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ORIGINAL RESEARCH

Logic of Justified Beliefs Based on Argumentation

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Abstract

This manuscript presents a *topological argumentation* framework for modelling notions of evidence-based (i.e., justified) belief. Our framework relies on so-called topological evidence models to represent the pieces of evidence that an agent has at her disposal, and it uses abstract argumentation theory to select the pieces of evidence that the agent will use to define her beliefs. The tools from abstract argumentation theory allow us to model agents who make decisions in the presence of contradictory information. Thanks to this, it is possible to define two new notions of beliefs, *grounded beliefs* and *fully grounded beliefs*. These notions are discussed in this paper, analysed and compared with the existing notion of *topological justified belief*. This comparison revolves around three main issues: closure under conjunction introduction, the level of consistency and their logical strength.

Keywords Doxastic logic · Modelling arguments · Justified belief · Grounded belief · Evidence models · Topological models · Abstract argumentation theory

1 Introduction

Within formal epistemology, different formal frameworks have been proposed for representing and analysing (different variations of) the notion of *belief*. Among them

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one can find not only proposals using quantitative tools [e.g., the theory of subjective probabilities (Easwaran 2011) and the ranking-based plausibility representation [Spohn 1988] but also approaches using qualitative structures [e.g., standard epistemic logic, either with a *KD45* relational representation (Hintikka 1962; Fagin et al. 1995; Meyer and van Der Hoek 1995) or via plausibility models (Board 2004; Baltag and Smets 2008]). Most of the mentioned proposals have been successful in modelling different doxastic notions and/or their dynamics. Yet, they are not necessarily adapted to model the *reasons*, *justifications* or *evidence* on which these beliefs are based.[1]

In answer to this general development, the last decade witnessed a number of different frameworks that can portray the justifications or evidence on which different epistemic attitudes are based. Within the qualitative camp, there are at least two different lines of work. The first has followed syntactic approaches. An example of this are the proposals in Baltag et al. (2012, 2014) and Égré et al. (2014), which combine dynamic epistemic logic (van Ditmarsch et al. 2008; van Benthem 2011) with the *justification terms* from justification logic (Artemov and Nogina 2005; Artemov 2008). Informally, these terms represent reasons or justifications. When incorporated to the formal language, one can build formulas as $t{:}\varphi$, expressing that t is a justification for φ. The second line of work relies on semantic tools. An example of this is the *evidence model* developed in van Benthem and Pacuit (2011); van Benthem et al. (2014), which represents evidence as a set of possible worlds. This proposal has opened new alternatives, as it allows the use of topological tools, leading to the *topological evidence model* of Baltag et al. (2016). The use of topological tools within logic is not new, and in fact has a long tradition.[2] In particular, there have been important proposals within epistemic logic. Among them, one can mention Georgatos (1992), Moss and Parikh (1992), Dabrowski et al. (1996), which use topological tools for describing a notion of *increase in knowledge*, that is, a form of *knowability*. The use of these tools for representing evidence establishes further connections between epistemic concepts and mathematical notions of measurement.

This paper extends the line of work on topological evidence structures. It does so by adding tools from *abstract argumentation theory* (Dung 1995), a framework that allows us to make decisions in the presence of contradictory information. As its name suggests, the abstract argumentation framework abstracts away from the actual content of a given set of arguments, focusing instead on the conflicts that occur among them. Based on this, it provides different criteria for selecting ‘appropriate’ sets of arguments, regardless of the detailed content-description of each individual argument. By using these tools when also discussing evidence, the agent can make sense of the potentially conflicting pieces of evidence she has obtained, selecting collections of evidence on which different notions of belief can be based.[3] The resulting setting,

[1] There are exceptions, as some earlier developments do represent doxastic notions and their supporting evidence. An example is the Dempster-Shafer theory of belief functions (Dempster 1968; Shafer 1976).

[2] Indeed, McKinsey (1941) already relates topology with modal logic. An overview of the work in this direction can be found in Aiello et al. (2007).

[3] Note how this idea of defining beliefs in terms of evidence and argumentation is different from, e.g., the proposal Grossi and van der Hoek (2014), which takes beliefs and arguments as two primitive and independent notions and studies their relationship.

called the *topological argumentation framework* in this paper, can be understood as realising an idea that was informally already present in the literature:

> [...] a statement is believable if it can be argued successfully against attacking arguments. In other words, whether or not a rational agent believes in a statement depends on whether or not [an] argument supporting this statement can be successfully defended against the counterarguments. (Dung 1995, p. 323)

The use of abstract argumentation within the topological evidence setting allows one to define different forms of evidence-based (i.e., justified) belief. This paper focuses on two of them, *grounded belief* and *fully grounded belief*, studying thoroughly their relationship as well as their relation with the notion of *topological justified belief* proposed in Baltag et al. (2016). The comparison reveals that *topological justified belief* and *grounded belief* represent different ways of dealing with a tension between consistency and informativeness, a tension that emerges when dealing with conflicting information: should the agent look for full consistency, even when this produces a flat notion of belief that contains mostly uninformative propositions? Or should she aim for more informative beliefs, even when this leads to a weaker form of consistency? The comparison also reveals that *fully grounded belief* strikes a balance between consistency and informativeness. Our formal analysis will make precise in what sense this notion is a balanced form of justified belief.

This paper is structured as follows. Section 2 provides a step-wise presentation and justification of the tools the paper relies on, leading to the definition of the main structure, the topological argumentation model. Then, in Sect. 3 we present the three notions of belief that the paper focuses on, *topological justified belief*, *grounded belief* and *fully grounded belief*, discussing their basic logical properties. In Sect. 4 we present a formal language for describing topological argumentation models, together with a sound and complete axiom system characterising its validities. All these tools are put to work in Sect. 5, in which we conduct a comprehensive comparative study of the relationship between the three doxastic notions. Finally, Sect. 6 summarises the paper and outlines some directions for further work.

Note that this paper can be seen as both a generalisation and an extension of the conference papers (Shi et al. 2017, 2018). As such, *some* results presented here are recalled from the mentioned papers so we have the basis to build further on them. The main novel contributions of the present paper, when compared to the mentioned conference papers, can be summarized as follows:

(i) The axiom system of Sect. 4 is proven to be sound and complete *for a class of models that imposes additional constraints on its components*. Hence, here it is shown that the same axiom system as in Shi et al. (2018) is now sound and complete for the more general notion introduced in this text (see Definition 5 and the footnote therein). Additionally, the present paper includes the details and subtleties of the new completeness proof.

(ii) Section 5 contains a novel discussion that makes fully precise the relationship between the three forms of justified belief examined here. In this way, it rounds up the analysis on justified belief that motivated this series of papers. The discussion makes use of some results presented in our earlier conference papers, but it also contains new ones. In particular, the results used in Sect. 5.3 to show the relationship between topological justified belief and fully grounded belief are new. Moreover, the proof of Proposition 15 makes use of the logic presented in Sect. 4, which demonstrates how our technical results contribute to our conceptual analysis.

2 A Formal Representation of Evidence

The notion of evidence has been discussed extensively in the literature, both in the logic and philosophy communities. Because of this, "*it is far from obvious that any one thing could play all of the diverse roles that evidence has at various times been expected to play*" (Kelly 2016). Still, some of the properties that are discussed in the literature can be lifted out, and that is what we will do for the purpose of this paper. This section focuses on these properties, introducing along the way the formal tools that will be used throughout the text.

2.1 Evidence Provides Information

Example 1 Nora is collecting information about a small meteor that crashed in a forest long time ago. Right now, she is interested on the day the meteor crashed. The people from a village nearby the crashing site say that the meteor crashed on September of 1914. Then, while looking at old newspapers, she finds a note stating that it crashed in that year on September the 12th.

During her research, Nora collects evidence in different ways: from the testimony of people and from a document. Each one of those pieces of evidence provides Nora with information which, if true, would exclude, respectively, all possible states of affairs in which the meteor crashed in a month other than September 1914, and all possible state of affairs in which it crashed on a date other than September the 12th, 1914. More generally, for an agent whose information is represented as the uncertainty that is present among a collection of possible states of affairs (e.g., the possible worlds model of Hintikka 1962), a *piece of evidence* can be understood as a subset of the initial collection. In this sense, a piece of evidence is a (defeasible) indication of the possible range of states of affairs to which the real world may belong.[4]

Throughout this text, let At be a countable set of atomic propositions.

Definition 1 [*Evidence model* (van Benthem and Pacuit 2011)] A (uniform) *evidence model* is a tuple $(W, \mathcal{E}_0, V)$ in which (i) $W \neq \varnothing$ is a set of possible worlds; (ii) $\mathcal{E}_0 \subseteq 2^W \setminus \{\varnothing\}$ is a collection of non-empty subsets of W (satisfying $W \in \mathcal{E}_0$), called the collection of *basic pieces of evidence*; (iii) $V : \mathsf{At} \to 2^W$ is a valuation function.

In an evidence model, $\mathcal{E}_0$ contains the basic pieces of evidence that the agent has acquired. By representing a piece of evidence simply as a set of worlds, evidence in

[4] For a further philosophical discussion on the notion of information-as-range, we refer to Adriaans and van Benthem (2008) and Martinez and Sequoiah-Grayson (2019).

an evidence model is simplified to its bare bones: its informational content (i.e., the alternatives the evidence discards). In particular, the model does not keep track of the evidence's sources.[5]

Now, observe the requirements over $\mathcal{E}_0$. The first states that a piece of evidence cannot be contradictory on its own ($\varnothing \notin \mathcal{E}_0$); the second states that the agent 'knows' which is the full range of possibilities ($W \in \mathcal{E}_0$). Note how there are no further restrictions over $\mathcal{E}_0$. In particular, pieces of evidence are assumed to be, in general, fallible: they do not need to contain 'the real world'. Thus, the agent might have two basic pieces of evidence contradicting each other: there may be $E_1, E_2 \in \mathcal{E}_0$ with $E_1 \cap E_2 = \varnothing$. Still, this means neither that the agent believes contradictions nor that she believes propositions that are at odds with one another. More precisely, having evidence *supporting* a certain proposition $P \subseteq W$ (i.e., having an $E \in \mathcal{E}_0$ such that $E \subseteq P$) is not enough to *believe* P.[6] A single piece of evidence is not enough; what matters is the bearing of the agent's total evidence.

2.2 Evidence Provides Affirmative Information

Identifying evidence as non-self-contradictory information is a good starting point for a more fine-grained analysis of evidence and its relation with belief. Yet, not all types of such information qualify as evidence. A further requirement is that evidence needs to be *affirmative*, that is, it needs to be *verifiable* when it is true and "true precisely in the circumstances when it can be affirmed" (Vickers 1989, p. 6).[7]

Example 2 The meteor has been kept in a museum, and Nora is now interested in its weight. Her different measuring devices provide different results. But each one of them has a measuring error.

Each one of the measuring devices provides Nora with an interval within which the meteor's weight might be. Each one of these intervals (a, b) provides affirmative evidence because, regardless of which is the precise weight of the meteor in the interval, she can look for a more precise device and affirm what the evidence asserts. But Nora can do more: she can combine the different intervals (i.e., the different pieces of evidence) to get a more precise result. How ought she do this?

It has been argued in the literature that the tools of mathematical topology can be used to model affirmative/verifiable information (Vickers 1989; Kelly 1996).[8] A *topology* over a non-empty domain X is a family $\tau \subseteq 2^X$ containing both X and $\varnothing$, and closed under both *finite* intersections and *arbitrary* unions. As argued in the references,

[5] Thus, a representation of Nora's scenario includes neither the people of the village nor the newspaper: it only includes the information that these sources provide.

[6] In other words, we are stating that a piece of evidence $E \in 2^W$ supports a proposition $P \subseteq W$ if and only if P holds whenever E is truthful (i.e., whenever E contains the real world).

[7] For a philosophical discussion on the use of these concepts in the context of scientific inquiry, we point to Kelly's work in Kelly (1996).

[8] Note that it is the idea of evidence as affirmative information what makes topology a suitable structure for modelling evidence. Other structures may be suitable as well (and even better suited) if some other aspects of evidence are taken into account. Here, no attempt is made to establish a topological structure as "the" structure for evidence.

this structure fits the need because it captures the logic of affirmative information. First, an infinite disjunction of pieces of affirmative information is affirmative: it can be affirmed by any of its affirmative disjunts. With respect to finite conjunctions of pieces of affirmative information, they are also affirmative: only a finite number of pieces of information needs to be affirmed, and this takes only finite time. But an infinite conjunction of pieces of affirmative information cannot necessarily be affirmed: it may take infinite time.[9] These ideas yield the following structure.

Definition 2 [*Topological evidence model* (Baltag et al. 2016)] A *topological evidence model* $(W, \mathcal{E}_0, \tau_{\mathcal{E}_0}, V)$ extends an evidence model $(W, \mathcal{E}_0, V)$ (Definition 1) with $\tau_{\mathcal{E}_0}$, the topology over W generated by $\mathcal{E}_0$.[10] Following the literature, the elements of a topology will be called *open sets*.

Based on what we have explained, $\mathcal{E}_0$ in a topological evidence model should be taken as a set of not only consistent but also affirmative pieces of information. All non-empty opens in the generated topology thus constitute all possible pieces of evidence which are available to the agent if she reasons about her basic set of evidence $\mathcal{E}_0$.

Once the agent has obtained all possible pieces of combined evidence, she can move on to the next step: using her evidence to define her beliefs. As it is discussed below, this is straightforward when all her evidence is truthful, but requires making decisions when, as in our case, it is not.

2.3 Evidence Weighted Against Other Evidence

Once the agent has combined all her available evidence (thus obtaining the topology $\tau_{\mathcal{E}_0}$), she is left with the task of selecting those pieces of combined information that will define her beliefs. If all basic pieces of evidence were truthful (technically, if every set $U \in \mathcal{E}_0$ contains 'the real world'), then their combination would have produced, among other things, a single *and consistent* strongest piece of combined evidence: intuitively, the *conjunction* of all basic pieces. In these cases, the agent can define her beliefs simply in terms of what is supported by this strongest piece of evidence.

But, in general, the agent might have obtained fallible data. Among other things, this means she might have two basic pieces of evidence contradicting each other (there might be $E_1, E_2 \in \mathcal{E}_0$ with $E_1 \cap E_2 = \varnothing$), and thus there might not be a single and consistent strongest piece of combined evidence (the conjunction of all basic pieces would not be consistent, and the consistent combinations should have left out E_1 or E_2 or both). In these cases, the agent is left with several consistent but mutually exclusive 'maximal' pieces of combined evidence. How can she define her beliefs? In other words, how can she make sense of this conflicting information? One alternative [taken by Baltag et al. (2016) and discussed in more detail in the next section] is to remain neutral: do not give priority to any 'maximal' piece of combined evidence. This is

9 Another classical example of an assertion which is not affirmative is "all ravens are black". It is clear that we should never take a piece of information which is impossible to be verified to be our evidence. Readers are referred to Vickers (1989, Chapter 2) for a deeper discussion.

