Playing with knowledge and belief
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Citation for published version (APA):

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Chapter 7

Bonanno’s belief revision logic in a dialogical setting

Aim: In this chapter our aim is to provide an argumentative study of belief revision logic. In particular we focus on the branching-time belief revision logic of Bonanno $L_{PLS*}$ as introduced in Definition 2.8.15. To fulfil our purpose we provide a dialogical approach to this logic $L_{PLS*}$.

Summary: In this chapter we motivate our choice to provide a dialogical approach to $L_{PLS*}$ and we precisely define this dialogical setting. We provide our dialogical approach to $L_{PLS*}$ as well as its soundness and completeness proof, establishing a formal relation with its model-theoretic approach. The main points are:

- we motivate the investigation of $L_{PLS*}$ in an argumentative setting. We provide the main characteristics of the dialogical setting we use as well as its historical background.

- we precisely define some important dialogical notions we use in this chapter.

- we provide a dialogical approach to the logic $L_{PLS*}$ providing the language and the rules. We apply our dialogical system to specific examples to illustrate its mechanisms.

- we prove that our dialogical approach to Bonanno’s logic of belief revision is sound and complete with respect to $L_{PLS*}$ showing that there exists a winning strategy for the Proponent in a dialogue with a thesis $\Delta$ if and only if $\Delta$ is a valid formula in $L_{PLS*}$.
Background: The dialogical approach to logic is a two-person game in which one party defends a proposition while the other party challenges it. The game is defined by a set of rules. Some rules stipulate how the logical constants can/have to be challenged and defended and some other rules define the process of the game itself (which player starts, can move, wins...). Two notions are fundamental in the dialogical approach to logic namely, the notion of choice which leads to the notion of strategy. Indeed players make choices among the available moves allowed by the rules. If a player wins, whatever the choices of the other player, he has a winning strategy in the corresponding dialogical game.

We have to underline that dialogical games are not played on a given model. There is no model in the dialogical approach to logic. The concept of validity has its counterpart in the concept of winning strategy. If the player defending the proposition has a winning strategy then this proposition is considered to be valid.

Paul Lorenzen and Kuno Lorenz were first to introduce the concept of formal dialogues.\footnote{In the spirit of our work on game semantics in this thesis in Chapter 5, let us mention that the dialogical logic founded by Lorenzen and Lorenz and the game theoretical semantics of Hintikka \cite{42}, have been compared with each other. For this comparison we refer to the work of \cite{70} and for an analysis of the importance of the differences between these game styles in the context of the philosophical (anti)-realism debate, we refer to \cite{58}.} This first dialogical approach was concerned with intuitionistic and classical logic \cite{55}. Later on, the dialogical framework has been developed and applied to non-classical logics by one of their students Shahid Rahman \cite{66}. In particular Shahid Rahman and Helge Rückert developed the first modal dialogues \cite{69,68,45}. They introduce new sets of rules in relation with modal operators. Very recently, a dialogical approach to Dynamic Epistemic Logic has been developed \cite{56} by Sébastien Magnier. He offers an argumentative study of Dynamic Epistemic Logic focusing mainly on Public Announcement Logic. New rules are given not only for epistemic operators but also for public announcement operators. We believe it is now time to extend the dialogical approach to belief revision logic in order to provide the first argumentative study of belief revision logic.\footnote{Note that \cite{67} provides a dialogical approach to the first logic of belief revision introduced by Bonanno in \cite{17} (which is actually a belief expansion logic rather than a belief revision logic). We would like to emphasise that this first version of Bonanno’s belief revision is very different from $L_{PLS}^\ast$.} In this thesis, we choose to study the branching-time belief revision logic of Bonanno $L_{PLS}^\ast$ in a dialogical setting (see Chapter 2).
7.1 General background on the dialogical approach to logic

After motivating our interest in the argumentative study of logic, we precisely define the dialogical framework we use in concrete terms.

7.1.1 Motivation

We first have to distinguish between two different notions of dynamics namely, the dynamics of an argumentative practice and the dynamics of semantics (through dynamic modal operators). We are taking up an idea put forward by Magnier [56] where he establishes two distinct levels of dynamical changes:

- A logic is called “dynamic” because of its object language. Some operations introduced in the logical language force its dynamic nature.

- A logic becomes “dynamic” because it is implemented in an argumentative context.

So Magnier distinguishes between a dynamic logical language and a dynamic practice of logic. Magnier names the first type of dynamics “internal dynamics” and the second one “external dynamics”. The argumentative practice of logic is dynamic in an external sense because the argumentative process is dynamic in itself but not necessarily the language of the logic investigated through this argumentative practice. Thus the argumentative practice is grounded in an irreducible form of dynamics. Contrary to some dynamic modal languages such as the language of dynamic epistemic logics that can be rewritten into static epistemic languages by means of appropriate reduction axioms (see the reduction axioms in [29]). Indeed the static language pre-encodes the dynamics.

We are interested in investigating the relation between beliefs and information over time from an external dynamic perspective. We claim that this external dynamic perspective will shed some new light on this relation. In particular we will interpret this relation through the notion of choice. In an argumentative process, players challenge each other’s claims. Thus players have to make choices. Implementing a belief revision logic inside an argumentative framework will produce choices about belief and information. Moreover we see now that the internal dynamics of the information operator will become more apparent in the light of this external dynamic point of view. The goal is to explore what we can learn about Bonanno’s logic of belief revision $L_{PLS^*}$ from an argumentative framework. But we also carefully examine what the interpretation of the information operator through the argumentative notion of choice provides to the argumentative framework itself. Indeed the non normality of Bonanno’s information operator will bring some new interesting developments in this framework.
7.1.2 Dialogues

There exist different types of dialogues. Our aim is not to present these dialogues in details, we only mention that they have different purposes and thus different sets of rules. Which type of dialogue would be suitable to investigate $L_{PLS^*}$ through an argumentative practice? We have to define the main characteristics of the type of dialogues we are looking for to answer this question. First we list what kind of argumentative practice we do not want. We do not want:

- exchanges of unrelated statements,
- exchanges linked to some particular background,
- arguments to be about players (challenging players as personal attacks),
- inequality between players as different inference rules,
- infinite argumentative processes (without a winning player),
- possible or plausible conclusions (no certain conclusions).

Since we want to reconstruct $L_{PLS^*}$ into an argumentative framework, we are only interested in formal dialogues. In formal dialogues, the players are objective and impartial. They can only challenge the arguments of the other player (and not the other player himself), and they have to cooperate even if they are engaged in a competitive argumentative process because they comply with the rules. Finally we look for dialogues regulated by symmetric rules, providing them with an objective aspect. Since we want to establish a counterpart of the notion of validity of a formula, namely the notion of winning strategy, we require finite dialogues providing at least one and only one winning player. This notion of winning strategy involves the notion of competitive players. The dialogical approach to logic meets all these criteria, that’s why we choose to provide a dialogical approach to $L_{PLS^*}$.

7.1.3 The dialogical approach to logic

The dialogical approach to logic was first introduced by Lorenzen in the 1950’s and then developed by Lorenz for classical and intuitionistic logic. Rahman, one of Lorenz’s students, has further developed the dialogical approach to logic to allow for the development and the combination of different logics in this framework (free logic, normal modal logic, non normal modal logic and so on) [66, 44]. The aim was to propose a semantics based on argumentation games as a new alternative to model theory and proof theory: the dialogical approach to logic is neither model theory nor proof theory. The main concept of this approach is “meaning as use” namely use in an argumentative process.

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3 For a complete taxonomy see [82].
4 The most important early papers on the dialogical approach to logic are collected in [55].
7.1. General background on the dialogical approach to logic

Dialogical game In a dialogical game two players confront each other. The Proponent proposes a thesis that he will defend against the challenges of the Opponent who aims to find a counter-argument for it. So the game starts with the Proponent (P) stating a formula from some given language \( \mathcal{L} \). Then the Opponent (O) challenges the formula and the Proponent defends: they interact by alternately choosing moves according to some rules. The notion of choice plays an essential role in the dialogical approach to logic. We will further develop this point when we will present the dialogical rules. Thus some rules are needed in order to define how players can challenge/defend logical constants but also to define when players can make a move, what kind of move they are allowed to make, when the game ends, which player wins and so on.

Dialogical rules The dialogue is a game which obeys two kinds of rules: particle rules and structural rules. Particle rules constitute the local semantics of a logic: it determines the meaning of each logical constant in terms of use (how players can/have to use them) in an argumentative process. Thus particle rules define the way in which connectives are played. These rules are symmetric that is, they are necessarily the same for the Opponent as well as the Proponent. As a consequence, the logical meaning of a given constant is independent of the players. That’s why we use \( X \) and \( Y \) as variables ranging on \( \{O, P\} \), always assuming that \( X \neq Y \) in their definition. This symmetry of the particle rules provides an objective aspect to dialogues. Structural rules determine the global semantics of a logic: they define the way in which the dialogue proceeds.

Dialogical language A dialogical language for propositional logic is obtained from the standard propositional language by the addition of one metalogical symbol “?” standing for “challenge”, and two labels \( O \) and \( P \), standing for the players (Opponent, Proponent) of the dialogue.

Dialogical approach to modal logic Modal dialogues are developed by Rahman and Rückert [69, 68, 45]. While dialogical propositional logic investigates the meaning in terms of use in an argumentative process, dialogical modal logic contextualises this meaning in terms of use. Thus the meaning of a logical constant depends on its contextual use. Technically, dialogical modal logic needs the introduction of contextual points allowing to specify the contextual nature of the moves, i.e. the context in which the moves are made. They are the counterpart of possible worlds in the model-theoretic approach.

7.1.1. Definition. A contextual point is a positive integer \( i \) indexing a statement in a dialogue.

Players can choose new contextual points when challenging a modal operator according to the rules. Indeed a particle rule must be added for each modal operator to define the way in which it is played and some structural rules must be added.
to define what contextual points can be chosen by the players to challenge the corresponding modal operator. In that case, structural rules define the conditions of particular use of particle rules for modal operators. They can be interpreted as the counterpart of conditions imposed on frames in the model-theoretic approach. A contextual point is new if and only if it is chosen by a player in some move and there is no previous move in the same game where the contextual point is chosen. The play (challenge/defence) on a modal operator creates a chain of contextual points.

7.1.2. Definition. If a player successively challenges modal operators from the contextual point $i$ choosing successively the contextual points $j$, $k$, ..., $n$, then $i.j.k....n$ is a chain of contextual points.

A chain of contextual points reflects the choices the players made. Thus $i.j.k$ means that the contextual point $k$ has been chosen from the contextual point $j$ to challenge a modal operator such that this contextual point $j$ has been itself chosen to challenge a modal operator from the contextual point $i$.

7.2 Original dialogical approach to Bonanno’s logic for belief revision

In this section we provide a dialogical approach to $L_{PLS}$ providing some important definitions and the corresponding language, particle and structural rules, based on [34] and [33]. Next, we illustrate our dialogical system for $L_{PLS}$ through some concrete examples. We name this dialogical approach **Dialogical Temporal Doxastic Logic (DTDL)**.

7.2.1 **DTDL Framework**

We first define the language of DTDL.

7.2.1. Definition. The language of $DTDL$ $\mathcal{L}_{DTDL}$ is obtained from the language of $L_{PLS}$ by the addition of:

- the symbols $O$ and $P$,
- the symbol for challenge “?”,
- two new symbols “!” and “?∗” respectively for request and confirmation.

The new symbols are introduced in relation to the non normality of the information operator.
Since the language of DTDL is a multimodal language, we need two different types of contextual points. Indeed we can distinguish between two types of modal operators: temporal operators (\(\bigcirc^{-1}\) and \(\bigcirc\)) and non temporal operators (\(B, I, A\)). Then we use the contextual points \(i\) and \(t\) such that moves are made in a context \((i, t)\). In that case, contextual points \(i\) are the counterpart of possible worlds \(s\) while \(t\) is the counterpart of instants \(t\) in the model-theoretic approach.

7.2.2. Definition. A move is a tuple \((X - i, t : e)\) where:

- \(X \in \{O, P\}\),
- \(i\) and \(t\) are contextual points that is, positive integers or sequences of positive integers such that \((i, t)\) is a context,
- \(e\) is a statement of the language of DTDL

We now have to draw a sharp distinction between some dialogical terms.

7.2.3. Definition. We define the notions of dialogue, play, close play and terminal play:

- a dialogical game or dialogue \(D_\Delta\) is the set of all the possible plays for a formula \(\Delta\),
- a play \(d_\Delta\) is a sequence of moves allowed by the rules. This sequence starts with a move \((P - 0, 0 : \Delta)\),
- a play \(d_\Delta\) is close if and only if it contains two moves such that \((O - i, z : p)\) and \((P - i, t : p)^5\); \((O - i, t : I_j)\) and \((P - i, t : I_j^*); (O - i, t : ?B_j)\) and \((P - i, t : ?I_j^*)\),
- a play \(d_\Delta\) is terminal if there are no more moves allowed by the rules.

7.2.2  Particle rules

Particle rules define the way in which logical constants are played that is, how they should be challenged and defended. These rules are strictly the same for the Opponent and the Proponent. In other words, what matters is how a logical constant can be used regardless of the player who uses it. Thus the meaning of logical constants is given independently of the role of players (Opponent/Proponent).

