Playing with knowledge and belief

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Chapter 3

Belief contraction in total plausibility models

Aim In this chapter we study belief contraction in the framework of Dynamic Epistemic Logic. Our aim is to model different belief contraction operations in the particular setting of total plausibility models.

Summary: In this chapter we consider three different kinds of belief contraction operations. We first present the notions of severe withdrawal, conservative contraction and moderate contraction. Then we axiomatize these AGM-friendly versions of contraction in DEL. The main points are:

- we provide a brief presentation of the notions of belief contraction in the literature.
- we present the notion of severe withdrawal and discuss the limits of this belief contraction operation and we present two other types of contraction operations namely, conservative and moderate contraction.
- we define these three contracting operations as operations on total plausibility models, we associate to them epistemic actions in DEL style and axiomatize them in DEL style.

Background In this chapter we use both Grove spheres and (finite) total plausibility models as described in Chapter 2. These types of models are in fact equivalent. Thus we can generate a Grove model from a given total plausibility model and vice versa.

We illustrate this correspondence on the following example. Let us start from a given Grove model in Figure 3.1. This model contains four nested spheres. Since the central sphere represents the actual belief set of the agent and the world $s$ belongs to the central sphere, $s$ is the most plausible world in this model. Since the second sphere contains the world $v$ and the third sphere contains the world $t$, this means that the agent believes that $v$ is more plausible than $t$.
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$v$ is more plausible than $t$. Since both $u$ and $w$ belong to the last sphere, $u$ and $w$ are equiplausible and are the less plausible worlds in this model.

![Figure 3.1: Example of a sphere system](image)

From these observations, we can easily draw the corresponding total plausibility model in Figure 3.2.

![Figure 3.2: Corresponding total plausibility model](image)

In this chapter we will define different dynamic languages on top of two different static languages which we introduced in Chapter 2:

- the language $\mathcal{L}_{CDL}$ is obtained from the language of propositional logic by the addition of a conditional belief operator, and

- the language $\mathcal{L}_{KK_d}$ is obtained from the language of propositional logic by the addition of an irrevocable knowledge operator and a defeasible knowledge operator.
3.1 General background on belief contraction

We focus here on the notion of belief contraction. If an agent may revise her beliefs after receiving a new piece of information, she may also contract her beliefs in the light of new information. For example, consider an agent who believes that Aristotle is lying on my couch. Then this agent receives the information that it is not the case that Aristotle is lying on my couch. After receiving this information she no longer believes that Aristotle is lying on my couch. In other words, the agent removes her belief that Aristotle is lying on my couch. Indeed she considers it now possible that Aristotle is lying on my couch but also that Aristotle is not lying on my couch.

We already mentioned in Chapter 2 the work of Alchourrón, Gärdenfors and Makinson who provide some postulates for belief contraction in [1]. In this chapter, we are only interested in the AGM-friendly versions of belief contraction.

3.1.1 Severe withdrawal

The first notion of contraction we want to study is the notion called “mild contraction” by Levi [50], “severe withdrawal” by Pagnuco and Rott [63] and “Rott contraction” by Ferme and Rodriguez [31].

3.1.1. Definition. Severe withdrawal is a belief change operation that removes a belief from the belief set of an agent such that, after contracting this belief set with $\mathcal{W}$, the most plausible worlds are all the worlds at least as plausible as the best $\neg \varphi$-worlds.

Example We provide one example of severe withdrawal in Figure 3.3. As an example of a severe withdrawal scenario we introduce the following story. Consider an agent and a dice. Someone throws the dice such that the agent cannot see the upper face. We have 6 possible worlds in our sphere system: i where 1 is the upper face, ii where 2 is the upper face and so on. Assume that the agent initially believes that the upper face is 3 while in reality (unknown to our agent) the upper face is 4. Thus she believes that the upper face is odd that is, she believes $\varphi$ (the formula $\varphi$ means “the number on the upper face is odd”). Besides, the agent considers that it is more likely that the upper face is 5 than 1, and that it is more likely that the upper face is 1 than 6 while she considers that it is equally likely that the upper face is 1 or 2, and that it is equally likely that it is 4 or 6. Then according to the agent: $3 < 5 < 1 \equiv 2 < 6 \equiv 4$ ($3 < 5$ is read as it is more likely that the upper face is 3 than 5 and $1 \equiv 2$ is read as it is equally likely that the upper face is 1 or 2). In other words, she considers iii more plausible than v, v more plausible than both i and ii, and finally i and ii more plausible
than both iv and vi.

