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Universal autohomeomorphisms of $\mathbb{N}^*$

*Klaas Pieter Hart and Jan van Mill*

To the memory of Cor Baayen, who taught us many things

**Abstract**

We study the existence of universal autohomeomorphisms of $\mathbb{N}^*$. We prove that the Continuum Hypothesis (CH) implies there is such an autohomeomorphism and show that there are none in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

**Introduction**

This paper is concerned with universal autohomeomorphisms on $\mathbb{N}^*$, the Čech-Stone remainder of $\mathbb{N}$.

In very general terms we say that an autohomeomorphism $h$ on a space $X$ is *universal* for a class of pairs $(Y, g)$, where $Y$ is a space and $g$ is an autohomeomorphism of $Y$, if for every such pair there is an embedding $e : Y \rightarrow X$ such that $h \circ e = e \circ g$, that is, $h$ extends the copy of $g$ on $e[Y]$.

In [1, Section 3.4](https://www.ams.org/journals/bproc/2022-09-08/S2330-1511-2022-00106-8/viewer/#settings) one finds a general way of finding universal autohomeomorphisms. If $X$ is homeomorphic to $X^\infty$ then the shift mapping $\sigma : X^\mathbb{Z} \rightarrow X^\mathbb{Z}$ defines a universal autohomeomorphism for the class of all pairs $(Y, g)$, where $Y$ is a subspace of $X$. One embeds $Y$ into $X^\mathbb{Z}$ by mapping each $y \in Y$ to the sequence $\langle g^n(y) : n \in \mathbb{Z} \rangle$.

Thus, the Hilbert cube carries an autohomeomorphism that is universal for all autohomeomorphisms of separable metrizable spaces and the Cantor set carries one for all autohomeomorphisms of zero-dimensional separable metrizable spaces. Likewise the Tychonoff cube $[0, 1]^\kappa$ carries an autohomeomorphism that is universal for all autohomeomorphisms...
of completely regular spaces of weight at most \( \kappa \), and the Cantor cube \( 2^\kappa \) has a universal autohomeomorphism for all zero-dimensional such spaces.

Our goal is to have an autohomeomorphism \( h \) on \( \mathbb{N}^* \) that is universal for all autohomeomorphisms of all closed subspaces of \( \mathbb{N}^* \). The first result of this paper is that there is no trivial universal autohomeomorphism of \( \mathbb{N}^* \), and hence no universal autohomeomorphism at all in any model where all autohomeomorphisms of \( \mathbb{N}^* \) are trivial. On the other hand, the Continuum Hypothesis implies that there is a universal autohomeomorphism of \( \mathbb{N}^* \). The proof of this will have to be different from the results mentioned above because \( \mathbb{N}^* \) is definitely not homeomorphic to its power \((\mathbb{N}^*)^0\); it will use group actions and a homeomorphism extension theorem.

We should mention the dual notion of universality where one requires the existence of a surjection \( s : X \to Y \) such that \( g \circ s = s \circ h \). For the space \( \mathbb{N}^* \) this was investigated thoroughly in 2 for general group actions.

1. Some preliminaries

Our notation is standard. For background information on \( \mathbb{N}^* \) we refer to 5.

We denote by \( \text{Aut} \) the autohomeomorphism group of \( \mathbb{N}^* \). We call a member \( h \) of \( \text{Aut} \) trivial if there are cofinite subsets \( A \) and \( B \) of \( \mathbb{N} \) and a bijection \( b : A \to B \) such that \( h \) is the restriction of \( b \) to \( \mathbb{N}^* \).

In both sections we shall use the \( G_\delta \)-topology on a given space \((X, \tau)\); this is the topology \( \tau_\delta \) on \( X \) generated by the family of all \( G_\delta \)-subsets in the given space. It is well-known that \( \omega(X, \tau_\delta) \leq \omega(X, \tau)^{\aleph_0} \); we shall need this estimate in Section 3.

2. What if all autohomeomorphisms are trivial?

To begin we observe that fixed-point sets of trivial autohomeomorphism of \( \mathbb{N}^* \) are clopen. Therefore, to show that no trivial autohomeomorphism is universal it would suffice to construct a compact space that can be embedded into \( \mathbb{N}^* \) and that has an autohomeomorphism whose fixed-point set is not clopen.
The example

We let $L$ be the ordinal $\omega_1 + 1$ endowed with its $G_\delta$-topology. Thus all points other than $\omega_1$ are isolated and the neighbourhoods of $\omega_1$ are exactly the co-countable sets that contain it. Then $L$ is a $P$-space of weight $\aleph_1$ and hence, by the methods in 4, Section 2, its Čech-Stone compactification $\beta L$ can be embedded into $\mathbb{N}^*$. 