\(^5\)It is possible that \(z \notin t\).
Vocabulary  First we have to distinguish the dialogical terms *challenge*, *request* and *confirmation*. A player challenges a statement of the other player. A request is always about a contextual choice. A player requests the other player to confirm that he could choose a particular contextual point to challenge a modal operator in a given context. A player confirms that he could choose the required contextual point to challenge a modal operator in a given context.

Reading particle rules  A particle rules involves three steps:

- X utters a formula,
- Y challenges this formula,
- X defends the formula.

Particle rules for standard connectives  We first provide particle rules for the standard connectives in Figure 7.1.

<table>
<thead>
<tr>
<th>Standard connectives</th>
<th>X Utterance</th>
<th>Y Challenge</th>
<th>X Defence</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬, there is no possible defence</td>
<td>i,t: ¬φ</td>
<td>i,t: φ</td>
<td>⊙</td>
</tr>
<tr>
<td>∧, the challenger chooses a conjunct</td>
<td>i,t: φ ∧ ψ</td>
<td>i,t: ?∧₁, or i,t: ?∧₂</td>
<td>i,t: φ respectively i,t: ψ</td>
</tr>
<tr>
<td>∨, the defender chooses a disjunct</td>
<td>i,t: φ ∨ ψ</td>
<td>i,t: ?∨</td>
<td>i,t: φ or \ i,t: ψ</td>
</tr>
</tbody>
</table>

Figure 7.1: Particle rules for standard connectives

When X utters the negation of a formula, Y challenges by uttering the formula. There is no corresponding defence – denoted in a dialogue by the symbol ⊙. When X utters a conjunction, Y chooses the conjunct X has to defend; while when X utters a disjunction, X chooses the disjunct he wants to defend.

Particle rules for modal operators  Now we provide particle rules for modal operators in Figure 7.2. The contextual points play an essential role here. Indeed they become paramount when modal operators come into the language. For the sake of clarity, we always explicitly state the modal operator challenged in the challenge itself. Thus a challenge of a belief operator looks like “?B_j” and a challenge of an information operator looks like “?I_j”. The same applies to request and confirmation.

But first we have to clarify a fundamental distinction between the players of the dialogue and an agent. The players of the dialogue are the Proponent and
7.2. Original dialogical approach to Bonanno’s logic for belief revision

the Opponent. These players can discuss about the beliefs of an agent and/or the information received by an agent but they are not in any case the agent in question. The Proponent and the Opponent discuss about a third person, not about their own beliefs/information.

<table>
<thead>
<tr>
<th>Modal operators</th>
<th>X Utterance</th>
<th>Y Challenge</th>
<th>X Defence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Box^{-1}$, the challenger chooses a contextual point $u$</td>
<td>$i,t: \Box^{-1}\varphi$</td>
<td>$i,t: ?\Box^{-1}$</td>
<td>$i,t,u: \varphi$</td>
</tr>
<tr>
<td>$\Box$, the challenger chooses a contextual point $u$</td>
<td>$i,t: \Box\varphi$</td>
<td>$i,t: ?\Box$</td>
<td>$i,t,u: \varphi$</td>
</tr>
<tr>
<td>$B$, the challenger chooses a contextual point $j$</td>
<td>$i,t: B\varphi$</td>
<td>$i,t: ?B$</td>
<td>$i,j,t: \varphi$</td>
</tr>
<tr>
<td>$A$, the challenger chooses a contextual point $j$</td>
<td>$i,t: A\varphi$</td>
<td>$i,t: ?A$</td>
<td>$i,j,t: \varphi$</td>
</tr>
<tr>
<td>$I$, the challenger has the choice between two challenges</td>
<td>$i,t: I\varphi$</td>
<td>$i,t: ?I$</td>
<td>$i,j,t: \varphi$</td>
</tr>
</tbody>
</table>

Figure 7.2: Particle rules for modal operators

When $X$ utters a formula of the form $\Box^{-1}\varphi$ in $(i,t)$, he must be able to defend $\varphi$ in any contextual point $u$ chosen by $Y$ to challenge the $\Box^{-1}$ operator: in that case the context $(i,t,u)$ is called an immediate past context of the context $(i,t)$. Indeed if a player $X$ states that at the previous instant it was the case that $\varphi$, he is committed to defend $\varphi$ in all immediate past contexts.

When $X$ utters a formula of the form $\Box\varphi$ in $(i,t)$, he must be able to defend $\varphi$ in any contextual point $u$ chosen by $Y$ to challenge the $\Box$ operator: in that case the context $(i,t,u)$ is called an immediate future context of the context $(i,t)$. Indeed if a player $X$ states that at every next instant it will be the case that $\varphi$, he is committed to defend $\varphi$ in all immediate future contexts.

When $X$ utters a formula of the form $B\varphi$ in $(i,t)$, he must be able to defend $\varphi$ in any contextual point $j$ chosen by $Y$ to challenge the $B$ operator. Indeed if a player $X$ states that an agent believes a proposition $\varphi$, he is committed to defend $\varphi$ in all contexts that this agent conceives.

When $X$ utters a formula of the form $A\varphi$ in $(i,t)$, he must be able to defend $\varphi$ in any contextual point $j$ chosen by $Y$ to challenge the $A$ operator. Indeed if a player $X$ states that it is always the case that $\varphi$, he is committed to defend $\varphi$ in all contexts.

When $X$ utters a formula of the form $I\varphi$ in $(i,t)$, $Y$ has the choice between two different challenges. He can choose the standard challenge: he challenges choosing a contextual point $j$, and then $X$ must be able to defend $\varphi$ in any
contextual point \( j \) chosen by \( Y \). Or he can choose the non-standard challenge: he chooses a contextual point \( j \) and requests \( X \) to confirm that this contextual point \( j \) could be chosen to challenge an information operator at \((i, t)\), and then \( X \) must be able to confirm that the contextual point \( j \) chosen by \( Y \) can be chosen at \((i, t)\) to challenge an information operator. Indeed if a player \( X \) states that an agent is informed about a proposition \( \varphi \), he is committed to defend \( \varphi \) in all contexts of which the agent is informed and he is committed to defend that all contexts where \( \varphi \) holds are contexts of which the agent is informed.

The information operator is a non normal operator, that’s why its particle rule is far from being standard. The non standard challenge \( !I_j \) at \((i, t)\) can be read as “show me that the contextual point \( j \) can be chosen to challenge the \( I \) operator at \((i, t)\)”. Then the corresponding defence \( ?I_j^* \) at \((i, t)\) is the confirmation that indeed this contextual point \( j \) can be chosen to challenge the \( I \) operator at \((i, t)\): “I confirm that the contextual point \( j \) can be chosen to challenge the \( I \) operator at \((i, t)\)”. This is new in the dialogical approach to logic: players no longer deal with formulas but with choices. The non normality of the \( I \) operator introduces directly the notion of choice inside the dialogue. Players discuss about their own choices, more precisely about the choices they can make. We will further develop this point in the examples (see Section 7.2.4).

**Particle rules of** \( DTDL \) The particle rules of \( DTDL \) consist of the particle rules for standard connectives and for modal operators. These rules determine all the possible uses of the logical constants of the language of \( DTDL \). Now we have to define the structural rules of \( DTDL \) to determine the conditions under which some particle rules of \( DTDL \) can be used and how the game is played.

### 7.2.3 Structural rules

Structural rules regulate the process of the dialogue. We first introduce the structural rules defining how the play starts and ends, how the players can play, as well as the winning rule\(^6\).

- (SR-0) **(Starting rule)**: Any play \( d_\Delta \) of a dialogue \( D_\Delta \) starts with \( P \) uttering the thesis in an initial context \((i, t)\). The moves of a play are numbered such that the thesis has number 0. Then \( O \) and \( P \) respectively choose a natural number \( n \) and \( m \) allowing a number of repetitions (called repetition rank). \( O \) and \( P \) can repeat the same move (challenge or defence) respectively \( n \)

\(^6\)We provide some schemas for the reader who is not familiar with structural rules in Appendix A. We strongly recommend to first read the rules with the help of the schemas and then to read the explanations of the rules.
and \(m\) times\(^7\).

\(\diamond\) (SR-1) (Game-playing rule): Moves are made alternately by \(O\) and \(P\) according to the other rules. In any move each player may challenge any complex formula uttered by the other player, or he may defend himself against any challenge, including those which have already been defended according to his repetition rank.

\(\diamond\) (SR-2) (Formal rule for atomic formulas): \(P\) is allowed to utter an atomic formula at \((i,t)\) only if \(O\) has first uttered it at \((i,z)\).

\(\diamond\) (SR-3) (Winning rule): A player wins a play if and only if the other player cannot make a move.

Now we introduce the structural rules defining the conditions under which particle rules for modal operators can/have to be used.

\(\diamond\) (SR-5) (Formal rule for contextual points \(t\)): To challenge a move as \(\langle P - i, t : \Box \varphi \rangle\), \(O\) can choose any contextual point \(u\) whenever other rules allow him to do so. To challenge a move as \(\langle P - i, t : \Diamond \neg \varphi \rangle\), \(O\) can choose any contextual point \(u\) provided that he has not chosen a contextual point \(v\) before to challenge a move as \(\langle P - i, t : \Diamond \neg \varphi \rangle\).

To challenge a move as \(\langle O - i, t : \Box \varphi \rangle\), \(P\) can only choose a contextual point \(u\) already chosen by \(O\) to challenge a move as \(\langle P - i, t : \Box \varphi \rangle\). To challenge a move as \(\langle O - i, t : \Diamond \neg \varphi \rangle\), \(P\) can only choose a contextual point \(u\) already chosen by \(O\) to challenge a move as \(\langle P - i, t : \Diamond \neg \varphi \rangle\).

However \(P\) can choose the initial contextual point \(t\) to challenge a move as \(\langle O - i, t.u : \Box \varphi \rangle\) or \(\langle O - i, t.u : \Diamond \neg \varphi \rangle\) under some conditions:

\(\diamond\) (SR-5.1) \(P\) can choose the initial contextual point \(t\) to challenge a move as \(\langle O - i, t.u : \Box \varphi \rangle\) if \(O\) has chosen the contextual point \(u\) to challenge a move as \(\langle P - i, t : \Diamond \neg \varphi \rangle\).

\(\diamond\) (SR-5.2) \(P\) can choose the initial contextual point \(t\) to challenge a move as \(\langle O - i, t.u : \Diamond \neg \varphi \rangle\) if \(O\) has chosen the contextual point \(u\) to challenge a move as \(\langle P - i, t : \Box \varphi \rangle\).

\(^7\)See N. Clerbout for more details [25].
◇ (SR-6) (Formal rule for contextual points j): To challenge a move as \( \langle O - i, t : B\varphi \rangle \), \( P \) can only choose a contextual point \( j \) already chosen by \( O \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \); if \( O \) did not choose any contextual point to challenge a move as \( \langle P - i, t : B\varphi \rangle \), \( P \) can choose a new contextual point \( j \).

To challenge a move as \( \langle O - i, t : I\varphi \rangle \), \( P \) can only choose a contextual point \( j \) already chosen by \( O \) to challenge a move as \( \langle P - i, t : I\varphi \rangle \) or \( \langle P - i, t : B\varphi \rangle \).

To challenge a move as \( \langle O - i, t : A\varphi \rangle \), \( P \) can only choose a contextual point \( j \) already chosen by \( O \) to challenge a move as \( \langle P - i, z : I\varphi \rangle \) or \( \langle P - i, z : B\varphi \rangle \) or \( \langle P - i, z : A\varphi \rangle \); or he can choose the contextual point \( i \).

However \( P \) can choose more contextual points \( j \) to challenge a move as \( \langle O - i, t : B\varphi \rangle \) under some conditions: let three contextual points \( t, u \) and \( v \) be such that \( u \) and \( v \) have been chosen by \( O \) to challenge a move as \( \langle P - i, t : O\varphi \rangle \) and consider three contextual points \( i, j \) and \( k \):

◇ (SR-6.1) \( P \) can choose a contextual point \( j \) to challenge a move as \( \langle O - i, t : B\varphi \rangle \) if \( O \) has chosen the contextual point \( k \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \) and to challenge a move as \( \langle P - i, t : I\varphi \rangle \) or if he has stated \( \langle O - i, t : ?I^*_k \rangle \) and if \( O \) has chosen the contextual point \( j \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \).

◇ (SR-6.2) \( P \) can choose a contextual point \( j \) to challenge a move as \( \langle O - i, t : B\varphi \rangle \) if \( O \) has chosen the contextual point \( j \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \) and to challenge a move as \( \langle P - i, t : I\varphi \rangle \) or if he has stated \( \langle O - i, t : ?I^*_j \rangle \).

◇ (SR-6.3) \( P \) can choose a contextual point \( j \) to challenge a move as \( \langle O - i, t : B\varphi \rangle \) if \( O \) has chosen the contextual point \( j \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \) and if every contextual points \( j \) chosen to challenge a move as \( \langle X - i, t : I\varphi \rangle \) can also be chosen to challenge a move as \( \langle X - i, t : I\varphi \rangle \).