10 The *topology generated by a given* $\mathcal{Y} \subseteq 2^X$ is the smallest topology $\tau_{\mathcal{Y}}$ over X such that $\mathcal{Y} \subseteq \tau_{\mathcal{Y}}$. When no confusion arises, $\tau_{\mathcal{E}_0}$ will be denoted simply by τ.

reasonable in some scenarios, but it leaves out a large and very important group of situations: those in which some evidence is, for some reason, more persuasive than other.

There are different ways to define some form of priority among pieces of evidence. From a purely mathematical perspective, one can define a ‘priority’ ordering over them (technically, assume a priority ordering over 2^W, or even assume a priority ordering over W and lift it). One could also expand the model to represent *sources* of evidence, assume some form of priority among them, and lift it to an ordering over the information they provide.

This paper follows a different approach. It relies on a well-known framework that allows us to make decisions in the presence of contradictory information: *abstract argumentation theory* (Dung 1995). The abstract argumentation framework abstracts away from the actual contents of a given set of arguments, focussing instead on the conflicts (attacks) that occur among them. Then, it uses this notion of attack to define different criteria for selecting ‘appropriate’ sets of arguments. Thus, one can understand the combined pieces of evidence the agent has obtained as arguments [as done already in Baltag et al. (2016)], and then use abstract argumentation tools for selecting sets of arguments on which notions of belief can be based. As it will be discussed below, the use of abstract argumentation is more general than defining/inducing an ordering over pieces of evidence.

Here are the basic definitions of the abstract argumentation setting.

Definition 3 (*Attack graph*) An *attack graph* is a tuple $(A, \leftarrowtail)$ in which (i) $A \neq \varnothing$ is a set of arguments, and (ii) $\leftarrowtail \subseteq (A \times A)$ is a binary relation, the *attack relation*, with $a_1 \leftarrowtail a_2$ read as “a_2 *attacks* a_1”.

The crucial component of an attack graph is the attack relation, which defines some form of priority among arguments. Still, the approach is more general than defining an ordering among arguments. The reason is that the relative priority between two arguments induced by the attack relation (e.g., by defining $a < b$ when $a \leftarrowtail b$ and $b \not\leftarrowtail a$) is independent of the arguments’ relative priority with respect to a third argument. In particular, transitivity does not need to hold: it is possible to have $a < b$ and $b < c$ without having $a < c$, thus allowing cycles of strictly more important pieces (e.g., the attack graph $(A = \{a, b, c\}, \leftarrowtail = \{(a, b), (b, c), (c, a)\})$).

In basic abstract argumentation, a key assumption is that a notion of *defense* between (sets of) arguments can be defined in terms of the notion of attack.[11]

Definition 4 (*Characteristic (defense) function*) Let $\mathcal{A} = (A, \leftarrowtail)$ be an attack graph. A set of arguments $B \subseteq A$ is said to *defend* argument $a \in A$ if and only if every argument attacking a (for all $x \in A$ such that $a \leftarrowtail x$) is itself attacked by some argument in B (there is $b \in B$ such that $x \leftarrowtail b$). Then, the *characteristic function of* $\mathcal{A}$, denoted by $d_{\mathcal{A}}$ and also called the *defense function*, returns the set of arguments

[11] In fact, the argumentation framework is general enough to allow an argument to be another argument’s attacker and defender simultaneously. Still, there are proposals introducing new relations between arguments, including different forms of attack and *support* (e.g., Oren and Norman 2008; Cayrol and Lagasquie-Schiex 2009; Prakken 2010).

defended by a given set $B \subseteq A$:

$$d_{\mathcal{A}}(B) := \{c \in A \mid c \text{ is defended by } B\}$$

When $c \in d_{\mathcal{A}}(B)$, it is said that c is *acceptable* with respect to B.

As mentioned, abstract argumentation provides different criteria to single out sets of arguments with appealing properties. For example, one might want for a given set to be *conflict-free* (there are no attacks between its elements) or *self-defended* (if someone in the set is attacked, someone in the set attacks the attacker). These and other options will be useful for selecting sets of arguments on which a notion of belief will be based. But before doing that (Sect. 3.2), here is how the argumentation framework is seamlessly integrated into a topological evidence model.

Definition 5 (*Topological argumentation model*) A *topological argumentation* (*TA*) *model* $(W, \mathcal{E}_0, \tau, \longleftarrow\!\!\prec, V)$ extends a topological evidence model $(W, \mathcal{E}_0, \tau, V)$ (Definition 2) with a relation $\longleftarrow\!\!\prec \subseteq (\tau \times \tau)$, the *attack relation* on τ, thus defining the attack graph $(\tau, \longleftarrow\!\!\prec)$ where $T_1 \longleftarrow\!\!\prec T_2$ is read as "T_2 *attacks* T_1". This attack relation is required to satisfy the following conditions[12]:

(i) for every $T_1, T_2 \in \tau$: $T_1 \cap T_2 = \varnothing$ if and only if $T_1 \longleftarrow\!\!\prec T_2$ or $T_2 \longleftarrow\!\!\prec T_1$;
(ii) for every $T \in \tau \backslash \{\varnothing\}$: $\varnothing \longleftarrow\!\!\prec T$ and $T \not\longleftarrow\!\!\prec \varnothing$.

In a *TA* model, the elements of τ are the arguments the attack graph talks about. This makes sense, as they are the components of the model representing potentially conflicting information. Still, we will be a bit more precise, using the term *arguments* to refer only to the non-empty elements of τ.[13]

The two conditions that $\longleftarrow\!\!\prec$ is required to satisfy define the notion of attack between two pieces of evidence. The first states (right to left) that 'attack implies conflict' (i.e., empty intersection), but also (left to right) that, while 'conflict implies attack', the attack does not need to be mutual. The second establishes that, while the empty set is attacked by all non-empty opens, it does not attack any of them.[14]

With the topological argumentation model defined, it is time to look into some of the different notions of belief that can be defined within this structure.

3 Three Notions of Justified Belief

This section discusses three notions of justified belief, all of them definable on *TA* models (Definition 5). While they differ on the ways they deal with conflicting infor-

[12] Previous presentations of this structure (Shi 2018; Shi et al. 2017, 2018) asked for a further condition on the attack relation: for every $T, T_1, T_1' \in \tau$, if $T_1 \longleftarrow\!\!\prec T$ and $T_1' \subseteq T_1$, then $T_1' \longleftarrow\!\!\prec T$. The definition provided here is a generalisation of the previous proposal. As it will be proved later, this does not change the resulting logic.

[13] The non-empty elements of τ have already been called *arguments* in Baltag et al. (2016). For a deeper discussion on this, see Özgün (2017, Section 5.2).

[14] Thus, from the first condition, it follows that the empty set only attacks itself. In particular, this says that no subset of τ containing $\varnothing$ can be conflict-free.

mation. these notions are closely related to one another. Their comparison, provided in Sect. 5, will bring along some interesting issues about justified belief.

3.1 Topological Justified Belief

As mentioned above, one alternative for dealing with conflicting information is to remain neutral: do not give priority to any 'maximal' piece of combined evidence. This is essentially the approach followed by Baltag et al. (2016), which defines a notion of belief, *topological justified belief*, in the following way.

Definition 6 [*Topological justified belief* (Baltag et al. 2016)] Let $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ be a *TA* model. The agent has a *topological justified belief* on a proposition $P \subseteq W$ in M (notation: $\mathsf{B}^t\, P$) if and only if every argument T can be *strengthened* into an argument T' that supports P, that is,

$$\mathsf{B}^t\, P \quad \textit{iff}_{def} \quad \text{for all } T \in \tau\backslash\{\varnothing\} \text{ there is } T' \in \tau\backslash\{\varnothing\} \text{ s.t. } T' \subseteq T \text{ and } T' \subseteq P$$

If one wants to allow the agent to have beliefs about her own beliefs, $\mathsf{B}^t\, P$ itself should be a proposition (that is, a set of worlds). In a *TA* model, the set of basic pieces of evidence $\mathcal{E}_0$ is global, and thus so is the topology τ. Then, it makes sense to make the notion of topological justified belief global, assigning it either the full domain or else the empty set:

$$\mathsf{B}^t\, P := \begin{cases} W & \text{for all } T \in \tau\backslash\{\varnothing\} \text{ there is } T' \in \tau\backslash\{\varnothing\} \text{ s.t. } T' \subseteq T \text{ and } T' \subseteq P \\ \varnothing & \text{otherwise} \end{cases}$$

As shown in Baltag et al. (2016), topological justified belief B^t is a *KD45*-operator: it is closed under material implication (*K*), consistent (*D*), and both positively (*4*) and negatively (*5*) introspective.

As discussed in the original proposal, one of the most interesting features of topological justified belief is that it can be characterised in terms of different topological notions [see Baltag et al. (2016), Proposition 2 for some of them]. One of these characterisations is useful to make clear the way topological justified belief deals with conflicting information. Recall that, given a topology τ over a set X, an open $T \in \tau$ is said to be *dense* if and only if it has a non-empty intersection with every other non-empty open. Then, as stated in Baltag et al. (2016, Prop. 2, item 4),

Proposition 1 *Let $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ be a TA model. Then, $\mathsf{B}^t\, P$ holds in M if and only if there is a dense open $T \in \tau$ such that $T \subseteq P$.*

This proposition tells us that the agent has a topological justified belief in P if and only if P is supported by an argument that is consistent with every other argument. Thus, when defining a topological justified belief, every argument is equally important. Although this is reasonable in some scenarios, it is not good enough for dealing with situations in which some evidence is more important than other.

3.2 Grounded Belief

As mentioned above, abstract argumentation allows us to make sense of conflicting information, and it does so by providing different criteria to single out sets of arguments with appealing properties. These sets can be used to identify arguments on which different notions of belief can be based on. One can look for arguments belonging to *conflict-free* or *self-defended* sets. One can also look for arguments in sets that are *admissible* (those that are conflict-free and whose members are all acceptable with respect to it), *complete* (those that are admissible and contain every acceptable argument with respect to it), *preferred* (maximal admissible sets) or *stable* (conflict-free sets that attack every argument not in it).[15] Each one of these alternatives yields a notion of belief whose properties are worthwhile to investigate.

This paper focuses on a particular set of arguments, defined in terms of the characteristic function d_τ that arises from the attack graph $(\tau, \longleftarrow\!\!\prec)$. Note that d_τ's domain and range, 2^W, defines a complete lattice (with the order given by inclusion); moreover, it has been shown that d_τ is monotonic (Dung 1995, Lemma 19). Then, it follows (Knaster 1928; Tarski 1955) that d_τ has a least fixed point: a smallest set $\mathsf{LFP}_\tau \subseteq \tau$ satisfying $\mathsf{LFP}_\tau = d_\tau(\mathsf{LFP}_\tau)$. The properties of this set of arguments LFP_τ, called within abstract argumentation theory the *grounded extension*, makes it an excellent candidate for the set of arguments over which beliefs can be defined. First, it can be proved that it is conflict-free (i.e., there are no $T, T' \in \mathsf{LFP}_\tau$ such that $T \longleftarrow\!\!\prec T'$).[16] Moreover, LFP_τ is a set that can defend all ($\mathsf{LFP}_\tau \subseteq d_\tau(\mathsf{LFP}_\tau)$) and only ($d_\tau(\mathsf{LFP}_\tau) \subseteq \mathsf{LFP}_\tau$) its members against any attackers. There might be other conflict-free subsets of τ containing exactly the arguments they defend, but LFP_τ is the common part shared by all of them.

Here are two more reasons that make the grounded extension appealing. The first is a property the set has in any attack graph: while the other mentioned alternatives provide more than one set (there is, in general, more than one admissible/complete/preferred/stable set in an attack graph), the grounded extension is always unique. The second is a property the set has in our specific setting. Although the grounded extension is unique, in general it might be empty. However, this is not the case in our framework. Indeed, from the restrictions imposed on $\longleftarrow\!\!\prec$, it follows that dense opens are never attacked (they are in conflict only with the empty set, which does not attack anybody), and thus they are always in LFP_τ. But $W \in \tau$ is a dense open, and thus LFP_τ always has at least one element.

Definition 7 [*Grounded belief* (Shi et al. 2017)] Let $M = (W, \mathcal{E}_0, \tau, \longleftarrow\!\!\prec, V)$ be a *TA* model. The agent *has a grounded belief on* $P \subseteq W$ in M (notation: $\mathfrak{B}^g\, P$) if and only if there is an argument in LFP_τ supporting P, that is

$$\mathfrak{B}^g\, P \quad \textit{iff}_{def} \quad \text{there exists } F \in \mathsf{LFP}_\tau \text{ such that } F \subseteq P.$$

[15] For more details and other alternatives, the reader is referred to Baroni and Giacomin (2009); Baroni et al. (2011).

[16] Thus, $\varnothing \notin \mathsf{LFP}_\tau$, and hence LFP_τ contains only *arguments*.

When $\mathfrak{B}^g\ P$ is understood as a proposition,

$$\mathfrak{B}^g\ P := \begin{cases} W & \text{there exists } F \in \mathsf{LFP}_\tau \text{ such that } F \subseteq P. \\ \varnothing & \text{otherwise} \end{cases}$$

The definition pins down LFP_τ as the set of arguments qualified for justifying an agent's belief. More precisely, the chosen arguments are those belonging to the smallest set which is conflict-free and defends all and only its members.

Here are the basic properties of grounded belief.

Proposition 2 *(*Properties of $\mathfrak{B}^g$*) Let* $(W, \mathcal{E}_0, \tau, \longleftarrow\!\!\!\prec, V)$ *be a TA model.*

(i) *Grounded beliefs are upward-closed, that is,* $\mathfrak{B}^g\ P$ *and* $P \subseteq Q$ *imply* $\mathfrak{B}^g\ Q$*. Thus, they are closed under conjunction elimination.*
(ii) *Moreover,* LFP_τ *itself is closed upwards: if* $F \in \mathsf{LFP}_\tau$ *and* $F' \in \tau$ *is such that* $F \subseteq F'$*, then* $F' \in \mathsf{LFP}_\tau$*.*
(iii) *Grounded beliefs are mutually consistent:* $\mathfrak{B}^g\ P$ *and* $\mathfrak{B}^g\ Q$ *imply* $P \cap Q \neq \varnothing$*.*
(iv) *Grounded beliefs are positively and negatively introspective.*

Proof We prove only the second and third statements in this proposition. The others are trivial.

