Pricing swaptions and coupon bond options in affine term structure models

Schrager, D.F.; Pelsser, A.A.J.

Publication date
2005

Citation for published version (APA):
Pricing Swaptions and Coupon Bond Options in Affine Term Structure Models

David F. Schrager\textsuperscript{1,2,†}, Antoon A.J. Pelsser\textsuperscript{3,4}

This version:
August 16, 2005

Abstract
We propose an approach to find an approximate price of a swaption in Affine Term Structure Models. Our approach is based on the derivation of approximate dynamics in which the volatility of the Forward Swap Rate is itself an affine function of the factors. Hence we remain in the Affine framework and well known results on transforms and transform inversion can be used to obtain swaption prices in ways similar to bond options (i.e. caplets). We demonstrate that we can also obtain a closed form formula for the approximate price which is based on square-root dynamics for the swap rate. The latter approximation is extremely fast while remaining accurate. The method can be easily generalized to price options on coupon bonds. Computational time compares favorably with other approximation methods. Numerical results on the quality of the approximation are excellent. Our results show that in Affine models, analogously to the LIBOR Market Model, LIBOR and Swap rates are driven by approximately the same type of (in this case affine) dynamics.

Keywords: Swaption, Coupon Bond Option, Affine Term Structure Models, Change of Numeraire, Swap Measure, Conditional Characteristic Function, Option Pricing using Transform Inversion

\textsuperscript{1}ABN AMRO Bank, Market Risk Modelling and Product Analysis, PO Box 283, 1000 EA Amsterdam, The Netherlands
\textsuperscript{2}University of Amsterdam, Dept. of Quantitative Economics, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands, Tel. +31 - (0)20 525 4125, e-mail: d.f.schrager@uva.nl
\textsuperscript{3}Erasmus University Rotterdam, Econometric Institute, PO Box 1738, 3000 DR Rotterdam, The Netherlands, Tel. +31 - (0)10 408 1259, Fax. +31 - (0)10 408 9162, e-mail: pelsser@few.eur.nl
\textsuperscript{4}ING, Corporate Insurance Risk Management, PO Box 810, 1000 AV Amsterdam, The Netherlands, Tel. +31 - (0)20 541 8858, Fax. +31 (0)20 541 8552
†Corresponding author
1 Introduction

Swaptions are among the most liquid derivatives being traded in the financial markets. The theoretical price of these instruments depends on the term structure model being used. From a different point of view it is critical for term structure models to be able to fit swaption prices well. The Affine Term Structure Models (ATSMs) by Duffie and Kan (1996) are a popular framework for analyzing term structure movements and pricing interest rate derivatives. However in these models swaption prices cannot be obtained in closed form. In this paper we introduce a fast and an accurate approximation to the price of a swaption in ATSMs. This results in two related approximation methods. The first, the slower more accurate one, accomplishes that the price of a swaption can be obtained by exactly the same techniques needed to price bond options or caplets (see Duffie, Pan and Singleton, 2000). The second results in a closed form formula which only requires a single numerical integration.

To our knowledge this is the fastest swaption pricing formula in ATSMs which also generalizes to complicated multifactor models. Furthermore the method can be easily generalized to price options on coupon bonds. Hence throughout the paper we write swaptions when we could have written both swaptions and coupon bond options.

The price of a swaption can be written in terms of the distribution of the swap rate under the relevant swap measure. Our methods are based on approximating the dynamics of the swap rate under this measure. We derive the dynamics of the swap rate and the underlying factors under the swap measure for a general affine term structure model. Then we suggest to approximate the dynamics by replacing some low variance martingales (LVM) by their time zero values. After this approximation we are again in the affine setting but now under the swap measure (and with time dependent drift for the factors and time dependent volatility for the swap rate). This approach allows us to remain in the affine setup and use results on transforms and transform inversion of ATSMs to price the swaption. The accuracy of this approximation is excellent. Building on the approximate stochastic differential equation (SDE) for the swap rate we can also derive a square root process for the swap rate. This yields a closed form expression for the approximate price of a swaption in a general ATSM. The resulting approximation is faster than the previous one while remaining accurate. In a Gaussian framework this simplifies considerably and analytical pricing formulas can be derived.

The technique of replacing LVM by their martingale values is also used in the context of the LIBOR Market Model (LMM) to derive swap volatilities in this model, see Brace et al. (2001) and Hull and White (2000). Our results contribute to the discussion on the existence of a central interest rate model by showing that in the affine class of term structure models, swap and LIBOR rates have (approximately) distributions of the same type. It seems that this property is not exclusive for LMM but rather is common for every term structure model.

Other papers considering swaption pricing in ATSMs are Munk (1999), Singleton and Umantsev (2002) and Collin-Dufresne and Goldstein (2002). The paper by Singleton and Umantsev introduces a method which is based on approximation of the exercise region in the space of underlying factors by line segments. This reduces the exercise probability of the swaption to one of the form of that of a caplet. They then use transform inversion to calculate the required exercise probabilities. Similar to our method they require a simplifying assumption to return to the affine framework. Our method differs from theirs as it doesn’t approximate the exercise region but finds approximate affine dynamics for the swap rate. In this way we decrease computational time and simplify implementation in general affine models.

Collin-Dufresne and Goldstein (2002) propose to approximate the price of a swaption based on an Edgeworth
expansion of the density of the coupon bond price (i.e. swap rate). This requires the calculation of the moments of the coupon bond, through the joint moments of the individual zero coupon bonds, which are available in closed form. Instead, our results on approximate swap rate dynamics enable us to calculate the approximate Conditional Characteristic Function (CCF) of the swap rate directly. This allows us to use well known transform inversion techniques and avoid the computation of joint moments of zero coupon bonds which could become very time consuming. Therefore, contrary to their approach, our implementation easily generalizes to more general ATSMs without an increase in computational effort.

Munk (1999) shows that the price of a European option on a coupon bond (i.e. a swaption) is approximately proportional to the price of a European option on a zero coupon bond with maturity equal to the stochastic duration of the coupon bond. The stochastic duration approximation is closely related to our method as it uses a similar approach to approximate the volatility of a coupon bond / swaption. However our method is based on a formal derivation of the factor and swap rate dynamics under the swap measure, whereas stochastic duration is more of an ad hoc approximation.

We improve upon the existing literature in either speed, accuracy or both. Most importantly, our methods are easier to apply to models with a higher number of factors or correlated factors than the approximations by Munk (1999), Singleton and Umantsev (2002) and Collin-Dufresne and Goldstein (2002).

The outline of the paper is the following. In section 2 we shortly review the mechanics of swaptions and ATSMs. In section 3 we review the alternative approaches by Munk (1999), Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2002). Section 4 introduces our approximation method in the context of the Vasicek model. In section 5 we present our approximations for general ATSMs and discuss extensions in several directions. The special case of Gaussian TSMs is discussed in an appendix. Section 6 discusses the quality, speed and implementation of the approximation method in comparison with other methods in the literature. Section 7 concludes.

2 Preliminaries: Swaps and Swaptions, Affine Term Structure Models

Let \( D(t, T) \) be the time \( t \) price of a zero coupon bond with maturity \( T \). A forward LIBOR rate \( L_{TS}(t) \) is the interest rate one can contract for at time \( t \) to put money in a money-market account for the time period \([T, S]\). We define,

\[
L_{TS}(t) = \frac{1}{\Delta_{TS}} \frac{D(t, T) - D(t, S)}{D(t, S)}
\]

Where \( \Delta_{TS} \) is the LIBOR market convention for the calculation of the daycount fraction for the period \([T, S]\). In the market, the tenor of the LIBOR rate, \( S - T \) is usually fixed at three or six months. Note that the LIBOR rate is fixed at time \( T \), but is not paid until time \( S \).

An interest rate swap is a contract in which two parties agree to exchange a set of fixed cash flows, consisting of a fixed rate \( K \) on the swap principal \( A \), for a set of floating rate payments, consisting of the LIBOR rate on the principal \( A \). In a payer swap you pay the fixed side and receive floating, in a receiver swap you receive the
fixed side (and pay floating). Given a set of dates \( T_i, i = n+1, \ldots, N \), at which swap payments are to be made, the value at time \( t \) of a (payer) swap contract starting at time \( T_n \) (paying out for the first time at \( T_{n+1} \)) and lasting until \( T_N \) with a principal of 1 and fixed payments at rate \( K \), is given by,

\[
V_{n,N}^{\text{pay}} (t) = V_{n,N}^{flo} (t) - V_{n,N}^{fix} (t) = \{ D(t, T_n) - D(t, T_N) \} - K \sum_{i=n+1}^{N} \Delta Y_{i-1} D(t, T_i)
\]  

(1)

Where \( \Delta Y_{i-1} \) is the market convention for the calculation of the daycount fraction for the swap payment at \( T_i \). Again given a set of payment dates \( T_i \), a Forward Par Swap Rate \( y_{n,N} (t) \) is defined by the fixed rate for which the value of the (forward starting) swap equals zero, solving (1) gives,

\[
y_{n,N} (t) = \frac{D(t, T_n) - D(t, T_N)}{\sum_{i=n+1}^{N} \Delta Y_{i-1} D(t, T_i)}
\]  

(2)

The rate \( y_{n,N} (t) \) is the (arbitrage free) rate at which at time \( t \) a person would like to enter into a swap contract starting at time \( T_n \) (paying out for the first time at \( T_{n+1} \)) and lasting until \( T_N \). If we look carefully we see that the 1 year forward LIBOR rate is equal to the 1 year forward par swap rate. The denominator of the swap rate, \( P_{n+1,N} (t) \), is called the Present Value of a Basis Point (PVBP) as it corresponds to the increase in value of the fixed side of the swap if the swap rate increases.

A swaption gives the holder the right to enter into a particular swap contract. A swaption with option maturity \( S_1 \) and swap maturity \( S_2 \) is termed a \( S_1 \times S_2 \)-swaption. The total timespan associated with the swaption is then \( S_1 + S_2 \). When the strike equals the forward par swap rate the option is At-The-Money-Forward (ATMF). A payer swaption gives the holder the right to enter into a payer swap and can be seen as a call option on a swap rate. The option has a payoff at time \( T_n \), the option maturity, of,

\[
[V_{n,N}^{\text{pay}} (T_n)]^+ = \left[ \{ D(T_n, T_n) - D(T_n, T_N) \} - K \sum_{i=n+1}^{N} \Delta Y_{i-1} D(T_n, T_i) \right]^+
\]  

(3)

where \( K \) denotes the strike rate of the swaption. It can be seen from the first line of (3) the price of a payer swaption can be interpreted as a European put option with strike 1 on a coupon bond with coupon \( K \). The second line follows from the definition of the forward swap rate. Equivalently a receiver swaption can be seen as a put option on a swap rate. Let \( M_t = \exp \left( \int_0^t r_s ds \right) \) be the money market account at time \( t \). Assuming absence of arbitrage, the value of a (payer) swaption with strike \( K \) at time \( t < T_n \), denoted by \( \text{PS}_t (K) \), can be expressed by the following risk-neutral conditional expectation,

\[
\text{PS}_t (K) = M_t E_t^Q \left\{ \frac{P_{n+1,N}(T_n)}{M_{T_n}} [y_{n,N}(T_n) - K]^+ \right\}
\]  

(4)

which can be rewritten using familiar Change of Numeraire Techniques (see Geman et al., 1995) as,

\[
\text{PS}_t (K) = P_{n+1,N} (t) E_t^{Q_{n+1,N}} \left\{ [y_{n,N}(T_n) - K]^+ \right\}
\]  

(5)

Where we let \( Q_{n+1,N} \) be the swap measure corresponding to a particular PVBP, \( P_{n+1,N} \), as numeraire. The price of the corresponding receiver swaption, \( \text{RS}_t (K) \), is then,

\[
\text{RS}_t (K) = P_{n+1,N} (t) E_t^{Q_{n+1,N}} \left\{ [K - y_{n,N}(T_n)]^+ \right\}
\]  

(6)
Note that under this swap measure the corresponding swap rate, $y_{n,N}$, is a martingale. The Radon-Nikodym derivative for this change of numeraire equals the ratio of numeraires, i.e. $\frac{dQ^{n+1,N}}{dQ} = \frac{P_{n+1,N}(T_n)}{P_{n+1}(T_n)}$. The change of numeraire shows explicitly why swaptions can be viewed as options on swap rates. This particular choice of numeraire can be attributed to Jamshidian (1998).