◇ (SR-6.4) \( P \) can choose a contextual point \( j \) to challenge a move as \( \langle O - i, t : B\varphi \rangle \) if \( O \) has chosen the contextual point \( j \) to challenge a move as \( \langle P - i, t : B\varphi \rangle \) and to challenge a move as \( \langle P - i, t : I\varphi \rangle \) or if he has stated \( \langle O - i, t : ?I^*_j \rangle \).

◇ (SR-7) (Request rule): \( Y \) can choose a contextual point \( j \) and request \( X \) to confirm that this contextual point \( j \) could be chosen to challenge a move as \( \langle X - i, t : I\varphi \rangle \), only if \( \langle O - i, j, z : \varphi \rangle \in d_\Delta \).

Now we have to explain our structural rules, more precisely our formal rules: why do they make sense? What does this mean for players? What does this mean for the notion of beliefs and information? First note that players can discuss about facts, time (immediate past/future), the beliefs of an agent or the information received by an agent.
Formal rule for atomic formulas  The main point of this rule is that the Proponent can only state atomic formulas in a particular context if the Opponent has already done it in some context. Indeed we saw that the Opponent tries to build a counter-argument to the thesis of the Proponent. Then he is the only one who can introduce (uttering first) atomic formulas in a context in a dialogue. The Proponent can only reuse them. But the important thing here is the contextual point \( i \) in which the Opponent states the atomic formulas, not the contextual point \( t \). When the Opponent utters an atomic formula, in fact he states a proposition describing a fact that holds in some particular context. And we only consider here facts that do not change during the play that is, the facts described by the propositions stated by the players do not change during the discussion. For example if the fact “Earth revolves around the sun” holds in the context \((i, t)\), it will also hold in the context \((i, z)\) for any \( t \) and \( z \). That’s why it is sufficient that Opponent utters atomic formulas at \((i, z)\) to be reused by Proponent at \((i, t)\). This is the counterpart of the non-changing worlds (facts describing the world do not change over time) in the model-theoretic approach.

Formal rule for contextual points \( t \) On one hand, the main point of this rule is that Opponent can choose several contextual points \( u \) to challenge a move as \( \langle P - i, t : \square \varphi \rangle \) according to the other rules (his repetition rank), but only one contextual point \( v \) to challenge a move as \( \langle P - i, t : \square^{-1} \varphi \rangle \) operator. When players talk about (immediate) past, they only deal with one immediate past context from the actual context. Indeed they discuss about a fixed and determined (immediate) past. There are no several possibilities of (immediate) past. For example, if “at the previous instant it was the case that some peanuts lie all over the table” holds in the context \((i, t)\), there is exactly one context \((i, t, u)\) where “it is the case that some peanuts lie all over the table” holds. On the contrary, when players discuss about the (immediate) future, they talk about an undetermined future and so several possibilities of an (immediate) future. For example, if “at every next instant it will be the case that some peanuts will be cleaned up” holds in the context \((i, t)\), there are several possible (immediate) contexts \((i, t, u)\) where “some peanuts are actually cleaned up” holds. This is the counterpart of the branching-time frame in the model-theoretic approach.

On the other hand, the Proponent cannot introduce contextual points \( u \) for the same reason he cannot introduce atomic formulas: only the Opponent can do this because the Opponent tries to build a counter-argument to the thesis of the Proponent. Note that when players discuss about the (immediate) past and future, they have to be consistent with the notion of time. Thus they have to be consistent with respect to the choices of contextual points \( u \) they make. When a player discusses about the (immediate) past in the context \((i, t)\) and then explicitly considers a(n) (immediate) past context \((i, t, u)\), then \((i, t)\) is a(n) (immediate)
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future context of \((i,t,u)\). Conversely, if he discusses about the (immediate) future in the context \((i,t)\) and explicitly considers a(n) (immediate) future context \((i,t,u)\), then \((i,t)\) is a(n) (immediate) past context of \((i,t,u)\). That’s why even if the initial contextual point \(t\) is not technically chosen by the Opponent, if the Opponent challenges a move as \(\langle P - i, t : \Diamond \varphi \rangle\) or \(\langle P - i, t : \Diamond^{-1} \varphi \rangle\) with the contextual point \(u\), then the Proponent can choose \(t\) to challenge respectively a move as \(\langle O - i, t.u : \Diamond^{-1} \varphi \rangle\) or \(\langle O - i, t.u : \Diamond \varphi \rangle\).

Formal rule for contextual points \(j\) The Proponent cannot introduce contextual points \(j\) for the same reason that he cannot introduce atomic formulas and contextual points \(u\): only the Opponent can do this because the Opponent tries to build a counter-argument to the thesis of the Proponent.

However, the Proponent can introduce a contextual point \(j\) to challenge a move as \(\langle O - i, t : B \varphi \rangle\) if the Opponent did not introduce such a contextual point \(j\) to challenge a move as \(\langle P - i, t : B \varphi \rangle\). When players discuss about beliefs, they discuss about the beliefs of an agent such that these beliefs are consistent. Thus if a player states that an agent believes a proposition \(\varphi\), there should be at least one context this agent conceives (see particle rules in Section 7.2.2) where \(\varphi\) holds. Otherwise, the beliefs of the agent would be inconsistent. For example, if “an agent believes that the Earth revolves around the sun” holds in \((i,t)\), there exists at least one context \((i,j,t)\) considered by the agent where “Earth revolves around the sun” holds. This is the counterpart of the seriality of beliefs in the model-theoretic approach.

Besides, the Proponent can choose a contextual point \(j\) already chosen by \(O\) to challenge a move as \(\langle P - i, t : B \varphi \rangle\), to challenge a move as \(\langle O - i, t : I \varphi \rangle\). In other words, the Proponent can use a contextual point \(j\) initially chosen to challenge a belief operator, to challenge an information operator in the same context. There is an interplay between the contextual points that can be chosen to challenge a belief operator and the contextual points that can be chosen to challenge an information operator in the same context. Indeed players discuss about the beliefs of an agent as well as the information she receives, such that if the agent has been informed about a proposition, she believes this proposition. That’s why all contextual points chosen to challenge a belief operator in a context \((i,t)\) can also be chosen to challenge an information operator in \((i,t)\). For example, if “an agent is informed that the Earth revolves around the sun” holds in \((i,t)\) and “an agent believes that the Earth revolves around the sun” holds in \((i,t)\), then every context \((i,j,t)\) that the agent conceives is also a context of which she has been informed. This is the counterpart of the acceptance of information in the model-theoretic approach.
Finally, \( P \) can choose a contextual point \( j \) already chosen by \( O \) to challenge a move as \( (P - i, z : I\varphi) \) or \( (P - i, z : B\varphi) \) or \( (P - i, z : A\varphi) \), to challenge a move as \( (O - i, t : A\varphi) \). When players discuss about a proposition that is always the case (in a non-temporal sense), they state that whatever the context they already discussed about, the proposition is the case in that context. For example, if “it is always the case that Amsterdam is the capital city of the Netherlands” holds at \((i, t)\), then “Amsterdam is the capital city of the Netherlands” holds in all contexts of the dialogue. Then whatever the modal operator for which the contextual point \( j \) has been introduced, it can be chosen to challenge a universal operator. Choices to challenge a universal operator are transitive and symmetric. The additional condition that \( P \) can choose the initial contextual point \( i \) to challenge \( (O - i, t : A\varphi) \) ensures there is reflexivity. This is the counterpart of the \( S5 \) frame in the model-theoretic approach.

But note that once again, the contextual point \( t \) is not important here. \( O \) can introduce the contextual point \( j \) in the context \((i, z)\) and \( P \) can choose this contextual point \( j \) to challenge a universal operator in \((i, t)\). Indeed when players introduce contexts in a dialogue, they can discuss about them all along the dialogue. Contexts do not disappear, they are constant throughout the whole discussion. This is the counterpart of the constant worlds over time in the model-theoretic approach (see Definition 2.8.3).

Now we have to explain the four exceptions we notice with respect to the choices of contextual points \( j \) \( P \) can make to challenge a move as \( (O - i, t : B\varphi) \).

We are dealing here with the interplay between information and beliefs as well as the interplay between beliefs themselves – namely, initial beliefs and revised beliefs – that involve an interplay between the choices of the players. In other words, some choices of a player allow for other choices for the other player to be made.

The first exception states that if there exists a context the agent conceives in \((i, t)\) and of which she is informed in \((i, t, u)\) such that \((i, t, u)\) is an immediate future context of \((i, t)\), then all the contexts the agent conceives in \((i, t, u)\) were already conceived by the agent in \((i, t)\). Indeed players discuss about an agent who receives a piece of information, such that if she receives an information compatible (that is, consistent) with her beliefs, then she does not add beliefs about which she is not informed. All the contexts the agent conceives after the information were already conceived before the information. That’s why under these conditions, \( P \) can choose a contextual point \( j \) initially chosen to challenge a belief operator in \((i, t, u)\), to challenge a belief operator in \((i, t)\). This is the counterpart of the No Add property in the model-theoretic approach.

The second exception states that if there exists a context the agent conceives in \((i, t)\) and of which she is informed in \((i, t, u)\) such that \((i, t, u)\) is an immediate future context of \((i, t)\), then the agent also conceives this context in \((i, t, u)\). Indeed players discuss about an agent who receives a piece of information, such that if
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she receives an information compatible (that is, consistent) with her beliefs, then
she does not drop these beliefs. She still conceives contexts compatible with
the information is received. That’s why under these conditions, \( P \) can choose
a contextual point \( j \) initially chosen to challenge a belief operator in \((i, t)\), to
challenge a belief operator in \((i, t.u)\). This is the counterpart of the No Drop
property in the model-theoretic approach.

The third exception states that if an agent is informed of the same contexts in
contexts \((i, t.u)\) and \((i, t.v)\) such that both contexts are immediate future contexts
of \((i, t)\), then she conceives the same contexts in \((i, t.u)\) and \((i, t.v)\). In that case,
\( P \) can choose a contextual point \( j \) initially chosen to challenge a belief operator
in \((i, t.u)\), to challenge a belief operator in \((i, t.v)\). Indeed players discuss about
an agent who is consistent with respect to the information she receives. If the
beliefs of the agent change and differ over time, it is only because she receives
different information. This is the counterpart of the Equivalence property in the
model-theoretic approach.

The last exception states that if an agent conceives a context in \((i, t.u)\) and is
informed about the same context in \((i, t.v)\) such that both contexts are immediate
future contexts of \((i, t)\), then the agent also conceives this context in \((i, t.v)\).
Indeed the players discuss about the beliefs of the agent such that these beliefs
are rationalized with respect to the information received. This is the counterpart
of the PLS property in the model-theoretic approach.

Request Rule  This rule ensures that players can only choose the non-standard
challenge on an information operator that is, request the other player to confirm
that a particular contextual point \( j \) could be chosen to challenge a move as \((X - i, t : I_\varphi)\), if the Opponent already stated that \( \varphi \) holds in the contextual point
\( j \)\(^8\). Indeed a player who states that an agent is informed about a proposition
\( \varphi \), is committed to defend that all and only contexts where \( \varphi \) holds are contexts
of which the agent is informed. For example, if “an agent is informed that the
President is dead” holds in \((i, t)\), then he is informed of all and only contexts
where “the President is dead” holds. This is the counterpart of the non-normality
of the information operator.

Structural rules of \( DTDL \)  The structural rules of \( DTDL \) consist of (SR-0),
(SR-1), (SR-2), (SR-3), (SR-4), (SR-5), (SR-5.1), (SR-5.2), (SR-6), (SR-6.1),
(SR-6.2), (SR-6.3), (SR-6.4), and (SR-7).

7.2.4. DEFINITION. \( DTDL \) is defined by the set of particle rules and structural
rules.

\(^8\)Remember that \( \varphi \) must be Boolean so the contextual point \( t \) does not matter.
7.2. Original dialogical approach to Bonanno’s logic for belief revision

An argumentative interpretation When investigating the relation between beliefs and information over time from an external dynamic perspective, we interpret this relation through the notion of choice. Players make choices when they discuss about the beliefs of an agent and the information she receives. We notice a genuine interplay between the choices of the players. Under specific conditions, some choices initially made to challenge belief operators can also be made to challenge information operators; some choices initially made to challenge belief operators in a particular context can also be made to challenge belief operators in different contexts and so on. Some choices allow some other choices otherwise prohibited by the other rules. Not only the meaning of belief and information operators is defined in terms of choice, but the belief revision policy itself is defined in terms of choice. Indeed the restrictions on the possible choices players can make to challenge belief and information operators define a particular belief revision policy. In other words DTDL allows an argumentative interpretation of the belief revision policy of Bonanno.

7.2.4 Applications

In the Figures 7.3, 7.4, 7.5 the number in the outer column corresponds to the number of the move whereas the one in the inner column corresponds to the number of the move challenged.