Now we consider the case where the agent receives a piece of information saying that the number on the upper face is even. The agent has to remove her belief that the number on the upper face is odd. So according to our definition for severe withdrawal, the agent will believe that the upper face is 1, 2, 3 or 5. Indeed, the agent considers i, ii, iii and v to be more plausible after the severe withdrawal. Then according to the agent: $1 = 2 = 3 = 5 < 6 = 4$.

In Figure 3.3, the numbers represent the spheres of the new Grove system after the revision. Thus all regions labelled with 1 form the first sphere of the new Grove system, the regions labelled with 2 form the second sphere and so on. Finally, the regions labelled with $\omega$ contain the states that are outside the union of all the spheres of the Grove system that is, the impossible states.

![Figure 3.3: Example of severe withdrawal $-\varphi$](image)

### 3.1.2 Other AGM-type contractions

Severe withdrawal is not the only AGM-friendly semantic contraction operation in the literature. Other options include conservative contraction $-cP$ and moderate contraction $-mP$ [72].

**3.1.2. Definition.** Conservative contraction is a belief change operation that removes a belief from the belief set of an agent such that, after contracting this
belief set with \( \varphi \), the most plausible worlds are the best \( \neg \varphi \)-worlds plus the initial best \( \varphi \) worlds.

3.1.3. Definition. **Moderate contraction** is a belief change operation that removes a belief from the belief set of an agent such that, after contracting this belief set with \( \varphi \), the most plausible worlds are the best \( \neg \varphi \)-worlds plus the initial best \( \varphi \) worlds and all the \( \neg \varphi \)-worlds are promoted. They become better than the rest of the \( \varphi \)-worlds.

**Examples** We provide two examples in Figures 3.4, 3.5 respectively for conservative contraction and moderate contraction. As an example of a conservative contraction and moderate contraction scenario, we come back to our story with an agent and a dice that we developed above.

After the agent receives the information that the number on the upper face is even, she has to remove her belief that the number on the upper face is odd. According to the definition for conservative contraction, the agent will believe that the upper face is 2 or 3. Indeed, the agent considers ii and iii to be more plausible after the conservative contraction. The agent still considers that it is more likely that the upper face is 5 rather than 1, and more likely that the upper face is 1 rather than 6 while she still considers that it is equally likely that the upper face is 4 or 6. Then according to the agent: \( 2 \equiv 3 \prec 5 \prec 1 \approx 6 \equiv 4 \).

According to the definition for moderate contraction, the agent will believe that the upper face is 2 or 3. Indeed, the agent considers ii and iii to be more plausible after the conservative contraction. The agent still considers that it is more likely that the upper face is 5 than 1; she now considers that both 4 and 6 are more likely than 5. Then according to the agent: \( 2 \equiv 3 \prec 6 \equiv 4 \prec 5 \prec 1 \).

3.2 A new approach to severe withdrawal

Now we will model these notions of contraction on (finite) total plausibility models. The language for belief contraction will consist of a base language equipped with a dynamic contraction modality. There are now several options and depending both on the choice of the static base language and the type of contraction operator a different language can be provided.

3.2.1 Language for the logic of severe withdrawal

We define two languages for severe withdrawal. The first language \( L_{\text{Sev1}} \) will be defined on top of \( L_{\text{CDL}} \) equipped with dynamic modalities for contraction \( \Box \neg \varphi \). The second language \( L_{\text{Sev2}} \) will be defined on top of \( L_{\text{KKD}} \) equipped with dynamic modalities for contraction \( \Box \neg \varphi \).
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3.2.2 Semantics for the logic of severe withdrawal

In Dynamic Epistemic Logic, belief contraction is modelled as a model-changing operation. In particular it will take as input a given total plausibility model $\mathcal{M} = (S, \leq, V)$ and produces as output $\mathcal{M}^{-\varphi} = (S, \leq^{-\varphi}, V)$. Note that the current ordering relation $\leq$ of a given total plausibility model will be replaced by the following relation $\leq^{-\varphi}$ in the new model after the severe withdrawal with $\varphi$. The intended meaning of this is that all the worlds at least as plausible as the best $\neg\varphi$-worlds have become the best worlds, but apart from that, the old ordering remains the same.