Affine Term Structure Models were introduced by Duffie and Kan (1996). Other publications include, among others, Dai and Singleton (2000) and Duffee (2002). Recently the Affine framework was extended to include jump diffusions by Duffie, Pan and Singleton (2000), from hereon DPS. In this paper we follow Singleton and Umantsev (2002) in our definition of the family of affine term structure models. In these models the short rate is modeled as an affine function of some latent factors, which is a diffusion process.

$$r_t = \omega_0 + \omega'_X X_t$$

(7)

where $\omega_0$ is a scalar and $\omega'X$ is an $M$ vector. The $M$-dimensional factor dynamics are given by the following diffusion,

$$dX_t = A(\theta - X_t)dt + \Sigma\sqrt{V_t}dW_t^Q$$

(8)

where $W_t^Q$ is an $M$-dimensional Brownian Motion under the risk-neutral measure, $A$ and $\Sigma$ are $M \times M$ matrices and $\theta$ is an $M$ vector. The matrix $V_t$ is a diagonal matrix holding the diffusion coefficients of the factors on the diagonal, i.e.

$$V_{t,(ii)} = \alpha_i + \beta'_i X_t \quad i = 1, 2, ..., M$$

(9)

where the $\beta_i$ are $M$ vectors. Or directly in matrix notation, defining the matrix $\beta = \left[ \beta_1 \cdots \beta_M \right]'$ and the vector $\alpha = \left[ \alpha_1 \cdots \alpha_M \right]'$, we have$^1$,

$$V_t = diag(\alpha + \beta X_t)$$

The instantaneous drift and variance of the factors are again affine functions of the factors. As a result bond prices are exponentially affine in the factors, with coefficients which can be obtained by solving a system of ODE’s, known as Riccati equations. Thus in ATSMs we have, $D(t, T) = \exp \left( A(t, T) - B(t, T) \cdot X_t \right)$. Applying Itô’s lemma to $D(t, T)$ gives,

$$dD(t, T) = r_t D(t, T)dt - B(t, T) D(t, T) \Sigma \sqrt{V_t}dW_t^Q$$

Usually when setting up an affine model one also specifies a vector of market prices of risk for the factors. Since we are concerned with option pricing, we require only knowledge of the $Q$ dynamics of the factors and hence leave the market prices of risk unspecified. Discussion on specification of these market prices is given in i.a. Duffee (2002).

Besides closed form solutions for bond prices, options on zero coupon bonds (i.e. caplets) are easily priced in affine models using transform inversion techniques (DPS, Carr and Madan, 1998). To see this, let $Q^T$ denote the $T$-Forward measure, write for the time $t$ price $C_t(K, T_0, T)$ of a call option on a zero coupon bond with strike $K$, maturity $T$ and option maturity $T_0$,

$$C_t(K, T_0, T) = E_t^Q \left[ e^{-\int_0^{T_0} r_s ds} (D(T_0, T) - K)^+ \right]$$

$$= D(t, T_0) E_t^{Q^T_0} \left[ (D(T_0, T) - K)^+ \right]$$

$^1$If $x$ is an $M$-vector then we define $\text{diag}(x)$ to be the $M \times M$ diagonal matrix with the elements of $x$ on the diagonal.
Now define the CCF of the log bond price,

\[ \varphi (v, t, T_0) = E^{Q_{T_0}} \left[ \exp \{ iv \ln (D (T_0, T)) \} \right] \] (10)

Because \( D (T_0, T) \) is exponentially affine in the factors \( \varphi \) is known in closed form, see DPS. Further define the dampened call price \( c_t (K) = \exp (\alpha \ln (K)) C_t (K, T_0, T) \) with dampening coefficient \( \alpha > 0 \) and the dampened call transform,

\[ \xi (u, t, T_0) = \int_{-\infty}^{\infty} \exp (iu \ln (K)) c_t (K) dK \] (11)

Then (10) and (11) are related by

\[ \xi (u, t, T_0) = \frac{D (t, T_0)}{\alpha^2 + \alpha - u^2 + i (2\alpha + 1) u} \] (12)

see Carr and Madan (1999). The price of the call option can now be calculated by a single integration,

\[ C_t (K, T_0, T) = \exp \left( -\frac{\alpha \ln (K)}{\pi} \right) \int_{-\infty}^{\infty} \text{Re} \left[ \exp \left( -iu \ln (K) \right) \xi (u, t, T_0) \right] du \]

It is important to note that if we know the CCF of \( y_{n, N} \) we could also price swaptions through transform inversion\(^2\). This would for instance be the case if the dynamics of \( y_{n, N} \) were itself to be affine in the factors. In section 4 we propose an approximation method which is partly based on this observation.

### 3 Literature Review

In this section we give a short technical review of currently known approximation methods to the price of a swaption. Whereas we base our approximation method on (5), both Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2002) rewrite (4) as,

\[ \text{PS}_t (K) = D (t, T_n) P^{Q_{T_n}}_t (y_{n, N} (T_n) > K) - (1 + \Delta_{N-1}^{Y} D (t, T_N) P^{Q_{T_N}}_t (y_{n, N} (T_n) > K) \] (13)

using a change of measure to several forward measures. The difficulty is that \( y_{n, N} \) is not a martingale under any of the forward measures. Now both methods are based on the approximation of the exercise probabilities under the different forward measures. The method by Collin-Dufresne and Goldstein is based on an Edgeworth expansion of the distribution of a coupon bond with coupon \( K \). As the \( D (t, T_i) \) are exponential affine functions of the factors, the moments of the coupon bond are exponential affine as well and the coefficients can be obtained by solving the familiar Riccati equations. The use of an Edgeworth expansion is validated by the low volatility of fixed income instruments and the resulting approximation is tight. However note that to determine the \( k \)-th

---

\(^2\)In the bond option case we know the CCF of the logarithm of the underlying in closed form. When the CCF of the underlying itself is known in closed form the procedure is similar and can be found in Lee (2004).
moment of $\sum_{i=1}^{j} c_i D (t, T_i)$ one has to determine (and sum) $\binom{1+k-1}{k}$ joint moments of zero coupon bonds. This would equal 11,628 for the fifth moment of a 15 year swap rate. Since these individual moments have to be determined by solving Riccati equations this could slow the procedure down considerably when no analytical solutions are available.

Taking a different point of view Singleton and Umantsev try to approximate the exercise probability through an approximation of the exercise region itself. They propose to linearize the exercise region, i.e. approximate it by a hyperplane, and use the approximation $P_{Q^T_N} (y_n, N (T_n) > K) \approx P_{Q^T_N} (a \cdot X_{T_n} > b)$, where the vector $a$ and the constant $b$ are to be determined using some procedure involving least squares.

The stochastic duration approximation by Munk postulates the following approximate price for a payer swaption (i.e. a put option on a coupon bond with coupon $K$ and strike 1),

$$P_{S_t} (K) = D (t, T_n) E_{Q^T_n} [1 - \xi D (T_n, t + SD (t))]^+$$

where $\xi = \frac{K \cdot PV BP_{p+1, N} (t) + D (t, T_n)}{D (t, t + SD (t))}$ and $SD (t)$ is the stochastic duration of the coupon bond with payments of $K \Delta^Y_i$ at time $T_i$, $i = n + 1, ..., N$. The approach aims at approximating the coupon bond process by a more simple process with similar volatility, hence we can interpret $CB_t = \xi D (t, t + SD (t))$ as an approximation to $CB_t = K \cdot PV BP_{p+1, N} (t) + D (t, T_N)$. We can now derive the price of a swaption using a change of measure from $Q^T$ to $Q^{t+SD (t)} = Q^{CB}$ to obtain,

$$P_{S_t} (K) = D (t, T_n) P_{Q^T_n} \left( CB (T_n) < 1 \right) - CB (t) P_{Q^{CB}} \left( CB (T_n) < 1 \right)$$

(14)

The previous equation implies two approximation errors. First, the volatility of the coupon bond is approximated and second, comparing (14) with (13) the exercise probabilities are taken under the wrong forward measure.

4 Swaption pricing within the Vasicek model: Approximation and Error

Contrary to the methods of the previous section we approximate the true factor and swap rate dynamics under the relevant swap measure by affine dynamics. Before we present the method in full generality, we will illustrate the approximation method within the context of the Vasicek model\(^3\). The simplicity and analytical structure of the model allows us to focus more on the approximation and less on technical details. Moreover within the Vasicek model we can estimate the approximation error quite easily. For completeness, Vasicek (1977) assumes the short rate, $r_t$, follows an Ornstein-Uhlenbeck process,

$$dr_t = a (\theta - r_t) dt + \sigma dW^Q_t$$

(15)

where $W^Q_t$ is one dimensional Brownian motion under the risk neutral measure. The parameter $\theta$ is the risk neutral mean of $r_t$ and $a$ is the mean reversion rate. Note this is is the ATSM with $M = 1$, $\omega_0 = 0$, $\omega_X = 1$, $A = a$, $\beta = 0$, $\alpha = 1$ and $\Sigma = \sigma$. Bond prices are given by $D (t, T) = \exp (A (t, T) - B (t, T) r_t)$, where $B (t, T) = [1 - \exp (-a (T - t))] / a$.

\(^3\)An exact swaption price formula for this model can be found in Jamshidian (1989).
4.1 Approximate Swap Rate and Factor Dynamics

In section 2 we observed that if we know the characteristic function, or equivalently the distribution, of the swap rate under the swap measure we can price a swaption using (5). We will exploit this observation to derive an approximation of the distribution of the swap rate using the underlying SDE. First, we derive the SDE of the swap rate under the swap measure applying Itô’s lemma to (2),

\[ dy_{n,N}(t) = \frac{\partial y_{n,N}(t)}{\partial r_t} dW_Q^{n+1,N} + \]  

(16)

where \( W_Q^{n+1,N} \) is Brownian motion under the swap measure and,

\[ \frac{\partial y_{n,N}(t)}{\partial r_t} = -B(t, T_n) D^P(t, T_n) + B(t, T_N) D^P(t, T_N) + y_{n,N}(t) \sum_{i=n+1}^{N} \Delta_i B(t, T_i) D^P(t, T_i) \]  

(17)

where \( D^P(t, T_n) = D(t, T_n) / P_{n+1,N}(t) \), the bond price normalised by the numeraire, the PVBP.