Non surprising information We illustrate a play where two players discuss about an agent who receives a piece of information that does not contradict her beliefs and revises this beliefs in the light of this new information. The thesis of this play described in Figure 7.3 is the formula $\neg [\neg B \neg q \land B p] \lor [\bigcirc (\neg I q \lor B (p \land q))].$

Explanations of Figure 7.3 In accordance with the starting rule (SR-0), the Proponent states the thesis at move 0. At move 1, the Opponent challenges the disjunction and the Proponent chooses to defend the first disjunct at move 2. At move 3, the Opponent challenges the negation and the Proponent has no corresponding defence. Then he chooses to challenge the conjunction of move 3 choosing respectively the first conjunct at move 4 and the second conjunct at move 6 in accordance with his repetition rank $n := 2$. The Opponent defends the corresponding conjunct at moves 5 and 7. At move 8, the Proponent challenges the negation of move 5 and the Opponent counter-attacks the move 8 since he has no possible defence. So he challenges the belief operator choosing the contextual point 2 at move 9 and the Proponent defends $\neg q$ in the context $(1, 2, 1)$ at move 10. The Opponent challenges the negation of move 10 and the Proponent has no possible defence. Then he chooses to change the defence against the challenge of move 1, choosing the second disjunct at move 12 in accordance with his repetition rank $n := 2$. At move 13, the Opponent challenges the $\bigcirc$ operator choosing a contextual point 2 and the Proponent defends $\neg I q \lor B (p \land q)$ in the context $(1, 1, 2)$. 
Then the Opponent challenges the disjunction and the Proponent chooses to defend the first disjunct. The Opponent challenges the negation of move 16 and the Proponent has no corresponding defence. He changes his defence against the challenge of move 15 choosing to defend the second disjunct. At move 19, the Opponent challenges the belief operator choosing the contextual point 3 and the Proponent defends $p \land q$ in the context $(1,1)$. The Opponent then chooses the first conjunct when he challenges the conjunction of move 20. In accordance with the formal rule (SR-2), the Proponent cannot defend now since he cannot utter first an atomic formula in a particular context. But he can challenge the information operator of move 17 choosing the non standard challenge and the contextual point 2. Indeed the Opponent has stated that $q$ holds in the contextual point 2 at move 11 so the request rule (SR-7) allows him to choose this contextual point 2 for his non standard challenge. Then the Opponent defends at move 23, confirming that this contextual point 2 can be chosen to challenge an information operator in the context $(1,1,2)$. The move 9, move 19 and move 23 allow the Proponent to challenge the belief operator of move 7 choosing the contextual point 3 in accordance with the formal rule for contextual point $j$ (SR-6.1). Indeed the Opponent has chosen the contextual point 2 to challenge a belief operator in $(1,1)$ and has stated that this contextual point 2 can be chosen to challenge an information operator in the context $(1,1,2)$, and he has also chosen
7.2. *Original dialogical approach to Bonanno’s logic for belief revision*

In move 22, the Proponent requests the Opponent to confirm that the contextual point 2 can be chosen to challenge an information operator in the context (1,1,2). In other words, the Proponent requests the Opponent to confirm a choice he could do. In moves 22 and 23, the players are actually dealing about the Opponent choices with respect to his previous choices and statements. And previously, the Opponent has stated that $q$ holds in the contextual point 2 and that the agent is informed about $q$ in the context (1,1,2) (moves 11 and 17). If the Opponent is consistent with himself, he must stated that (1,2,1,2) is a context of which the agent is informed. Indeed a player who states that an agent is informed about a proposition $\varphi$, is committed to defend that all and only contexts where $\varphi$ holds are contexts of which the agent is informed (see Request Rule p 168). So the Opponent must confirm that 2 is an available choice to challenge an information operator in the context (1,1,2) with respect to his argumentation otherwise he contradicts himself.

What happens now if the Opponent chooses the second conjunct when he challenges the conjunction of move 20? We consider a play with the same thesis as in Figure 7.3.

**Explanations of Figure 7.4** The play proceeds in the same way as in Figure 7.3 until move 21. Indeed the Opponent chooses the second conjunct when he challenges the conjunction of move 20. In that case, the Proponent challenges the information operator of move 17 choosing the standard challenge. Since the Opponent has chosen the contextual point 3 to challenge the belief operator of move 18 (move 19), the Proponent can choose this contextual point to challenge the information operator of move 17 in accordance with the formal rule for contextual point $j$ (SR-6) at move 22. At move 23, the Opponent defends stating $q$ in the context (1,3,1,2) allowing the Proponent to state $q$ in (1,3,1,2) at move 24 to defend against the challenge of move 21 in accordance with the formal rule (SR-2). In that play, in accordance with the winning rule (SR-3) the Proponent wins too.

In the Figures 7.3 and 7.4, the Opponent first states that the agent considers $q$ possible (move 5) and believes $p$ (move 7), and then is informed about $q$ (move 17). In that case and according to the belief revision policy described by the rules
of DTDL, it is impossible for players to argue that the agent does not believe $p$ and $q$ after she receives the information.

**Surprising information** What happens if the information is surprising? We illustrate a play where two players discuss about an agent who receives a piece of information that does contradict her beliefs and revises this beliefs in the light of this new information. The thesis of this play described in Figure 7.5 is the formula $\neg[B \neg q \land B p] \lor [\bigcirc (\neg I q \lor B (p \land q))]$.

**Explanations of Figure 7.5** The play starts and proceeds as in Figure 7.3. At move 8, the Proponent cannot challenge the moves 5 or 7 in accordance with the formal rule for contextual point $j$ (SR-6), so he changes his defence against the challenge of move 1, choosing the second disjunct in accordance with his repetition rank $n := 2$. Once again the play proceeds as in Figure 7.3. In accordance with the formal rule (SR-2), the Proponent cannot defend against the challenge on move 16 since he cannot utter first an atomic formula in a particular context. The only move the Proponent can then make is to challenge the move 13 choosing the standard challenge. Since the Opponent has chosen the contextual point 3 to challenge the belief operator of move 14 (move 15), the Proponent can choose this contextual point to challenge the information operator of move 13 in accordance
with the formal rule for contextual point $j$ (SR-6) at move 18. At move 19, the Opponent defends stating $q$ in the context $(1,1,2,1)$. Then the Proponent cannot make a move.

The main difference with Figure 7.3 is that the Proponent cannot challenge the move 7 with contextual point 3 in accordance with the rules. The choices of the Opponent do not allow this choice to the Proponent. So in that play, the Proponent loses in accordance with the winning rule (SR-3).

In the Figure 7.5, the Opponent first states that the agent believes $\neg q$ (move 5) and believes $p$ (move 7), and then is informed about $q$ (move 13). In that case and according to the belief revision policy described by the rules of DTDL, it is possible for the Opponent to argue that the agent does not believe $p$ after she receives the information and then does not believe $(p \land q)$.

**Stubborn agent** What happens if the agent is stubborn? We illustrate a play where two players discuss about a stubborn agent who receives a piece of information that does not contradict her beliefs and revises this beliefs in the light of this new information such that she does not believe the information. The thesis of this play described in Figure 7.6 is the formula $\neg[\neg Bq \land Bp] \lor [\lozenge(\neg Iq \lor B(p \land q))]$.

**Explanations of Figure 7.6** The play starts and proceeds as in Figure 7.5. At move 17, the Opponent chooses the second conjunct when he challenges the conjunction of move 16. The Proponent defends stating $\neg q$ in the context
Figure 7.6: Stubborn agent

(1.3, 1.2). At move 19, the Opponent challenges the negation stating $q$ in the context (1.3, 1.2). The only move the Proponent can then make is to challenge the move 13 choosing the standard challenge. Since the Opponent has chosen the contextual point 3 to challenge the belief operator of move 14 (move 15), the Proponent can choose this contextual point to challenge the information operator of move 13 in accordance with the formal rule for contextual point $j$ (SR-6) at move 20. At move 21, the Opponent defends stating $q$ in the context (1.3, 1.2). Then the Proponent cannot make a move.

In the Figure 7.6, the Opponent first states that the agent believes $\neg q$ (move 5) and believes $p$ (move 7), and then is informed about $q$ (move 13). In that case and according to the belief revision policy described by the rules of DTDL, it is possible for the Opponent to argue that the agent does not believe $\neg q$ after she receives the information and then does not believe $(p \land q)$.  

Irrational agent What happens if the agent is irrational? We illustrate a play where two players discuss about an irrational agent who receives a piece of information that does not contradict her beliefs and revises this beliefs in the light of this new information in an irrational way. Here we interpret “irrational way” as changing her beliefs regardless the information received. The thesis of this play described in Figure 7.7 is the formula $\neg[\bigcirc(\neg (Ip \land Bq)) \lor \bigcirc(\neg Ip \lor B \neg q)]$.  

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### Explanations of Figure 7.7

At move 1, the Opponent challenges the disjunction and the Proponent chooses to defend the first disjunct at move 2. At move 3, the Opponent challenges the negation and the Proponent has no corresponding defence. Then he counter-attacks and challenges the negation of move 3. At move 5 the Opponent challenges the $\Box$ operator choosing a contextual point $2$ and the Proponent defends $\neg (Ip \land Bq)$ in the context $(1,1,2)$. The Opponent challenges the negation at move 7 and the Proponent counter-attacks challenging the conjunction. He chooses respectively the first conjunct at move 8 and the second conjunct at move 10. The Opponent defends the corresponding conjunct at move 9 and 11. The Proponent cannot challenge these moves so he decides to change his defence against the challenge of move 1, choosing the second disjunct at move 12 in accordance with his repetition rank $n = 2$. At move 13 the Opponent challenges the $\Box$ operator choosing a contextual point 3 and the Proponent defends $\neg Ip \lor B \neg q$ in the context $(1,1,3)$. At move 15, the Opponent challenges the disjunction and the Proponent chooses to defend the first disjunct at move 16. At move 17, the Opponent challenges the negation and the Proponent has no corresponding defence. He decides to change his defence against the challenge of move 15, choosing the second disjunct at move 18 in accordance with his repetition rank $n = 2$. At move 19, the Opponent challenges the belief operator...
choosing the contextual point 2 and the Proponent defends \( \neg q \) in the context (1,2,1.3). The Opponent then challenges the negation of move 20 stating \( q \) in the context (1,2,1.3). The Proponent challenges the information operator of move 17 choosing the standard challenge and the contextual point 2 in accordance with the formal rule for contextual point \( j \) (SR-6). Indeed the Opponent has chosen the contextual point 2 to challenge a belief operator in (1,1,3) at move 19. Since the Opponent defends stating \( p \) in (1,2,1.3), the Proponent can challenge the information operator of move 9 choosing the contextual point 2 for his non standard challenge at move 24 in accordance with the request rule (SR-7). Then the Opponent defends at move 25, confirming that this contextual point 2 can be chosen to challenge an information operator in the context (1,1,2). The move 19, move 22 and move 25 allow the Proponent to challenge the belief operator of move 11 choosing the contextual point 2 in accordance with the formal rule for contextual point \( j \) (SR-6.3). Indeed the Opponent has chosen the contextual point 2 to challenge a belief operator in (1,1,3) and so contextual point 2 can be chosen by the Proponent to challenge an information operator in (1,1,3), and the Opponent has also stated that this contextual point 2 can be chosen to challenge an information operator in the context (1,1,2). The Opponent then defends stating \( q \) in the context (1,2,1,2). In accordance with the winning rule (SR-3) the Proponent loses since he cannot move.

Note  In move 24, the Proponent requests the Opponent to confirm that the contextual point 2 can be chosen to challenge an information operator in the context (1,1,2) that is, he requests the Opponent to confirm a choice he could do. In moves 24 and 25, the players are actually dealing about the Opponent choices with respect to his previous choices and statements: the Opponent has stated that the agent is informed about \( p \) in the context (1,1,2) and that \( p \) holds in the contextual point 2 (moves 9 and 23). If the Opponent is consistent with himself, he must stated that (1,2,1,2) is a context of which the agent is informed (see Request Rule). So he must confirm that 2 is an available choice to challenge an information operator in the context (1,1,2) with respect to his argumentation otherwise he contradicts himself.

In the Figure 7.7, the Opponent first states that the agent is informed about \( p \) (move 9) and believes \( q \) (move 11) in a particular context and that this agent is also informed about \( p \) in another context (move 17). In that case and according to the belief revision policy described by the rules of DTDL, it is possible for the Opponent to argue that the agent does not believe \( \neg q \) after she receives the information in the second context.
Winning strategy In the Figures 7.3 and 7.4, we provide two different plays of the dialogue \( D_{\neg(B \land \neg q \land \neg B \land p) \lor (\neg I \lor B(p \land q))} \) corresponding to different choices of the Opponent when he challenges the conjunction of move 20. Whatever he chooses the first –Figure 7.3– or the second –Figure 7.4– conjunct, the Proponent wins the play. In other words, the Proponent can win whatever the choices of the Opponent if he plays optimally that is, if he makes the choices allowing his victory (among the available ones).

Being able to win whatever the choices of the player \( Y \) means not only winning a play \( d_\Delta \in D_\Delta \) but also all the possible plays for \( \Delta \) that is, winning \( D_\Delta \). In other words, this means that \( X \) has a winning strategy.

**7.2.5. Definition.** A player has a *winning strategy* if he can win whatever the choices of the other player.

This notion of winning strategy is the counterpart of the notion of validity in the model-theoretic approach. Since DTDL is a dialogical approach to \( L_{PLS^*} \), there exists a correspondence between having a winning strategy for a formula \( \varphi \) in DTDL and being valid in \( L_{PLS^*} \) for the same formula \( \varphi \).