We define $f : L \to L$ such that $\omega_1$ is the only fixed point of $\beta f$. We put

$$
\begin{align*}
    f(\omega_1) &= \omega_1, \\
    f(2 \cdot \alpha) &= 2 \cdot \alpha + 1, \text{ and} \\
    f(2 \cdot \alpha + 1) &= 2 \cdot \alpha.
\end{align*}
$$

This defines a continuous involution on $L$.

If $p \in \beta L \setminus L$ then $p \in \text{cl} \alpha$ for some $\alpha < \omega_1$ and then either $E = \{2 \cdot \beta : \beta < \alpha\}$ or $O = \{2 \cdot \beta + 1 : \beta < \alpha\}$ belongs to the ultrafilter $p$. But $f[E] \cap E = \emptyset = f[O] \cap O$, hence $\beta f(p) \neq p$.

Since $\omega_1$ is not an isolated point of $\beta L$, no matter how this space is embedded into $\mathbb{N}^*$ there is no trivial autohomeomorphism of $\mathbb{N}^*$ that would extend $\beta f$.

3. The Continuum Hypothesis

Under the Continuum Hypothesis the space $\mathbb{N}^*$ is generally very well-behaved and one would expect it to have a universal autohomeomorphism as well. We shall prove that this is indeed the case. We need some well-known facts about closed subspaces of $\mathbb{N}^*$.

First we have Theorem 1.4.4 from 5 which characterizes the closed subspaces of $\mathbb{N}^*$ under $\text{CH}$: they are the compact zero-dimensional $F$-spaces of weight $c$, and, in addition: every closed subset of $\mathbb{N}^*$ can be re-embedded as a nowhere dense closed $P$-set.

Second we have the homeomorphism extension theorem from 3: $\text{CH}$ implies that every homeomorphism between nowhere dense closed $P$-sets of $\mathbb{N}^*$ can be extended to an autohomeomorphism of $\mathbb{N}^*$. 


Step 1.

We consider the natural action of Aut on $\mathbb{N}^*$; the map $\sigma : \text{Aut} \times \mathbb{N}^* \to \mathbb{N}^*$ given by $\sigma(f, p) = f(p)$. This action is continuous when Aut carries the compact-open topology $\tau$ and hence also when Aut carries the $G_\delta$-modification $\tau_\delta$ of $\tau$. For the rest of the construction we consider the topology $\tau_\delta$.

Using this action we define an autohomeomorphism $h : \text{Aut} \times \mathbb{N}^* \to \text{Aut} \times \mathbb{N}^*$ by $h(f, p) = (f, f(p))$. The map $h$ is continuous because its two coordinates are and it is a homeomorphism because its inverse $(f, p) \mapsto (f, f^{-1}(p))$ is continuous as well.

Now if $X$ is a closed subset of $\mathbb{N}^*$ and $g : X \to X$ is an autohomeomorphism then we can re-embed $X$ as a nowhere dense closed $P$-set and we can then find an $f \in \text{Aut}$ such that $f \upharpoonright X = g$. We transfer this embedded copy of $X$ to $\{f\} \times \mathbb{N}^*$ in $\text{Aut} \times \mathbb{N}^*$; for this copy of $X$ we then have $h \upharpoonright X = g$. It follows that $h$ satisfies the universality condition.

Step 2.

We embed $\text{Aut} \times \mathbb{N}^*$ into $\mathbb{N}^*$ in such a way that there is an autohomeomorphism $H$ of $\mathbb{N}^*$ such that $H \upharpoonright (\text{Aut} \times \mathbb{N}^*) = h$. Then $H$ is the desired universal autohomeomorphism of $\mathbb{N}^*$.

To this end we list a few properties of this product.

Weight

The weight of the product is equal to $\omega$, as both factors have weight $\omega$. For $\mathbb{N}^*$ this is clear and for Aut this follows because the topology $\tau$ has weight $\omega$ and one obtains a base for $\tau_\delta$ by taking the intersections of all countable subfamilies of a base for $\tau$.