3.2.1. DEFINITION. The initial total plausibility model $\mathcal{M} = (S, \leq, V)$ after a severe withdrawal with $\varphi$ is transformed into the following model $\mathcal{M}^{-\varphi} = (S, \leq^{-\varphi}, V)$ in which $t \leq^{-\varphi} s$ iff:

- $t \leq s$ or

- $t \leq w$ for some $w \in \text{best} \| \neg\varphi \|_\mathcal{M}$

The semantic clause for the dynamic operator $[-\varphi]$ is:

$$\mathcal{M}, s \models [-\varphi] \psi \text{ iff } \mathcal{M}^{-\varphi}, s \models \psi$$
3.2. A new approach to severe withdrawal

Example In Figure 3.6 we present a sphere model composed of 6 states $s, t, u, v, w, x$ such that $\varphi$ is true in $s, t, v, x$ and $\neg \varphi$ is true in $u, w$.

From this sphere system we can generate the corresponding total plausibility model given in Figure 3.7.

Next, we present the total plausibility model resulting from the severe withdrawal with $\varphi$ in Figure 3.8. In this model, the states $s, t, u$ and $v$ are equi-plausible, all of them being on top. Thus in the resulting model, the agent does not believe $\varphi$ anymore and does not believe $\neg \varphi$ either.

3.2.3 Axiom system for the logic of severe withdrawal

3.2.2. Theorem. A sound and complete proof system for the logic SEV with the language $\mathcal{L}_{Sev}$ is given by the axioms and rules of $\mathcal{L}_{KKD}$ plus the following reduction axioms:
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3.2.3. PROOF. The soundness of the axioms of $\mathcal{L}_{KK\theta}$ is proved in [8]. All that remains is to show the soundness of the Reduction axioms. The proof of the first Reduction axioms is straightforward, we focus here only on the proof of the
3.2. A new approach to severe withdrawal

Figure 3.8: Severe withdrawal in the initial total plausibility model of Figure 3.7

reduction axiom for defeasible knowledge.

Suppose \( s \models \mathcal{M} \neg \varphi \). Then \( s \models \mathcal{M} \neg \neg \varphi \). So for all \( t \) such that \( t \leq \neg \varphi \), \( t \models \mathcal{M} \neg \varphi \). This means that for all \( t \) such that \( t \leq \neg \varphi \), \( t \models \mathcal{M} \neg \varphi \).

We know that \( t \leq \neg \varphi \) iff \( t \leq w \) for some \( w \in \text{best} \) or \( t \leq s \). Then:

- for all \( t \) such that \( t \leq w \) for some \( w \in \text{best} \), \( t \models \mathcal{M} \neg \varphi \) and,

- for all \( t \) such that \( t \leq s \), \( t \models \mathcal{M} \neg \varphi \).

This means that \( s \models \mathcal{M} \neg \neg \varphi \rightarrow \neg \neg \neg \varphi \rightarrow \neg \neg (\neg \varphi \wedge \mathcal{K} \mathcal{D} \neg \varphi) \) and \( s \models \mathcal{M} \neg \mathcal{K} \mathcal{D} \neg \varphi \).

Sketch of the proof for completeness From the axiom system for the logic SEV with the language \( \mathcal{L}_{\text{Sev2}} \), we note that what is the case after a severe withdrawal can be expressed by saying what is the case before the severe withdrawal. Using the Reduction Laws, the severe withdrawal operator can be step by step “pushed through” all other operators and at the end, completely eliminated using the Reduction Law for atomic formulas.

3.2.4. Lemma. Every formula of \( \mathcal{L}_{\text{Sev2}} \) is provably equivalent in the above proof system with another formula in \( \mathcal{L}_{\mathcal{KKD}} \).

The completeness of the axioms of \( \mathcal{L}_{\text{Sev2}} \) follows from the completeness of the axioms of \( \mathcal{L}_{\mathcal{KKD}} \) that is proved in [8] and Lemma 3.2.4.