(ii) Assume that $F \in \mathsf{LFP}_\tau$ and $F \subseteq F_1 \in \tau$. It will be shown that, for any $F_2 \in \tau$ such that $F_1 \longleftarrow\!\!\!\prec F_2$, there is $F_3 \in \mathsf{LFP}_\tau$ such that $F_2 \longleftarrow\!\!\!\prec F_3$. Take an arbitrary $F_2 \in \tau$ such that $F_1 \longleftarrow\!\!\!\prec F_2$. From $F \subseteq F_1$ and $F_1 \cap F_2 = \varnothing$, it follows that $F \cap F_2 = \varnothing$. By the first condition of the attack relation, it follows that either $F \longleftarrow\!\!\!\prec F_2$ or else $F_2 \longleftarrow\!\!\!\prec F$. In the first case, $F \longleftarrow\!\!\!\prec F_2$, from $F \in \mathsf{LFP}_\tau$ it follows that there must be another argument $F_3 \in \mathsf{LFP}_\tau$ such that $F_2 \longleftarrow\!\!\!\prec F_3$. So we find the required argument F_3. In the second case, $F_2 \longleftarrow\!\!\!\prec F$, the argument $F \in \mathsf{LFP}_\tau$ is the required one.
(iii) Suppose $\mathfrak{B}^g\ P$ and $\mathfrak{B}^g\ Q$ hold in the model. Then, there are $F_P, F_Q \in \mathsf{LFP}_\tau$ such that both $F_P \subseteq P$ and $F_Q \subseteq Q$. Since LFP_τ is conflict-free, both $F_Q \longleftarrow\!\!\!\prec F_P$ and $F_P \longleftarrow\!\!\!\prec F_Q$ fail; this implies $F_P \cap F_Q \neq \varnothing$ (first condition on $\longleftarrow\!\!\!\prec$), from which the required $P \cap Q \neq \varnothing$ follows. □

Despite mutual consistency (previous proposition) and closure under conjunction elimination (which follows from upwards-closure), grounded belief is not closed under conjunction introduction. The following use of a *TA* model for reasoning about a specific scenario (Example 3.1 in Shi et al. 2017) illustrates this.

Example 3 The zoo in Tom's town bought a new animal and will show it soon to the public. Tom is curious about what species the animal is, so he asks his colleagues. However, he gets different answers. Some tell him that the animal is a penguin ($\{1\}$), some tell him that the animal is a pterosaur ($\{2\}$) and some tell him that the animal is a bat ($\{3\}$). Moreover, two other colleagues, who he really trusts, tell him that the animal can fly ($\{2, 3\}$) and the animal is not a mammal ($\{1, 2\}$). After receiving all these pieces of information, Tom is puzzled. Although "the animal can fly" and "the animal is not a mammal" imply that the animal is a neither a penguin nor a bat, it is still

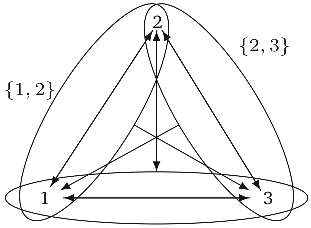

Fig. 1 Grounded beliefs are not closed under conjunction

hard to imagine that there can be a pterosaur living in this modern world. Intuitively, in such a situation, Tom comes to believe that the animal can fly and the animal is not a mammal. However, it seems that his evidence is not strong enough to support the claim that the animal is a pterosaur.

Let $\mathsf{At} = \{p, t, b\}$ be a set of atomic propositions (p: "the animal is a penguin"; t: "the animal is a pterosaur"; b: "the animal is a bat"). The following *TA* model M describes Tom's evidence, arguments and doxastic situation.

$$(\{1, 2, 3\}, \mathcal{E}_0 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}, \tau = 2^{\{1,2,3\}}, \longleftarrow\!\!\prec, V) \tag{1}$$

with $V = \{(p, \{1\}), (t, \{2\}), (b, \{3\})\}$ and the attack relation $\longleftarrow\!\!\prec$ given by the union of (i) singletons attacking one another, (ii) every open attacking the empty set, and (iii) $\{\{3\} \longleftarrow\!\!\prec \{1, 2\}, \{1\} \longleftarrow\!\!\prec \{2, 3\}, \{2\} \longleftarrow\!\!\prec \{1, 3\}, \{1, 3\} \longleftarrow\!\!\prec \{2\}\}$, as shown in Fig. 1.[17] Following its definition, $\mathsf{LFP}_\tau = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ (a set that is not closed under intersection); together with the definition of grounded belief, this confirms the intuition that Tom can come to believe that the animal can fly ($\{2, 3\} \in \mathsf{LFP}_\tau$) and that the animal is not a mammal ($\{1, 2\} \in \mathsf{LFP}_\tau$) without coming to believe that the animal is a pterosaur (no subset of $\{2\}$ is in LFP_τ).

It is important to notice that the lack of closure under conjunction does not indicate that the agent lacks reasoning abilities. Her set of available arguments (the topology) is the power set of the domain, thus showing that she can put all her pieces of evidence together in a logical way. However, the agent uses a very specific strategy for selecting the arguments (i.e., non-empty pieces of evidence) on which her beliefs will be based. It turns out that the set of chosen arguments, LFP_τ, is in general not closed under intersection; thus, the agent's *grounded* beliefs are, in general, not closed under conjunction introduction.

As a consequence of the failure of closure under conjunction introduction, grounded beliefs are consistent only up to a certain level.

Proposition 3 *Let $(W, \mathcal{E}_0, \tau, \longleftarrow\!\!\prec, V)$ be a TA model, and let $B \subseteq \{P \subseteq W \mid \mathfrak{B}^g P = W\}$ be any set of propositions the agent groundedly believes. Then, $\bigcap B \neq \varnothing$ holds only when $|B| \leq 2$; in other words, grounded beliefs are only mutually consistent but not necessarily fully consistent.*

17 Attack edges involving the empty set are not drawn.

An interesting question is, then, under which conditions are grounded beliefs closed under conjunction introduction? Or, in other words, under which conditions is LFP_τ closed under intersections? The next proposition provides two such scenarios.

Proposition 4 *Let $(W, \mathcal{E}_0, \tau, \leftarrow\!\!\prec, V)$ be a TA model, with LFP_τ the least fixed point of its characteristic function.*

- *If $\leftarrow\!\!\prec$ is symmetric on the set of arguments, then LFP_τ is closed under intersections.*
- *If $\leftarrow\!\!\prec$ is unambiguous (i.e. for all $T_1, T_2, T_3 \in \tau$, if $T_1 \leftarrow\!\!\prec T_2$ and $T_2 \leftarrow\!\!\prec T_3$, then $T_1 \not\leftarrow\!\!\prec T_3$ and $T_3 \not\leftarrow\!\!\prec T_1$), then LFP_τ is closed under intersections;*

Proof See Sect. A.1. □

3.3 Fully Grounded Belief

Both topological justified belief and grounded belief rely on the topology generated by the agent's basic pieces of evidence, but they do so in a different way; hence the discrepancy with respect to the closure under conjunction introduction. Still, it is possible to combine the ideas behind their respective definitions in order to get a notion of belief that, while remaining neutral about potential conflicts, it does so with respect to a potentially smaller set of arguments: those in LFP_τ. This is achieved by taking the original definition of topological justified belief (Definition 6), and adjusting the range over which the quantifiers work.

Definition 8 (*Fully grounded belief*) Let $M = (W, \mathcal{E}_0, \tau, \leftarrow\!\!\prec, V)$ be a topological argumentation model. The agent has a *fully grounded belief* on a proposition $P \subseteq W$ in M (notation: $\mathcal{B}^f\ P$) if and only if every argument in LFP_τ can be strengthened to an argument in LFP_τ which supports P. That is,

$$\mathcal{B}^f\ P \quad \textit{iff}_{\,def} \ \text{ for all } F \in \mathsf{LFP}_\tau \text{ there is } F' \in \mathsf{LFP}_\tau \text{ s.t. } F' \subseteq F \text{ and } F' \subseteq P.$$

Fully grounded belief can be turned into a proposition $\mathcal{B}^f\ P \subseteq W$ just as before.

Thus, the only difference between the definitions of topological justified belief and fully grounded belief is that, in the latter, all involved arguments should be members of LFP_τ.

Example 4 Recall the scenario of Example 3, in which $\mathsf{LFP} = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, and in which Tom's grounded beliefs are the propositions in $\{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ (the supersets of some argument in LFP). Note how the set of Tom's *fully* grounded beliefs contains $\{1, 2, 3\}$ (every element of LFP can be shrunk into an element of LFP that is also a subset of $\{1, 2, 3\}$) but nothing else: for example, $\{1, 2\}$ is not a fully grounded belief because $\{2, 3\} \in \mathsf{LFP}$ cannot be shrunk into a subset of $\{1, 2\}$ that is also in LFP, and $\{2, 3\}$ is not a fully grounded belief because $\{1, 2\} \in \mathsf{LFP}$ cannot be shrunk into a subset of $\{2, 3\}$ that is also in LFP.[18]

[18] Alternatively, $\{1, 2, 3\}$ is the only set supported by every element of LFP.

In this example, Tom's fully grounded beliefs coincide with his topological justified beliefs. Still, this does not mean that both notions are logically equivalent. To see the difference, consider a variation of the situation in which the attack relation from $\{1, 3\}$ to $\{2\}$ has been deleted. In this modified TA model, Tom's topological justified beliefs remain $\{\{1, 2, 3\}\}$, as the underlying topology does not change. However, LFP changes, becoming the set $\{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, which happens to be also the set containing the propositions about which Tom has a fully grounded belief.

Although fully grounded belief and topological justified belief are not logically equivalent, the similarity in their definitions (a $\forall\exists$ quantification pattern) suggests that they may share logical properties. Indeed, this is the case. As mentioned before, it has been shown that topological justified belief is a *KD45* operator; the proposition below does the same for fully grounded belief.

Proposition 5 *Fully grounded belief $\mathcal{B}^f$ is a KD45 operator (closed under both conjunction introduction and conjunction elimination, consistent, and with both positive and negative introspection) TA models.*

Proof The D axiom is valid for fully grounded belief because LFP_τ is conflict-free. Axioms 4 and 5 are valid because fully grounded belief is defined globally in the model. The necessitation rule and closure under conjunction elimination hold because LFP_τ contains the model's domain (recall: W is never attacked) and the definition of fully grounded belief is upward closed (if the argument F' in the definition satisfies $F' \subseteq P$, then for any $Q \supseteq P$ it follows that $F' \subseteq Q$).

It is only left to show that fully grounded beliefs are closed under conjunction introduction. Take a *TA* model, with $P, Q \subseteq W$, and suppose both $\mathcal{B}^f\ P$ and $\mathcal{B}^f\ Q$ hold. Then, for any $F \in \mathsf{LFP}_\tau$, not only there is $F' \subseteq F$ such that $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P$ (the first), but also there is $F'' \subseteq F$ such that $F'' \in \mathsf{LFP}_\tau$ and $F'' \subseteq Q$ (the second). Now, take an arbitrary $T \in \mathsf{LFP}_\tau$. From the first, there is $T' \subseteq T$ such that $T' \in \mathsf{LFP}_\tau$ and $T' \subseteq P$. From the second, $T' \in LFP_\tau$ implies that there is $T'' \subseteq T'$ such that $T'' \in \mathsf{LFP}_\tau$ and $T'' \subseteq Q$. But $T'' \subseteq T'$ and $T' \subseteq P$ imply that $T'' \subseteq P$. Hence, $T'' \subseteq P \cap Q$. Thus, for all $F \in \mathsf{LFP}_\tau$, we can find an argument $F' \subseteq F$ such that $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P \cap Q$. Therefore, $\mathcal{B}^f(P \cap Q)$ holds in the model. □

As a consequence, fully grounded beliefs are, indeed, 'fully' consistent.

Proposition 6 *Let $(W, \mathcal{E}_0, \tau, \leftarrow, V)$ be a TA model, for any finite set $B \subseteq \{P \subseteq W \mid B^t\ P = W\}$, $\bigcap B \neq \varnothing$. That is, fully grounded beliefs are fully consistent.*

This section has introduced three notions of justified belief, discussing their basic logical properties. Still, their relationship has not been made completely precise. Section 5 will provide a more systematic analysis of these three notions, including the way they relate to each other. The most important tool for such analysis will be provided in the next section.

4 The Logic of Argumentation, Justified Belief and Knowledge

This section provides a detailed account of the language $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$, introduced in Shi et al. (2018) to describe *TA* models. Since a *TA* model is based on a topological evidence model, $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$ follows the language presented in Baltag et al. (2016) in including an operator (U) for *infallible/irrevocable knowledge* (knowledge that, once acquired, it cannot be defeated by any further evidence) and an operator ($\Box$) for *factive argument* (an argument that is true in the actual world). These two operators are both conceptually meaningful and technically needed for a sound and complete axiomatization. Still, this is not enough for our purposes: a *TA* model also includes an attack graph ($\tau, \leftarrowtail$), from which the crucial least fixed point LFP_τ is defined. For talking about the information LFP_τ provides, $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$ includes two additional operators: $\mathfrak{B}^g$ for describing *grounded belief*, and $\mathfrak{T}$ for describing *grounded knowledge* (or, in more detail, factive grounded belief, that is, a grounded belief based on a factive argument).[19]

Definition 9 (*Language* $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$) The language $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$ is generated by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathrm{U}\,\varphi \mid \Box\,\varphi \mid \mathfrak{B}^g\,\varphi \mid \mathfrak{T}\,\varphi$$

with $p \in \mathsf{At}$. As mentioned above, formulas of the form $\mathrm{U}\,\varphi$ describe the agent's *infallible knowledge*, those of the form $\Box\,\varphi$ expresses that *"the agent has a factive argument for* φ", and those of the form $\mathfrak{B}^g\,\varphi/\mathfrak{T}\,\varphi$ express that *"the agent has grounded belief/knowledge of* φ". Other Boolean operators ($\vee, \rightarrow, \leftrightarrow$) as well as the modal duals of U, $\Box$, $\mathfrak{T}$ and $\mathfrak{B}^g$ are defined as usual (for the latter: $\widehat{\mathrm{U}}\,\varphi := \neg\,\mathrm{U}\,\neg\varphi$, $\Diamond\,\varphi := \neg\,\Box\,\neg\varphi$, $\widehat{\mathfrak{B}^g}\,\varphi := \neg\,\mathfrak{B}^g\,\neg\varphi$ and $\widehat{\mathfrak{T}}\,\varphi := \neg\,\mathfrak{T}\,\neg\varphi$).

Formulas in $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$ are semantically evaluated in *pointed TA* models, pairs (M, w) with $M = (W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ a *TA* model and $w \in W$ a world in it. The semantic interpretation of atoms and Boolean operators is as usual. For the modal operators,

$$\begin{array}{lll}
M, w \models \mathrm{U}\,\varphi & \textit{iff}_{\,def} & W \subseteq [\![\varphi]\!] \\
M, w \models \Box\,\varphi & \textit{iff}_{\,def} & \text{there exists } T \in \tau_{\mathcal{E}_0} \setminus \{\varnothing\} \text{ such that } w \in T \text{ and } T \subseteq [\![\varphi]\!] \\
M, w \models \mathfrak{B}^g\,\varphi & \textit{iff}_{\,def} & \text{there exists } F \in \mathsf{LFP}_\tau \text{ such that } F \subseteq [\![\varphi]\!] \\
M, w \models \mathfrak{T}\,\varphi & \textit{iff}_{\,def} & \text{there exists } F \in \mathsf{LFP}_\tau \text{ such that } w \in F \text{ and } F \subseteq [\![\varphi]\!]
\end{array}$$

with $[\![\varphi]\!] = \{w \in W \mid M, w \models \varphi\}$. A formula is valid (notation: $\models \varphi$) when $M, w \models \varphi$ holds for every world w of every *TA* model M.