### 7.3 Soundness and Completeness for DTDL

We prove that DTDL is sound and complete with respect to \( L_{PLS^*} \) showing that there exists a winning strategy for the Proponent in \( D_\Delta \) iff \( \Delta \) is a valid formula in \( L_{PLS^*} \).

We start with one hypothesis.

**7.3.1. Hypothesis.** *Both Players always play the best move that is, they are ideal players able to choose the best move to win the play. Thus we can always consider plays where the Opponent chooses \( m := 1 \) and the Proponent chooses \( n := 2 \) as repetition ranks. Indeed if the the Opponent plays in an optimal way, it is sufficient for him to have \( m := 1 \) because if he has a winning strategy and follows it, he does not have to change his defences or challenges. In Theorem 7.3.16, we show that the repetition ranks \( m := 1 \) and \( n := 2 \) are optimal respectively for the Opponent and the Proponent.*

**7.3.1 Soundness**

We prove that our dialogical approach is sound with respect to \( L_{PLS^*} \), showing that if the Proponent has a winning strategy in \( D_\Delta \) then the formula \( \Delta \) is valid in \( L_{PLS^*} \). We prove the contrapositive: we prove that if there exists a model satisfying \( \neg \Delta \) then the Proponent cannot win any play with \( \Delta \) as thesis.

We start from one hypothesis.
7.3.2. Hypothesis. A dialogical move \( \langle X - i, t : \varphi \rangle \) means that:

- \( M, (i, t) \models \varphi \) if \( X = O \)
- \( M, (i, t) \models \neg \varphi \) if \( X = P \)

A dialogical move \( \langle X - i, t : ?I_j^* \rangle \) means that:

- \( j \in I_t(i) \) if \( X = O \)
- \( j \notin I_t(i) \) if \( X = P \)

We first need to prove that our particle rules preserve satisfaction that is, our Hypothesis 7.3.2 is preserved after the use of any particle rule.

7.3.3. Lemma. Given a branching time belief revision model \( M \), all the particle rules preserve satisfiability.

7.3.4. Proof. We show that our 8 particle rules preserve satisfiability.

\[ \diamond \] Particle rule for negation:

- if \( \langle X - i, t : \neg \varphi \rangle \in d_\Delta \)
- then \( \langle Y - i, t : \varphi \rangle \in d_\Delta \)

1. if \( X = O \), by Hypothesis 7.3.2, \( M, (i, t) \models \neg \varphi \) iff \( M, (i, t) \not\models \varphi \) (by Definition 2.8.11).

2. if \( X = P \), by Hypothesis 7.3.2, \( M, (i, t) \models \neg \varphi \) iff \( M, (i, t) \not\models \varphi \) (by Definition 2.8.11).

\[ \diamond \] Particle rule for conjunction:

- if \( \langle X - i, t : \varphi_1 \land \varphi_2 \rangle \in d_\Delta \)
- then \( \langle Y - i, t : ?\alpha_1 \rangle \in d_\Delta \), or \( \langle Y - i, t : ?\alpha_2 \rangle \in d_\Delta \)
- so \( \langle X - i, t : \varphi_1 \rangle \in d_\Delta \), or \( \langle X - i, t : \varphi_2 \rangle \in d_\Delta \)

1. if \( X = O \), \( P \) can change his challenge since \( n := 2 \), then by Hypothesis 7.3.2, \( M, (i, t) \models (\varphi_1 \land \varphi_2) \) iff \( M, (i, t) \not\models \varphi_1 \) and \( M, (i, t) \not\models \varphi_2 \) (by Definition 2.8.11).

2. if \( X = P \), \( O \) cannot change his challenge since \( m := 1 \), then by Hypothesis 7.3.2, \( M, (i, t) \models \neg (\varphi_1 \land \varphi_2) \) iff \( M, (i, t) \not\models \neg \varphi_1 \) or \( M, (i, t) \not\models \neg \varphi_2 \) (by Definition 2.8.11).

\[ \diamond \] Particle rule for disjunction:

- if \( \langle X - i, t : \varphi_1 \lor \varphi_2 \rangle \in d_\Delta \)
- then \( \langle Y - i, t : ?\lor \rangle \in d_\Delta \)
- so \( \langle X - i, t : \varphi_1 \rangle \in d_\Delta \), or \( \langle X - i, t : \varphi_2 \rangle \in d_\Delta \)
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1. if $X = O$, $O$ cannot change his defence since $m := 1$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models (\varphi_1 \lor \varphi_2)$ iff $\mathcal{M}, (i, t) \models \varphi_1$ or $\mathcal{M}, (i, t) \models \varphi_2$ (by Definition 2.8.11).

2. if $X = P$, $P$ can change his defence since $n := 2$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \not\models (\varphi_1 \lor \varphi_2)$ iff $\mathcal{M}, (i, t) \models \neg \varphi_1$ and $\mathcal{M}, (i, t) \models \neg \varphi_2$ (by Definition 2.8.11).

- Particle rule for $\Box^{-1}$ operator:
  
  - if $\langle X - i, t : \Box^{-1} \varphi \rangle \in d_{\Delta}$
  - then $\langle Y - i, t : ?\Box^{-1}_{u} \rangle \in d_{\Delta}$
  - so $\langle X - i, t, u : \varphi \rangle \in d_{\Delta}$

  1. if $X = O$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models \Box^{-1} \varphi$ iff $\mathcal{M}, (i, t, u) \models \varphi$ for every $u$ such that $u \sim t$ (by Definition 2.8.11).

  2. if $X = P$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models \neg \Box^{-1} \varphi$ iff $\mathcal{M}, (i, t, u) \models \neg \varphi$ for at least one $u$ such that $u \sim t$ (by Definition 2.8.11).

- Particle rule for $\Box$ operator:
  
  - if $\langle X - i, t : \Box \varphi \rangle \in d_{\Delta}$
  - then $\langle Y - i, t : ?\Box_{u} \rangle \in d_{\Delta}$
  - so $\langle X - i, t, u : \varphi \rangle \in d_{\Delta}$

  1. if $X = O$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models \Box \varphi$ iff $\mathcal{M}, (i, t, u) \models \varphi$ for every $u$ such that $t \sim u$ (by Definition 2.8.11).

  2. if $X = P$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models \neg \Box \varphi$ iff $\mathcal{M}, (i, t, u) \models \neg \varphi$ for at least one $u$ such that $t \sim u$ (by Definition 2.8.11).

- Particle rule for $B$ operator:
  
  - if $\langle X - i, t : B \varphi \rangle \in d_{\Delta}$
  - then $\langle Y - i, t : ?B_{j} \rangle \in d_{\Delta}$
  - so $\langle X - i, j, t : \varphi \rangle \in d_{\Delta}$

  1. if $X = O$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models B \varphi$ iff $\mathcal{M}, (i, j, t) \models \varphi$ for every $j \in B_{i}(i)$ (by Definition 2.8.11).

  2. if $X = P$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models \neg B \varphi$ iff $\mathcal{M}, (i, j, t) \models \neg \varphi$ for at least one $j \in B_{i}(i)$ (by Definition 2.8.11).

- Particle rule for $A$ operator:
  
  - if $\langle X - i, t : A \varphi \rangle \in d_{\Delta}$
  - then $\langle Y - i, t : ?A_{j} \rangle \in d_{\Delta}$
  - so $\langle X - i, j, t : \varphi \rangle \in d_{\Delta}$

  1. if $X = O$, then by Hypothesis 7.3.2, $\mathcal{M}, (i, t) \models A \varphi$ iff $\mathcal{M}, (i, j, t) \models \varphi$ for every $j \in S$ (by Definition 2.8.11).
2. if \( X = P \), then by Hypothesis 7.3.2, \( \mathcal{M},(i,t) \models \neg P \) iff \( \mathcal{M},(i,j,t) \models \neg \varphi \) for at least one \( j \in S \) (by Definition 2.8.11).

\[ \text{Particle rule for } I \text{ operator:} \]
\[ \begin{align*}
\text{if } (X - i,t : I\varphi) & \in d_\Delta \\
\text{then } (Y - i,t : ?I\varphi) & \in d_\Delta, \text{ or } (Y - i,t : !I\varphi) \in d_\Delta \\
\text{so } (X - i,j,t : \varphi) & \in d_\Delta, \text{ or } (X - i,t : ?!I\varphi) \in d_\Delta
\end{align*} \]

1. if \( X = O \), \( P \) can change his challenge since \( n := 2 \), then by Hypothesis 7.3.2, \( \mathcal{M},(i,t) \models I\varphi \) iff \( \mathcal{M},(i,j,t) \models \varphi \) for every \( j \in I_t(i) \); and \( \mathcal{M},(i,j,t) \models \varphi \) and \( j \in I_t(i) \) (by Definition 2.8.11).

2. if \( X = P \), \( O \) cannot change his challenge since \( m := 1 \), then by Hypothesis 7.3.2, \( \mathcal{M},(i,t) \models \neg I\varphi \) iff \( \mathcal{M},(i,j,t) \models \neg \varphi \) for at least one \( j \in I_t(i) \); or \( \mathcal{M},(i,j,t) \models \varphi \) and \( j \notin I_t(i) \) (by Definition 2.8.11).

\[ \Box \]

7.3.5. Lemma. The Proponent wins \( d_\Delta \) iff he states an atomic formula or a context choice confirmation.

7.3.6. Proof. Note that the structural rule SR-3 states that a player wins iff he plays the last move of the play\(^9\).

1. If the Proponent wins \( d_\Delta \), then his last move is \( \langle P - i,t : e \rangle \) such that \( e \) is an atomic formula or a context choice confirmation.

7.3.7. Hypothesis. The Proponent wins \( d_\Delta \) such that the last move is \( \langle P - i,t : e \rangle \), where \( e \) is not an atomic formula or a context choice confirmation.

From Hypothesis 7.3.7 it follows that:

(a) \( \langle P - i,t : e \rangle \) is a defence of the Proponent where \( e \) is a complex formula. Then the Opponent can challenge that formula, consequently the previous move of the Proponent is not the last one, contradicting the Hypothesis 7.3.7; or

(b) \( \langle P - i,t : e \rangle \) is a challenge of the Proponent. Then the Opponent is always able to produce a defence (consequently the previous move of the Proponent is not the last one, contradicting the Hypothesis 7.3.7) unless it is a challenge against a negation (since there is no possible defence in that case). In that case:

\(^9\)See structural rule SR-3 in Section 7.2.
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i. either \( e \) is an atomic formula, contradicting the Hypothesis 7.3.7,

ii. or \( e \) is not an atomic formula but a complex formula, and the Opponent is always able to produce a counter-attack (consequently the previous move of the Proponent is not the last one, contradicting the Hypothesis 7.3.7).

2. If the Proponent states an atomic formula or a context choice confirmation then he wins \( d_\Delta \) with the corresponding move.

7.3.8. Hypothesis. The Proponent states an atomic formula or a context choice confirmation in a move \( \alpha \in d_\Delta \), but he does not win \( d_\Delta \).

From Hypothesis 7.3.8 it follows that:

(a) there exists a move \( \beta \) of the Opponent immediately following \( \alpha \) (by SR-1 and SR-3).

(b) Since there is no possible challenge on atomic formula or context choice confirmation, \( \beta \) cannot be a challenge on \( \alpha \).

(c) \( \beta \) cannot be a defence against \( \alpha \). Indeed \( \alpha \) should be a challenge, consequently it would be a challenge on a negation (this is the only case where the challenge amounts to state a formula) and there is no possible defence in that case.

(d) Consequently \( \beta \) must be a challenge or a defence against a previous move of the Proponent.

\[ \diamond \text{ If } \beta \text{ is a challenge :} \]

- there exists a move \( \gamma \) of the Proponent challenged by \( \beta \).
- After \( \gamma \), the Opponent had the choice between \( \beta \) and the move \( \delta \) immediately following \( \gamma \) in \( d_\Delta \).
- But since \( m := 1 \) and \( \delta \) is already a challenge on \( \gamma \), \( \beta \) cannot be another challenge on this move: \( \beta \) cannot be a challenge on \( \gamma \).

Consequently, if \( \beta \) is a challenge on a previous move of the Proponent, this challenge is on a move \( \varepsilon \) preceding \( \gamma \).

\[ \diamond \text{ If } \beta \text{ is a defence :} \]

- there exists a move \( \gamma \) of the Proponent of which \( \beta \) is the defence.
- After \( \gamma \), the Opponent had the choice between \( \beta \) and the move \( \delta \) immediately following \( \gamma \) in \( d_\Delta \).
- But since \( m := 1 \) and \( \delta \) is already a defence against \( \gamma \), \( \beta \) cannot be another defence against this move: \( \beta \) cannot be a challenge against \( \gamma \).
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Consequently, if $\beta$ is a defence against a previous move of the Proponent, this defence is against a move $\varepsilon$ preceding $\gamma$.