Let \( \varphi \) be a formula in \( \mathcal{L}_{\mathcal{KKD}} \) such that it is satisfiable in a total plausibility model and let \( \varphi' \) be a formula in \( \mathcal{L}_{\text{Sev2}} \) equivalent to \( \varphi \). Then \( \varphi' \) is also satisfiable in the total plausibility model.

We cannot provide Reduction Laws in the language \( \mathcal{L}_{\text{Sev1}} \), that is, if the static base language is the language of Conditional Doxastic Logic. Indeed we do not have a reduction axiom for conditional belief.
3.2.4 Objections against severe withdrawal

Many authors consider severe withdrawal to be a bad candidate for modelling contraction. In addition to not satisfying the Recovery principle\(^1\), it does satisfy a highly implausible property, called Expulsiveness. For ontic facts \(p, q\), we have that \(\neg Bp \land \neg Bq\) implies \([\neg p]Bq \lor [\neg q]Bp\). This property does not allow unrelated beliefs to be undisturbed by each other’s contraction.

Conservative contraction and moderate contraction are much better behaved than severe withdrawal. Indeed they satisfy the Recovery postulate.

3.3 A new approach to conservative contraction

3.3.1 Language for the logic of conservative contraction

We define two languages for conservative contraction. The first language \(\mathcal{L}_{\text{Cons1}}\) will be defined on top of the static base language \(\mathcal{L}_{\text{CDL}}\) equipped with dynamic modalities for contraction \([\neg c\varphi]\). The second language \(\mathcal{L}_{\text{Cons2}}\) will be defined on top of the static base language \(\mathcal{L}_{\text{KKD}}\) equipped with dynamic modalities for contraction \([\neg c\varphi]\).

3.3.2 Semantics for the logic of conservative contraction

The operation of conservative contraction has as an effect that a given total plausibility model \(M = (S, \preceq, V)\) will be transformed into a model \(M^{\neg c\varphi} = (S, \preceq^{\neg c\varphi}, V)\). In this setting the current ordering relation \(\preceq\) of a given total plausibility model will be replaced by the following relation \(\preceq^{\neg c\varphi}\) in the new model after the conservative contraction with \(\varphi\). The intended meaning of this is that the best \(\neg \varphi\)-worlds have become equi-plausible with the best worlds initially on top, but apart from that, the old ordering remains the same.

3.3.1. Definition. The initial total plausibility model \(M = (S, \preceq, V)\) after a conservative contraction with \(\varphi\) is transformed into the following model \(M^{\neg c\varphi} = (S, \preceq^{\neg c\varphi}, V)\) in which \(t \preceq^{\neg c\varphi} s\) iff:

\[ t \in \text{best} \models \neg \varphi \models_M \text{ or } \]
\[ t \preceq s. \]

3.3.2. Definition. We define \(\text{best}^{\neg c\varphi}P\):

\[ \text{best}^{\neg c\varphi}P := P \text{ if } P \cap \text{best} \models \neg \varphi \models_M \emptyset \text{ or,} \]

\(^1\)We provide an explanation of the Recovery principle in Chapter 2 when we introduce the AGM postulates for belief contraction.
3.3. A new approach to conservative contraction

\[
\text{best}^{-\varphi} P := (P \cap \text{best} \parallel \neg \varphi \upharpoonright_M) \cup (P \cap \text{best} S) \text{ if } P \cap \text{best} \parallel \neg \varphi \upharpoonright_M \neq \emptyset.
\]

The semantic clause for the dynamic operator \([-c\varphi]\) is:

\[M, s \vDash [-c\varphi] \psi \text{ iff } M^{-c\varphi}, s \vDash \psi\]

**Example** In Figure 3.9 we present a sphere model composed of 6 states \(s, t, u, v, w, x\) such that \(\varphi\) is true in \(s, t, v, x\) and \(\neg \varphi\) is true in \(u, w\).

![Figure 3.9: Grove system](image)

From this sphere system we can generate the corresponding total plausibility model given in Figure 3.10.

![Figure 3.10: Initial total plausibility model](image)
Next we present the total plausibility model resulting from the conservative contraction with $\varphi$ in Figure 3.11. In this model, the state $u$ is equi-plausible with the state $s$, both being on top. Thus the agent does not believe $\varphi$ anymore and does not believe $\neg\varphi$ either.