For clarity, let us fix some notation. In the remainder of this paper we let \( t \) denote running time and let \( t = 0 \) denote the specific point in time at which we want to value swaptions. The term structure at the time of valuation is thus given by \( D(0; T) \).

Now we consider the actual approximation. Since bond prices in this model are stochastic processes the volatility of the swap rate is stochastic as well. However the volatility of the swap rate is a function of asset prices normalised by the PVBP. Hence the volatility is a function of martingales. We conjecture these martingales have sufficiently low variance to be approximated by their expectations, i.e. time zero values. This is similar to the approach taken in the literature on market models.

It is argued in Brace et al. (1998), Brace and Womersley (2000) and d’Aspremont (2003) that \( D(t, T_i) / P_{n+1,N}(t) \) are low variance martingales, in the context of a LIBOR Market Model (Miltersen, Sandmann and Sonderman, 1997, Jamshidian, 1998 or Brace, Gatarek and Musiela, 1997). We conjecture this is also the case in the affine class of term structure models. We will approximate the random terms \( D(t, T_i) / P_{n+1,N}(t) \) by their conditional expected value under the swap measure, \( D(0; T_i) / P_{n+1,N}(0) \). We also approximate the swap rate itself in the expression for the volatility of the swap rate by its martingale value. To be specific we approximate the swap rate volatility by \( \sigma \frac{\partial y_{n,N}(t)}{\partial r_t} \), where,

\[ \frac{\partial y_{n,N}(t)}{\partial r_t} = -B(t, T_n) D^P(0, T_n) + B(t, T_N) D^P(0, T_N) + y_{n,N}(0) \sum_{i=n+1}^{N} \Delta_i B(t, T_i) D^P(0, T_i) \]  

(18)

With this approximation swap rate volatility is deterministic. Therefore the swap rate is a Gaussian martingale. However, we can take it a step further. If we take a closer look at the function \( B(t, T) \) we can write,

\[ B(t, T) = \frac{1}{a} - \frac{e^{-aT}}{a} e^{at} \]

we see that \( B(t, T) \) can be split into three separate functions of which only one is really time dependent. Furthermore the constant \( \frac{1}{a} \) cancels in (18). Using this in the approximate swap rate volatility, we can rewrite

---

\( ^4 \)Assuming the absence of arbitrage, the swap rate is a martingale under the swap measure. Hence we have zero drift.
(18) as,
\[
\frac{\partial \widehat{y}_{n,N}(t)}{\partial t} = \frac{1}{a} e^{at} \left[ e^{-aT_n} D^P(0, T_n) - e^{-aT_N} D^P(0, T_N) - y_{n,N}(0) \sum_{i=n+1}^{N} \Delta_i^{Y_i} e^{-aT_i} D^P(0, T_i) \right]
\]
\[
\equiv \frac{1}{a} e^{at} \widehat{C}_{n,N}
\]
Where the first line defines \(\widehat{C}_{n,N}\). Now if we define the integrated variance of \(y_{n,N}\) (associated with a \(T_n \times T_N\) swaption) over the interval \([0, T_n]\) to be \(\sigma_{n,N}\), we have,
\[
\sigma_{n,N} = \sqrt{\int_0^{T_n} \left( \frac{\sigma}{a} e^{at} \widehat{C}_{n,N} \right)^2 \, dt}
\]
\[
= \sigma \widehat{C}_{n,N} \sqrt{\frac{e^{2aT_n} - 1}{2a}}
\]
All this leads to simple analytical pricing formulas for a swaption in the Vasicek model. Recall that,
\[
PS_0(K) = P_{n+1,N}(0) E_0^{Q_{n+1,N}} (y_{n,N}(T_n) - K)^+
\]
and,
\[
RS_0(K) = P_{n+1,N}(0) E_0^{Q_{n+1,N}} (K - y_{n,N}(T_n))^+
\]
Now for an ATMF swaption we have the following special result for the approximate price,
\[
PS_0(y_{n,N}(0)) = RS_0(y_{n,N}(0))
\]
\[
= P_{n+1,N}(0) \int_{y_{n,N}(0)}^{\infty} \frac{(x - y_{n,N}(0))}{\sigma_{n,N} \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{x - y_{n,N}(0)}{\sigma_{n,N}} \right)^2 \right) dx
\]
\[
= P_{n+1,N}(0) \frac{\sigma_{n,N}}{\sqrt{2\pi}}
\]  
(19)
For the swaption price approximation when the strike is not ATMF we need to calculate the following integral, where \(\varphi_{\mu,\sigma}(\cdot)\) is the density of a Gaussian r.v. with mean \(\mu\) and s.d. \(\sigma\), \(\Phi_{\mu,\sigma}\) is the corresponding distribution function and \(\Phi = \Phi_{0,1}\),
\[
PS_0(K) = P_{n+1,N}(0) \int_K^{\infty} (x - K) \varphi_{y_{n,N}(0),\sigma_{n,N}} (x) dx
\]
\[
= P_{n+1,N}(0) \left[ (y_{n,N}(0) - K) \Phi \left( \frac{y_{n,N}(0) - K}{\sigma_{n,N}} \right) + \sigma_{n,N} \varphi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) \right]
\]  
(20)
and for a receiver swaption,
\[
RS_0(K) = P_{n+1,N}(0) \left[ (K - y_{n,N}(0)) \Phi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) + \sigma_{n,N} \varphi \left( \frac{K - y_{n,N}(0)}{\sigma_{n,N}} \right) \right]
\]  
(21)
For a general ATSM a square root term, containing the factors, shows up in the swap rate volatility in (16). This complicates matters considerably. Approximations in this case are the subject of section 5.
4.2 Numerical Results

We calculated approximate and exact prices at parameter values in table 1. Numerical results on the quality of the approximation are given in table 2. As one can see these results are excellent and compare very favourably with the stochastic duration approximation in Munk (1999) and all other results on swaption price approximations in Gaussian Term Structure models, like for example Collin-Dufresne and Goldstein (2002). We show ATMF swaptions together with In-The-Money (ITM) and Out-of-The-Money (OTM) swaptions. We also think it is important to show how the approximation performs over different option and swap maturities. These results essentially support our conjecture that we can approximate the swap rate dynamics under the swap measure by freezing some low variance martingales at their martingale value\(^5\). As one would expect, as we approximate swap rate volatility, the quality of the approximation declines with total volatility and hence maturity. To compare with stochastic duration, we obtain an error 0.0236\% for a \(1 \times 10\) ATMF swaption whereas the stochastic duration has an error of approximately 1\% for an ATMF 6 month call on a 10 year coupon bond (Table 2 in Munk, 1999). Collin-Dufresne and Goldstein report errors of 0.1\% or less for a \(2 \times 10\) Swaption. This number should be compared with the element in the fourth row of the second column in tables 2.

In the appendix we generalize the approach sketched above to multi-factor Gaussian Term Structure Models. We also find an analytical formula for the approximate price in this general case. This result will speed up calibration of a multi factor Gaussian model to ATMF swaption prices considerably. Besides for derivative pricing purposes obtaining parameters for these models could be of interest when a Gaussian TSM is used in the pricing of stock options under stochastic interest rates, or for ALM purposes where a multi-factor Gaussian TSM is used in the pricing of insurance liabilities.

It is clear that for Gaussian TSM our method is the fastest one available in the literature as it is the only one which gives analytical formulas for the approximate price of a swaption. But not only CPU performance is good. The quality of the approximation seems at least as good as the methods of Munk (1999), Singleton and Umantsev (2002) and Collin-Dufresne and Goldstein (2002).

4.3 Approximation Error

Within our approximation the swap rate in the Vasicek model is a Gaussian martingale under the swap measure and hence its terminal distribution under this measure is completely determined by its current value and integrated volatility. Now consider expressions (16), (17) and (18). We see that in approximating swap rate volatility we make two (as it turns out, partially offsetting) types of errors. First, we replace a stochastic quantity by a constant, hence we underestimate swap rate volatility. Roughly speaking we use a deterministic 'estimate' for the stochastic swap rate volatility. Second, the expected value of \(\frac{\partial y}{\partial r}\) is different from \(\frac{\partial y}{\partial r}\). Thus we use a biased estimate of \(\frac{\partial y}{\partial r}\). Given these observations and the analytical structure of the Vasicek model we can find an error estimate for our approximation.

To ease the notational burden in the remainder of this section, we will sometimes drop time subscripts completely and denote dependence on maturity with a subscript (e.g. we will write \(D_P\) and \(B_N\) instead of

\(^5\)Results for multi-factor Gaussian models are similar and hence omitted.
$D^P(t, T_i)$ and $B(t, T_N)$ respectively. We also write $\tilde{D}^P_i$ instead of $D^P(0, T_i)$.

To analyse the error we compute the mean squared error of the approximation in (18),

$$MSE_{n+1, N} = E^{Q^{n+1, N}} \left[ \int_0^{T_n} \left( \frac{\partial y}{\partial r} - E^{Q^{n+1, N}} \left[ \frac{\partial y}{\partial r} \right] \right)^2 dt \right] + E^{Q^{n+1, N}} \left[ \int_0^{T_n} \left( \frac{\partial y}{\partial r} - E^{Q^{n+1, N}} \left[ \frac{\partial y}{\partial r} \right] \right)^2 dt \right]$$

where we’ve split the error into a part due to bias and to volatility of $\frac{\partial y}{\partial r}$ respectively. By an application of Itô’s lemma we can write,

$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial r}(0) + \int_0^t \nu(s) \, ds + \sigma \int_0^t \frac{\partial^2 y}{(\partial r)^2}(s) \, dW_s$$

where $\nu$ is a drift term, which we leave unspecified and $W$ is Brownian motion under the swap measure. Now we can simplify the volatility component of the $MSE$ as follows,

$$E^{Q^{n+1, N}} \left[ \int_0^{T_n} \left( \frac{\partial y}{\partial r} - E^{Q^{n+1, N}} \left[ \frac{\partial y}{\partial r} \right] \right)^2 dt \right] = \int_0^{T_n} \left[ \sigma^2 \int_0^t \left( \frac{\partial^2 y}{(\partial r)^2} \right)^2 ds dt \right]$$

where we assume the conditions for the equality in the first line to hold and the second line follows from Itô’s isometry. Summarizing, the mean squared error of the approximation in (18) is given by,

$$MSE_{n+1, N} = E^{Q^{n+1, N}} \left[ \int_0^{T_n} \left( \frac{\partial y}{\partial r} - E^{Q^{n+1, N}} \left[ \frac{\partial y}{\partial r} \right] \right)^2 dt \right] + \int_0^{T_n} \sigma^2 \int_0^t \left( \frac{\partial^2 y}{(\partial r)^2} \right)^2 ds dt$$

We will analyze bias and volatility seperately.