Zero-dimensional and $F$

The product is a zero-dimensional $F$-space as the product of the $P$-space Aut and the compact zero-dimensional $F$-space $\mathbb{N}^*$, see 6, Theorem 6.1.
Strongly zero-dimensional

The product $\text{Aut} \times \mathbb{N}^*$ is not compact, but we shall construct a compactification of it that is also a zero-dimensional $F$-space of weight $\mathfrak{c}$.

For this we need to prove that $\text{Aut} \times \mathbb{N}^*$ is actually strongly zero-dimensional. We prove more: the product is ultraparacompact, meaning that every open cover has a pairwise disjoint open refinement.

Let $\mathcal{U}$ be an open cover of the product consisting of basic clopen rectangles.

For each $f \in \text{Aut}$ there is a finite subfamily $\mathcal{U}_f$ of $\mathcal{U}$ that covers $\{f\} \times \mathbb{N}^*$, say $\mathcal{U}_f = \{C_i \times D_i : i < k_f\}$. Let $C_f = \bigcap_{i < k_f} C_i$ and $D_{f,i} = D_i \setminus \bigcup_{j < i} D_j$ for $i < k_f$. Then $\mathcal{C}_f = \{C_f \times D_{f,i} : i < k_f\}$ is a disjoint family of clopen rectangles that covers $\{f\} \times \mathbb{N}^*$ and refines $\mathcal{U}$.

Because $\text{Aut}$ has weight $\mathfrak{c}$, and we assume CH, there is a sequence $(f_\alpha : \alpha \in \omega_1)$ in $\text{Aut}$ such that $\{C_{f_\alpha} : \alpha \in \omega_1\}$ covers $\text{Aut}$. Next we let $V_\alpha = C_{f_\alpha} \setminus \bigcup_{\beta < \alpha} C_{f_\beta}$ for all $\alpha$. Because $\text{Aut}$ is a $\mathcal{P}$-space the family $\{V_\alpha : \alpha \in \omega_1\}$ is a disjoint open cover of $\text{Aut}$.

The family $\{V_\alpha \times D_{f_\alpha,i} : i < k_{f_\alpha}, \alpha \in \omega_1\}$ then is a disjoint open refinement of $\mathcal{U}$.

A compactification

To complete Step 2 we construct a compactification of $\text{Aut} \times \mathbb{N}^*$ that is a zero-dimensional $F$-space of weight $\mathfrak{c}$ and that has an autohomeomorphism that extends $h$. The Čech-Stone compactification would be the obvious candidate, were it not for the fact that its weight is equal to $2^\mathfrak{c}$. More precisely, using some continuous onto function from $(\text{Aut}, \tau)$ onto $[0, 1]$ one obtains a clopen partition of $(\text{Aut}, \tau_\delta)$ of cardinality $\mathfrak{c}$. This shows that $\beta(\text{Aut} \times \mathbb{N}^*)$ admits a continuous surjection onto the space $\beta\mathfrak{c}$ (where $\mathfrak{c}$ carries the discrete topology).

To create the desired compactification we build, either by transfinite recursion or by an application of the Löwenheim-Skolem theorem, a subalgebra $\mathcal{B}$ of the algebra of clopen subsets of $\text{Aut} \times \mathbb{N}^*$ that is closed under $h$ and $h^{-1}$, of cardinality $\mathfrak{c}$, and that has the property that for every pair of countable subsets $A$ and $B$ of $\mathcal{B}$ such that $a \cap b = \emptyset$ whenever $a \in A$.
and \( b \in B \) there is a \( c \in B \) such that \( a \subseteq c \) and \( c \cap b = \emptyset \) for all \( a \in A \) and \( b \in B \). The latter condition can be fulfilled because \( \text{Aut} \times \mathbb{N}^* \) is an \( F \)-space — \( \bigcup A \) and \( \bigcup B \) have disjoint closures — and strongly zero-dimensional — the closures can be separated using a clopen set.

The Stone space \( \text{St}(B) \) of \( B \) is then a compactification of \( \text{Aut} \times \mathbb{N}^* \) that is a compact zero-dimensional \( F \)-space of weight \( \iota \), with an autohomeomorphism \( \tilde{h} \) that extends \( h \). We embed \( \text{St}(B) \) into \( \mathbb{N}^* \) as a nowhere dense \( P \)-set and extend \( \tilde{h} \) to an autohomeomorphism \( H \) of \( \mathbb{N}^* \).

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