Figure 3.11: Conservative contraction in the initial total plausibility model of Figure 3.10

### 3.3.3 Axiom system for the logic of conservative contraction

**3.3.3. Theorem.** A sound and complete proof system for the logic $\text{CONS}_1$ with the language $\mathcal{L}_{\text{Cons}1}$ is given by the axioms and rules of $\mathcal{L}_{\text{CDL}}$ plus the following reduction axioms:

$$
\begin{align*}
[-c\varphi]p & \iff p \\
[-c\varphi]\neg\theta & \iff \neg[-c\varphi]\theta \\
[-c\varphi](\theta \land \psi) & \iff [-c\varphi]\theta \land [-c\varphi]\psi \\
[-c\varphi]B^\psi\theta & \iff B([-c\varphi]\psi \rightarrow [-c\varphi]\theta) \land B\neg\varphi([-c\varphi]\psi \rightarrow [-c\varphi]\theta) \\
& \land (B\neg\varphi[-c\varphi]\neg\psi \rightarrow B[-c\varphi]\psi[-c\varphi]\theta)
\end{align*}
$$

**3.3.4. Proof.** The soundness of the axioms of $\mathcal{L}_{\text{CDL}}$ is proved in [6]. All that remains is to show the soundness of the Reduction axioms. The proof of the first Reduction axioms is straightforward, we focus here only on the proof of the reduction axiom for conditional belief.

Suppose $s \models \mathcal{M} [-c\varphi]B^\psi\theta$. Then $s \models \mathcal{M} \neg\varphi B^\psi\theta$. So $\text{best}^{-\varphi} \parallel \psi \parallel \mathcal{M}^{-\varphi} \subseteq \parallel \theta \parallel \mathcal{M}^{-\varphi}$.

This means that $\text{best}^{-\varphi} \parallel [-c\varphi]\psi \parallel \mathcal{M} \subseteq \parallel [-c\varphi]\theta \parallel \mathcal{M}$.

We know that:

- $\text{best}^{-\varphi} \parallel [-c\varphi]\psi \parallel \mathcal{M} := \text{best} \parallel [-c\varphi]\psi \parallel \mathcal{M}$ if $\parallel [-c\varphi]\psi \parallel \mathcal{M} \cap \text{best} \parallel \neg\varphi \parallel \mathcal{M} = \emptyset$

or,
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Theorem. 3.3.6.

In the total plausibility model.

Reduction axioms: the language model and let the axioms of system to another formula in CONS logic.

Lemma. 3.3.5.

Completely eliminated using the Reduction Law for atomic formulas. Operator can be step by step “pushed through” all other operators and at the end, conservative contraction can be expressed by saying what is the case before the CONS.

Then:

- if best || ¬ϕ ||M ∈ || [ϕ]ψ ||M then best || [ϕ]ψ ||M ∈ || [ϕ]θ ||M and,
- best || ¬ϕ ||M ∩ || [ϕ]ψ ||M ∈ || [ϕ]θ ||M and

This means that:

- s = M (B¬ϕ[[ϕ]ψ → B[[ϕ]ψ[¬ϕ]θ]) and
- s = M B¬ϕ([¬ϕ]ψ → [¬ϕ]θ) and

□

Sketch of the proof for completeness From the axiom system for the logic CONS1 with the language \( L_{\text{Cons1}} \), we note that what is the case after a conservative contraction can be expressed by saying what is the case before the conservative contraction. Using the Reduction Laws, the conservative contraction operator can be step by step “pushed through” all other operators and at the end, completely eliminated using the Reduction Law for atomic formulas.

3.3.5. Lemma. Every formula of \( L_{\text{Cons1}} \) is provably equivalent in the above axiom system to another formula in \( L_{\text{CDL}} \).

The completeness of the axioms of \( L_{\text{Cons1}} \) follows from the completeness of the axioms of \( L_{\text{CDL}} \) that is proved in [6] and Lemma 3.3.5.

Let \( \varphi \) be a formula in \( L_{\text{CDL}} \) such that it is satisfiable in a total plausibility model and let \( \varphi' \) be a formula in \( L_{\text{Cons1}} \) equivalent to \( \varphi \). Then \( \varphi' \) is also satisfiable in the total plausibility model.