To analyze the bias first we calculate the expected value of $\frac{\partial y}{\partial r}$,

$$E^{Q^{n+1, N}} \left[ \frac{\partial y}{\partial r} \right] = \frac{\partial y}{\partial r} - \text{Cov} \left( y; \sum_{i=n+1}^{N} \Delta Y_{i-1}e^{-aT_i}D^P_i \right)$$

$$= \frac{\partial y}{\partial r} - \sum_{i=n+1}^{N} \Delta Y_{i-1}e^{-aT_i} \text{Cov} \left( D^P_i; D^P_i \right)$$

$$+ \sum_{i=n+1}^{N} \Delta Y_{i-1}e^{-aT_i} \text{Cov} \left( D^P_i; D^P_i \right)$$

To calculate the covariance between $D^P_i$ and $D^P_j$ note that we can easily derive an (approximate) SDE for a bond price normalised by the PVBP,

$$dD^P_j = \sigma \left\{ \sum_{i=n+1}^{N} \Delta Y_{i-1}B_iD^P_i - B_j \right\} D^P_j dW$$

$$\approx \sigma \left\{ \sum_{i=n+1}^{N} \Delta Y_{i-1}B_i\tilde{D}^P_i - B_j \right\} D^P_j dW \equiv \sigma \tilde{B}_j D^P_j dW$$
where $\tilde{B}_j = e^{at} \left[ \frac{e^{-aT_j}}{a} - \frac{1}{a} \sum_{i=n+1}^{N} \Delta Y_{i-1} e^{-aT_i} \tilde{D}_i^P \right]$. Thus, using the same arguments as before, the volatility of a normalised bond is approximately deterministic, hence we can accurately approximate the covariance between two normalised bonds by,

$$\text{Cov} (D_i^P; D_j^P) \approx \tilde{D}_i^P \tilde{D}_j^P \exp \left( \sigma^2 \int_0^t \tilde{B}_i \tilde{B}_j ds \right)$$

Finally we can approximate the bias of $\frac{\partial y}{\partial r}$, $\frac{\partial y}{\partial r} - E \left[ \frac{\partial y}{\partial r} \right] = \sum_{i=n+1}^{N} \Delta Y_{i-1} e^{-aT_i} \tilde{D}_i^P \left\{ \tilde{D}_n^P \exp \left( \sigma^2 \int_0^t \tilde{B}_i \tilde{B}_n ds \right) - \tilde{D}_N^P \exp \left( \sigma^2 \int_0^t \tilde{B}_i \tilde{B}_N ds \right) \right\}$ (23)

Note that any errors in the covariance estimate are partially offset because the $D_n^P$ and $D_N^P$ terms have opposite signs, precisely because of this reason the bias will be small in general. More specifically, the bias caused by the terms in the summation with $i$ close to $N$ will be offset against terms with $i$ close to $n$. The sign of the bias is negative since the exponents are of order $\sigma^2$ the main determinant of the sign is $e^{D_P n} e^{D_P N}$ which is negative.

To shed some more light on (23) first consider the expression for $\tilde{B}_j$, we have,

$$\tilde{B}_j = e^{at} \left[ \frac{e^{-aT_j}}{a} - \frac{1}{a} \sum_{i=n+1}^{N} \Delta Y_{i-1} e^{-a(T_i-T_{i+1})} \tilde{D}_i^P \right]$$

$$\approx e^{at} \left[ \frac{e^{-aT_j} - e^{-aT_{i+1}}}{a} \right]$$

$$\approx e^{at} \left[ T_{n+1} - T_j + O \left( \frac{aT_{i+1}^2}{2} \right) \right]$$

since $e^{-a(T_i-T_{i+1})}$ is close to one for reasonable $a$ and $\sum_{i=n+1}^{N} \Delta Y_{i-1} \tilde{D}_i^P = 1$. The second to third line follows from a Taylor approximation. Intuitively, from the above formulas we expect our approximation to deteriorate when the option maturity increases. When we look at $\exp \left( \sigma^2 \int_0^t \tilde{B}_i \tilde{B}_j ds \right)$ using the previous approximation we expect this to be close to zero since,

$$\sigma^2 << \int_0^t e^{2as} [T_{n+1} - T_j] [T_{n+1} - T_i] ds < \frac{e^{2a} - 1}{a} [T_N - T_{n+1}]^2$$

Hence,

$$\text{Cov} (D_i^P; D_j^P) \approx \tilde{D}_i^P \tilde{D}_j^P$$

Given the previous considerations on $\tilde{B}_i$ and $\text{Cov} (D_i^P; D_j^P)$ consider (23) once more,
where

\[
\frac{\partial^2 y}{(\partial r)^2} = B_n^2 D_n^P - B_N^2 D_N^P - y \sum_i \Delta Y_{i-1} D_i^P - 2 (B_n D_n^P - B_N D_N^P) \left( \sum_i \Delta Y_{i-1} B_i D_i^P \right)
\]

(24)

\[
+ 2y \left( \sum_i \Delta Y_{i-1} B_i D_i^P \right)^2
\]

this determines the volatility of \( \frac{\partial y}{\partial r} \). If we define, \( g(z) = z T_n D_n^P - z T_N D_N^P - y \sum_{j=n+1}^{N} \Delta Y_{j-1} z T_j D_j^P \) then after some tedious calculations we can simplify (24) to,

\[
\frac{\partial^2 y}{(\partial r)^2} = \frac{1}{a^2} g \left( e^{-2a t} \right) e^{2at} - \frac{2}{a^2} \left[ \sum_{i=n+1}^{N} \Delta Y_{i-1} e^{-a T_i} D_i^P \right] g \left( e^{-a} \right) e^{2at}
\]

(25)

which can be approximated in now familiar fashion by \( \frac{\partial^2 y}{(\partial r)^2} \). Now we have the following expression for the \( MSE \) in the volatility of the swap rate, with separate bias and volatility terms,

\[
MSE_{n+1,N} = \int_0^{T_n} \sigma^2 \tilde{B}^2_{n,N} dt + \sigma^4 D_{n,N}^2 \left[ \frac{e^{4a T_n}}{4a^2} - \frac{1}{(4a)^2} - \frac{T_n}{4a} \right]
\]

(26)

From (26) we can see that the volatility error is of order \( \sigma^4 \) while the swap rate volatility itself is of order \( \sigma^2 \). It is important to note that although this is not directly clear from the \( MSE \), in practice the bias (overestimation) will offset the lower volatility (underestimation) of \( \frac{\partial y}{\partial r} \). Formulas (23) and (25) can be used to obtain an improved estimate of the swap rate volatility. This can be plugged into the pricing formulas (19), (20) and (21) to get an improved price approximation. Combining this price with the original approximation gives us an estimate of the error. Formulas (23) and (26) summarize the behavior of our approximation error in the Vasicek model. This analysis is difficult to carry out for the general affine case. The properties of the error, e.g. the offsetting effects of bias and volatility errors, however will remain since the approximation technique is essentially the same.
5 Approximation Method for general Affine Models

In a general ATSM we cannot infer the approximate distribution of the swap rate directly from the approximate SDE. However, it turns out that if we, again, replace some LVM by their martingale values, the swap rate and factor dynamics (under the swap measure) can be cast in the affine framework. We can then view a swaption as an option on one particular factor, namely the swap rate itself. This enables us to use the valuation approach based on the CCF, see Carr and Madan (1998), DPS and Lee (2004). The approximation method develops according to three steps. We first derive the Radon-Nikodym (RN) derivative for the change from the risk neutral measure to the swap measure, \( Q^{n+1,N} \). Second we derive the swap rate and factor dynamics under \( Q^{n+1,N} \). Third, we propose the approximate dynamics and the pricing formula. In 5.3., we propose a “quick and dirty” approximation method which is extremely fast while remaining reasonably accurate. This last method can for example be used to generate starting values for calibration.

5.1 Approximate Swap Rate and Factor Dynamics

Recall that 
\[
M_t = \exp \left( \int_0^t r_s ds \right)
\]
denotes the money market account. We will now derive the swap rate and factor dynamics under the martingale measure associated with the PVBP as a numeraire (‘swap measure’).

The RN-derivative for a change from the risk-neutral to the swap measure is given by,
\[
\frac{dQ^{n+1,N}}{dQ} = \frac{P^{n+1,N}(T)}{P^{n+1,N}(t)} \frac{M_T}{M_t}
\]

Applying Ito’s lemma to this expression and combining with (8) gives,
\[
\frac{dP^{n+1,N}(t)}{M_t} = \frac{1}{M_t} dP^{n+1,N}(t) + P^{n+1,N}(t) d \frac{1}{M_t} + \frac{1}{M_t} dP^{n+1,N}(t)
\]
\[
= - \sum_{i=n+1}^N \Delta Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{M_t} \Sigma \sqrt{V_i} dW^{Q^{n+1,N}}_t
\]
\[
= - \sum_{i=n+1}^N \left( \Delta Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{P^{n+1,N}(t)} \Sigma \sqrt{V_i} \right) \frac{P^{n+1,N}(t)}{M_t} dW^{Q^{n+1,N}}_t
\]

where \( W^{Q^{n+1,N}}_t \) is an \( M \)-dimensional Brownian Motion under the swap measure and \( B(t, T_i) \) is an \( M \)-vector of loadings on the factor volatility to determine bond volatility. So the RN-kernel for a Change of Measure from the risk neutral to the swap measure, \( Q^{n+1,N} \), is given by
\[
\sqrt{V_i} \frac{\partial \ln P^{n+1,N}(t)}{\partial X_t} = - \sqrt{V_i} \Sigma \left[ \sum_{i=n+1}^N \Delta Y_{i-1} B(t, T_i) \frac{D(t, T_i)}{P^{n+1,N}(t)} \right].
\]

This implies for the \( Q^{n+1,N} \) dynamics of the

---

6 We leave the drift term unspecified. Since the RN-derivative is a martingale it is irrelevant in our calculations.

7 Applying Itô’s lemma to \( D(t, T) = \exp \left( A(t, T) - B(t, T) \cdot X_t \right) \) gives,
\[
dD(t, T) = ... dt - B(t, T) D(t, T) \Sigma \sqrt{V_i} dW^{Q^{n+1,N}}_t
\]
factors,

\[ dX_t = \left[ A(\theta - X_t) + \Sigma V_t \Sigma' \frac{\partial \ln P_{n+1,N}(t)}{\partial X_t} \right] dt + \Sigma \sqrt{V_t} dW^Q_{n+1,N} \]

\[ = \left[ A(\theta - X_t) - \Sigma V_t \Sigma' \left\{ \sum_{i=n+1}^{N} \Delta Y_{i-1} B(t, T_i) \frac{D(t,T_i)}{P_{n+1,N}(t)} \right\} \right] dt + \Sigma \sqrt{V_t} dW^Q_{n+1,N} \]  

(28)

If \( \frac{\partial \ln P_{n+1,N}(t)}{\partial X_t} \) would be deterministic (but time dependent) this implies an affine structure (with time varying coefficients). Note that the randomness in the RN-kernel is caused by: \( \frac{D(t,T_i)}{P_{n+1,N}(t)} \), \( i = n + 1, \ldots, N \). However these terms are asset prices normalised by the numeraire associated with the swap measure. Hence they are martingales under this measure. The dynamics of the swap rate itself can be obtained by a simple application of Itô’s lemma, while we use that the swap rate is a martingale under the swap measure,

\[ dy_{n,N}(t) = \left\{ \sum_{i=n}^{N} q_i^Y(t) B(t, T_i') \right\} \Sigma \sqrt{V_t} dW^Q_{n+1,N} \]  

(29)

where \( q_i^Y(t) = -\frac{D(t,T_n)}{P_{n+1,N}(t)} \), \( q_i^Y(t) = \Delta Y_{i-1} y_{n,N}(t) \frac{D(t,T_i)}{P_{n+1,N}(t)} \) for \( i = n + 1, \ldots, N - 1 \) and \( q_Y^N(t) = \{1 + \Delta Y_{N-1} y_{n,N}(t)\} \frac{D(t,T_N)}{P_{n+1,N}(t)} \). From (29) the swap rate volatility can be interpreted as a weighted bond volatility.