This reasoning can be applied until the start of $d_\Delta$ showing that $\beta$ cannot be a challenge or a defence against $\varepsilon$ but has to be (on the same grounds as above) a challenge or a defence against a previous move $\zeta$ and so on. Finally, $\beta$ would be a challenge on $\Delta$ or a defence against the first challenge of the Proponent but since $m := 1$, the actual choice and $\beta$ cannot both belong to $d_\Delta$.

Consequently, $\beta$ cannot be a challenge or a defence against a previous move of the Proponent. In other words, (d) leads to a contradiction. Then it follows that (a) leads to a contradiction: there does not exist a move $\beta$ following $\alpha$, contradicting – in accordance with the structural rule SR-3 – the defeat of the Proponent in $d_\Delta$ (Hypothesis 7.3.8).

\[\square\]

Soundness Theorem

7.3.9. Theorem. If the Proponent wins $d_\Delta$ with the rules of DTDL, then $\Delta$ is a valid formula in $L_{PLS^*}$.

7.3.10. Proof. We prove soundness by showing the contrapositive that is, we show that if there exists one $(M, (i,t))$ such that $\neg\Delta$ is satisfiable in $(M, (i,t))$, then the Proponent loses $d_\Delta$. It follows from Lemma 7.3.3 and Lemma 7.3.5 that if $\neg\Delta$ is satisfiable then the Proponent loses:

7.3.11. Hypothesis. Let a play $d_\Delta$ be such that $\neg\Delta$ is satisfiable in $(M, (i,t))$ and the Proponent wins $d_\Delta$.

1. By SR-3 and Hypothesis 7.3.11, it follows that the Proponent plays the last move.

2. By (1) the Proponent plays the last move in $d_\Delta$. By Lemma 7.3.5, the last move of the Proponent is an atomic formula or a context choice confirmation. By SR-2, the Proponent can state an atomic formula only if this atomic formula has been stated by the Opponent first. By SR-5 and SR-5.2 The Proponent can only state a context choice confirmation if the corresponding context has been chosen by the Opponent first to challenge a $B$ or $I$ operator.

3. From (2) and Definition 7.2.3, $d_\Delta$ is close.

4. By Lemma 7.3.3 et (3) it follows that it exists a branching time belief revision model $M$ such that $M,(i,t) \models p$ and $M,(i,t) \models \neg p$, or such that $j \in I_t(i)$ and $j \notin I_t(i)$, which is a contradiction.
Consequently, if the Proponent wins $d_\Delta$, there is no branching time belief revision model satisfying $\neg \Delta$.

\[\square\]

### 7.3.2 Completeness

We prove that $DTDL$ is complete with respect to $L_{PLS}$, showing that if $\Delta$ is valid in $L_{PLS}$, then the Proponent has a winning strategy in $D_\Delta$ with the rules of $DTDL$. We prove the contrapositive: we prove that if the Proponent loses $d_\Delta$ with the rules of $DTDL$ then $\Delta$ is not valid in $L_{PLS}$. 

Note that we still assume Hypothesis 7.3.1.

We start providing two definitions.

**7.3.12. Definition.** An extended dialogue $D_\Delta$ is a play $d_\Delta$ where the Proponent can challenge modal operators as many times as he needs. In other words, repetition ranks do not concern modal operators anymore. A branching time belief revision model $M$ can be built from an extended dialogue: $M$ is defined as $<T, \sim, S, \{B_i, I_i\}_{i \in T}, V>$, where:

- $T = \{t$ such that $(X - i, t : \varphi) \in D_\Delta \}$, or $(X - i, s : ?\Box) \in D_\Delta \}$ for $\Box$ any kind of temporal operator $(\Box, \Box^{-1})$,
- $t^\sim = \{u$ such that $(X - i, t : ?\Box_u) \in D_\Delta \}$,
- $S = \{i$ such that $(X - i, t : \varphi) \in D_\Delta \}$, or $(X - h, t : ?\Box_i) \in D_\Delta \}$ for $\Box$ any kind of non temporal operator $(A, B, I)$,
- $B_i(i) = \{j$ such that $(X - i, t : ?B_j) \in D_\Delta \}$,
- $I_i(i) = \{j$ such that $(X - i, t : ?I_j) \in D_\Delta \}$ or $(O - i, t : ?I_j^*) \in D_\Delta \}$,
- $V_p = \{i$ such that $(O - i, t : p) \in D_\Delta \}$.

It seems that an extended dialogue can then be infinite. However, we noticed in Definition 2.8.7 that the set of states $S$ of branching-time belief revision frames is finite as well as the set $t^\sim$ of all immediate successors of an instant $t$ for all instants $t$. Then we can consider a finite number of choices of contextual points to challenge modal operators. So we can only consider finite extended dialogues.

\[\text{In Theorem 7.3.16, we show that if the Proponent can win, } n = 2 \text{ is enough to win } d_\Delta \text{ that is, he does not need to challenge all contextual points.}\]

\[\text{Note that we cast all the relations in terms of maps.}\]
7.3.13. **Definition.** The length of \( \varphi \) is defined as:

- \( \text{len}(p) = 1 \)
- \( \text{len}(-\varphi) = 1 + \text{len}(\varphi) \)
- \( \text{len}(\varphi \land \psi) = 1 + \text{len}(\varphi) + \text{len}(\psi) \)
- \( \text{len}(\neg 1 \varphi) = 2 + \text{len}(\varphi) \)
- \( \text{len}(\neg 1 \varphi) = 2 + \text{len}(\varphi) \)
- \( \text{len}(B \varphi) = 1 + \text{len}(\varphi) \)
- \( \text{len}(I \varphi) = 1 + \text{len}(\varphi) \)
- \( \text{len}(A \varphi) = 1 + \text{len}(\varphi) \)

7.3.14. **Lemma.** If \( \mathcal{D}_\Delta \) is terminal and the Proponent loses \( \mathcal{D}_\Delta \), then it exists a model \((\mathcal{M}, (i, t))\) such that:

- \( \langle O - i, t : \varphi \rangle \in \mathcal{D}_\Delta \) means \( \mathcal{M}, (i, t) \models \varphi \), and
- \( \langle P - i, t : \varphi \rangle \in \mathcal{D}_\Delta \) means \( \mathcal{M}, (i, t) \models \neg \varphi \).
- \( \langle O - i, t : ?I^*_j \rangle \in \mathcal{D}_\Delta \) means \( j \in I_t(i) \), and
- \( \langle P - i, t : ?I^*_j \rangle \in \mathcal{D}_\Delta \) means \( j \notin I_t(i) \).

7.3.15. **Proof.** We proceed by induction on the length of \( \varphi \). The basic case is about atomic formulas.

1. **Base:** \( \varphi := p \)

   If \( \langle X - (i, t) : p \rangle \in \mathcal{D}_\Delta \), then either:

   1. \( X = O \) and \( \mathcal{M}, (i, t) \models p \) (Definition 7.3.12); or
   2. \( X = P \) and consequently, in accordance with Lemma 7.3.5, the Proponent wins \( \mathcal{D}_\Delta \) since he states an atomic formula, contradicting Lemma 7.3.14.

2. **Induction Hypothesis:**

   If \( \text{len}(\varphi) \leq n \) then if \( \langle X - i, t : \varphi \rangle \in \mathcal{D}_\Delta \) and the Proponent loses \( \mathcal{D}_\Delta \), there exists a model \((\mathcal{M}, (i, t))\) such that:

   - \( \langle O - i, t : \varphi \rangle \in \mathcal{D}_\Delta \) means \( \mathcal{M}, (i, t) \models \varphi \), and
   - \( \langle P - i, t : \varphi \rangle \in \mathcal{D}_\Delta \) means \( \mathcal{M}, (i, t) \models \neg \varphi \).
– \langle O - i, t : ?I_j^+ \rangle \in \mathfrak{D}_\Delta \text{ means } j \in I_t(i), \text{ and}
– \langle P - i, t : ?I_j^+ \rangle \in \mathfrak{D}_\Delta \text{ means } j \notin I_t(i).

3. Inductive step:
Let us assume that \text{len}(\varphi) = n + 1. We consider 8 different cases, one for each logical constant in our dialogical language.

\textbf{Case 1: } \varphi := \lnot \psi

If \langle X - i, t : \lnot \psi \rangle \in \mathfrak{D}_\Delta, \text{ then: }
\langle Y - i, t : \psi \rangle \in \mathfrak{D}_\Delta.

1. \text{ If } Y = O \text{ then } M, (i, t) \vdash \psi \text{ (by Induction Hypothesis – H. I.)}; \text{ or }
2. \text{ If } Y = P \text{ then either: }
   (a) \psi \notin \Phi: \langle P - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } M, (i, t) \vdash \lnot \psi \text{ (by H. I.)}; \text{ or }
   (b) \psi \in \Phi:
      \begin{enumerate}
      i. \langle O - i, z : \psi \rangle \notin \mathfrak{D}_\Delta \text{ for any contextual point } z,^{12} \text{ so } i \notin V_{\psi} \text{ then }
         M, (i, t) \vdash \lnot \psi \text{ (by Definition 7.3.12)},
      ii. \langle O - i, z : \psi \rangle \in \mathfrak{D}_\Delta \text{ and } \langle P - i, t : \psi \rangle \in \mathfrak{D}_\Delta, \text{ then by Lemma 7.3.5 the Proponent wins } \mathfrak{D}_\Delta, \text{ contradicting the hypothesis of Lemma 7.3.14.}
   \end{enumerate}

\textbf{Case 2: } \varphi := \psi \land \chi

If \langle X - i, t : \psi \land \chi \rangle \in \mathfrak{D}_\Delta, \text{ then: }
\langle Y - i, t : ?A_1 \rangle \in \mathfrak{D}_\Delta \text{ or } \langle Y - i, t : ?A_2 \rangle \in \mathfrak{D}_\Delta.

1. \text{ If } X = O, \text{ the Proponent can change his challenge since } n := 2. \text{ Consequently: }
   \langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } M, (i, t) \vdash \psi \text{ (by H. I.)}; \text{ and }
   \langle O - i, t : \chi \rangle \in \mathfrak{D}_\Delta \text{ then } M, (i, t) \vdash \chi \text{ (by H. I.)}
   \text{ iff } M, (i, t) \vdash (\psi \land \chi) \text{ (by Definition 2.8.11).}
2. \text{ If } X = P, O \text{ can only challenge once since } m := 1. \text{ We only deal with the case where the Opponent challenges the first conjunct}^{13}.
   \begin{enumerate}
   \item \text{ If } \psi \notin \Phi \text{ then: }
       \langle P - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } M, (i, t) \vdash \lnot \psi \text{ (by H. I.)}
       \text{ iff } M, (i, t) \vdash \lnot (\psi \land \chi) \text{ (by Definition 2.8.11).}
   \end{enumerate}

^{12}\text{Indeed it is possible that } z \neq t.
^{13}\text{The same reasoning can be applied in the case where the Opponent challenges the second conjunct.}
(b) If $\psi \in \Phi$ then either:

i. $\langle O - i, z : \psi \rangle \notin \mathcal{D}_\Delta$ for any contextual point $z$, so $i \notin V_\psi$ then $\mathcal{M}_i, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg(\psi \land \chi)$ (by Definition 2.8.11); or

ii. $\langle O - i, z : \psi \rangle \in \mathcal{D}_\Delta$ and $\langle P - i, t : \psi \rangle \in \mathcal{D}_\Delta$, then by Lemma 7.3.5 the Proponent wins $\mathcal{D}_\Delta$, contradicting the hypothesis of Lemma 7.3.14.

**Case 3:** $\varphi := \psi \lor \chi$

If $\langle X - i, t : \psi \lor \chi \rangle \in \mathcal{D}_\Delta$, then:

$\langle Y - i, t : ?\varphi \rangle \in \mathcal{D}_\Delta$.

1. If $X = O$, $O$ can only produce one defence since $m := 1$. Consequently:

   i. $\langle O - i, t : \psi \rangle \in \mathcal{D}_\Delta$ then $\mathcal{M}_i, (i, t) \models \psi$ (by H. I.) and $\mathcal{M}_j, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg(\psi \land \chi)$ (by Definition 2.8.11).