3.3.6. Theorem. A sound and complete proof system for the logic CONS2 with the language \( L_{\text{Cons2}} \) is given by the axioms and rules of \( L_{\text{KKD}} \) plus the following reduction axioms:

\[
\begin{align*}
[-\phi]p & \iff p \\
[-\phi]\neg \theta & \iff \neg[-\phi]\theta \\
[-\phi](\theta \land \psi) & \iff [-\phi]\theta \land [-\phi]\psi \\
[-\phi]K\psi & \iff K[-\phi]\psi \\
[-\phi]K_D\psi & \iff K_D[-\phi]\psi \land B^-[-\phi]\psi
\end{align*}
\]
Note that in the last reduction axiom, we use the following conditional operator $B\neg c\varphi[\neg c\varphi]\psi$ as an abbreviation as defined in Proposition 2.5.12.

3.3.7. PROOF. The soundness of the axioms of $\mathcal{L}_{KKD}$ is proved in [8]. This leaves us to focus on the Reduction axioms. The proof of the first Reduction axioms is straightforward, we focus here only on the proof of the reduction axiom for defeasible knowledge.

Suppose $s \models_M [\neg c\varphi]K_D\psi$. Then $s \models_{M^c\varphi} K_D\psi$. So for all $t$ such that $t \leq \neg c\varphi s$, $t \models_{M^c\varphi} \psi$. This means that for all $t$ such that $t \leq \neg c\varphi s$, $t \models_M [\neg c\varphi]\psi$.

We know that $t \leq \neg c\varphi s$ iff $t \in \text{best} \parallel \neg \varphi \parallel_M$ or $t \leq s$. Then:

- for all $t$ such that $t \in \text{best} \parallel \neg \varphi \parallel_M$, $t \models_M [\neg c\varphi]\psi$ and,
- for all $t$ such that $t \leq s$, $t \models_M [\neg c\varphi]\psi$.

This means that $s \models_M B\neg c\varphi[\neg c\varphi]\psi$ and $s \models_M K_D[\neg c\varphi]\psi$.

Sketch of the proof for completeness From the axiom system for the logic CONS$_2$ with the language $\mathcal{L}_{\text{Cons2}}$, we note that what is the case after a conservative contraction can be expressed by saying what is the case before the conservative contraction. Using the Reduction Laws, the conservative contraction operator can be step by step “pushed through” all other operators and at the end, completely eliminated using the Reduction Law for atomic formulas.

3.3.8. LEMMA. Every formula of $\mathcal{L}_{\text{Cons2}}$ is provably equivalent in the above proof system to another formula in $\mathcal{L}_{KKD}$.

The completeness of the axioms of $\mathcal{L}_{\text{Cons2}}$ follows from the completeness of the axioms of $\mathcal{L}_{KKD}$ that is proved in [8] and Lemma 3.3.8.

Let $\varphi$ be a formula in $\mathcal{L}_{KKD}$ such that it is satisfiable in a total plausibility model and let $\varphi'$ be a formula in $\mathcal{L}_{\text{Cons2}}$ equivalent to $\varphi$. Then $\varphi'$ is also satisfiable in the total plausibility model.

3.4 A new approach to moderate contraction

3.4.1 Language for the logic of moderate contraction

We define two languages for moderate contraction. The first language $\mathcal{L}_{\text{Mod1}}$ will be defined on top of the static base language $\mathcal{L}_{\text{CDL}}$ equipped with dynamic modalities for contraction $[\neg m\varphi]$. The second language $\mathcal{L}_{\text{Mod2}}$ will be defined on top of the static base language $\mathcal{L}_{KKD}$ equipped with dynamic modalities for contraction $[\neg m\varphi]$. 
3.4.2 Semantics for the logic of moderate contraction

The operation of moderate contraction has as an effect that a given total plausibility model $\mathcal{M} = (S, \leq, V)$ will be transformed into a model $\mathcal{M}^{\neg m \varphi} = (S, \leq^{m \varphi}, V)$. In this setting the current ordering relation $\leq$ of a given total plausibility model will be replaced by the following relation $\leq^{m \varphi}$ in the new model after the moderate contraction with $\varphi$. The intended meaning of this is that the best $\neg \varphi$-worlds become equi-plausible with the best worlds initially on top, then the rest of the $\neg \varphi$-worlds become better than the rest of the $\varphi$-worlds, and within these two zones the old ordering remains.