Similar to the approach in section 4 and motivated by the excellent results we approximate the dynamics of the factors and the swap rate under the swap measure by substituting \( \frac{D(0,T_n)}{P_{n+1,N}(0)} \) for \( \frac{D(t,T_i)}{P_{n+1,N}(t)} \) in (28) and \( q_i^Y(0) \) for \( q_i^Y(t) \) in (29)\(^8\). We obtain the following approximate dynamics,

\[ dy_{n,N}(t) = \sum_{i=n}^{N} q_i^Y(0) B(t, T_i') \Sigma \sqrt{V_t} dW^Q_{n+1,N} \]  

(30)

\[ dX_t = \left[ A(\theta - X_t) - \Sigma V_t \Sigma' \left\{ \sum_{i=n+1}^{N} \Delta Y_{i-1} B(t, T_i) \frac{D(0,T_i)}{P_{n+1,N}(0)} \right\} \right] dt + \Sigma \sqrt{V_t} dW^Q_{n+1,N} \]  

(31)

These approximations under the swap measure result in an affine model with time dependent coefficients for the joint dynamics of the swap rate and the factors. Note that the drift function of the approximate factor dynamics under the swap measure is affine, however the drift change (which in our approximation we assume to be a deterministic function of time) influences the joint restrictions on \( A, \theta, \Sigma \) and \( V \), see Duffie and Kan (1996) and Dai and Singleton (2000). This drift change is small however so will most likely not cause any problems in practice.

We can regard \( y_{n,N} \) as a pseudo factor. The swaption payoff can then be written as a trivial linear combination of the ‘factors’ minus the strike, if positive. This means that even after a change to the swap measure we can calculate all quantities related to swaption pricing using the apparatus of the affine setup. For instance a swaption is just an option on the first “factor” of the affine model in (30) and (31). Hence swaptions can be priced in a similar manner as bond options. We will now elaborate on this.

\(^8\)The approximation of the swap rate volatility is not the crucial assumption for ATSMs since this doesn’t effect the factor dynamics and seems to work extremely well in Gaussian models.
5.2 Levy Inversion

From the approximate dynamics we can infer an approximation for the conditional characteristic function of \( y_{n,N} \) under the swap measure, see Duffie, Pan and Singleton (2000). Then, as outlined at the end of section 2, via a single one-dimensional Levy inversion, we can find the price of a payer swaption. Recall that contrary to other approaches we interpret a swaption as an option on the swap rate. It is important to note that this particular approach enables us to use the elegant inversion approach of Carr and Madan (1998). Not only is this approach faster than an approach based on “exercise” probabilities\(^9\) but it is also more accurate. No accumulation of errors takes place. This latter point is especially relevant for options with strikes away from the forward. See Carr and Madan (1998) and Lee (2004) for a full discussion.

Our approximation in (30) and (31) results in a degenerate affine diffusion for the swap rate \( y_{n,N} \) and the factors \( X \). Stacking those variables in a vector, defining \( \mathbf{w}_t = \left[ \sum_{i=n}^{N} \Delta \gamma_i B (t, T_i) \right] \) and \( \mathbf{k}_t = \left[ \sum_{i=n}^{N} q_i^Y (0) B (t, T_i) \right] \) (both \( M \times 1 \)-vectors), we get the following dynamics for \( \tilde{X}_t = \left[ y_{n,N} (t) \ X'_t \right]^\prime \),

\[
\begin{align*}
\frac{d\tilde{X}_t}{dt} &= \left\{ \begin{array}{c}
0 \\
A \theta - \Sigma \text{diag}(\alpha) \Sigma' \mathbf{w}_t \\
-AX_t - \Sigma \text{diag}(\beta X_t) \Sigma' \mathbf{w}_t
\end{array} \right\} dt + \begin{bmatrix} k'_t \\ \mathbf{I}_m \end{bmatrix} \Sigma' \sqrt{\mathbf{v}_t} dW^{Q_{n+1,N}}_t \\
&= \tilde{A} (t) \left( \tilde{\theta} (t) - \tilde{X}_t \right) dt + \tilde{\Sigma} (t) \sqrt{\mathbf{v}_t} dW^{Q_{n+1,N}}_t
\end{align*}
\]

We let \( \tilde{A} (t), \tilde{\theta} (t) \) and \( \tilde{\Sigma} (t) \) be defined implicitly. Furthermore we define \( \tilde{\beta}_i = \left[ \begin{array}{c} 0 \\ \beta'_i \end{array} \right] \) and \( \tilde{\alpha}_i = 0, \tilde{\alpha}_i = \alpha_{i-1} \) for \( i = 2, \ldots, M \). Now we let \( \psi \) denote the CCF of the swap rate,

\[
\psi (u, t, T_n) \equiv E^{Q_{n+1,N}}_t \left[ \exp (iu y_{n,N} (T_n)) \right]
\]

Furthermore, let \( \phi \) be the transform of the dampened payer price, \( \mathbf{p}_{s,t} (K) = \exp (\alpha K) \mathbf{P}_{s,t} (K) \), define

\[
\phi (u, t, T_n) \equiv \int_{-\infty}^{\infty} \exp (iu K) \mathbf{p}_{s,t} (K) dK
\]

Then from results in Carr and Madan (1999) and Lee (2004) it follows that,

\[
\phi (u, t, T_n) = P_{n+1,N} \frac{\psi (u - i \alpha, t, T_n)}{(\alpha + i u)^2}
\]

(32)

Furthermore the results in DPS state that,

\[
\psi (u, t, T_n) = \exp \left( \gamma_t + \delta_t \cdot \tilde{X}_t \right)
\]

where \( \delta, \gamma \) are solutions to the following system of complex valued Riccati equations, with initial conditions, \( \delta_{T_n} = -i v \mathbf{e}_1, \gamma_{T_n} = 0, \)

\[
\begin{align*}
\frac{d\delta_t}{dt} &= \tilde{A} (t)' \delta_t - \frac{1}{2} \sum_{i=1}^{M+1} \left[ \tilde{\Sigma} (t)' \delta_t \right]_i \beta'_i \\
\frac{d\gamma_t}{dt} &= \tilde{\theta} (t)' \tilde{A} (t)' \delta_t - \frac{1}{2} \sum_{i=1}^{M+1} \left[ \tilde{\Sigma} (t)' \delta_t \right]_i \tilde{\alpha}_i
\end{align*}
\]

\(^9\)The classic use of transform inversion in option pricing, introduced by Heston (1993), uses two transform inversion to find “\( N (d_1) \)” and “\( N (d_2) \)” in the price formula of a call with strike \( K \) on a stock \( S \), \( C_0 = D (0, T) [S_0 N (d_1) - K N (d_2)] \).
where \( \mathbf{e}_1 \) is the first basis vector of the \( M + 1 \) dimensional Euclidean space. Note that \( \delta \) is an \( (M + 1) \)-vectors and \( \gamma \) a scalars.

We can summarize the approach in the following proposition. We label the resulting approximation “TransformApprox”.

**Proposition 1** Under the approximate dynamics for the swap rate and the factors in (30) and (31), the price of a payer swaption in terms of the dampened payer transform, is given by,

\[
\mathbf{P}_{S_0}(K) = \exp \left( \frac{-\alpha K}{\pi} \right) \int_0^\infty \text{Re} \left[ \exp \left( -iu \, K \right) \phi (u, 0, T_n) \right] du
\]

where \( \phi (u, 0, T_n) \) is linked to the CCF of the swaprate by (32).

**Proof.** This follows from Duffie, Pan and Singleton (2000) and Lee (2004) in combination with our approximate dynamics. ■

### 5.3 Square Root Swap Rate process: A CEV Option Pricing Formula

We can take another step in approximating the dynamics of the swap rate. This will result in a square root process for the swap rate under the swap measure and will allow us to determine swaption prices using results on option prices in square root models (or more general Constant Elasticity of Variance, CEV, models). Let us proceed with the approximation, rewrite equation (29) as,

\[
dy_{n,N}(t) = \left\{ \sum_{i=n}^{N} q_i y(t, T_i) B(t, T_i) \right\} \frac{\sqrt{\gamma_t}}{\sqrt{\gamma_{n,N}(t)}} \sqrt{y_{n,N}(t)} dW_{t}^{Q_{n+1,N}}
\]

Now we have the following,

**Proposition 2** If \( \sigma_y(t, n, N) = \left\{ \sum_{i=n}^{N} q_i y(t, T_i) B(t, T_i) \right\} \frac{\sqrt{\gamma_t}}{\sqrt{\gamma_{n,N}(t)}} \) was deterministic we would have the following time 0 price for a payer swaption,

\[
\mathbf{P}_{S_0} = P_{n+1,N} (0) \left[ y_{n,N} (0) \left( 1 - \chi^2 (d_K, 4, d_y) \right) - K \chi^2 (d_y, 2, d_K) \right]
\]

where \( \chi^2 (a, b, c) \) is the non-central \( \chi^2 \) distribution function (cf. Ding, 1992) and,

\[
\begin{align*}
    d_K &= \frac{4K}{\sigma_y^2 (0, n, N)} \\
    d_y &= \frac{4y_{n,N} (0)}{\sigma_y^2 (0, n, N)} \\
    \sigma_y^2 (0, n, N) &= \int_0^T \| \sigma_y (t, n, N) \|^2 dt
\end{align*}
\]
Proof. See Theorem 3 in Andersen and Andreasen (1998).}

We propose to approximate the total volatility by taking the expected values of \( q_i^Y, y_{n,N} \) and \( X \) over the time interval \([0, T_n]\). From (31) we see that the process for \( X_t \) can be written as,

\[
dX_t = A^* (t) \left[ \theta^* (t) - X_t \right] dt + \Sigma \sqrt{V_t} dW_t^{Q^{n+1,N}}
\]

where \( A^* \) and \( \theta^* \) are defined implicitly through (31). Hence we have that \( E_{Q^{n+1,N}} [X_{T_n}] \) is a solution to the ODE,

\[
\dot{x}_t = A^* (t) \left[ \theta^* (t) - x_t \right] \tag{34}
\]

with boundary condition \( x_0 = X_0 \). All the other terms in the swap rate volatility are martingales so we have the following corollary to proposition 2,

**Corollary 3** By approximating the swap rate volatility by

\[
\tilde{\sigma}_y(t, n, N) = \left\{ \sum_{i=n}^{N} q_i^Y (0) B (t, T_i)^\prime \right\} \frac{\Sigma \sqrt{\text{diag} (\alpha + \beta E_{Q^{n+1,N}} [X_i])}}{\sqrt{y_{n,N} (0)}},
\]

we can accurately approximate the time 0 price for a payer swaption by,

\[
PS_0 = P_{n+1,N} (0) \left[ y_{n,N} (0) \left( 1 - \chi^2 \left( \tilde{d}_K, 4, \tilde{d}_y \right) \right) - K \chi^2 \left( \tilde{d}_y, 2, \tilde{d}_K \right) \right]
\]

where \( E_{Q^{n+1,N}} [X_{T_n}] \) solves (34) and,

\[
\tilde{d}_K = \frac{4K}{\tilde{\sigma}_y^2 (0, n, N)}
\]

\[
\tilde{d}_y = \frac{4y_{n,N} (0)}{\tilde{\sigma}_y^2 (0, n, N)}
\]

\[
\tilde{\sigma}_y^2 (0, n, N) = \int_0^{T_n} \frac{\| \tilde{\sigma}_y (t, n, N) \|^2 dt}{T_n}
\]

We label this approximation “CEVApprox”. Not only is this approximation extremely fast and easy to implement, it requires only a single numerical integration and a routine for the cumulative non-central chi-square distribution, it also attains a workable level of accuracy. Note that this not only yields an extremely fast formula for swaptions but, as the LIBOR rate is essentially a one-period swap rate, also a pricing formula for caplets in ATSMs.