Note how infallible knowledge U is given by a global universal modality. Then, the modality $\Box$ allows us to build formulas of the form $\Box\,\varphi$, indicating the existence of an argument ($T \in \tau_{\mathcal{E}_0} \backslash \{\varnothing\}$) that is *factive* (the evaluation point w is in T) and that supports φ ($T \subseteq [\![\varphi]\!]$). Combining these two, the notion of topological justified belief can be defined (Baltag et al. 2016, Proposition 2) in the following way:

$$\mathsf{B}^t\,\varphi := \mathrm{U}\,\Diamond\,\Box\,\varphi. \tag{2}$$

It is also useful to compare the truth conditions of $\mathfrak{B}^g$ and $\mathfrak{T}$. The difference is simple: while making $\mathfrak{B}^g\,\varphi$ true at w requires the existence of an argument F in LFP_τ

[19] This paper does not elaborate on the interpretation of $\mathfrak{T}$ as grounded *knowledge*. Interested readers are referred to Shi (2020), which makes a fine-grained comparative analysis of three different notions of knowledge that arise within *TA* models.

Table 1 Axiom system $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$, for $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ w.r.t. *TA* models

• Propositional Tautologies and Modus Ponens	
• The S5 axioms and rules for U	• The S4 axioms and rules for $\Box$
• $\vdash \mathfrak{T}\top$	• $\vdash \mathfrak{T}\varphi \to \mathfrak{T}\mathfrak{T}\varphi$
• $\vdash \mathfrak{T}\varphi \to \varphi$	• From $\vdash \varphi \to \psi$ infer $\vdash \mathfrak{T}\varphi \to \mathfrak{T}\psi$
• $\vdash \mathfrak{B}^g\varphi \to \neg\mathfrak{B}^g\neg\varphi$	• $\vdash (\mathfrak{T}\varphi \wedge \mathrm{U}\psi) \to \mathfrak{T}(\varphi \wedge \mathrm{U}\psi)$
• $\vdash \mathfrak{T}\varphi \to \Box\varphi$	
• $\vdash \mathfrak{T}\varphi \to \mathrm{U}(\Box\varphi \to \mathfrak{T}\varphi)$	• $\vdash \mathrm{U}\Diamond\Box\varphi \to \mathfrak{B}^g\varphi$
• $\vdash \big(\mathfrak{B}^g\varphi \wedge \neg\mathfrak{B}^g\psi \wedge \mathrm{U}((\varphi\wedge\psi) \to \Box(\varphi\wedge\psi))\big) \to \widehat{\mathrm{U}}\Box(\varphi \wedge \neg\psi)$	

that supports φ, making $\mathfrak{T}\varphi$ true at w requires, additionally, for the argument to be *factive* ($w \in F$). Notice how, given their semantic interpretation, $\mathfrak{B}^g\varphi$ and $\mathfrak{T}\varphi$ are mutually definable in the presence of conjunction, U and $\Box$. Indeed, first, grounded belief $\mathfrak{B}^g\varphi$ can be defined by simply adding a global existential quantification $\widehat{\mathrm{U}}$ before grounded knowledge $\mathfrak{T}$. In this way, the given argument $F \in \mathsf{LFP}_\tau$ does not need to be factive in the evaluation point; it is enough for it to be factive somewhere in the model, a tautological statement as every argument is non-empty. Thus,

$$\models \mathfrak{B}^g\varphi \leftrightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\varphi.$$

Second, by item (ii) of Proposition 2 and the truth conditions,

$$\models \mathfrak{T}\varphi \leftrightarrow (\mathfrak{B}^g\varphi \wedge \Box\varphi).$$

Thus, the languages $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$, $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ and $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g}$ are all equally expressive. Both 'grounded operators' $\mathfrak{B}^g\varphi$ and $\mathfrak{T}\varphi$ are included in $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$ as primitive operators because of their philosophical significance. Yet, one can work within $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ or $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g}$ without losing expressivity because the 'missing' primitive operator can be defined in terms of the others.[20] Still, this does not mean that any of the grounded operators can be defined within $\mathcal{L}_{\Box,\mathrm{U}}$[or, for that matter, within the languages discussed in Baltag et al. (2016)]. If both $\mathfrak{B}^g\varphi$ and $\mathfrak{T}\varphi$ are removed, the language cannot talk about the information provided by the attack graph (τ, $\leftarrow\!\!\!\prec$), and thus it cannot talk about arguments in LFP_τ.

As a final detail, the mentioned equivalence $\mathfrak{T}\varphi \leftrightarrow (\mathfrak{B}^g\varphi \wedge \Box\varphi)$ also shows how grounded knowledge $\mathfrak{T}\varphi$ does not need to coincide with the similar but different notion of *grounded true belief*, expressed in the language as $\mathfrak{B}^g\varphi \wedge \varphi$. The latter only asks for the grounded belief to be true, regardless of which is the LFP-argument it relies on, but the former requires a special kind of LFP-argument: a factive one.

Axiom system for $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$. It has been proved (Baltag et al. 2016, Thm. 4) that the validities of $\mathcal{L}_{\Box,\mathrm{U}}$ with respect to topological evidence models (Definition 2) are

[20] The completeness proof takes advantage of this fact, working only with $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$.

characterised by (i) propositional tautologies and Modus Ponens, (ii) the S4 axioms and rules for $\Box$; (iii) the S5 axioms and rules for U, (iv) $\mathrm{U}\,\varphi \rightarrow \Box\,\varphi$. Thus, the challenge in finding an axiom system for $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with respect to *TA* models consists essentially in finding additional axioms that characterise not only the grounded knowledge operator $\mathfrak{T}$ (which refers, semantically, to the least fixed point of a function defined in terms of a relation $\leftarrowtail$ with special properties) but also its relationship with the other modalities U and $\Box$.

Table 1 shows our proposal. Axioms and rules in the upper block should be familiar to modal logicians, and those in the second block describe the essential properties of grounded knowledge: it contains validities ($\mathfrak{T}\top$), it is truthful ($\mathfrak{T}\,\varphi \rightarrow \varphi$) and positively introspective ($\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\mathfrak{T}\,\varphi$), and it is monotonic ($\varphi \rightarrow \psi$ implies $\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\psi$).

The axioms in the third block describe the interaction between $\mathfrak{T}$ and U (recall that $\mathfrak{B}^g\,\varphi \leftrightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\varphi$). For the first, $\mathfrak{B}^g\,\varphi \rightarrow \neg\,\mathfrak{B}^g\,\neg\varphi$ states that grounded beliefs are mutually consistent. The second, $(\mathfrak{T}\,\varphi \wedge \mathrm{U}\,\psi) \rightarrow \mathfrak{T}(\varphi \wedge \mathrm{U}\,\psi)$, called "pullout axiom", states that infallible knowledge can be merged into grounded knowledge. Semantically, it is not hard to understand why this is valid.[21] Conceptually, this is also intuitive because infallible knowledge is in fact the agent's commitment to what is assumed to be true in the model, and this commitment accompanies all her other epistemic and doxastic attitudes. For a better grasp of the difference between the two notions of knowledge U and $\mathfrak{T}$, it is helpful to contrast the pullout axiom with the formula $\mathfrak{T}\,\varphi \wedge \mathfrak{T}\,\psi \rightarrow \mathfrak{T}(\varphi \wedge \psi)$, which is not valid.

The pullout axiom can be used to derive other validities describing the interaction between $\mathfrak{T}$ and U. An example is the following one, a variation of the famous *K* axiom. Note how the validity is different from the formula $\mathfrak{T}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\psi)$, which is not valid.

Proposition 7 $\vdash \mathrm{U}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\psi)$.

Proof

(1) $\vdash (\mathfrak{T}\,\varphi \wedge \mathrm{U}(\varphi \rightarrow \psi)) \rightarrow \mathfrak{T}(\varphi \wedge \mathrm{U}(\varphi \rightarrow \psi))$	Instance of the 'pullout' axiom
(2) $\vdash (\varphi \wedge \mathrm{U}(\varphi \rightarrow \psi)) \rightarrow \psi$	Axiom T for U; Modus Ponens
(3) $\vdash \mathfrak{T}(\varphi \wedge \mathrm{U}(\varphi \rightarrow \psi)) \rightarrow \mathfrak{T}\,\psi$	(2) and rule for $\mathfrak{T}$
(4) $\vdash \mathrm{U}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\,\varphi \rightarrow \mathfrak{T}\,\psi)$	(1), (3) and Modus Ponens

□

The axioms in the forth block of Table 1 describe the relationship between different modalities. Axiom $\mathfrak{T}\,\varphi \rightarrow \Box\,\varphi$ expresses that grounded knowledge of φ implies the existence of a factive argument for φ. Axiom $\mathrm{U}\,\Diamond\,\Box\,\varphi \rightarrow \mathfrak{B}^g\,\varphi$ characterises the relation between the two notions of justified belief (recall that $\mathsf{B}^t\,\varphi := \mathrm{U}\,\Diamond\,\Box\,\varphi$).

Axiom $\mathfrak{T}\,\varphi \rightarrow \mathrm{U}(\Box\,\varphi \rightarrow \mathfrak{T}\,\varphi)$ describes part of the relationship between an agent's grounded knowledge, infallible knowledge and factive arguments. It says that if the agent has grounded knowledge of φ, then she infallibly knows that for any of her arguments for φ, if it is factive then she has grounded knowledge of φ. Williamson

[21] When ψ is globally true, $\mathrm{U}\,\psi$ is also globally true, and thus is true in the factive argument that makes $\mathfrak{T}\,\varphi$ true.

(Williamson 2000) argues that evidence should not only be true but also be knowledge. The axiom can be seen as a weaker statement: after knowing φ, the agent can equate having true evidence for φ with having knowledge of φ. But, in general, true evidence does not equal knowledge.

Finally, axiom $\big(\mathfrak{B}^g\,\varphi \wedge \neg\,\mathfrak{B}^g\,\psi \wedge \mathrm{U}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))\big) \to \widehat{\mathrm{U}}\,\Box(\varphi \wedge \neg\psi)$ expresses the following intuition: if an agent that has the argument $\varphi \wedge \psi$ (the $\mathrm{U}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))$ part[22]) and has a grounded belief in φ but not in ψ, then she must have an argument for $\varphi \wedge \neg\psi$.

As usual, the soundness of the axiom system is proved by verifying that the axioms are valid (i.e., true in every world of every *TA* model) and the rules preserve validity. Most of the cases are relatively simple; here we focus on the last three axioms, to give the reader a better grasp of the involved modalities' semantic interpretation.

Proposition 8 $\models \mathfrak{T}\,\varphi \to \mathrm{U}(\Box\,\varphi \to \mathfrak{T}\,\varphi)$.

Proof Let $M = (W, \mathcal{E}_0, \tau, \leftarrow\!\!\prec, V)$ be a *TA* model; take $w \in W$, and suppose $M, w \models \mathfrak{T}\,\varphi$. Then, there exists $F \in \mathsf{LFP}_\tau$ such that $w \in F$ and $F \subseteq [\![\varphi]\!]$, that is, there is $F \in \mathsf{LFP}_\tau$ such that $F \subseteq [\![\varphi]\!]$. Now, take any $u \in W$ and suppose there is $T_u \in \tau_{\mathcal{E}_0} \backslash \{\varnothing\}$ such that both $u \in T_u$ and $T_u \subseteq [\![\varphi]\!]$ (i.e., suppose, $M, u \models \Box\,\varphi$). Then, $F \cup T_u$ is not only a factive (at u, as $u \in (F \cup T_u)$) argument supporting φ (clearly, $(F \cup T_u) \subseteq [\![\varphi]\!]$); it is also in LFP_τ, as $F \in \mathsf{LFP}_\tau$ and LFP_τ is closed upwards (Proposition 2). Therefore, $M, u \models \mathfrak{T}\,\varphi$. □

Proposition 9 $\models \mathrm{U}\,\Diamond\,\Box\,\varphi \to \mathfrak{B}^g\,\varphi$.

Proof By Proposition 1, $\mathrm{U}\,\Diamond\,\Box\,\varphi$ (recall: topological justified belief in φ) implies the existence of a dense open T supporting φ. But dense opens intersect with all non-empty opens, so they are not attacked at all; hence, T must be in LFP_τ, which gives us grounded belief in φ. □

Proposition 10 $\models \big(\mathfrak{B}^g\,\varphi \wedge \neg\,\mathfrak{B}^g\,\psi \wedge \mathrm{U}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))\big) \to \widehat{\mathrm{U}}\,\Box(\varphi \wedge \neg\psi)$.

Proof Let $M = (W, \mathcal{E}_0, \tau, \leftarrow\!\!\prec, V)$ be a *TA* model; take $w \in W$. Use the unfolded version of $\mathfrak{B}^g$, i.e., suppose

$$M, w \models \widehat{\mathrm{U}}\,\mathfrak{T}\,\varphi \wedge \neg\,\widehat{\mathrm{U}}\,\mathfrak{T}\,\psi \wedge \mathrm{U}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))$$

From the first conjunct, there is $F \in \mathsf{LFP}_\tau$ such that $F \subseteq [\![\varphi]\!]$. But, from the second, no $F' \in \mathsf{LFP}_\tau$ is such that $F' \subseteq [\![\psi]\!]$; in particular, $[\![\varphi \wedge \psi]\!] \notin \mathsf{LFP}$. However, by the third conjunct, $[\![\varphi \wedge \psi]\!] \in \tau$. So there must be an argument keeping $[\![\varphi \wedge \psi]\!]$ out of LFP, that is, there is a non-empty $T \in \tau$ such that $[\![\varphi \wedge \psi]\!] \leftarrow\!\!\prec T$ and T intersects with all arguments in LFP_τ, with the former implying that $T \subseteq W \backslash [\![\varphi \wedge \psi]\!]$ and the latter implying that $T \cap F \neq \varnothing$. It is precisely $T \cap F$ the argument that will be used to prove that $\widehat{\mathrm{U}}\,\Box(\varphi \wedge \neg\psi)$ holds at w. It is indeed an argument, as $T \cap F \neq \varnothing$ and $T \cap F \in \tau$; moreover, it supports $\varphi \wedge \neg\psi$, as from $F \subseteq [\![\varphi]\!]$ and $T \subseteq W \backslash [\![\varphi \wedge \psi]\!]$ it follows that $T \cap F \subseteq (W \backslash [\![\varphi \wedge \psi]\!]) \cap [\![\varphi]\!] = [\![\varphi \wedge \neg\psi]\!]$. Then, $M, w \models \widehat{\mathrm{U}}\,\Box(\varphi \wedge \neg\psi)$. □

[22] Note that "the agent has the argument φ" is different from "the agent has *an* argument *for* φ". The former is expressed by $\mathrm{U}(\varphi \to \Box\,\varphi)$, semantically stating that there is an argument T satisfying $T = [\![\varphi]\!]$; the latter corresponds to $\widehat{\mathrm{U}}\,\Box\,\varphi$, semantically stating that there is an argument T satisfying $T \subseteq [\![\varphi]\!]$.

For the strong completeness of the system, the reader can find a detailed proof in Appendix B.[23] For dealing with the operators U and $\Box$, the proof follows the general strategy used in Baltag et al. (2016). However, dealing with the additional the operator $\mathfrak{T}$ requires a by no means routine strategy. Semantically, this operator relies on the least fixed-point of the characteristic function d_τ, which is defined in terms of the attack relation $\leftarrowtail$. However, disentangling the subtle structure of LFP_τ in the constructed canonical model is not straightforward, as neither d_τ nor $\leftarrowtail$ are dealt with explicitly by the language.