2. If $X = P$, the Proponent can change his defence since $n := 2$.

   a. If $\psi \notin \Phi$ and $\chi \notin \Phi$ then:

      i. $\langle P - i, t : \psi \rangle \in \mathcal{D}_\Delta$ then $\mathcal{M}_i, (i, t) \models \neg \psi$ (by H. I.) and $\mathcal{M}_j, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg(\psi \land \chi)$ (by Definition 2.8.11).

   b. If $\psi \in \Phi$ and $\chi \notin \Phi$ then either:

      i. $\langle O - i, z : \psi \rangle \notin \mathcal{D}_\Delta$ for any contextual point $z$, so $i \notin V_\psi$ then $\mathcal{M}_i, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg(\psi \land \chi)$ (by Definition 2.8.11); or

      ii. $\langle O - i, z : \psi \rangle \in \mathcal{D}_\Delta$ and $\langle P - i, t : \psi \rangle \in \mathcal{D}_\Delta$, then by Lemma 7.3.5 the Proponent wins $\mathcal{D}_\Delta$, contradicting the hypothesis of Lemma 7.3.14.  

   c. If $\psi \in \Phi$ and $\chi \in \Phi$ then either:

      i. $\langle O - i, z : \psi \rangle \notin \mathcal{D}_\Delta$ for any contextual point $z$, so $i \notin V_\psi$ then $\mathcal{M}_i, (i, t) \models \neg \psi$ (by Definition 7.3.12) and $\mathcal{M}_j, (i, t) \models \neg(\psi \land \chi)$ (by Definition 2.8.11); or

      ii. $\langle O - i, z : \psi \rangle \in \mathcal{D}_\Delta$ and $\langle P - i, t : \psi \rangle \in \mathcal{D}_\Delta$, then by Lemma 7.3.5 the Proponent wins $\mathcal{D}_\Delta$, contradicting the hypothesis of Lemma 7.3.14.

\[ \text{The same reasoning can be applied in the case where } \psi \notin \Phi \text{ and } \chi \in \Phi. \]

\[ \text{We only show the reasoning for } \psi. \text{ The same reasoning can be applied for } \chi. \]
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Case 4: \( \varphi := \bigcirc^{-1}\psi \)

If \( \langle X - i, t : \bigcirc^{-1}\psi \rangle \in \mathfrak{D}_\Delta \), then:
\[
\langle Y - i, t : ?\psi_u \rangle \in \mathfrak{D}_\Delta \text{ for all contextual points } u.
\]

1. If \( X = O \), then:
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, u) \models \psi \text{ (by H. I.)}. \text{ By Hypothesis Lemma 7.3.14, } \mathfrak{D}_\Delta \text{ is terminal, consequently:}
\]
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, v) \models \psi \text{ (by H. I.), and}
\]
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, w) \models \psi \text{ (by H. I.)}
\]
for all contextual points \( u \) that respect SR-4, SR-4.1 and SR-4.2

iff \( \mathcal{M}, (i, t) \models \bigcirc^{-1}\psi \) (by Definition 2.8.11).

2. If \( X = P \), then:

(a) If \( \psi \notin \Phi \) then:
\[
\langle P - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, u) \models \neg \psi \text{ (by H. I.)}
\]
for at least one contextual point \( u \) that respect SR-4

iff \( \mathcal{M}, (i, t) \models \neg \bigcirc^{-1}\psi \) (by Definition 2.8.11).

(b) If \( \psi \in \Phi \) then either:

i. \( \langle O - i, z : \psi \rangle \notin \mathfrak{D}_\Delta \) for any contextual point \( z \),

 so \( i \notin V_\psi \) then
\[
\mathcal{M}, (i, u) \models \neg \psi \text{ (by Definition 7.3.12)}
\]
iff \( \mathcal{M}, (i, t) \models \neg \bigcirc^{-1}\psi \) (by Definition 2.8.11); or

ii. \( \langle O - i, z : \psi \rangle \in \mathfrak{D}_\Delta \) and \( \langle P - i, t : \psi \rangle \in \mathfrak{D}_\Delta \) then by Lemma 7.3.5 the Proponent wins \( \mathfrak{D}_\Delta \), contradicting the hypothesis of Lemma 7.3.14.

Case 5: \( \varphi := \bigcirc \psi \)

If \( \langle X - i, t : \bigcirc \psi \rangle \in \mathfrak{D}_\Delta \), then:
\[
\langle Y - i, t : ?\psi_u \rangle \in \mathfrak{D}_\Delta \text{ for all contextual points } u.
\]

1. If \( X = O \), then:
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, u) \models \psi \text{ (by H. I.)}. \text{ By Hypothesis Lemma 7.3.14, } \mathfrak{D}_\Delta \text{ is terminal, consequently:}
\]
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, v) \models \psi \text{ (by H. I.), and}
\]
\[
\langle O - i, t : \psi \rangle \in \mathfrak{D}_\Delta \text{ then } \mathcal{M}, (i, w) \models \psi \text{ (by H. I.)}
\]
for all contextual points \( u \) that respect SR-4, SR-4.1 and SR-4.2

iff \( \mathcal{M}, (i, t) \models \bigcirc \psi \) (by Definition 2.8.11).

\[16\]Indeed it is possible that \( z \neq u \).
2. If $X = P$, then:

(a) If $\psi \notin \Phi$ then:
   $$\langle P - i, t : \psi \rangle \in \mathcal{D}_{\Delta} \text{ then } \mathcal{M},(i,t) \models \neg \psi \text{ (by H. I.)}$$
   for at least one contextual point $u$ that respect \textbf{SR-4}
   iff $\mathcal{M},(i,t) \models \neg \Box \psi$ (by Definition 2.8.11).

(b) If $\psi \in \Phi$ then either:
   i. $\langle O - i, z : \psi \rangle \notin \mathcal{D}_{\Delta}$ for any contextual point $z$, so $i \notin V_\psi$ then $\mathcal{M},(i,u) \models \neg \psi$ (by Definition 7.3.12)
   iff $\mathcal{M},(i,t) \models \neg \Box \psi$ (by Definition 2.8.11); or
   ii. $\langle O - i, z : \psi \rangle \in \mathcal{D}_{\Delta}$ and $\langle P - i, t : \psi \rangle \in \mathcal{D}_{\Delta}$ then by Lemma 7.3.5 the Proponent wins $\mathcal{D}_{\Delta}$, contradicting the hypothesis of Lemma 7.3.14.

\textbf{Cas 6: $\varphi := B\psi$}

If $\langle X - i, t : B\psi \rangle \in \mathcal{D}_{\Delta}$ then:

$$\langle Y - i, t : ?B_j \rangle \in \mathcal{D}_{\Delta} \text{ for all contextual points } j.$$

1. If $X = O$, then:

   $$\langle O - i, j, t : \psi \rangle \in \mathcal{D}_{\Delta} \text{ then } \mathcal{M},(j,t) \models \psi \text{ (by H. I.)}. \text{ By Hypothesis of Lemma 7.3.14, } \mathcal{D}_{\Delta} \text{ is terminal, consequently:}$$
   $$\langle O - i, k, t : \psi \rangle \in \mathcal{D}_{\Delta} \text{ then } \mathcal{M},(k,t) \models \psi \text{ (by H. I.), and}$$
   $$\vdots$$
   $$\langle O - i, l, t : \psi \rangle \in \mathcal{D}_{\Delta} \text{ then } \mathcal{M},(l,t) \models \psi \text{ (by H. I.)}$$
   for all contextual points $j$ that respect \textbf{SR-5, SR-5.4, SR-5.5, SR-5.6, SR-5.7 and SR-5.8}
   iff $\mathcal{M},(i,t) \models B\psi$ (by Definition 2.8.11).

2. If $X = P$, then:

   (a) If $\psi \notin \Phi$ then:
   $$\langle P - i, j, t : \psi \rangle \in \mathcal{D}_{\Delta} \text{ then } \mathcal{M},(j,t) \models \neg \psi \text{ (by H. I.)}$$
   for at least one contextual point $j$ that respect \textbf{SR-5}
   iff $\mathcal{M},(i,t) \models \neg B\psi$ (by Definition 2.8.11).

   (b) If $\psi \in \Phi$ then either:
      i. $\langle O - i, j, z : \psi \rangle \notin \mathcal{D}_{\Delta}$ for any contextual point $z$,\footnote{Indeed it is possible that $z \neq t$.} so $j \notin V_\psi$ then $\mathcal{M},(j,t) \models \neg \psi$ (by Definition 7.3.12)
      iff $\mathcal{M},(i,t) \models \neg B\psi$ (by Definition 2.8.11); or
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ii. \( \langle O - i, j, z : \psi \rangle \) \( \in \mathcal{D}_\Delta \) and \( \langle P - i, j, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then by Lemma 7.3.5 the Proponent wins \( \mathcal{D}_\Delta \), contradicting the hypothesis of Lemma 7.3.14.

**Cas 7:** \( \varphi := A\psi \)

If \( \langle X - i, t : A\psi \rangle \) \( \in \mathcal{D}_\Delta \) then:
\( \langle Y - i, t : ?A_j \rangle \) \( \in \mathcal{D}_\Delta \) for all contextual points \( j \).

1. If \( X = O \), then:
\( \langle O - i, j, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (j, t) \models \psi \) (by H. I.). By Hypothesis of Lemma 7.3.14, \( \mathcal{D}_\Delta \) is terminal, consequently:
\( \langle O - i, k, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (k, t) \models \psi \) (by H. I.), and
\( \langle O - i, l, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (l, t) \models \psi \) (by H. I.)
for all contextual points \( j \) that respect SR-5, SR-5.1, SR-5.3

iff \( M, (i, t) \models A\psi \) (by Definition 2.8.11).

2. If \( X = P \), then:

(a) If \( \psi \notin \Phi \) then:
\( \langle P - i, j, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (j, t) \models \neg \psi \) (by H. I.)
for at least one contextual point \( j \) that respect SR-5

iff \( M, (i, t) \models \neg A\psi \) (by Definition 2.8.11).

(b) If \( \psi \in \Phi \) then either:
   i. \( \langle O - i, j, z : \psi \rangle \) \( \notin \mathcal{D}_\Delta \) for any contextual point \( z \), so \( j \notin V_\psi \) then
\( M, (j, t) \models \neg \psi \) (by Definition 7.3.12)
   iff \( M, (i, t) \models \neg A\psi \) (by Definition 2.8.11); or
   ii. \( \langle O - i, j, z : \psi \rangle \) \( \in \mathcal{D}_\Delta \) and \( \langle P - i, j, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then by Lemma 7.3.5
the Proponent wins \( \mathcal{D}_\Delta \), contradicting the hypothesis of Lemma 7.3.14.

**Cas 8:** \( \varphi := I\psi \)

If \( \langle X - i, t : I\psi \rangle \) \( \in \mathcal{D}_\Delta \) then:
\( \langle Y - i, t : ?I_j \rangle \) \( \in \mathcal{D}_\Delta \) or \( \langle Y - i, t : !I_j \rangle \) \( \in \mathcal{D}_\Delta \) for all contextual points \( j \).

1. If \( X = O \), the Proponent can change his challenge since \( n := 2 \). Consequently:
\( \langle O - i, j, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (j, t) \models \psi \) (by H. I.). By Hypothesis of Lemma 7.3.14, \( \mathcal{D}_\Delta \) is terminal, consequently:
\( \langle O - i, k, t : \psi \rangle \) \( \in \mathcal{D}_\Delta \) then \( M, (k, t) \models \psi \) (by H. I.), and
\( \vdots \).
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\[ \{ O - i, l, t : \psi \} \in D_{\Delta} \text{ then } M, (l, t) \models \psi \text{ (by H. I.)} \]

for all contextual points \( j \) that respect \( \text{SR-5, SR-5.2} \); and

\[ \{ O - i, t : ?I_j^* \} \in D_{\Delta} \text{ then } j \in I_i(i) \text{ (by H. I.)} \]

for all contextual points \( j \) that respect \( \text{SR-5, SR-5.2, SR-5.X} \) that is,

\[ M, (i, j, t) \not\models \psi \text{ (by Definition 2.8.11).} \]

2. If \( X = P, O \) can only challenge once since \( m := 1 \).

(a) If \( \psi \notin \Phi \) then:

\[ \{ P - i, j, t : \psi \} \in D_{\Delta} \text{ then } M, (j, t) \models \neg \psi \text{ (by H. I.)} \]

for at least one contextual point \( j \) that respect \( \text{SR-5} \)

\[ M, (i, t) \models \neg I\psi \text{ (by Definition 2.8.11).} \]

(b) If \( \psi \in \Phi \) then either:

i. \( \{ O - i, j, z : \psi \} \notin D_{\Delta} \) for any contextual point \( z \), so \( j \notin V_{\psi} \) then

\[ M, (j, t) \models \neg \psi \text{ (by Definition 7.3.12) } \]

\[ M, (i, t) \models \neg I\psi \text{ (by Definition 2.8.11); or} \]

ii. \( \{ O - i, j, z : \psi \} \in D_{\Delta} \) and \( \{ P - i, j, t : \psi \} \in D_{\Delta} \) then by Lemma 7.3.5

the Proponent wins \( D_{\Delta} \), contradicting the hypothesis of Lemma 7.3.14; or

(a) \( \{ O - i, t : ?I_j \} \notin D_{\Delta} \) and \( \{ O - i, t : ?B_j \} \notin D_{\Delta} \) and \( \{ O - i, t : ?I_j^* \} \notin D_{\Delta} \)

then \( j \notin I_i(i) \) for any contextual points \( j \) that respect \( \text{SR-5, SR-5.X} \)

that is, \( M, (i, j, t) \not\models \psi \) (by Definition 7.3.12)

\[ M, (i, t) \models \neg I\psi \text{ (by Definition 2.8.11); or} \]

(b) \( \{ O - i, t : ?I_j \} \in D_{\Delta} \) or \( \{ O - i, t : ?B_j \} \in D_{\Delta} \) or \( \{ O - i, t : ?I_j^* \} \in D_{\Delta} \),

and \( \{ P - i, t : ?I_j^* \} \in D_{\Delta} \) then by Lemma 7.3.5 the Proponent wins \( D_{\Delta} \),

contradicting the hypothesis of Lemma 7.3.14.

\[ \square \]

In Lemma 7.3.14, we assumed players can change their challenges or defences
on modal operators as many time as needed. This allows to establish a symmetric
link between the moves in \( D_{\Delta} \) and a model \( M \) satisfying \( \Delta \) since it allows to
check every possible situations of a model. In the next theorem, we show that it
is enough to consider a play \( d_{\Delta} \) where \( m := 1 \) and \( n := 2 \).