### 5.4 Extensions

We envision straightforward extensions of the outlined approximation procedure in the following directions. First, the pricing of coupon bonds instead of swaptions can be handled with a simple modification of the approximation method. The payoff of a put option with option maturity \( T_n \) and strike \( K^* \) on a coupon bond (with coupon payments \( C \) at \( T_i, i = n + 1, ..., N \) and \( \Delta_i \) the relevant daycount fraction) is given by,

\[
\left[ K^* - C \sum_{i=n+1}^{N} \Delta_i D (T_n, T_i) - D (T_n, T_N) \right]^+
\]
this expression can be rewritten as an option on a fictitious interest rate \( y_{n,N}^{K^*} \), we obtain,

\[
\left[ K^* - C \sum_{i=n+1}^{N} \Delta_i D(T_n, T_i) - D(T_n, T_N) \right]^+ = P_{n+1,N}(T_n) \left[ \frac{K^* D(T_n, T_n) - D(T_n, T_N) - C}{\sum_{i=n+1}^{N} \Delta_i D(T_n, T_i)} \right]^+
\]

where \( y_{n,N}^{K^*}(t) = \frac{K^* D(t, T_n) - D(t, T_N)}{\sum_{i=n+1}^{N} \Delta_i D(t, T_i)} \) and note that \( P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_i D(t, T_i) \). This shows that an option on a coupon bond is equivalent to an option on the fictitious interest rate \( y_{n,N}^{K^*} \). Our approximation method can be modified to accommodate this slightly different interest rate. To be precise, the choice of numeraire remains the same, only \( q_n^Y(t) \) in (29) changes to \( q_n^Y(t)^* = \frac{-K^* D(t, T_n)}{P_{n+1,N}(t)} \).

Second, using a simple modification in the pricing procedures swaptions can be priced according to the current term structure instead of the model generated term structure. Modifications of this procedure should first extend the results from Brigo and Mercurio (2001) to a multivariate setting. One can easily show that to price swaptions in the model which fits the initial term structure perfectly we simply replace all model generated bond prices in (28) and (29) by bond prices generated by the extended model. Furthermore, and more important for the actual pricing algorithm, we replace all bond prices in (30) and (31) generated by our original model by bond prices from the initial term structure.

Third, we could use our approximation technique to price options on life annuities, a so called Guaranteed Annuity Option or GAO (see Pelsser, 2003 among others). In the case of an option on a life annuity, i.e. pension, the holder has the right to exchange a number of fixed cash flows, \( CF_i \), at future dates, \( T_i \), \( i = 1, ..., n \) for a fixed payment \( K \) at the exercise date \( T \). The value of the annuity at a certain date \( t \) is given by

\[
a_n(t) = \sum_{i=1}^{n} CF_i D(t, T_i)
\]

The value of the option is given by,

\[
GAO_t = a_n(t) E^{Q^4} \left( D(T, T) K / a_n(T) - 1 \right)^+
\]

where \( Q^4 \) is the measure associated with numeraire \( a_n(t) \). Now we can use the techniques of sections 4 and 5 to derive the approximate dynamics of \( D(t, T) / a(t) \) for any ATSM and hence calculate an approximate price for the option.

Fourth, we could apply our results to the pricing of a Credit Default Swaption. Roughly speaking a Credit Default Swap starting at \( T_n \) with final payment at \( T_N \), swaps two payment streams \( S \), the “survival annuity stream” payed by the buyer of default insurance and \( P \), the “default protection stream” payed by the seller of insurance in case of default of the “reference entity”. The cash flows in these payment streams are usually modelled as fixed conditional upon no default. Furthermore no payments are made if default occurs before the first payment date, i.e. both are \( T_n \)-survival claims. Jamshidian (2004) shows that under general conditions the value of a CDS option in a subfiltration setting\(^{10}\) with subfiltration \( \mathcal{H}_t \) is given by,

\[
CDS_t = S_t E^{\mathbb{P}} \left[ \left( \frac{V_{T_n}(P)}{V_{T_n}(S)} - K \right)^+ \middle| \mathcal{H}_t \right]
\]

where \( V_T(S) \) denotes the risk neutral expectation\(^{11}\) of the discounted payoff in the payment stream \( S \) conditional on \( \mathcal{H}_T \). Now if we have a model for which \( V_{T_n}(S) \) and \( V_{T_n}(P) \) are sums of affine functions of a number of

\(^{10}\)The subfiltration contains information on the economy except the fact of default itself.

\(^{11}\)Other numeraires and associated measures are allowed as well. See theorem 3.4. and 3.8. in Jamshidian (2004).
factors this result reduces the pricing of a CDS option to the pricing of a normal swaption and the results in section 4 and 5 can be applied\textsuperscript{12}. For a quite general model which satisfies these latter requirements (but not in a subfiltration setting), see Chen and Filipovic (2004).

Furthermore our procedure could easily deal with options on swaps in which fixed payments are made at the start of the period (adjustment of choice of numeraire).

\section{Quality and Computational Speed}

Both our approximation methods are partly based on Andersen and Andreasen (1998), Brace et al. (2000) and Hull and White (2000). These authors show that in the lognormal version of the LMM we can approximate swap rates by lognormal martingales by replacing quantities similar to $D(t, t_i) / P_{n+1,N}(t)$ by their time zero values. Variations around this martingale value should not be too important, when small, since we are aiming at finding the average volatility of the swap rate over the interval from the current time to the option maturity. This approximation yields swaption prices which are accurate up to a (couple of) basis point (in absolute terms) or a couple of tenths of a percent (in relative terms).

Singleton and Umantsev (2002) simplify the problem of pricing a swaption to pricing several caplets. However several drawbacks to their method exist which are mainly related to their algorithm to approximate the exercise region. First, to find the approximate exercise region one requires complete knowledge of the density of the factors. This density can be computed analytically in the case they consider, that of a 2-factor CIR model with uncorrelated factors. In the more general case, e.g. models which feature correlated factors and volatility components which are driven by multiple factors, this density must be computed by Levy inversion. More precise, for each evaluation of the density one needs to perform a full transform inversion procedure. Their algorithm is based on finding the region outside which the density of the factors is negligible. To find this region one would need at least a couple of inversions for each factor. In comparison, our method requires only a single inversion to obtain the price. Second, for each different strike they need to find a new approximation to the exercise region which can be troublesome in high dimensions. Third, they need to perform as much transform inversions as there are cash flows in the swap or coupon bond, whereas we need to perform only a single inversion. From a numerical point of view the end result of these multiple inversions is less accurate than the inversion method used by our method which is based on a single inversion. Summarizing, we can say that our approach is more easily implemented for multi-factor models and is computationally less intensive.

Implementation for multi-dimensional models is certainly difficult for the method by Collin-Dufresne and Goldstein. Their method based on Edgeworth expansions requires calculation of the moments of the swap rate under several forward measures. These moments are a summation of moments of products of bond prices. However when zero bond prices are not available analytically (but only by solving a system of ODEs) this method becomes computationally very intensive. To illustrate, for the calculation of the 5th moment of a 15 year swap rate (annual payments) this requires $(15+5-1)^5 = 11,628$ solutions to a system of Riccati equations. Again our approach extends much easier to the general case and is computationally less intensive as we calculate the approximate CCF of the swap rate directly.

\textsuperscript{12}The CDS rate is then defined by $V_{T_n}(S) / V_{T_n}(P)$. 

20
Our procedure for calculating swaption prices nicely fits in the affine framework so should not give to many implementation problems. To deal with time dependence in the coefficients observe that this originates from the bond volatilities of the Affine model, \( B(t,T_i) \). These bond volatilities (which can be calculated by solving the familiar Riccati equations) need to be calculated only once and can then be stored for further use. However for the use of the affine model the procedure to calculate bond volatilities should be in place anyway.

Next, we present results on the quality of our approximation. The quality is excellent for both the Gaussian and the CIR case. Tables 2, 3 and 4 show the performance of our approximation method relative to exact prices for \( n \times m \) ATMF, ITM and OTM payer swaptions, where \( n = 1,...,5 \) is the option maturity and \( m = 1,...,10 \) is the swap maturity, in the single factor Vasicek model, the two factor CIR model and the three factor Gaussian term structure model (see also Collin-Dufresne and Goldstein, 2002). The performance is measured both in absolute and relative deviations of both price and implied Black volatility. The ITM strike is set at 85% of the ATMF and the OTM strike is set at 115% of the ATMF. For sake of completeness we give the model equations. The two factor CIR model,

\[
\begin{align*}
    r_t &= \omega + X_{1t} + X_{2t} \\
    dX_{it} &= a_i (\theta_i - X_{it}) dt + \sigma_i \sqrt{X_{it}} dW_{it}, \quad X_{i0} = x_i, \quad i = 1, 2
\end{align*}
\]

where \( W_{1t} \) and \( W_{2t} \) are independent Brownian motions under \( Q \). The three factor Gaussian model,

\[
\begin{align*}
    r_t &= \omega + X_{1t} + X_{2t} + X_{3t} \\
    dX_{it} &= -a_i X_{it} dt + \sigma_i dW_{it}, \quad X_{i0} = x_i, \quad i = 1, 2, 3
\end{align*}
\]

where the \( W_{it} \) are correlated Brownian motions under \( Q \) with \( dW_{it} dW_{jt} = \rho_{ij} dt \).

For the CIR model and the three factor Gaussian model the true prices are calculated by simulation. The parameter values at which these results are obtained are taken from Collin-Dufresne and Goldstein (2002) and are given in Table 1. Parameter values are close to realistic levels encountered in practice. Table 3 shows results on the approximation in the 2 factor CIR model (see Collin-Dufresne and Goldstein, 2002). Table 4 shows results on the approximation error in the 3 factor Gaussian model. Monte Carlo results are obtained using 500,000 simulations with standard antithetic variables. The quality of both approximations in the CIR case is also excellent and comparable to stochastic duration and the approximation methods by Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2002). This again confirms our conjecture that we can freeze the LVM at their martingale values.