Theorem 1 *The axiom system of Table 1 is sound and strongly complete for the language $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ w.r.t. topological argumentation models.*

Fully grounded belief? The language $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ can describe both the agent's topological justified beliefs (B^t) and her grounded beliefs ($\mathfrak{B}^g$). Here we use the axiom system $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$ discuss whether fully grounded belief is expressible in $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$.

Recall that topological justified belief is syntactically characterised within $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ as $\mathrm{U}\Diamond\Box\varphi$, with $\Diamond$ and $\Box$ referring to arbitrary arguments. Considering that the definition of fully grounded belief follows the same quantification pattern as that of topological justified belief, differing only on the quantification's domain (from arbitrary arguments to grounded arguments), this suggests that fully grounded belief could be syntactically characterised within $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ as $\mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$. However, this is not the case, witness the following derivation within the axiom system $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$.

Proposition 11 $\vdash \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi \leftrightarrow \widehat{\mathrm{U}}\mathfrak{T}\varphi$

Proof Each column proves one direction.

(1) $\vdash \widehat{\mathfrak{T}}\mathfrak{T}\varphi \to \widehat{\mathrm{U}}\mathfrak{T}\varphi$	(1) $\vdash \widehat{\mathrm{U}}\mathfrak{T}\varphi \to \widehat{\mathrm{U}}\mathfrak{T}\mathfrak{T}\varphi$
(2) $\vdash \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi \to \mathrm{U}\widehat{\mathrm{U}}\mathfrak{T}\varphi$	(2) $\vdash \widehat{\mathrm{U}}\mathfrak{T}\mathfrak{T}\varphi \to \neg\widehat{\mathrm{U}}\mathfrak{T}\neg\mathfrak{T}\varphi$
(3) $\vdash \mathrm{U}\widehat{\mathrm{U}}\mathfrak{T}\varphi \to \widehat{\mathrm{U}}\mathfrak{T}\varphi$	(3) $\vdash \neg\widehat{\mathrm{U}}\mathfrak{T}\neg\mathfrak{T}\varphi \to \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$
(4) $\vdash \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi \to \widehat{\mathrm{U}}\mathfrak{T}\varphi$	(4) $\vdash \widehat{\mathrm{U}}\mathfrak{T}\varphi \to \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$

$\square$

From this syntactic equivalence and the soundness of $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$, it follows that $\models \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi \leftrightarrow \mathfrak{B}^g\varphi$. But, as discussed before (Example 4), fully grounded belief $\mathcal{B}^f$ and grounded belief $\mathfrak{B}^g$ are not semantically equivalent (see also Sect. 5.2); thus, fully grounded belief cannot be syntactically defined by $\mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$. The reason for the discrepancy is that, despite following a similar quantification patter, topological justified belief B^t and fully grounded belief $\mathcal{B}^f$ require arguments from domains that have different closure properties: the one used in the definition of B^t (the topology τ) is closed under finite intersections, but the one used in the definition of $\mathcal{B}^f$ (the least fixed point LFP_τ) is not (Example 3).

The axiom system presented in this section allow us to reason about the different epistemic notions that can be expressed within $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ (and thus within $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{B}^g,\mathfrak{T}}$). In the next section, the system will be put to work to make precise the relationship between the three discussed notions of belief: topological justified belief, grounded belief and fully grounded belief.

[23] The completeness result proved in Appendix B is actually stronger than what Theorem 1 states (see Theorem 2).

5 Comparing the Discussed Notions of Belief

Section 3 has presented three different notions of justified belief that can be defined within a TA model. The first one, topological justified belief (B^t), relies only on the topology generated by the agent's set of basic evidence. The second and third, grounded belief ($\mathfrak{B}^g$) and fully grounded belief ($\mathcal{B}^f$), make additional use of the attack relation between arguments. The logical properties of each one of these notions have been also discussed: while both B^t and $\mathcal{B}^f$ are *KD45* operators, $\mathfrak{B}^g$ is not: it is consistent, fully introspective and closed under conjunction elimination, but it is not closed under conjunction introduction.

This section discusses the relationship between these three notions, making crucial use of the axiom system $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$. The discussion will reveal a tension between informativeness and consistency within an agent's justified beliefs.

5.1 Topological Justified Belief and Grounded Belief

The crucial connection between topological justified belief B^t and grounded belief $\mathfrak{B}^g$ has been already presented and proved (Proposition 9): in *TA* models, topological justified belief implies grounded belief. The reason is that every argument defining the first is also an argument defining the second: every dense open is in LFP_τ. Nevertheless, there might be arguments defining the second that are not arguments defining the first: LFP_τ might have opens that are not dense. Indeed, in the situation described by Example 3, $\{1, 2\}$ and $\{2, 3\}$ belong to LFP_τ, despite the fact that they are not dense opens. Thus, e.g., $\{1, 2\}$ is part of the agent's grounded beliefs, but not part of her topological justified ones (the unique dense open is $\{1, 2, 3\}$). This shows that grounded belief is strictly weaker than topological justified belief, allowing more propositions to be believed.

More interestingly, grounded belief is a generalization of topological justified belief in the sense that the latter is the special case of the former that arises when the attack relation in the *TA* model is *symmetric*.

Proposition 12 *Let* $(W, \mathcal{E}_0, \tau, \leftarrowtail, V)$ *be a TA model in which the attack relation* $\leftarrowtail$ *is symmetric. Then, for all* $P \subseteq W$, $\mathsf{B}^t\, P = \mathfrak{B}^g\, P$.

Proof Observe that, when the attack relation is symmetric, LFP_τ is actually given by $\{T \in \tau \mid \forall x \in \tau\backslash\{\varnothing\} : x \cap T \neq \varnothing\}$. The proposition follows immediately from this observation, Proposition 1 and the definition of $\mathfrak{B}^g$. □

In summary, while grounded belief accepts more propositions than topological justified belief (and hence it is a weaker notion), it does so at the cost of lacking closure under conjunction introduction (Example 3). It is worthwhile to emphasise, once again, that this should not be taken as an indication that the agent lacks reasoning abilities. The agent represented by a *TA* model is in fact a powerful reasoner, as she can combine the pieces of evidence in $\mathcal{E}_0$ she has collected. The 'issue', if one wants to call it that way, is the behaviour of the set of arguments she has deemed as 'good enough' for defining her beliefs, LFP_τ: the existence of both a 'good enough' argument for believing φ and a 'good enough' argument for believing ψ does not imply the existence of a 'good enough' argument for believing $\varphi \wedge \psi$.

Still, the lack of closure under conjunction introduction implies that grounded beliefs are potentially inconsistent, something that does not occur with topological justified beliefs. These two notions of belief can be seen then as what result from two different strategies for keeping consistency in the face of conflicting information. On the one hand, topological justified belief keeps consistency by not using the conflicting pieces at all (thus 'believing less'). On the other hand, grounded belief uses the conflicting pieces (thus 'believing more'), keeping consistency because of its lack of a closure property. For example, in Example 3, Tom's topological justified belief is only W, which seems to be too prudent.

Being aware of such a tension between topological justified belief and grounded belief, it is meaningful to ask whether there is a notion of belief which is still weaker than topological justified belief, and yet satisfies the closure property. It turns out that fully grounded belief serves as a positive answer to the question.

5.2 Grounded Belief and Fully Grounded Belief

Both grounded belief $\mathfrak{B}^g$ and fully grounded belief $\mathcal{B}^f$ are defined in terms of the arguments in the least fixed point LFP_τ in a topological argumentation model. An immediate observation about the logical relationship between these two notions is that fully grounded belief implies grounded belief.

Proposition 13 *Let $(W, \mathcal{E}_0, \tau, \hookleftarrow, V)$ be a TA model. Then,*

$$\{P \subseteq W \mid \mathcal{B}^f\ P = W\} \subseteq \{P \subseteq W \mid \mathfrak{B}^g\ P = W\}.$$

It is also clear that the two notions are not logically equivalent: fully grounded belief is closed under conjunction introduction (Proposition 5), but grounded belief is not (Example 3). From this and the previous proposition, it follows that, grounded belief is strictly weaker than fully grounded belief.

A more interesting and not so obvious fact is that fully grounded belief is the key to the question of the sufficient and necessary condition for grounded belief to be closed under conjunction introduction.[24]

Proposition 14 *Given a TA model, $\mathcal{B}^f\ P$ and $\mathfrak{B}^g\ P$ are semantically equivalent for any $P \subseteq W$ if and only if LFP_τ is closed under finite intersections.*

Proof From left to right: if grounded belief and fully grounded belief are equivalent in the given model, then grounded beliefs should be closed under conjunction introduction: for any $P, Q \subseteq W$, if $\mathfrak{B}^g\ P \wedge \mathfrak{B}^g\ Q$ holds then $\mathfrak{B}^g(P \wedge Q)$ also holds. This implies that LFP_τ is closed under finite intersection, as otherwise it is easy to find P and Q such that $\mathfrak{B}^g$'s closure under conjunction introduction fails.

From right to left: by the alternative characterisation of grounded belief as $\mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$ (Proposition 11), we only need to prove that $\mathcal{B}^f\ P \leftrightarrow \mathrm{U}\widehat{\mathfrak{T}}\mathfrak{T}\varphi$ holds when LFP_τ is

[24] Note that Proposition 4 only provides two sufficient conditions.

closed under finite intersection. For the first direction, assume $\mathcal{B}^f\ P$; then, for all $F \in \mathsf{LFP}_\tau$ there is $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$. Now take an arbitrary $w \in W$ and an arbitrary $F \in \mathsf{LFP}_\tau$ with $w \in F$; then there is $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$. But $\varnothing \notin \mathsf{LFP}_\tau$ so $F' \neq \varnothing$; there is $v \in F'$ with $F' \in \mathsf{LFP}_\tau$ and $F' \subseteq P$. Hence, for all $w \in W$ and all $F \in \mathsf{LFP}_\tau$ such that $w \in F$, we can find a $v \in F'$ such that $\mathfrak{T}\,P$ holds on v; then, $\mathrm{U}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,P$ holds in the model. Note that we did not use LFP_τ's closure under finite intersections.

For the second direction, assume $\mathrm{U}\,\widehat{\mathfrak{T}}\,\mathfrak{T}\,P$; then, for all $w \in W$ and all $F \in \mathsf{LFP}_\tau$ with $w \in F$, there is $v \in F$ such that there is an argument $F' \in LFP_\tau$ such that $v \in F'$ and $F' \subseteq P$. Note that F' is not required to be a subset of F; still, LFP_τ is closed under finite intersections, so $F \cap F'$ is also in LFP_τ, which gives us an argument in LFP_τ that is a subset of F ($F \cap F' \in \mathsf{LFP}_\tau$) and supports P ($F \cap F' \subseteq P$). So, for all $F \in \mathsf{LFP}_\tau$, we can find an $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and $F' \subseteq P$, which implies that $\mathcal{B}^f\ P$ holds in the model. □

5.3 Topological Justified Belief and Fully Grounded Belief

Sections 5.1 and 5.2 show that both topological justified belief B^t and fully grounded belief $\mathcal{B}^f$ are strictly stronger and more consistent (in the sense of satisfying closure under conjunction introduction) than grounded belief $\mathfrak{B}^g$. Then, what is the relationship between B^t and $\mathcal{B}^f$? The answer is that topological justified belief is stronger than fully grounded belief.

Proposition 15 *Take a TA model with domain W. For any proposition $P \subseteq W$, if $\mathsf{B}^t\ P$ holds, then so does $\mathcal{B}^f\ P$.*

The proof of this proposition is not so easy as one may think. It requires the following definition, fact and lemma, with the latter being the crucial step. The axiom system $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$ of Sect. 4 plays a pivotal role in the proofs.

Definition 10 (*Subspace topology*) Take a *TA* model with W its domain and $\tau_{\mathcal{E}_0}$ its topology; take a proposition $P \subseteq W$. The subspace topology on P is defined as $\tau|P := \{T \cap P \mid T \in \tau\}$.

Fact 1 *Given a TA model, if T is a dense open in $\tau_{\mathcal{E}_0}$ and P is also an open in $\tau_{\mathcal{E}_0}$, then $T \cap P$ is a dense open in $\tau|P$.*

Lemma 1 *Take a TA model with topology τ. For any $F \in LFP_\tau$, if there exists $F' \in LFP_\tau$ such that $F' \subseteq F$ and there is a dense open $T_{F'}$ in $\tau|F'$ satisfying $T_{F'} \subseteq P$, then there exists $F'' \in LFP_\tau$ such that $F'' \subseteq F$ and $F'' \subseteq P$.*

Proof The proof of the lemma hinges on the following validity:

$$\models \mathfrak{T}\,\Diamond\,\Box\,\varphi \to \widehat{\mathfrak{T}}\,\mathfrak{T}\,\varphi.$$

To prove the formula's semantic validity, it will be shown that it can be derived within $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$; then, the system's soundness will guarantee that the formula holds in every world of every *TA* model.

Proof of $\vdash \mathfrak{T} \Diamond \Box \varphi \rightarrow \widehat{\mathfrak{T}} \mathfrak{T} \varphi$. From axiom T for U (its contraposition in the first case), it follows that $\vdash \mathfrak{T} \Diamond \Box \varphi \rightarrow \widehat{\mathrm{U}} \mathfrak{T} \Diamond \Box \varphi$ and $\vdash \mathrm{U} \widehat{\mathfrak{T}} \mathfrak{T} \varphi \rightarrow \widehat{\mathfrak{T}} \mathfrak{T} \varphi$. Then, it is enough to 'bridge the gap', proving $\vdash \widehat{\mathrm{U}} \mathfrak{T} \Diamond \Box \varphi \rightarrow \mathrm{U} \widehat{\mathfrak{T}} \mathfrak{T} \varphi$. The equivalence stated in Proposition 11, $\vdash \mathrm{U} \widehat{\mathfrak{T}} \mathfrak{T} \varphi \leftrightarrow \widehat{\mathrm{U}} \mathfrak{T} \varphi$, provides an alternative for the consequent; then, it is enough to prove

$$\vdash \widehat{\mathrm{U}} \mathfrak{T} \Diamond \Box \varphi \rightarrow \widehat{\mathrm{U}} \mathfrak{T} \varphi$$

The proof goes through six stages. The derivations below use the standard names of axioms and rules, and CPL indicates the use of tautologies in classical propositional logic.