3.4.1. Definition. The initial total plausibility model $\mathcal{M} = (S, \leq, V)$ after a moderate contraction with $\varphi$ is transformed into the following model $\mathcal{M}^{\neg m \varphi} = (S, \leq^{m \varphi}, V)$ in which $t \leq^{m \varphi} s$ iff:
- $t \in \text{best} \neg \varphi |_{M}$ or,
- $s \in \varphi |_{M}$, $s \notin \text{best} \varphi |_{M}$ and $t \in \neg \varphi |_{M}$ or,
- $s \in \varphi |_{M}$, $t \in \varphi |_{M}$ and $t \leq s$ or,
- $s \in \neg \varphi |_{M}$, $t \in \neg \varphi |_{M}$ and $t \leq s$.

3.4.2. Definition. We can define $\text{best}^{\neg m \varphi} P$:
- $\text{best}^{\neg m \varphi} P := \text{best} P$ if $P \cap \neg \varphi |_{M} = \emptyset$ or,
- $\text{best}^{\neg m \varphi} P := \text{best}(\neg \varphi \cap P) \cup (P \cap \text{best} S)$ if $P \cap \neg \varphi |_{M} \neq \emptyset$.

The semantic clause for the dynamic operator $[-m \varphi]$ is:

$\mathcal{M}, s \models [-m \varphi] \psi$ iff $\mathcal{M}^{m \varphi}, s \models \psi$

Example In Figure 3.12 we present a sphere model composed of 6 states $s, t, u, v, w, x$ such that $\varphi$ is true in $s, t, v, x$ and $\neg \varphi$ is true in $u, w$.

From this sphere system we can generate the corresponding total plausibility model given in Figure 3.13.

Next we present the total plausibility model resulting from the moderate contraction with $\varphi$ in Figure 3.14. In this model, the state $u$ is equi-plausible with the state $s$, both being on top while $w$ is more plausible than $t, v, x$. Thus the agent does not believe $\varphi$ anymore and does not believe $\neg \varphi$ either. However she has a propensity to consider $\neg \varphi$ more plausible than $\varphi$. If she further revises/contracts her beliefs, she will finally come to believe $\neg \varphi$ more easily than $\varphi$. 

3.4.3 Axiom system for the logic of moderate contraction

3.4.3. Theorem. A sound and complete proof system for the logic \( \text{MOD}_1 \) with the language \( \mathcal{L}_{\text{Mod1}} \) is given by the axioms and rules of \( \mathcal{L}_{\text{CDL}} \) plus the following reduction axioms:

\[
\begin{align*}
[-m\varphi] p & \iff p \\
[-m\varphi] \neg \theta & \iff \neg [-m\varphi] \theta \\
[-m\varphi] (\theta \land \psi) & \iff [-m\varphi] \theta \land [-m\varphi] \psi \\
[-m\varphi] B^\psi \theta & \iff B([-m\varphi] \psi \rightarrow [-m\varphi] \theta) \land B^{\neg \varphi \land [-m\varphi] \psi} [-m\varphi] \theta \\
& \land (K^{\neg \varphi} [-m\varphi] \neg \psi \rightarrow B^{[-m\varphi] \psi} [-m\varphi] \theta)
\end{align*}
\]

3.4.4. Proof. The soundness of the axioms of \( \mathcal{L}_{\text{CDL}} \) is proved in [6]. All that remains is to show the soundness of the Reduction axioms. The proof of the
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Figure 3.14: Moderate contraction in the initial total plausibility model of Figure 3.13

first Reduction axioms is straightforward, we focus here only on the proof of the reduction axiom for conditional belief.

Suppose \( s \models_{\mathcal{M}} [-m \varphi]B^\psi \theta \). Then \( s \models_{\mathcal{M}^{m \varphi}} B^\psi \theta \). So \( best^{-m \varphi} \models_{\mathcal{M}^{m \varphi}} B^\psi \theta \). This means that \( best^{-m \varphi} \models_{\mathcal{M}^{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} \psi \models_{\mathcal{M}_{m \varphi}} \). This means that \( best^{-m \varphi} \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \).