We have compared computational times for our two methods with the methods by Munk (1999), Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2002) in table 5. As could be expected, our TransformApprox is slightly slower than stochastic duration. The comparison with the numbers in Singleton and Umantsev and Collin-Dufresne and Goldstein is made by assuming the reported numbers are from 2001 (a year before publication) and further assuming computing speed doubles every two years. Effectively we divide the reported numbers by four. Although the computation of the method by Collin-Dufresne and Goldstein is extremely fast, as we argued above, this is largely due to the analytical structure of the 2 factor CIR model. Contrary to their formula our CEV approximation requires only a single numerical integration regardless of the model structure. As one can see the CEV approximation is the fastest among known approximations with a computational time of a few hundredths of a second.

21
7 Conclusion

In this paper we have introduced two related methods to obtain an accurate and fast approximation to the prices of swaptions in Affine Term Structure Models. The approximation is based on approximate dynamics of the swap rate and the latent interest rate factors under the associated swap measure. Based on these approximate dynamics, our first approximation uses techniques familiar to the affine setup, such as transforms and transform inversion, to calculate swaption prices. Contrary to other approaches we write a swaption as an option on the swap rate. This enables us to use the faster and more accurate transform inversion method of Carr and Madan (1998). The resulting approximation is comparable in speed and superior in accuracy to the stochastic duration approximation of Munk (1999). Approximating the swap rate dynamics even further, by a square root process, we can use familiar results on option pricing in CEV models. This yields a closed form pricing formula which requires only a single numerical integration while remaining accurate. The resulting approximation has similar accuracy as the methods currently known in the literature but is superior in speed. The approach results in analytical formulae for Gaussian models.

This shows that using our approximate swap rate dynamics results in swaption price formulas which improve on the existing methods in either accuracy, speed or both. Furthermore the implementation for general affine models is easily accomplished.

References


22
Appendix: Results for Multi-Factor Gaussian Term Structure Models

In this appendix we will generalize the results in section 4, on the single factor Vasicek model to Multivariate Gaussian Term Structure models. Closed form formulas for the prices of swaptions in single (Jamshidian, 1989) and two factor models (through numerical integration of the single factor result) are readily available. However
using the results in this section, calculations are sped up considerably while preserving the required accuracy. We can obtain the class of Gaussian Term Structure Models from equations (7), (8) and (9) by setting $\beta = 0$. Without loss of generality we can assume that $\omega_X = 1$, $\alpha = 1$ and that $A$ is a diagonal matrix. To simplify the derivation we first perform a double change of variables. That is, we will analyse the following model of the short rate, $r_t$,

$$
\begin{align*}
    r_t &= \alpha_t^* + 1'Y_t \\
    dY_t &= -AY_t dt + \tilde{\Sigma}dZ_t
\end{align*}
$$

Where $\alpha_t^* = \omega_0 + 1' \{ \theta - \varepsilon^{-A t} (X_0 - \theta) \}$ is a deterministic function of time, $\tilde{\Sigma} = \text{diag} (\Sigma \Sigma')$ and $Z_t = \Gamma^{1/2} W_t$ is correlated $M$-dimensional Brownian Motion under the risk neutral measure, where $\Gamma$ is the instantaneous correlation matrix of $X_t$ (i.e. $\Gamma = \tilde{\Sigma}^{-1/2} \Sigma \Sigma' \tilde{\Sigma}^{-1/2}$). To address the fit to the initial term structure one could choose $\alpha_t^*$ such that this fit is perfect. Then the model becomes in fact a multi-factor Hull-White (1990) model.

In the remainder of this section we will not differentiate between these approaches and with $D(t,T)$ just refer to the model generated price at time $t$ of a zero coupon bond with maturity $T$. Bond prices in this model are given by,

$$
D(t,T) = \exp \left( A(t,T) - \sum_{i=1}^{M} B(i,t,T) Y_t^{(i)} \right)
$$

where $B(i,t,T) = 1/A_{ii} \left( 1 - e^{-A_{ii}(T-t)} \right)$. We leave $A(t,T)$ unspecified. This is not important as long as the term structure generated by the model at the time of valuation is known.

We again start by deriving the SDE of a swap rate under its own swap measure. From the Change of Numeraire theorem we know that the swap rate is a martingale associated with the PVBP as a numeraire. Since the interest rate volatility doesn’t contain a square root term, applying Itô’s lemma to the swap rate gives,

$$
dy_{n,N}(t) = \frac{\partial y_{n,N}(t)}{\partial Y_t} \tilde{\Sigma}dZ_t^{n+1,N}
$$

where $Z_t^{n+1,N}$ is $M$-dimensional correlated Brownian Motion under the risk neutral measure $Q^{n+1,N}$ corresponding to the numeraire $P_{n+1,N}$. Now for each element of the vector of derivatives we have,

$$
\frac{\partial y_{n,N}(t)}{\partial Y_t^{(i)}} = -B(i,t,T_n) D^P(t,T_n) + B(i,t,T_N) D^P(t,T_N) + y_{n,N}(t) \sum_{j=n+1}^{N} \Delta_j B(i,t,T_j) D^P(t,T_j)
$$

recall that $D^P(t,T_n) = D(t,T_n) / P_{n+1,N}(t)$, the bond price normalised by the numeraire, the PVBP.

As in sections 4 and 5 our approximation consists of replacing the stochastic terms $D(t,T_i) / P_{n+1,N}(t)$ by their martingale values. We obtain for each partial derivative of the swap rate,

$$
\frac{\partial y_{n,N}(t)}{\partial Y_t^{(i)}} \approx -B(i,t,T_n) D^P(0,T_n) + B(i,t,T_N) D^P(0,T_N) + y_{n,N}(0) \sum_{j=n+1}^{N} \Delta_j B(i,t,T_j) D^P(0,T_j)
$$

$$
\quad = \frac{\partial y_{n,N}(t)}{\partial Y_t^{(i)}}
$$
This makes the swap rate volatility deterministic. Like in section 4, we can write,

\[ B^{(i)} (t, T) = \frac{1}{A_{(ii)}} e^{-A_{(ii)} T} \]

Using this in the approximate swap rate volatility (37) gives,

\[
\frac{\partial y_{n,N} (t)}{\partial Y_t} = \frac{1}{A_{(ii)}} e^{A_{(ii)} t} \left[ e^{-A_{(ii)} T_n} D^P (0, T_n) - e^{-A_{(ii)} T_N} D^P (0, T_N) - y_{n,N} (0) \sum_{j=n+1}^{N} \Delta Y_{j-1} e^{-A_{(ii)} T_j} D^P (0, T_j) \right]
\]

\[ = e^{A_{(ii)} t} \tilde{C}_{n,N}^{(i)} \]

Where the first line defines \( \tilde{C}_{n,N}^{(i)} \). So in the approximate model the swap rate at time \( T_n \) is given by,

\[
\int_{0}^{T_n} dy_{n,N} (s) = \int_{0}^{T_n} \frac{\partial y_{n,N} (s)}{\partial Y_s} \tilde{\Sigma} dZ_{s}^{n+1,N} \]

\[ = \sum_{i=1}^{M} \tilde{C}_{n,N}^{(i)} \int_{0}^{T_n} e^{A_{(ii)} s} dW_{s}^{(i), n+1,N} \]

Which leads to an analytic expression for the volatility of a \( T_n \times T_N \) swaption,

\[
\sigma_{n,N} = \sqrt{\sum_{i=1}^{M} \tilde{\Sigma}^{2}_{(ii)} \left( \tilde{C}_{n,N}^{(i)} \right) \left[ \frac{e^{2A_{(ii)} T_n} - 1}{2A_{(ii)}} \right] + 2 \sum_{i=1}^{M} \sum_{j=i+1}^{M} \rho_{ij} \tilde{\Sigma}_{(ii)} \tilde{\Sigma}_{(jj)} \tilde{C}_{n,N}^{(i)} \tilde{C}_{n,N}^{(j)} \left[ \frac{e^{A_{(ii)} + A_{(jj)} T_n} - 1}{A_{(ii)} + A_{(jj)}} \right]}
\]

Or if we introduce \( e^{At} = \begin{pmatrix} e^{A_{(11)} t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ e^{A_{(MM)} t} & \cdots & \tilde{C}_{n,N}^{(M)} \end{pmatrix} \), \( \text{diag} (\tilde{C}_{n,N}) = \begin{pmatrix} \tilde{C}_{n,N}^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{C}_{n,N}^{(M)} \end{pmatrix} \) and \( \Omega = \tilde{\Sigma} \tilde{\Sigma}' \) the covariance matrix of \( W_{t}^{n+1,N} \), we can write in vector notation,

\[
\sigma_{n,N} = \left( \int_{0}^{T_n} [e^{As}]' \text{diag} \left( \tilde{C}_{n,N} \right) \Omega \text{diag} (C_{n,N}) e^{As} ds \right)^{\frac{1}{2}}
\]

We have established that, similar to the Vasicek case, in a Gaussian TSM, the swap rate is an approximately Gaussian Martingale with volatility as in (39) or equivalently (40). Approximate pricing formulas in the case of Multi-Factor Gaussian TSMs are equivalent to those in (19), (20) and (21) with (18) replaced by (39).
<table>
<thead>
<tr>
<th>Hull-White</th>
<th>2 Factor CIR</th>
<th>3 Factor Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0.05$</td>
<td>$a_1 = 0.2$</td>
<td>$a_1 = 1.0$</td>
</tr>
<tr>
<td>$\sigma = 0.01$</td>
<td>$a_2 = 0.2$</td>
<td>$a_2 = 0.2$</td>
</tr>
<tr>
<td>$\theta = 0.05$</td>
<td>$\theta_1 = 0.03$</td>
<td>$a_3 = 0.5$</td>
</tr>
<tr>
<td>$r_0 = 0.05$</td>
<td>$\theta_2 = 0.01$</td>
<td>$\sigma_1 = 0.01$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_1 = 0.04$</td>
<td>$\sigma_2 = 0.005$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_2 = 0.02$</td>
<td>$\sigma_3 = 0.002$</td>
</tr>
<tr>
<td></td>
<td>$\omega = 0.02$</td>
<td>$\rho_{12} = -0.2$</td>
</tr>
<tr>
<td></td>
<td>$X_1(0) = 0.04$</td>
<td>$\rho_{13} = -0.1$</td>
</tr>
<tr>
<td></td>
<td>$X_2(0) = 0.02$</td>
<td>$\rho_{23} = 0.3$</td>
</tr>
<tr>
<td></td>
<td>$\omega = 0.06$</td>
<td>$X_1(0) = 0.01$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X_2(0) = 0.005$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X_3(0) = -0.02$</td>
</tr>
</tbody>
</table>

Table 1. *Parameter values of the Vasicek model, the 2-Factor CIR model and the 3 Factor Gaussian model.*

The results on the quality of the approximation are produced using parameter values given here.
### Table 2.