(i) The first step is to prove

$$\vdash \mathrm{U}((\Box \Diamond \Box \varphi \wedge \Box \varphi) \rightarrow \Box(\Box \Diamond \Box \varphi \wedge \Box \varphi)).$$

(1)	$\Box \Diamond \Box \varphi \rightarrow \Box \Box \Diamond \Box \varphi$	Axiom $4_\Box$
(2)	$\Box \varphi \rightarrow \Box \Box \varphi$	Axiom $4_\Box$
(3)	$(\Box \Diamond \Box \varphi \wedge \Box \varphi) \rightarrow \Box(\Box \Diamond \Box \varphi \wedge \Box \varphi)$	(1), (2), axiom $K_\Box$ and CPL
(4)	$\mathrm{U}((\Box \Diamond \Box \varphi \wedge \Box \varphi) \rightarrow \Box(\Box \Diamond \Box \varphi \wedge \Box \varphi))$	Nec_U

(ii) The second step is to prove

$$\vdash \neg \widehat{\mathrm{U}} \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$$

(1)	$\Box \Box \Diamond \Box \varphi \rightarrow \neg \Box \neg \Box \varphi$	Axiom $T_\Box$ and $\Diamond := \neg \Box \neg$
(2)	$\neg(\Box \Box \Diamond \Box \varphi \wedge \Box \neg \Box \varphi)$	(1) and CPL
(3)	$(\Box \Box \Diamond \Box \varphi \wedge \Box \neg \Box \varphi) \leftrightarrow \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$	Axiom $K_\Box$
(4)	$\neg \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$	(2), (3) and CPL
(5)	$\mathrm{U} \neg \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$	(4) and Nec_U
(6)	$\neg \widehat{\mathrm{U}} \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$	(5), $\widehat{\mathrm{U}} := \neg \mathrm{U} \neg$ and CPL

(iii) The third and crucial step uses the key axiom,

$$\vdash \widehat{\mathrm{U}} \mathfrak{T} \varphi \wedge \neg \widehat{\mathrm{U}} \mathfrak{T} \psi \wedge \mathrm{U}((\varphi \wedge \psi) \rightarrow \Box(\varphi \wedge \psi)) \rightarrow \widehat{\mathrm{U}} \Box(\varphi \wedge \neg \psi),$$

By taking φ to be $\Box \Diamond \Box \varphi$ and ψ to be $\Box \varphi$, it yields

$$\vdash \bigwedge \left\{ \begin{array}{l} \widehat{\mathrm{U}} \mathfrak{T} \Box \Diamond \Box \varphi, \\ \neg \widehat{\mathrm{U}} \mathfrak{T} \Box \varphi, \\ \mathrm{U}((\Box \Diamond \Box \varphi \wedge \Box \varphi) \rightarrow \Box(\Box \Diamond \Box \varphi \wedge \Box \varphi)) \end{array} \right\} \rightarrow \widehat{\mathrm{U}} \Box(\Box \Diamond \Box \varphi \wedge \neg \Box \varphi)$$

Item (i) above is the third conjunct in the antecedent; Item (ii) is the negation of the consequent. Thus, by CPL,

$$\vdash \widehat{\mathrm{U}}\,\mathfrak{T}\,\Box\,\Diamond\,\Box\,\varphi \rightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\Box\,\varphi$$

(iv) The fourth step is to prove

$$\vdash \widehat{\mathrm{U}}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\Box\,\Diamond\,\Box\,\varphi.$$

(1)	$\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \mathfrak{T}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi$	Axiom $4_{\mathfrak{T}}$
(2)	$\mathfrak{T}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \mathfrak{T}\,\Box\,\Diamond\,\Box\,\varphi$	Axiom $\vdash \mathfrak{T}\,\varphi \rightarrow \Box\,\varphi$ and Monotonicity$_{\mathfrak{T}}$
(3)	$\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \mathfrak{T}\,\Box\,\Diamond\,\Box\,\varphi$	(1), (2) and CPL
(4)	$\widehat{\mathrm{U}}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\Box\,\Diamond\,\Box\,\varphi$	Axioms and rules for U

(v) The fifth step is to prove that

$$\vdash \widehat{\mathrm{U}}\,\mathfrak{T}\,\Box\,\varphi \rightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\varphi$$

by using axiom $T_{\Box}$ and both Nec$_{\mathfrak{T}}$ and Nec$_{\widehat{\mathrm{U}}}$.

(vi) The sixth step uses Item (iv), Item (iii) and Item (v) to obtain the required

$$\vdash \widehat{\mathrm{U}}\,\mathfrak{T}\,\Diamond\,\Box\,\varphi \rightarrow \widehat{\mathrm{U}}\,\mathfrak{T}\,\varphi$$

For the actual proof of the lemma, take $F \in \mathsf{LFP}_\tau$ in a *TA* model M, and assume there exists $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and there is a dense open $T_{F'}$ in $\tau|F'$ satisfying $T_{F'} \subseteq [\![\varphi]\!]$. Then, for any $T \in \tau$ such that $T \cap F' \neq \varnothing$, we have $T \cap T_{F'} \neq \varnothing$ and $T \cap T_{F'} \subseteq [\![\varphi]\!] \cap F$. Therefore, for any $T \in \tau$ such that $T \cap F' \neq \varnothing$, there exists a world $v \in T$ such that $M, v \models \Box([\![\varphi]\!] \cap F)$. This implies that for any $w \in F'$, $M, w \models \mathfrak{T}\,\Diamond\,\Box([\![\varphi]\!] \cap F)$.[25]

By applying the proved validity, it follows that for any $w \in F'$, $M, w \models \widehat{\mathfrak{T}}\,\mathfrak{T}([\![\varphi]\!] \cap F)$. So for any $F'' \in \mathsf{LFP}_\tau$ such that $F'' \cap F' \neq \varnothing$, there exists $u \in F''$ such that $M, u \models \mathfrak{T}([\![\varphi]\!] \cap F)$. Recall that LFP_τ is conflict-free; thus, for any $F'' \in \mathsf{LFP}_\tau$ there exists $u \in F''$ such that $M, u \models \mathfrak{T}([\![\varphi]\!] \cap F)$, i.e. there is $F_u \in \mathsf{LFP}_\tau$ such that $F_u \subseteq [\![\varphi]\!] \cap F$.

Thus, under the assumptions of this proposition, any F can be strengthened to F_u such that $F_u \subseteq [\![\varphi]\!]$. □

Finally, Proposition 15 can be proved.

Proof (Proposition 15) Assume that $\mathsf{B}^t\,P$ holds in the given *TA* model. Then, by Proposition 1, there exists a dense open T in $\tau_{\mathcal{E}_0}$ such that $T \subseteq P$. By Fact 1, it follows that for any $F \in \mathsf{LFP}_\tau$, there is a dense open $T_F = T \cap F$ in $\tau|F$ satisfying

25 Note that we write P holds in w (i.e. $w \in P$) as $M, w \models P$.

$T_F \subseteq P$. So for any $F \in \mathsf{LFP}_\tau$ there exists $F' \in \mathsf{LFP}_\tau$ such that $F' \subseteq F$ and there is a dense open $T_{F'} \subseteq P$. By applying Lemma 1, it follows that for any $F \in \mathsf{LFP}_\tau$, there exists $F'' \in \mathsf{LFP}_\tau$ such that $F'' \subseteq F$ and $F'' \subseteq P$, which implies by the definition of fully grounded belief that $\mathcal{B}^f\ P$ holds in the given *TA* model. □

5.4 Discussion

The results in this section show that, for any *TA* model with domain W and for any $P \subseteq W$, we have $\mathsf{B}^t\ P \subseteq \mathcal{B}^f\ P \subseteq \mathfrak{B}^g\ P$. Moreover: as shown by Example 4, the opposite directions do not hold in general: $\mathfrak{B}^g\ P \nsubseteq \mathcal{B}^f\ P$ and $\mathcal{B}^f\ P \nsubseteq \mathsf{B}^t\ P$. Therefore, fully grounded belief is strictly weaker than topological justified belief, but strictly stronger than grounded belief.

$\mathsf{B}^t\ P$		$\mathcal{B}^f\ P$		$\mathfrak{B}^g\ P$
topological justified belief	$\rightarrow$	fully grounded belief	$\rightarrow$	grounded belief

As mentioned in the introduction, *topological justified belief* and *grounded belief* represent different ways of dealing with the tension between consistency and informativeness. The first chooses full consistency, even when this produces a ‘flat’ notion of belief; the second accepts more informative beliefs, even when this leads to a weaker form of consistency.

It is important to notice the crucial role the principle of closure under conjunction introduction (believing “P” and believing “Q” implies believing “P and Q”) plays. It is precisely by giving up this closure property that grounded beliefs manage to remain consistent up to a point (Item (iii) of Proposition 2; Proposition 3), even after allowing more informative propositions. In the literature, this principle is usually discussed in the context of probability-based beliefs, where belief is interpreted as probability above a given threshold (see, e.g., Huber and Schmidt-Petri 2009). Some philosophers have argued that it is necessary to give up the principle, as the lottery paradox (Kyburg 1961) demonstrates; some others insist on the necessity of “the closure of all-or-nothing belief under conjunction” (Leitgeb 2017).

Interestingly, some authors (e.g., Foley 2009) have proposed distinguishing between the closure principles governing beliefs and those characterising deductive reasoning. This position fits with what the results in this paper convey. Indeed, as it has been discussed, the lack of closure under conjunction introduction for grounded beliefs does not indicate that the agent lacks reasoning abilities. She is capable of combining her basic pieces of evidence as represented by the topology $\tau_{\mathcal{E}_0}$.[26] However, there is more than just combining evidence to obtain arguments: there is also the aspect of choosing arguments to define (potentially different) notions of beliefs. As it happens in the topological argumentation framework, while using arguments in certain ways produces notions of belief satisfying the principle (topological justified belief, fully grounded belief), using them in different ways produce epistemic notions that lack it (grounded belief).

The results also show how *fully grounded belief* strikes a balance between topological justified beliefs and grounded beliefs, that is, a balance between consistency and

[26] See Bjorndahl and Özgün (2019) and Balbiani et al. (2019) for proposals modelling this process.

informativeness. On the one hand, it is ‘more consistent’ than grounded belief, as it satisfies closure under conjunction introduction.[27] On the other hand, it is more informative than topological justified belief, as it allows more propositions to be believed.

6 Future Work

This paper combines a semantic representation of an agent’s evidence with tools from abstract argumentation theory. In the resulting structure, topological argumentation (*TA*) models, it is possible to define a wide spectrum of epistemic notions. This includes concepts already discussed in the literature on evidence-based beliefs, as *evidence*, *argument*, *infallible knowledge* and *topological justified belief*. Crucially, the use of tools from abstract argumentation makes it possible to define new doxastic attitudes, with this paper focussing on those of *grounded belief* and *fully grounded belief*. The main subject of this paper has been the study of the properties of these notions as well as the relationship they have with *topological justified belief*.

The *individual* analysis of the new notions has shown that, while grounded beliefs are mutually consistent, closed under conjunction elimination, and both positively and negatively introspective, they are not closed under conjunction introduction. The analysis has also shown that fully grounded belief has the properties of a *KD45* operator. Then, the *comparative* analysis has shown how, while topological justified beliefs can be seen as a notion that prioritises consistency, grounded beliefs can be seen as a notion that prioritises informativeness. More interestingly, the analysis has shown how fully grounded belief can be seen as the ‘middle point’ between the two previous doxastic notions: it is more informative than topological justified beliefs, but also more consistent than grounded beliefs. It is important to emphasise that the main points of discussion manifest themselves only because, in a *TA* model, there is an explicit distinction between the process of constructing arguments from evidence and the process of selecting arguments for defining beliefs.

The presented setting opens several interesting alternatives for further research. An immediate technical one follows from the fact that *fully grounded belief* has been semantically characterised but not syntactically defined. Further research on its syntactic definability is necessary, and it may as well require an extension of the language discussed here. It is also worth mentioning that the attack relation in a TA model is specified by the modeller, and not extracted from the model itself. In fact, although the model indicates the basic pieces of evidence the agent has, it does not say anything about their *source*. Thus, one could extend the setting by allowing an explicit representation of the sources of evidence.

There are also conceptual questions. The concept of grounded belief relies on the notion of grounded extension, but further doxastic notions may arise by using further tools from abstract argumentation theory. Indeed, other extensions might be considered, as *preferred* extension, *stable* extension and so on. They would give raise to further types of belief that can be compared with the ones studied here.

[27] In fact, fully grounded belief is *exactly* what emerges when the set of arguments on which grounded beliefs are based, LFP_τ, is closed under intersections (Proposition 14).

From a more general perspective, it is also interesting to move to a multi-agent scenario. In this topological-argumentation setting, this means not only for agents to have potentially different sets of basic pieces of evidence, but also for them to consider possibly different attack relations. This would give rise to a more ‘real’ argumentation setting, with argumentation taking place not only within an agent’s mind, but also between different agents. In turn, this emphasises the importance of a further dynamic layer, exploring the different epistemic actions that might affect the agent’s epistemic state. In line with other work on evidence-dynamics (e.g., van Benthem and Pacuit 2011), the emergence of new evidence is interesting (as is the dismissal of existing one); our setting also allows for changes in an agent’s attack relation (arising, e.g., from her interaction with others).

Finally, it is worth exploring the relation between our qualitative setting for evidence and belief and the quantitative approach developed in Dempster (1968) and Shafer (1976).

A Proofs

A.1 Proof of Proposition 4

For the first sufficient condition, assume that $\leftarrowtail$ is symmetric on the set of arguments. Observe that, then, $\mathsf{LFP}_\tau = \{T \in \tau \mid \forall x \in \tau \backslash \{\varnothing\} : x \cap T \neq \varnothing\}$, which is closed under conjunction.

The second sufficient condition requires more details, and the following lemma will be useful.

Lemma 2 *Let $M = (W, E_0, \tau_{E_0}, \leftarrowtail, V)$ be a TA model. Then, for all $F_1, F_2 \in \mathsf{LFP}_\tau$,*

$$F_1, F_2 \in \mathsf{LFP}_\tau \ \textit{implies}\ F_1 \cap F_2 \in \mathsf{LFP}_\tau$$
$$\textit{if and only if}$$
$$\textit{for all}\ T \in \tau,\ \textit{if}\ F_1 \cap F_2 \leftarrowtail T\ \textit{then there is}\ F \in \mathsf{LFP}_\tau\ \textit{such that}\ T \cap F = \varnothing.$$

Proof Take arbitrary $F_1, F_2 \in \mathsf{LFP}_\tau$. From left to right, consider the contrapositive, and suppose there is an open $T \in \tau$ such that T attacks $F_1 \cap F_2$ but is not in conflict with anybody in LFP_τ. From the latter it follows that nobody in LFP_τ attacks T, and thus the attacked $F_1 \cap F_2$ is not defended by LFP_τ; therefore, $F_1 \cap F_2$ is not in LFP_τ.

