Using Hierarchies to Efficiently Combine Evidence with Dempster's Rule of Combination (Supplementary material)

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A ADDITIONAL PROOFS

In this appendix, we provide proofs for the statements for which we omitted a proof from the main paper.

Proof (sketch) of Proposition 3.2. The main idea behind this proof is the following. Whenever you combine two b.p.a.'s m_1 and m_2 whose proper focal elements are all of size at most c using DRC, the resulting mass function only assigns positive mass to sets of size at most c. For any frame Θ of discernment of size n, the number of subsets of size at most c is upper bounded by $(n+1)^c$—which is a polynomial. Therefore, one can compute the result of DRC in a brute force fashion in polynomial time. □

Proof of Lemma 5.2. Let us see that two pairs $P_i = (B_i, \overline{B}_i)$ and $P_j = (B_j, \overline{B}_j)$ of $\mathcal{A}$ have at least one conflict. If $P_i \not\rightleftharpoons P_j$ then $B_i \not\rightleftharpoons B_j$ (1), $\overline{B}_i \not\rightleftharpoons B_j$ (2), $B_i \not\rightleftharpoons \overline{B}_j$ (3) and $\overline{B}_i \not\rightleftharpoons \overline{B}_j$ (4). For (1), at least one of these three conditions must hold:

1. $B_i \subseteq B_j$,
2. $B_j \subseteq B_i$ or
3. $B_i \cap B_j = \emptyset$.

If $B_i \subseteq B_j$, then $B_j \cap \overline{B}_i \neq \emptyset$ since the inclusion is strict. In addition, $B_j \not\subseteq \overline{B}_i$ and, if $B_j \neq \Theta$, $\overline{B}_i \not\subseteq B_j$. Therefore, $\overline{B}_i \rightleftharpoons B_j$, which contradicts (2).

A similar reasoning can show that if $B_j \subseteq B_i$, and $B_i \neq \Theta$, then $B_i \rightleftharpoons \overline{B}_j$, contradicting (3).

Finally, if $B_i \cap B_j = \emptyset$, then $B_j \subseteq \overline{B}_i$ and $B_i \subseteq \overline{B}_j$, so $\overline{B}_i \not\subseteq \overline{B}_j$ and $\overline{B}_j \not\subseteq \overline{B}_i$ respectively. Furthermore, as these inclusions are not strict, $\overline{B}_i \cap \overline{B}_j \neq \emptyset$. This means that $\overline{B}_i \rightleftharpoons \overline{B}_j$ and contradicts (4).

Due to all of the above three conditions implies a contradiction, we can conclude that there is at least one conflict between elements of P_i and P_j.

Now, let us prove that if there is a conflict between B_i, B_j and $((B_i, \overline{B}_i), (B_j, \overline{B}_j)) \notin C_4$ then $\overline{B_i} \cap \overline{B_j} = \emptyset$, $\overline{B_i} \subseteq B_j$ and $\overline{B_j} \subseteq B_i$, and as a consequence, $((B_i, \overline{B}_i), (B_j, \overline{B}_j)) \in C_1$.

On the one hand, $\overline{B_i} \cap \overline{B_j} = \emptyset$ implies $\overline{B_i} \subseteq B_j$ and $\overline{B_j} \subseteq B_i$, since that empty intersection implies that all the elements of $\overline{B}_i$ (resp. $\overline{B}_j$) are contained in the complement of $\overline{B}_j$ (resp. $\overline{B}_i$).

On the other hand, if $\overline{B_i} \cap \overline{B_j} \neq \emptyset$, then not only B_i has a conflict with B_j but also (a) B_i has a conflict with $\overline{B}_j$, (b) $\overline{B}_i$ has a conflict with B_j and (c) $\overline{B}_i$ has a conflict with $\overline{B}_j$.

(a) First, $B_i \rightleftharpoons B_j$ implies $B_i \not\subseteq B_j$, so there is an element in B_i which belong to $\overline{B}_j$ and $B_i \cap \overline{B}_j \neq \emptyset$. Secondly, $B_i \cap B_j \neq \emptyset$ so $B_i \not\subseteq \overline{B}_j$. Finally, $\overline{B_i} \cap \overline{B_j} \neq \emptyset$, so there is an element in $\overline{B_j}$ which is not in B_i, i.e., $\overline{B_j} \not\subseteq B_i$.

(b) The conflict $B_i \rightleftharpoons B_j$ also implies $B_i \cap B_j \neq \emptyset$ so $B_j \not\subseteq \overline{B}_i$. In addition, $\overline{B_i} \cap \overline{B_j} \neq \emptyset$ proves that $\overline{B_i} \not\subseteq B_j$. Lastly, if $\overline{B_i} \cap A_j = \emptyset$ then $B_j \subseteq B_i$ which is not possible since $B_i \rightleftharpoons B_j$.

(c) On the one hand, our hypotheses is that $\overline{B_i} \cap \overline{B_j} \neq \emptyset$. On the other hand, $\overline{B}_i \not\subseteq \overline{B}_j$ and $\overline{B}_j \not\subseteq \overline{B}_i$ for $B_i \not\subseteq B_j$ and $B_j \not\subseteq B_i$ respectively.

Therefore, if $B_i \rightleftharpoons B_j$ then $(B_i, \overline{B}_i) \stackrel{1}{\not\rightleftharpoons} (B_j, \overline{B}_j)$ or $(B_i, \overline{B}_i) \stackrel{4}{\not\rightleftharpoons} (B_j, \overline{B}_j)$. □

Proof (sketch) of Proposition 5.5. We describe the main lines of this reduction, and we omit a proof of correctness—which is analogous to the proof of Theorem 5.4. Let Θ be a frame of discernment, $\mathcal{A} = \{(B_i, \overline{B_i})\}_{i=1}^m$ a set of complementary pairs over Θ, and ℓ a positive integer. We construct $G = (V, E)$ by letting $V = \{v_1, \dots, v_m\}$ and $E = \{\{v_i, v_j\} \mid (B_i, \overline{B}_i) \stackrel{4}{\rightleftharpoons} (B_j, \overline{B}_j)\}$. Then $\mathcal{A}$ and k form a yes-instance for PARTIAL-HIERARCHY if and only if G has a vertex cover of size $k = m - \ell$, and solutions are in one-to-one correspondence. □

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