcurlyeq a$, then $a \sim b$) or asymmetric (if $a \succcurlyeq b$, then not $b \succcurlyeq a$). The discrepancy in the meaning of the weak order relation in terms of the pairings (probabilities) on the one hand and the objects (person, items) on the other is disquieting however. The interpretation of transitivity suffers from the same problem. Support for adherence to the axiom of weak order by the Guttman model is clearly up for discussion, but let us first consider the other axioms.

The second axiom is that of independence. It states that when any two objects from the first set are paired with any object from the second set, the ordering of the pairings should in no way depend on the object selected from the second set. So if person x has a higher or equal ability compared to person y, then for all items the probability of answering an item correctly should be higher or equal for person x compared to person y. The same goes for items of course, if item a is easier that item b, this item should have a higher or equal probability of being answered correctly, no matter the ability of the person that answers the two items. The independence axiom holds for both the Rasch and the Guttman model. In the Rasch model the item response curves stochastically dominate each other. The item curves that describe the relation between ability and probability of answering an item correctly are parallel, ensuring that the probability of answering the easier item correctly will always be higher for any given value of the ability (and vice versa for persons and abilities). In the Guttman model the probabilities for different items overlap in large parts, but they never 'flip over' because this is exactly the restriction that Guttman puts on the data. Suppose person x has a probability of 1 to answer item c correctly and person y has a probability of 0. Now for an easier item b they could have these same probabilities or both have a probability of 1, but the smarter person x will never have a lower probability than person y. The same goes for a harder item d; If the Guttman model applies, they could both answer this item incorrectly or the same as item c, but the less smart person y will never do better than person x.

The second axiom can be considered somewhat redundant, since it a special case of the third axiom, which is called the double cancellation axiom (Michell, 1988). We present the axioms separately however, because if one posits the axiom of independence, then only one specific instantiation of the double cancellation axiom, called the Luce-Tversky condition, needs to hold. The double cancellation axiom is less intuitive than the others and is easiest to understand graphically. The simple diagram in Figure 4.2 illustrates the special instance of the axiom. The Cartesian product of object (a, b, c) from set I and objects (x, y, z) from set P is displayed. The Luce-Tversky condition requires the pairings at the arrow-tails to be compared with the pairings at the arrowheads in Figure 4.2a. If in both cases the probabilities show a weak order relation in the same direction then there should also be a weak order relation in the same direction between the probabilities at the arrow-tail and arrowhead in Figure 4.2b. More formally, if and only if $(y, a) \succcurlyeq (x, b)$ and $(z, b) \succcurlyeq (y, c)$, then $(z, a) \succcurlyeq (x, c)$. It suffices to show that this requirement holds for the cells depicted in Figure 4.2 in all possible three-by-three matrices. There are other combinations of antecedents (Figure 4.2a) and consequents (Figure 4.2b) that form different instantiation of double cancellation, but these all logically follow from the axiom of independence (Michell, 1988).

What does the double cancellation condition entail when we pair persons with items for the Rasch model and Guttman model respectively? In Table 4.1 an example of probabilities for each model was given. When data perfectly fit the Rasch model, the probabilities resembling a pattern in Table 4.1b will directly show acceptance of double cancellation for all possible three-by-three sub-matrices. This is because in the Rasch model, the probabilities are a continuous, monotonically increasing function of the difference between the person ability and the item difficulty and because the item response curves are stochastically ordered. This automatically results in the more complicated consistent ordering required by the double cancellation axiom.

If Guttman probabilities (Table 4.1a) are put to the test however, double cancellation will be violated in many of the possible three-by-three matrices. Two problematic patterns are displayed in Table 4.2. In both cases the antecedents show an equivalence relation ($0 \sim 0$ or $1 \sim 1$) which conforms to $\succcurlyeq$. In contrast, the consequent shows a simple order relation ($1 \prec 0$) that contradicts the weak order found in the antecedents. In the Rasch model, differences between persons and items will always result in differences between the associated probabilities. In the Guttman model this is not necessarily the case. Two persons that differ greatly in their ability still have the same probability of 1 for all the items that the person with the lower ability dominates and the probability of 0 for all the items that dominate the person with the higher ability. The lumping together of persons or items in terms of probabilities leads to the rejection of double cancellation in the Guttman model. Other problematic patterns can occur, for example when the antecedents show an equivalence relation ($0 \sim 0$ or $1 \sim 1$) and the consequent shows a simple order relation (higher than: $1 \prec 0$). These Guttman patterns violate double cancellation as specified by Michell, who showed that equivalence relations in the antecedents and a simple order in the consequent constitutes a rejection of double cancellation (Michell, 1988).

Here the same issue arises as before, when we discussed the weak order axiom. The interpretation of the relation $\succcurlyeq$ is problematic. The binary coding creates a discrepancy between what the relation $\succcurlyeq$ means in terms of the latent variable (on which items and persons are compared) and what the relation means in terms of probabilities. In other words, if we observe the equivalence $(y, a) \sim (x, b)$, this does not mean that $f(y) - g(a)$ is equivalent to $f(x) - g(b)$. This means the equivalence relation for the pairings has a different interpretation than the equivalence relation for the objects. This is highly ambiguous and undesirable.

The fourth axiom of additive conjoint measurement concerns the solvability of the relation $\succcurlyeq$. It ensures that the set of pairings of objects does not contain gaps. This means that when we consider the relation between two pairings of the sets of objects, say (x, a) and (z, c), for any three of these four elements we can find a fourth that makes the relation between these pairings an equivalence $(x, a) \sim (z, c)$. This axiom is not empirically testable, but in both the Rasch and Guttman models it is obvious that one can imagine a person or item that would satisfy this axiom. Say person x is of low ability, person z is of high ability and item a is slightly easier compared to the ability of x. If we had endless items and persons that conform to the Rasch model, we should be able to find an item c that is just as easy for person z as item a is for person x. In the Guttman model the same applies, x will answer item a correctly, so we only have to find an item c that is easy enough for person z to answer correctly. It is noteworthy that this item could either be just a little easier compared to the ability of person z, or much easier. In the Rasch model, there will be only one item c that satisfies the condition; in the Guttman model there are an infinite number of items that will satisfy it. Again, the Guttman model meets the condition technically, but it is obvious that the ACM axiom system was not intended for discrete pairings of objects.

The fifth and final axiom is called the Archimedean condition and requires that no difference between objects in either set is infinitely small. In other words, it ensures that no item difficulty or person ability is infinitely small or large. This condition is like the solvability axiom in the sense that it cannot be tested empirically. More formally it requires that, when you order items and persons and denote this order by using natural numbers, if you pair person x_i with item a, and if you then take the next person x_{i+1}, it should take a different (more difficult) item b to get the same result. In the Rasch model, this is clearly the case. If a person with a certain ability answers an easy item, a person with a slightly higher ability would have to answer a more difficult item to get the same probability of answering the item correctly. In the Guttman model this is not the case. If a person with a certain ability has a probability of 1 to answer an easy item correctly, a person with slightly higher ability will have the same probability of answering the item correctly. It does not take a different (more difficult) item to get an equivalence relation between the first person-item pairing and the second. The Guttman model therefore fails the last axiom.