From right to left, take an argument $T \in \tau$ such that $F_1 \cap F_2 \leftarrowtail T$. Then, there is $F' \in \mathsf{LFP}_\tau$ such that $T \cap F' = \varnothing$, and thus either $T \leftarrowtail F'$ or else $F' \leftarrowtail T$. The first case gives us an argument in LFP_τ attacking T, namely F'; the second case does that too, as F' is in LFP_τ, and thus there should be $F'' \in \mathsf{LFP}_\tau$ such that $T \leftarrowtail F''$. Hence, for all $T \in \tau$ such that $F_1 \cap F_2 \leftarrowtail T$ there is $F \in \mathsf{LFP}_\tau$ such that $T \leftarrowtail F$: the set $F_1 \cap F_2$ is defended by LFP_τ. Thus, $F_1 \cap F_2 \in \mathsf{LFP}_\tau$. □

Now, take any $F_1, F_2 \in \mathsf{LFP}_\tau$ and, for the contrapositive, suppose $F_1 \cap F_2$ is not in LFP_τ. By Lemma 2, there is an open $T \in \tau$ which attacks $F_1 \cap F_2$ (i.e., $F_1 \cap F_2 \leftarrowtail T$) and which is not in conflict with any elements of LFP_τ (i.e., $F \in \mathsf{LFP}_\tau$

implies $T \cap F \neq \emptyset$). The goal is to show that $\longleftarrow$ is not unambiguous, and in order to achieve that define the following T_1, T_2 and T_3:

$$T_1 := F_1 \cap T, \quad T_2 := F_2 \cap T, \quad T_3 := F_1 \cap F_2.$$

Note how none of the sets are empty. Note also how, due to the fact that T attacks $F_1 \cap F_2$, T must be in conflict with $F_1 \cap F_2$ (that is, $(F_1 \cap F_2) \cap T = \emptyset$); hence, $T_1 \cap T_2 = T_2 \cap T_3 = T_3 \cap T_1 = \emptyset$: the sets are in conflict with one another, and thus there should be pairwise attacks in at least one direction. Consider two cases, $T_1 \longleftarrow T_2$ or $T_2 \longleftarrow T_1$.

- Suppose $T_1 \longleftarrow T_2$. If $T_2 \longleftarrow T_3$ is also the case, then the fact that either $T_1 \longleftarrow T_3$ or else $T_3 \longleftarrow T_1$ should hold make $\longleftarrow$ not unambiguous. Otherwise, $T_3 \longleftarrow T_2$ should hold and then, similarly, the fact that either $T_1 \longleftarrow T_3$ or else $T_3 \longleftarrow T_1$ hold make $\longleftarrow$ not unambiguous.
- Suppose $T_2 \longleftarrow T_1$. By an analogous argument, $\longleftarrow$ is not unambiguous.

Thus, the attack relation is not unambiguous.

B Proof of Theorem 1

The text has already argued for the soundness of the system of Table 1 within topological argumentation models (the validity of some of the interesting axioms is proved in Propositions 8, 9 and 10).

For completeness, It will be shown that any $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$-formulas is satisfiable. Satisfiability will be proved in an *Alexandroff qTA model* (see below), which is $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$-equivalent to its corresponding *TA* model.[28] In fact, we will prove a stronger completeness result than that stated in Theorem 1.

Theorem 2 *The axiom system of Table 1 is strongly complete for the language $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ w.r.t. topological argumentation models which satisfies the following extra condition: for every $T, T_1, T_1' \in \tau$: if $T_1 \longleftarrow T$ and $T_1' \subseteq T_1$, then $T_1' \longleftarrow T$.*

Next we prove the theorem.

Definition 11 (qTA model) A *quasi-topological argumentation model* (qTA) is a tuple $\mathcal{M} = (W, \mathcal{E}_0, \leq, \longleftarrow, V)$ in which $(W, \mathcal{E}_0, \tau, \longleftarrow, V)$ is a *TA* model (with τ generated by $\mathcal{E}_0$, as before) and $\leq \subseteq (W \times W)$ a preorder such that, for every $E \in \mathcal{E}_0$, if $u \in E$ and $u \leq v$, then $v \in E$.

Formulas in $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ are interpreted in qTA models just as in *TA* models. The only difference is $\Box$, which becomes a normal universal modality for $\leq$. More precisely, $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if $w \leq v$ then $\mathcal{M}, w \models \Box\varphi$. Now, two topological definitions, a refined qTA model, and the connection.

[28] A similar strategy is used in Baltag et al. (2016): the authors first showed show that any consistent set of formulas is satisfiable in a quasi-model, and then transformed such model into a modally-equivalent topological evidence model.

Definition 12 (*Specification preorder*) Let (X, τ) be a topological space. Its specification preorder $\sqsubseteq_\tau \subseteq (X \times X)$ is defined, for any $x, y \in X$, as $x \sqsubseteq_\tau y$ iff for all $T \in \tau$, $x \in T$ implies $y \in T$.

Definition 13 (*Alexandroff space*) A topological space (X, τ) is *Alexandroff* iff τ is closed under *arbitrary* intersections (i.e., $\bigcap T \in \tau$ for any $T \subseteq \tau$).

Definition 14 (*Alexandroff qTA model*) A qTA-model $\mathcal{M} = (W, \mathcal{E}_0, \leq, \leftarrow\!\!\prec, V)$ is called *Alexandroff* iff (i) $(W, \tau_{\mathcal{E}_0})$ is *Alexandroff*, and (ii) $\leq \,=\, \sqsubseteq_\tau$.

Proposition 16 *Given an Alexandroff qTA model* $\mathcal{M} = (W, \mathcal{E}_0, \leq, \leftarrow\!\!\prec, V)$, *take* $M = (W, \mathcal{E}_0, \tau, \leftarrow\!\!\prec, V)$. *Then,* $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_M$ *for every* $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$.

Proof Exactly as that of Özgün (2017, Prop. 5.6.14) for topological evidence models and $\mathcal{L}_{\Box,\mathrm{U}}$, as $\mathfrak{T}$ has the same truth condition in qTA and *TA* models. □

For notation, define $\Gamma^{\bigcirc} = \{\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}} \mid \bigcirc\varphi \in \Gamma\}$ for $\Gamma \subseteq \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ and $\bigcirc \in \{\Box, \mathrm{U}, \mathfrak{T}\}$. For the proof, let Φ_0 be a $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$-formulas. A slightly modified version of Lindenbaum Lemma shows that it can be extended to a maximal consistent one. Let MCS be the family of all maximally $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$-consistent sets of $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$-formulas; let Φ be an element of MCS extending Φ_0.

Definition 15 (*Canonical qTA model*) The canonical qTA model for Φ, $\mathcal{M}^\Phi = (W^\Phi, \mathcal{E}_0^\Phi, \leq^\Phi, \leftarrow\!\!\prec^\Phi, V^\Phi)$, is defined as follows.

- $W^\Phi := \{\Gamma \in \mathrm{MCS} \mid \Gamma^{\mathrm{U}} = \Phi^{\mathrm{U}}\}$ and $V^\Phi(p) := \{\Gamma \in W^\Phi \mid p \in \Gamma\}$.
- For $\Gamma, \Delta \in W^\Phi$, $\Gamma \leq^\Phi \Delta$ *iff*$_{def}$ for any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$, $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$.
- For any $\Gamma \in W^\Phi$, define the set $\leq^\Phi[\Gamma] := \{\Omega \in W^\Phi \mid \Gamma \leq^\Phi \Omega\}$. Then, let $\mathcal{E}_0^\Phi := \{\bigcup_{\Gamma \in U} \leq^\Phi[\Gamma] \mid U \subseteq W^\Phi\} \backslash \{\varnothing\}$.

While $\leq^\Phi$ and V^Φ are standard (recall: $\Box$ is a normal universal modality for $\leq$), each $E \in \mathcal{E}_0^\Phi$ is a non-empty union of the $\leq^\Phi$-upwards closure of the elements of some subset of W^Φ. The last component, the attack relation $\leftarrow\!\!\prec^\Phi$, is the novel one in this model, and it requires more care. First, define $\{\!|\varphi|\!\}_M := \{\Gamma \in W^\Phi \mid \varphi \in \Gamma\}$. Then, by taking τ^Φ to be the topology generated by $\mathcal{E}_0^\Phi$ define, for any $T, T' \in \tau^\Phi$,

$$\bullet\quad T \leftarrow\!\!\prec^\Phi T' \quad \mathit{iff}_{def} \quad \begin{cases} T = \varnothing & \text{if } T' = \varnothing \\ T \cap T' = \varnothing \text{ and there is no } \varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}} \text{ s.t.} & \text{otherwise} \\ \qquad\qquad \text{both } \{\!|\mathfrak{T}\varphi|\!\} \subseteq T \text{ and } \widehat{\mathrm{U}}\mathfrak{T}\varphi \in \Phi & \end{cases}$$

When no confusion arises, the superscript Φ will be omitted.

Note how $\mathcal{M}^\Phi$ is indeed a qTA model (Definition 11). First, it is clear that $\varnothing \notin \mathcal{E}_0$ and $W \in \mathcal{E}_0$. Moreover, $\leq$ is indeed a preorder (see its axioms) satisfying the extra condition. Finally, it can be proved that $\leftarrow\!\!\prec$ satisfies the three conditions.

Lemma 3 *Let* $\mathcal{M}^\Phi = (W, \mathcal{E}_0, \leq, \leftarrow\!\!\prec, V)$ *be the model of Definition* 15. *Then,*

(i) *for every* $T_1, T_2 \in \tau$*:* $T_1 \cap T_2 = \varnothing$ *if and only if* $T_1 \leftarrowtail T_2$ *or* $T_2 \leftarrowtail T_1$*;*
(ii) *for every* $T, T_1, T_1' \in \tau$*: if* $T_1 \leftarrowtail T$ *and* $T_1' \subseteq T_1$*, then* $T_1' \leftarrowtail T$*;*
(iii) *for every* $T \in \tau \backslash \{\varnothing\}$*:* $\varnothing \leftarrowtail T$ *and* $T \not\leftarrowtail \varnothing$*.*

Proof (i) The right-to-left direction is immediate. From left to right, assume $T_1 \cap T_2 = \varnothing$; moreover, for a contradiction, suppose both $T_1 \not\leftarrowtail T_2$ and $T_2 \not\leftarrowtail T_1$. Then, there is $\varphi_1 \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \subseteq T_1$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_1 \in \Phi$, and there is $\varphi_2 \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} \subseteq T_2$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_2 \in \Phi$. It follows that $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \cap \{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} = \varnothing$; to finish the proof, it is enough to show

Lemma 4 *For any* $\varphi_1, \varphi_2 \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$*, having* $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \cap \{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} = \varnothing$ *and both* $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_1 \in \Phi$ *and* $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_2 \in \Phi$ *leads to a contradiction.*

Proof From $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \cap \{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} = \varnothing$ and theorem $\mathfrak{T}\mathfrak{T}\varphi \rightarrow \mathfrak{T}\varphi$ it follows that $\{\!\!|\, \mathfrak{T}\mathfrak{T}\varphi_1 \,|\!\!\} \cap \{\!\!|\, \mathfrak{T}\mathfrak{T}\varphi_2 \,|\!\!\} = \varnothing$. Moreover: $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_1 \in \Phi$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_2 \in \Phi$ imply, respectively, $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \neq \varnothing$ and $\{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} \neq \varnothing$ (Proposition 17 below); this, together with axiom $\mathfrak{T}\varphi \rightarrow \mathfrak{T}\mathfrak{T}\varphi$, yields both $\{\!\!|\, \mathfrak{T}\mathfrak{T}\varphi_1 \,|\!\!\} \neq \varnothing$ and $\{\!\!|\, \mathfrak{T}\mathfrak{T}\varphi_2 \,|\!\!\} \neq \varnothing$, which imply (Proposition 17) $\widehat{\mathrm{U}}\,\mathfrak{T}\mathfrak{T}\varphi_1 \in \Phi$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\mathfrak{T}\varphi_2 \in \Phi$.

Observe how $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \cap \{\!\!|\, \mathfrak{T}\varphi_2 \,|\!\!\} = \varnothing$ also implies $\{\!\!|\, \mathfrak{T}\varphi_1 \,|\!\!\} \subseteq \{\!\!|\, \neg\mathfrak{T}\varphi_2 \,|\!\!\}$, so $\mathrm{U}(\mathfrak{T}\varphi_1 \rightarrow \neg\mathfrak{T}\varphi_2) \in \Phi$; then, from Proposition 7 it follows that $\mathrm{U}(\mathfrak{T}\mathfrak{T}\varphi_1 \rightarrow \mathfrak{T}\neg\mathfrak{T}\varphi_2) \in \Phi$. From the latter and $\widehat{\mathrm{U}}\,\mathfrak{T}\mathfrak{T}\varphi_1 \in \Phi$ we get $\widehat{\mathrm{U}}\,\mathfrak{T}\neg\mathfrak{T}\varphi_2 \in \Phi$; but then, axiom $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \rightarrow \neg\widehat{\mathrm{U}}\,\mathfrak{T}\neg\varphi$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\mathfrak{T}\varphi_2 \in \Phi$ imply $\neg\widehat{\mathrm{U}}\,\mathfrak{T}\neg\mathfrak{T}\varphi_2 \in \Phi$. Thus, we have a contradiction, as Φ is consistent.

(ii) Take $T, T_1, T_1' \in \tau$ with $T_1 \leftarrowtail T$ and $T_1' \subseteq T_1$. If $T = \varnothing$, the case is trivial (T_1 should be $\varnothing$, and so T_1'), so suppose $T \neq \varnothing$. From $T_1 \leftarrowtail T$, there is no $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!\!|\, \mathfrak{T}\varphi \,|\!\!\} \subseteq T_1$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$. But $T_1' \subseteq T_1$, so no such φ exists for T_1' either. Moreover, $T_1 \cap T = \varnothing$ so $T_1' \cap T = \varnothing$; hence, $T_1' \leftarrowtail T$.
(iii) Immediate. □

Thus, $\mathcal{M}^{\Phi}$ is a qTA model. The next proposition (standard proof) provides existence lemmas for the standard modality $\Box$ and the global modality $\widehat{\mathrm{U}}$.

Proposition 17 *For any* $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ *and any* $\Gamma \in W$*:*

- $\Diamond\varphi \in \Gamma$ *iff there is* $\Delta \in W$ *s.t.* $\Gamma \leq \Delta$ *and* $\varphi \in \Delta$*.*
- $\widehat{\mathrm{U}}\,\varphi \in \Gamma$ *iff there is* $\Delta \in W$ *s.t.* $\varphi \in \Delta$*;*

Now, tools to prove a similar result for the operator $\mathfrak{T}$, whose truth clause relies on LFP, given by $\leftarrowtail$. First, some useful properties of the model.

Fact 2 *(i)* $\tau = \mathcal{E}_0 \cup \{\varnothing\}$*. (ii) If* $\widehat{\mathrm{U}}\,\Box\varphi \in \Phi$*, then* $\{\!\!|\, \Box\varphi \,|\!\!\} \in \tau$*. (iii) If* $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$*, then* $\{\!\!|\, \mathfrak{T}\varphi \,|\!\!\} \in \tau$*. (iv) For any* $T \in \tau$ *and any* $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$*: if* $T \subseteq \{\!\!|\, \varphi \,|\!\!\}$*, then* $T \subseteq \{\!\!|\, \Box\varphi \,|\!\!\}$*.*

Proof (i) Immediate, as $\{\bigcup_{\Gamma \in U} \leq^{\Phi}[\Gamma] \mid U \subseteq W^{\Phi}\}$ is closed under intersections and unions.
(ii) Analogous to the next one.