We know that:

\[
\begin{align*}
- & best^{-m \varphi} \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \quad \text{if} \quad \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} = \emptyset \quad \text{or},
- & best^{-m \varphi} \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} = \text{best} (-m \varphi \models_{\mathcal{M}} \subseteq [-m \varphi] \models_{\mathcal{M}} \cap \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \cap best \{S\}) \quad \text{if} \quad \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \neq \emptyset.
\end{align*}
\]

Then:

\[
\begin{align*}
- & \text{if} \quad \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \text{and},
- & \text{if} \quad \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \text{and}
- & \text{if} \quad \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \subseteq \models_{\mathcal{M}_{m \varphi}} [-m \varphi] \models_{\mathcal{M}} \text{otherwise}.
\end{align*}
\]

This means that:

\[
\begin{align*}
& s \models_{\mathcal{M}} K^{-\varphi} [-m \varphi] \models_{\mathcal{M}} B^{-m \varphi} \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \text{ and}
& s \models_{\mathcal{M}} B^{-\varphi} [-m \varphi] \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \text{ and}
& s \models_{\mathcal{M}} B (-m \varphi) \models_{\mathcal{M}} [-m \varphi] \models_{\mathcal{M}} \text{.}
\end{align*}
\]

\[\square\]

Sketch of the proof for completeness  From the axiom system for the logic MOD1 with the language \( \mathcal{L}_{\text{Mod1}} \), we note that what is the case after a conservative contraction can be expressed by saying what is the case before the conservative contraction. Using the Reduction Laws, the moderate contraction operator can be step by step “pushed through” all other operators and at the end, completely eliminated using the Reduction Law for atomic formulas.

3.4.5. Lemma. Every formula of \( \mathcal{L}_{\text{Mod1}} \) is provably equivalent in the above proof system to another formula in \( \mathcal{L}_{\text{CDL}} \).
Chapter 3. Belief contraction in total plausibility models

The completeness of the axioms of $L_{Mod1}$ follows from the completeness of the axioms of $L_{CDL}$ that is proved in [6] and Lemma 3.4.5.

Let $\varphi$ be a formula in $L_{CDL}$ such that it is satisfiable in a total plausibility model and let $\varphi'$ be a formula in $L_{Mod1}$ equivalent to $\varphi$. Then $\varphi'$ is also satisfiable in the total plausibility model.

We cannot provide an axiom system for the logic $MOD_2$ with the language $L_{Mod2}$. Indeed we do not have a reduction axiom for defeasible knowledge. This reduction axiom would require a new operator or equivalently a new language. Let us consider what would be such an axiom:

$$[-m\varphi]K_D \psi \iff B^{\neg\varphi}[-m\varphi]\psi \land (\varphi \rightarrow K_D(\varphi \rightarrow [-m\varphi]\psi))$$

$$\land (\neg\varphi \rightarrow K_D(\neg\varphi \rightarrow [-m\varphi]\psi)) \land (\varphi \land \text{best}\varphi \rightarrow K^{\neg\varphi}[-m\varphi]\psi))$$

where the semantics of $\text{best}\varphi$ is given by $s \models \text{best}\varphi$ iff $s \in \text{best}|| \varphi ||_M$.

The problem here is the operator “best” which cannot be expressed in our language.

However, we can provide the following reduction axioms:

$$[-m\varphi]p \iff p$$

$$[-m\varphi]\neg \theta \iff \neg[-m\varphi]\theta$$

$$[-m\varphi](\theta \land \psi) \iff [-m\varphi]\theta \land [-m\varphi]\psi$$

$$[-m\varphi]K \psi \iff K[-m\varphi]\psi$$

Conclusion

Belief revision has been widely explored in $DEL$ contrary to belief contraction. However, belief contraction is also a very interesting notion and is worth being studied in the setting of $DEL$. A belief contraction operation really comes with its own reduction axioms. We explored in this chapter three different notions of contraction: severe withdrawal, conservative contraction and moderate contraction. We clarified the mechanism of each of these operations. We also explained the limits of the mechanism of severe withdrawal while stressing the advantages of conservative and moderate contraction.

In the next chapter, we continue to develop Soft $DEL$ by designing a formal setting allowing to make explicit the connections between plausibility models, evidence models and uniting some existing different settings.