7.3.16. THEOREM. The repetition ranks \( m := 1 \) and \( n := 2 \) are enough to check if
there exists a winning strategy for the Proponent in \( D_{\Delta} \).
7.3. Proof. We proceed by induction on the length of $\varphi$ in the scope of modal operators.

1. $\bigcirc^{-1}$ operator:

(a) **Base**: We show that if the Proponent wins $\mathcal{D}_\Delta$, then a repetition rank 1 is enough for a challenge on a formula $\bigcirc^{-1}\varphi$ of the Opponent where $\text{len}(\varphi) = 1$:

\[
\langle O - i, t : \bigcirc^{-1}\varphi \rangle \in \mathcal{D}_\Delta
\]

\[
\langle P - i, t : ?\bigcirc^{-1}_u \rangle \in \mathcal{D}_\Delta \text{ or } \langle P - i, t : ?\bigcirc^{-1}_v \rangle \in \mathcal{D}_\Delta \text{ for any contextual points } u \text{ and } v.\]

However these contextual points have to respect SR-4, SR-4.1 and SR-4.2 that is, $u = v$. The Proponent can only choose one single instant to challenge a $P$ operator whatever his repetition rank is.

(b) **Induction Hypothesis**: If the Proponent wins $\mathcal{D}_\Delta$ then a repetition rank 1 is enough for a challenge on a formula $\bigcirc^{-1}\varphi$ of the Opponent if $\text{len}(\varphi) = n$.

(c) **Inductive step**: The same reasoning as the basic case can be applied to show that if the Proponent wins $\mathcal{D}_\Delta$, a repetition rank 1 is enough for a challenge on a formula $\bigcirc^{-1}\varphi$ of the Opponent where $\text{len}(\varphi) \geq n + 1$.

2. $\bigcirc$ operator:

(a) **Base**: We show that if the Proponent wins $\mathcal{D}_\Delta$, then a repetition rank 1 is enough for a challenge on a formula $\bigcirc\varphi$ of the Opponent where $\text{len}(\varphi) = 1$:

\[
\langle O - i, t : \bigcirc\varphi \rangle \in \mathcal{D}_\Delta
\]

\[
\langle P - i, t : ?\bigcirc_u \rangle \in \mathcal{D}_\Delta \text{ or } \langle P - i, t : ?\bigcirc_v \rangle \in \mathcal{D}_\Delta \text{ for any contextual points } u \text{ and } v,\]

so

\[
\langle O - i, t.u : \varphi \rangle \in \mathcal{D}_\Delta \text{ or } \langle O - i, t.v : \varphi \rangle \in \mathcal{D}_\Delta.\]

The Proponent can then state the atomic formula $\varphi$ in the contextual point $i$ at every instant (by SR-2). The Proponent only needs to challenge once whatever the contextual point he chooses.

(b) **Induction Hypothesis**: If the Proponent wins $\mathcal{D}_\Delta$ then a repetition rank 1 is enough for a challenge on a formula $\bigcirc\varphi$ of the Opponent if $\text{len}(\varphi) = n$.

(c) **Inductive step**: We show that if the Proponent wins $\mathcal{D}_\Delta$, then a repetition rank 1 is enough for a challenge on a formula $\bigcirc\varphi$ of the Opponent where $\text{len}(\varphi) \geq n + 1$:

\[
\langle O - i, t : \bigcirc\varphi \rangle \in \mathcal{D}_\Delta
\]
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\(\langle P - i, t : ?\Omega_u \rangle \in \mathfrak{D}_\Delta\) or \(\langle P - i, t : \Omega_v \rangle \in \mathfrak{D}_\Delta\) for any contextual points \(u\) and \(v\), so
\(\langle O - i, t, u : \varphi \rangle \in \mathfrak{D}_\Delta\) or \(\langle O - i, t, v : \varphi \rangle \in \mathfrak{D}_\Delta\). Since the Proponent wins \(\mathfrak{D}_\Delta\), we only need to consider the choice \(u\) or \(v\) leading him to win.

3. **B operator:**

   (a) **Base:** We show that if the Proponent wins \(\mathfrak{D}_\Delta\), then a repetition rank 1 is enough for a challenge on a formula \(B\varphi\) of the Opponent where \(\text{len}(\varphi) = 1\):
   \(\langle O - i, t : B\varphi \rangle \in \mathfrak{D}_\Delta\)
   \(\langle P - i, t : ?B_j \rangle \in \mathfrak{D}_\Delta\) or \(\langle P - i, t : ?B_k \rangle \in \mathfrak{D}_\Delta\) for any contextual points \(j\) and \(k\), so
   \(\langle O - i, j, t : \varphi \rangle \in \mathfrak{D}_\Delta\) or \(\langle O - i, k, t : \varphi \rangle \in \mathfrak{D}_\Delta\). The Proponent can then state the atomic formula \(\varphi\) in the contextual point \(j\) or \(k\) at any instant (by SR-2). Since he wins \(\mathfrak{D}_\Delta\), we only need to consider the choice \(j\) or \(k\) leading him to win.

   (b) **Induction Hypothesis:** If the Proponent wins \(\mathfrak{D}_\Delta\) then a repetition rank 1 is enough for a challenge on a formula \(B\varphi\) of the Opponent if \(\text{len}(\varphi) = n\).

   (c) **Inductive step:** We show that if the Proponent wins \(\mathfrak{D}_\Delta\), then a repetition rank 1 is enough for a challenge on a formula \(B\varphi\) of the Opponent where \(\text{len}(\varphi) \geq n + 1\):
   \(\langle O - i, t : B\varphi \rangle \in \mathfrak{D}_\Delta\)
   \(\langle P - i, t : ?B_j \rangle \in \mathfrak{D}_\Delta\) or \(\langle P - i, t : ?B_k \rangle \in \mathfrak{D}_\Delta\) for any contextual points \(j\) and \(k\), so
   \(\langle O - i, j, t : \varphi \rangle \in \mathfrak{D}_\Delta\) or \(\langle O - i, k, t : \varphi \rangle \in \mathfrak{D}_\Delta\). Since the Proponent wins \(\mathfrak{D}_\Delta\), we only need to consider the choice \(j\) or \(k\) leading him to win.

4. **A operator:**

   (a) **Base:** We show that if the Proponent wins \(\mathfrak{D}_\Delta\), then a repetition rank 1 is enough for a challenge on a formula \(A\varphi\) of the Opponent where \(\text{len}(\varphi) = 1\):
   \(\langle O - i, t : A\varphi \rangle \in \mathfrak{D}_\Delta\)
   \(\langle P - i, t : ?A_j \rangle \in \mathfrak{D}_\Delta\) or \(\langle P - i, t : ?A_k \rangle \in \mathfrak{D}_\Delta\) for any contextual points \(j\) and \(k\), so
   \(\langle O - i, j, t : \varphi \rangle \in \mathfrak{D}_\Delta\) or \(\langle O - i, k, t : \varphi \rangle \in \mathfrak{D}_\Delta\). Then Proponent can then state the atomic formula \(\varphi\) in the contextual point \(j\) or \(k\) at any instant (by SR-2). Since he wins \(\mathfrak{D}_\Delta\), we only need to consider the choice \(j\) or \(k\) leading him to win.
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(b) **Induction Hypothesis:** If the Proponent wins \( D_\Delta \) then a repetition rank 1 is enough for a challenge on a formula \( A\varphi \) of the Opponent if \( \text{len}(\varphi) = n \).

(c) **Inductive step:** We show that if the Proponent wins \( D_\Delta \), then a repetition rank 1 is enough for a challenge on a formula \( A\varphi \) of the Opponent where \( \text{len}(\varphi) \geq n + 1 \):

\[
\langle O - i, t : A\varphi \rangle \in D_\Delta
\]

\[
\langle P - i, t : ?A_j \rangle \in D_\Delta \text{ or } \langle P - i, t : ?A_k \rangle \in D_\Delta \text{ for any contextual points } j \text{ and } k, \text{ so }
\]

\[
\langle O - i, j, t : \varphi \rangle \in D_\Delta \text{ or } \langle O - i, k, t : \varphi \rangle \in D_\Delta. \text{ Since the Proponent wins } D_\Delta, \text{ we only need to consider the choice } j \text{ or } k \text{ leading him to win.}
\]

5. \( I \) operator:

(a) **Base:** We show that if the Proponent wins \( D_\Delta \), then a repetition rank 1 is enough for a challenge on a formula \( I\varphi \) of the Opponent where \( \text{len}(\varphi) = 1 \):

\[
\langle O - i, t : I\varphi \rangle \in D_\Delta
\]

\[
\langle P - i, t : ?I_j \rangle \in D_\Delta \text{ or } \langle P - i, t : ?I_k \rangle \in D_\Delta, \text{ or } \langle P - i, t : !I_j \rangle \in D_\Delta \text{ or } \langle P - i, t : !I_k \rangle \in D_\Delta \text{ for any contextual points } j \text{ and } k, \text{ so }
\]

\[
\langle O - i, j, t : \varphi \rangle \in D_\Delta \text{ or } \langle O - i, k, t : \varphi \rangle \in D_\Delta, \text{ or } \langle O - i, t : ?I^*_j \rangle \in D_\Delta \text{ or } \langle O - i, t : ?I^*_k \rangle \in D_\Delta. \text{ The Proponent can then state the atomic formula } \varphi \text{ in the contextual point } j \text{ or } k \text{ at any instant (by SR-2), or he can then use the contextual point } j \text{ or } k \text{ at } t \text{ to challenge a } I \text{ operator.}
\]

Since he wins \( D_\Delta \), we only need to consider the choice leading him to win.

(b) **Induction Hypothesis:** If the Proponent wins \( D_\Delta \) then a repetition rank 1 is enough for a challenge on a formula \( I\varphi \) of the Opponent if \( \text{len}(\varphi) = n \).

(c) **Inductive step:** We show that if the Proponent wins \( D_\Delta \), then a repetition rank 1 is enough for a challenge on a formula \( I\varphi \) of the Opponent where \( \text{len}(\varphi) \geq n + 1 \):

\[
\langle O - i, t : I\varphi \rangle \in D_\Delta
\]

\[
\langle P - i, t : ?I_j \rangle \in D_\Delta \text{ or } \langle P - i, t : ?I_k \rangle \in D_\Delta, \text{ or } \langle P - i, t : !I_j \rangle \in D_\Delta \text{ or } \langle P - i, t : !I_k \rangle \in D_\Delta \text{ for any contextual points } j \text{ and } k, \text{ so }
\]

\[
\langle O - i, j, t : \varphi \rangle \in D_\Delta \text{ or } \langle O - i, k, t : \varphi \rangle \in D_\Delta, \text{ or } \langle O - i, t : ?I^*_j \rangle \in D_\Delta \text{ or } \langle O - i, t : ?I^*_k \rangle \in D_\Delta. \text{ Since the Proponent wins } D_\Delta, \text{ we only need to consider the choice leading him to win.}
\]

\[\square\]
If the Proponent wins $\mathfrak{D}_\Delta$, a repetition rank 1 is enough for a challenge on a modal formula $\square \varphi$ such that $\square$ is any $(\bigcirc^{-1}, \bigcirc, B, A, I)$ of the Opponent, whatever the length of $\varphi$ is. A repetition rank 2 is only required for Proponent to challenge a conjunction or change his defence against disjunction. Then we can deal with $d_\Delta$ instead of $\mathfrak{D}_\Delta$ for the completeness theorem.

Completeness Theorem

7.3.18. Theorem. If $\Delta$ is a valid formula in $L_{PLS^*}$, then the Proponent wins $d_\Delta$ with the rules of DTDL.

7.3.19. Proof. We prove completeness by showing the contrapositive that is, we show that if the Proponent loses $d_\Delta$ then $\Delta$ is not a valid formula in $L_{PLS^*}$. By Lemma 7.3.14 and Theorem 7.3.16, if the Proponent loses $d_\Delta$, then there exists a branching time belief revision model $(M,(i.t))$ satisfying $\neg \Delta$. Consequently, there exists a model $(M,(i.t))$ such that $\Delta$ is not satisfiable in $(M,(i.t))$ and then, $\Delta$ is not a valid formula in $L_{PLS^*}$. \qed

Conclusion

DTDL allows an argumentative interpretation of the belief revision policy of Bonanno. The notion of belief and information as well as their relation is then interpreted in terms of choice in a dialogical framework. We pointed out the interpretation of the interplay between information and beliefs as well as the interplay between the beliefs themselves – namely, initial beliefs and revised beliefs – as the interplay between the choices of the players. In particular we showed the interplay between the choices of contextual points to challenge belief and/or information operators. We noticed that this is the interplay that defines a particular belief revision policy, namely the belief revision policy of Bonanno. Finally, we underlined the originality of the interpretation of the information operator through the notion of choice. Indeed this notion of choice is directly implemented in the dialogue in the sense that players discuss explicitly the choices they can/should make with respect to their previous arguments.