<table>
<thead>
<tr>
<th>Swap Maturity</th>
<th>Option Maturity</th>
<th>ATMF</th>
<th>ITM</th>
<th>OTM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td></td>
<td>36.11 (0.00)</td>
<td>81.08 (0.10)</td>
<td>11.49 (-0.10)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19.5% (0.00%)</td>
<td>21.2% (0.08%)</td>
<td>18.2% (-0.07%)</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td></td>
<td>68.78 (0.00)</td>
<td>156.97 (0.19)</td>
<td>21.18 (-0.19)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19.1% (0.00%)</td>
<td>20.7% (0.08%)</td>
<td>17.8% (-0.07%)</td>
</tr>
<tr>
<td><strong>5</strong></td>
<td></td>
<td>149.30 (0.01)</td>
<td>357.27 (0.47)</td>
<td>41.65 (-0.45)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>17.8% (0.00%)</td>
<td>19.3% (0.09%)</td>
<td>16.6% (-0.07%)</td>
</tr>
<tr>
<td><strong>10</strong></td>
<td></td>
<td>239.86 (0.06)</td>
<td>618.15 (1.05)</td>
<td>56.73 (-0.98)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16.1% (0.00%)</td>
<td>17.4% (0.12%)</td>
<td>15.0% (-0.09%)</td>
</tr>
</tbody>
</table>

**Table 2.** Prices, implied volatilities and errors of the approximation method of section 4, in the Vasicek model for a set of ATMF swaptions at the parameters given in Table 1. To compare with stochastic duration, we obtain a relative error of 0.02% for a 1 × 10 ATMF swaption whereas the stochastic duration has an error of approximately 1% for an ATMF 6 month call on a 10 year coupon bond (Table 2 in Munk, 1999). Collin-Dufresne and Goldstein report errors of 0.1% or less for a 2 × 10 Swaption. The ITM results are for a payer swaption with a strike equal to 85% of the ATMF strike. The OTM results are for a payer swaption with a strike equal to 115% of the ATMF strike. For each swaption we report on the first line the approximate price in basis points next to the error in basis points. On the second line we report both approximate implied Black volatility in % and the error in implied Black volatility.
<table>
<thead>
<tr>
<th>ATMF</th>
<th>Option Maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>25.34 (-0.01)</td>
<td>29.88 (-0.01)</td>
<td>28.88 (-0.01)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9.5% (0.00%)</td>
<td>8.9% (0.00%)</td>
<td>7.3% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>44.57 (-0.01)</td>
<td>52.60 (-0.02)</td>
<td>50.94 (-0.03)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.9% (0.00%)</td>
<td>8.2% (0.00%)</td>
<td>6.7% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>78.98 (-0.02)</td>
<td>93.32 (-0.03)</td>
<td>90.61 (-0.05)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.2% (0.00%)</td>
<td>6.7% (0.00%)</td>
<td>5.3% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>99.82 (-0.04)</td>
<td>117.95 (-0.07)</td>
<td>114.55 (-0.10)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.6% (0.00%)</td>
<td>5.1% (0.00%)</td>
<td>4.0% (0.00%)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ITM</th>
<th>Option Maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>101.05 (0.03)</td>
<td>92.65 (0.08)</td>
<td>72.35 (0.08)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9.6% (0.06%)</td>
<td>8.9% (0.06%)</td>
<td>7.2% (0.04%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>190.75 (0.05)</td>
<td>174.51 (0.13)</td>
<td>136.24 (0.14)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.8% (0.06%)</td>
<td>8.2% (0.06%)</td>
<td>6.5% (0.04%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>411.03 (0.06)</td>
<td>375.12 (0.22)</td>
<td>293.35 (0.26)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.1% (0.08%)</td>
<td>6.5% (0.08%)</td>
<td>5.1% (0.05%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>675.49 (0.02)</td>
<td>618.51 (0.16)</td>
<td>487.83 (0.24)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.1% (0.29%)</td>
<td>5.0% (0.16%)</td>
<td>3.6% (0.07%)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>OTM</th>
<th>Option Maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.07 (-0.05)</td>
<td>5.28 (-0.08)</td>
<td>8.43 (-0.10)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9.4% (-0.05%)</td>
<td>8.8% (-0.04%)</td>
<td>7.3 (-0.04%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.86 (-0.08)</td>
<td>7.88 (-0.14)</td>
<td>13.17 (-0.19)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.8% (-0.05%)</td>
<td>8.2% (-0.04%)</td>
<td>6.7% (-0.03%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.27 (-0.13)</td>
<td>8.14 (-0.26)</td>
<td>15.74 (-0.41)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.3% (-0.07%)</td>
<td>6.8% (-0.05%)</td>
<td>5.5% (-0.04%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.65 (-0.10)</td>
<td>3.86 (-0.34)</td>
<td>9.73 (-0.71)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.7% (-0.10%)</td>
<td>5.3% (-0.08%)</td>
<td>4.2% (-0.07%)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3a. Prices, implied volatilities and errors of the approximation method of section 5.2 using transform inversion (TransformApprox), in a 2-factor CIR model for a set of swaptions at the parameters given in Table 1. The ITM and OTM are set at the same levels as before (85% and 115%) relative to the ATMF.
<table>
<thead>
<tr>
<th>Swap Maturity</th>
<th>Option Maturity</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>25.30</td>
<td>29.87</td>
<td>28.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.5%</td>
<td>8.9%</td>
<td>7.3%</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>44.52</td>
<td>52.59</td>
<td>51.09</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.8%</td>
<td>8.2%</td>
<td>6.7%</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>78.88</td>
<td>93.31</td>
<td>90.89</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.2%</td>
<td>6.7%</td>
<td>5.3%</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>99.71</td>
<td>117.95</td>
<td>114.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.6%</td>
<td>5.1%</td>
<td>4.0%</td>
</tr>
<tr>
<td><strong>ITM</strong></td>
<td><strong>Option Maturity</strong></td>
<td><strong>1</strong></td>
<td><strong>2</strong></td>
<td><strong>5</strong></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>101.25</td>
<td>93.14</td>
<td>73.24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.9%</td>
<td>9.2%</td>
<td>7.6%</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>191.08</td>
<td>175.40</td>
<td>137.91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.2%</td>
<td>4.3%</td>
<td>7.0%</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>411.41</td>
<td>376.51</td>
<td>296.38</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.5%</td>
<td>6.9%</td>
<td>2.8%</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>675.62</td>
<td>619.39</td>
<td>490.42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.9%</td>
<td>5.0%</td>
<td>4.1%</td>
</tr>
<tr>
<td><strong>OTM</strong></td>
<td><strong>Option Maturity</strong></td>
<td><strong>1</strong></td>
<td><strong>2</strong></td>
<td><strong>5</strong></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1.84</td>
<td>4.79</td>
<td>7.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.2%</td>
<td>8.6%</td>
<td>7.0%</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2.46</td>
<td>6.97</td>
<td>11.74</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.5%</td>
<td>7.9%</td>
<td>6.5%</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1.73</td>
<td>6.56</td>
<td>12.87</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.0%</td>
<td>6.4%</td>
<td>5.2%</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0.08</td>
<td>2.49</td>
<td>6.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.6%</td>
<td>4.9%</td>
<td>3.8%</td>
</tr>
</tbody>
</table>

Table 3b. Prices, implied volatilities and errors of the approximation method of section 5.3 using the closed form approximation (CEVApprox), in a 2-factor CIR model for a set of swaptions at the parameters given in Table 1. The ITM and OTM are set at the same levels as before (85% and 115%) relative to the ATMF.
<table>
<thead>
<tr>
<th></th>
<th>Option Maturity</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>21.07 (0.00)</td>
<td>23.85 (0.00)</td>
<td>23.52 (0.00)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.2% (0.00%)</td>
<td>8.4% (0.00%)</td>
<td>6.0% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>33.54 (0.02)</td>
<td>38.94 (0.02)</td>
<td>39.26 (0.02)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.2% (0.00%)</td>
<td>7.0% (0.00%)</td>
<td>5.2% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>54.03 (0.03)</td>
<td>64.57 (0.04)</td>
<td>66.61 (0.04)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.6% (0.00%)</td>
<td>5.0% (0.00%)</td>
<td>3.8% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>66.49 (0.04)</td>
<td>80.18 (0.05)</td>
<td>83.33 (0.06)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.9% (0.00%)</td>
<td>3.5% (0.00%)</td>
<td>2.8% (0.00%)</td>
<td></td>
</tr>
<tr>
<td>ITM</td>
<td>Option Maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>79.54 (0.03)</td>
<td>78.56 (0.05)</td>
<td>69.64 (0.06)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11.0% (0.05%)</td>
<td>9.1% (0.04%)</td>
<td>6.5% (0.03%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>154.66 (0.03)</td>
<td>151.11 (0.06)</td>
<td>132.23 (0.09)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.9% (0.04%)</td>
<td>7.6% (0.04%)</td>
<td>5.6% (0.03%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>361.50 (0.00)</td>
<td>346.40 (0.04)</td>
<td>295.44 (0.10)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6.1% (0.02%)</td>
<td>5.4% (0.03%)</td>
<td>4.1% (0.03%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>636.98 (-0.01)</td>
<td>604.83 (-0.00)</td>
<td>508.96 (0.05)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.3% (-0.44%)</td>
<td>4.0% (-0.01%)</td>
<td>3.0% (0.03%)</td>
<td></td>
</tr>
<tr>
<td>OTM</td>
<td>Option Maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Swap Maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.64 (-0.03)</td>
<td>2.93 (-0.05)</td>
<td>3.93 (-0.06)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9.5% (-0.04%)</td>
<td>7.8% (-0.03%)</td>
<td>5.6% (-0.02%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.12 (-0.04)</td>
<td>2.73 (-0.06)</td>
<td>4.50 (-0.08)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.7% (-0.04%)</td>
<td>6.5% (-0.03%)</td>
<td>4.8% (-0.2%)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.16 (-0.01)</td>
<td>0.95 (-0.06)</td>
<td>2.68 (-0.12)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.2% (-0.04%)</td>
<td>4.6% (-0.04%)</td>
<td>3.6% (-0.03%)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.00 (-0.00)</td>
<td>0.07 (-0.01)</td>
<td>0.51 (-0.08)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.7% (-0.05%)</td>
<td>3.3% (-0.05%)</td>
<td>2.6% (-0.04%)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Prices, implied volatilities and errors of the approximation method of section 4, generalized in the appendix, in the 3 factor Gaussian model for a set of swaptions at the parameters given in Table 1. The ITM and OTM are set at the same levels as before (85% and 115%) relative to the ATMF.
<table>
<thead>
<tr>
<th>Option Char.</th>
<th>StochDur</th>
<th>SU</th>
<th>CDG</th>
<th>TransformApprox</th>
<th>CEVApprox</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 × 30</td>
<td>0.06s</td>
<td>0.36s</td>
<td>NA</td>
<td>0.14s</td>
<td>0.02s</td>
</tr>
<tr>
<td>2 × 10</td>
<td>0.02s</td>
<td>NA</td>
<td>0.01s</td>
<td>0.05s</td>
<td>0.00s</td>
</tr>
</tbody>
</table>

Table 4. **Computational times for options with different option maturity and swap maturity for different approximation methods.** The numbers in the first line are scaled to represent the numbers reported in Singleton and Umantsev (2002).