(iii) Assume $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$; then, there is at least one $\Gamma \in W$ such that $\mathfrak{T}\varphi \in \Gamma$, that is, $\{\!|\, \mathfrak{T}\varphi \,|\!\} \neq \varnothing$. Now, note how $\{\!|\, \mathfrak{T}\varphi \,|\!\} = \bigcup_{\Gamma \in \{\!|\, \mathfrak{T}\varphi \,|\!\}} \leq[\Gamma]$.[29] Moreover, $\{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq W$ so, by $\mathcal{E}_0$'s definition, $\bigcup_{\Gamma \in \{\!|\, \mathfrak{T}\varphi \,|\!\}} \leq[\Gamma] = \{\!|\, \mathfrak{T}\varphi \,|\!\} \in \mathcal{E}_0$; then, by the first item, $\{\!|\, \mathfrak{T}\varphi \,|\!\} \in \tau$.

(iv) First, note how $T = \bigcup_{\Gamma \in T} \leq[\Gamma]$.[30] Then, from $T \subseteq \{\!|\varphi|\!\}$ it follows that $\bigcup_{\Gamma \in T} \leq[\Gamma] \subseteq \{\!|\varphi|\!\}$, that is, $\leq[\Gamma] \subseteq \{\!|\varphi|\!\}$ for every $\Gamma \in T$; therefore, $\Box\varphi \in \Gamma$ (Proposition 17) for every such Γ, and hence $T \subseteq \{\!|\, \Box\varphi \,|\!\}$.

□

Here are the first steps towards locating LFP.

Definition 16 (*Semi-acceptable and acceptable*) Define $\mathcal{C}_1$ as

$$\mathcal{C}_1 = \{T \in \tau \mid \text{there exists } \varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}} \text{ such that } \{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq T \text{ and } \widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi\}$$

- An open $T \in \tau$ is *semi-acceptable* if and only if, for any $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\, \Box\psi \,|\!\}$, there is $\xi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!|\, \mathfrak{T}\xi \,|\!\} \subseteq \{\!|\, \Box\psi \,|\!\}$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\xi \in \Phi$.
- An open $T \in \tau$ is *acceptable* if and only if T is semi-acceptable and there is no $T' \in \tau$ such that $T \cap T' = \varnothing$ and $T' \cap T'' \neq \varnothing$ for all $T'' \in \mathcal{C}_1$.

Define $\mathcal{C}_2$ as $\mathcal{C}_2 = \{T \in \tau \setminus \mathcal{C}_1 \mid T \text{ is acceptable}\}$.

Note that no element of $\mathcal{C}_1$ is attacked by elements of τ. Moreover,

Fact 3 *(i) For any $T \in \tau$, if $T \in \mathcal{C}_1$, then T is acceptable. (ii) If $T \in \tau$ is semi-acceptable, then $T \cap T' \neq \varnothing$ for all $T' \in \mathcal{C}_1$.*

Proof (i) If $T \in \mathcal{C}_1$, then there is $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $\{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq T$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$. For semi-acceptability, take any $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\, \Box\psi \,|\!\}$; the initial φ satisfies both $\{\!|\, \mathfrak{T}\xi \,|\!\} \subseteq \{\!|\, \Box\psi \,|\!\}$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\xi \in \Phi$. For the second condition of acceptability, $T \in \mathcal{C}_1$, so for any $T' \in \tau$ such that $T' \cap T'' \neq \varnothing$ for all $T'' \in \mathcal{C}_1$, it is the case that $T' \cap T \neq \varnothing$.

(ii) It will be proved that, for any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$, any semi-acceptable open $T \in \tau$ satisfies that $T \cap \{\!|\, \mathfrak{T}\varphi \,|\!\} \neq \varnothing$. Fact 3.(ii) follows, because every open in $\mathcal{C}_1$ should be a superset of $\{\!|\, \mathfrak{T}\varphi \,|\!\}$ for some $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$.
For a contradiction, suppose there is $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$ and $T \cap \{\!|\, \mathfrak{T}\varphi \,|\!\} = \varnothing$ for a semi-acceptable T. Now take any $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $T \subseteq \{\!|\, \Box\psi \,|\!\}$; then, $T \subseteq \{\!|\, \Box\psi \wedge \neg\mathfrak{T}\varphi \,|\!\}$. By Item (iv) of Fact 2, $T \subseteq \{\!|\, \Box(\Box\psi \wedge \neg\mathfrak{T}\varphi) \,|\!\}$. But T is semi-acceptable, so there is $\xi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!|\, \mathfrak{T}\xi \,|\!\} \subseteq \{\!|\, \Box(\Box\psi \wedge \neg\mathfrak{T}\varphi) \,|\!\}$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\xi \in \Phi$. Now, $\vdash \Box(\Box\psi \wedge \neg\mathfrak{T}\varphi) \rightarrow \neg\mathfrak{T}\varphi$, so $\{\!|\, \mathfrak{T}\xi \,|\!\} \subseteq \{\!|\neg\mathfrak{T}\varphi|\!\}$, that is, $\{\!|\, \mathfrak{T}\xi \,|\!\} \cap \{\!|\, \mathfrak{T}\varphi \,|\!\} = \varnothing$. The contradiction then follows from Lemma 4.

□

[29] For ($\subseteq$), suppose $\Delta \in \{\!|\, \mathfrak{T}\varphi \,|\!\}$; since $\leq$ is reflexive, $\Delta \in \leq[\Delta]$, and thus $\Delta \in \bigcup_{\Gamma \in \{\!|\, \mathfrak{T}\varphi \,|\!\}} \leq[\Gamma]$. For ($\supseteq$), take $\Delta \in \bigcup_{\Gamma \in \{\!|\, \mathfrak{T}\varphi \,|\!\}} \leq[\Gamma]$; then, $\Delta \in \leq[\Gamma]$ for some Γ in $\{\!|\, \mathfrak{T}\varphi \,|\!\}$, that is, for some Γ with $\mathfrak{T}\varphi \in \Gamma$. But then, from axioms $\mathfrak{T}\varphi \rightarrow \mathfrak{T}\mathfrak{T}\varphi$ and $\mathfrak{T}\varphi \rightarrow \Box\varphi$, we get $\Box\mathfrak{T}\varphi \in \Gamma$. Then, from $\leq$'s definition, $\Gamma \leq \Delta$ implies $\mathfrak{T}\varphi \in \Delta$. Hence, $\Delta \in \{\!|\, \mathfrak{T}\varphi \,|\!\}$.

[30] Indeed, $T \subseteq \bigcup_{\Gamma \in T} \leq[\Gamma]$ is immediate ($\leq$ is reflexive), and $T \supseteq \bigcup_{\Gamma \in T} \leq[\Gamma]$ follows from the fact that if $\Gamma \in T$ then $\leq[\Gamma] \subseteq T$.

Lemma 5 *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then, $\mathsf{LFP} = \mathcal{C}$.*

Proof ($\supseteq$) The proof of this direction consists of proving two cases. (i) If $T \in \mathcal{C}_1$, there is $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq T$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$. Hence, by $\leftarrowtail$'s definition, there is no $T' \in \tau$ with $T \cap T' = \varnothing$ such that $T \leftarrowtail T'$;[31] thus, $T \in \mathsf{LFP}$. Therefore, $\mathcal{C}_1 \subseteq \mathsf{LFP}$. (ii) Otherwise, $T \in \mathcal{C}_2$. Take any $T' \in \tau$ such that $T \leftarrowtail T'$. From $\leftarrowtail$'s definition, $T' \cap T = \varnothing$. From $\mathcal{C}_2$'s definition and $T \cap T' = \varnothing$, there is $T'' \in \mathcal{C}_1$ such that $T' \cap T'' = \varnothing$. From the previous case, T'' cannot be attacked, i.e., $T'' \not\leftarrowtail T'$; then, by the first (iii) on Lemma 3, $T' \leftarrowtail T''$. Summarising, for any $T' \in \tau$ attacking T, there is $T'' \in \mathcal{C}_1$ attacking T'; hence, $T \in d(\mathcal{C}_1) \subseteq d(\mathsf{LFP}) = \mathsf{LFP}$, that is, $T \in \mathsf{LFP}$. Therefore, $\mathcal{C}_2 \subseteq \mathsf{LFP}$.

($\subseteq$) Take now $T \in \tau$ such that $T \notin \mathcal{C}$; it will be shown that $T \notin \mathsf{LFP}$. The case with $T = \varnothing$ is immediate, as $\varnothing \leftarrowtail \varnothing$. Thus, suppose $T \neq \varnothing$.

From $T \notin \mathcal{C}$ it follows that $T \notin \mathcal{C}_1$, so there is no $\phi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!|\, \mathfrak{T}\phi \,|\!\} \subseteq T$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\phi \in \Phi$; hence, from $\leftarrowtail$'s definition, every $T' \in \tau$ with $T \cap T' = \varnothing$ satisfies that $T \leftarrowtail T'$.

Note that there is at least one $T' \in \tau$ with $T \cap T' = \varnothing$, for suppose otherwise, i.e., suppose $T' \in \tau$ implies $T \cap T' \neq \varnothing$. Now, take any $\Gamma \in W$; since $\leq[\Gamma] \in \tau$ (from $\leq[\Gamma] \in \mathcal{E}_0$ and Item (i) of Fact 2), it follows that $\leq[\Gamma] \cap T \neq \varnothing$, i.e., there is Δ with $\Gamma \leq \Delta$ and $\Delta \in T$. Furthermore, take any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\varphi|\!\}$; by Item (iv) of Fact 2, $T \subseteq \{\!|\, \Box\varphi \,|\!\}$ and then, from $\Gamma \leq \Delta$ and $\Delta \in \{\!|\, \Box\varphi \,|\!\}$, Proposition 17 gives us $\Gamma \in \{\!|\, \Diamond\Box\varphi \,|\!\}$, i.e., $\Diamond\Box\varphi \in \Gamma$. Thus, it has been shown that for any $\Gamma \in W$ and any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\varphi|\!\}$, we have $\Diamond\Box\varphi \in \Gamma$. Then, from Proposition 17 it follows that any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\varphi|\!\}$ satisfies that $\mathrm{U}\Diamond\Box\varphi \in \Phi$; thus, axiom $\mathrm{U}\Diamond\Box\varphi \to \widehat{\mathrm{U}}\,\mathfrak{T}\varphi$ implies $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$. Moreover, axiom $\mathfrak{T}\varphi \to \Box\varphi$ implies $\{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq \{\!|\, \Box\varphi \,|\!\}$. Thus, for any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\, \Box\varphi \,|\!\}$ we have found a formula in $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$, φ itself, such that $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$ and $\{\!|\, \mathfrak{T}\varphi \,|\!\} \subseteq \{\!|\, \Box\varphi \,|\!\}$; hence, T is semi-acceptable. But $T \cap T' \neq \varnothing$ for all $T' \in \tau$, so there is no $T' \in \tau$ such that both $T \cap T' = \varnothing$ and, for any $T'' \in \mathcal{C}_1$, $T \cap T'' \neq \varnothing$: in other words, T is acceptable. Hence, $T \in \mathcal{C}_2$, i.e., $T \in \mathcal{C}$: a contradiction. Thus, indeed there must be $T' \in \tau$ such that $T \cap T' = \varnothing$.

The rest of the proof is divided into two cases: either there is $T' \in \tau$ with $T \cap T' = \varnothing$ and $T' \in \mathcal{C}$ (at least one T' contradicting T is in $\mathcal{C}$), or else for any $T' \in \tau$ with $T \cap T' = \varnothing$ we have $T' \notin \mathcal{C}$ (no T' contradicting T is in $\mathcal{C}$). In both cases, it will be shown that $T \notin \mathsf{LFP}$.

(i) The first case is the simple one: take any $T' \in \tau$ such that $T \cap T' = \varnothing$ and $T' \in \mathcal{C}$. Then, as it has been argued, $T \leftarrowtail T'$; moreover, as it has been proved, $\mathcal{C} \subseteq \mathsf{LFP}$. Thus, $T \notin \mathsf{LFP}$, as LFP has to be conflict-free.

(ii) The second case requires more care. Since no element of $\mathcal{C}$ contradicts T, it follows that $C \in \mathcal{C}$ implies $T \cap C \neq \varnothing$. Now, consider the following two sub-cases: either T is semi-acceptable, or it is not. We will prove that, in both cases, $T \notin d(\mathcal{C})$.

If T is semi-acceptable, recall the initial assumption $T \notin C = \mathcal{C}_1 \cup \mathcal{C}_2$. Then T

[31] This includes $\varnothing$, as $T \in \mathcal{C}_1$ implies $\{\!|\, \mathfrak{T}\varphi \,|\!\} \neq \varnothing$ for T's corresponding φ; hence, $T \neq \varnothing$ and thus $T \not\leftarrowtail \varnothing$.

cannot be acceptable, that is, there must be $T'' \in \tau$ such that $T \cap T'' = \varnothing$ and, for any $C_1 \in \mathcal{C}_1$, we have $T'' \cap C_1 \neq \varnothing$. Now, take any such T''. From $T \notin C$, $T \cap T'' = \varnothing$ and $T'' \neq \varnothing$ (as its intersection with C_1 is non-empty), it follows that $T \leftarrowtail T''$: this T'' attacks T. Since $T'' \cap C_1 \neq \varnothing$ for any $C_1 \in \mathcal{C}_1$, there is no $C_1 \in \mathcal{C}_1$ such that $T'' \leftarrowtail C_1$. Moreover, there is no $C_2 \in \mathcal{C}_2$ such that $T'' \leftarrowtail C_2$ – otherwise, $T'' \cap C_2 = \varnothing$ would imply $C_2 \notin \mathcal{C}_2$. So, in summary, if T is semi-acceptable, then there is $T'' \in \tau$ such that $T \leftarrowtail T''$; there is no $C \in \mathcal{C}$ such that $T'' \leftarrowtail C$. Therefore, $T \notin d(\mathcal{C})$.

If T is not semi-acceptable, there is $\varphi_T \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $T \subseteq \{\!\!\{ \Box\varphi_T \}\!\!\}$ and there is no $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that both $\{\!\!\{ \mathfrak{T}\psi \}\!\!\} \subseteq \{\!\!\{ \Box\varphi_T \}\!\!\}$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\psi \in \Phi$. In particular, φ_T itself cannot be such ψ, so either $\{\!\!\{ \mathfrak{T}\varphi_T \}\!\!\} \nsubseteq \{\!\!\{ \Box\varphi_T \}\!\!\}$ or else $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_T \notin \Phi$. But axiom $\mathfrak{T}\varphi \to \Box\varphi$ implies $\{\!\!\{ \mathfrak{T}\varphi_T \}\!\!\} \subseteq \{\!\!\{ \Box\varphi_T \}\!\!\}$, so $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_T \notin \Phi$. Now, take any $C \in \mathcal{C}_1$; let $\varphi_C \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ be one of the formulas satisfying both $\{\!\!\{ \mathfrak{T}\varphi_C \}\!\!\} \subseteq C$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_C \in \Phi$ (by $\mathcal{C}$'s definition, there is at least one). From theorem $\mathfrak{T}\varphi \to \Box\mathfrak{T}\varphi$, it follows that $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to (\Box\varphi_T \wedge \Box\mathfrak{T}\varphi_C)$ is a theorem too, and thus so are $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to (\Box\Box\varphi_T \wedge \Box\mathfrak{T}\varphi_C)$ (by axiom $\Box\varphi \to \Box\Box\varphi$) and $(\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)$ (axiom K for $\Box$). Hence, by Proposition 17, $\mathrm{U}\big((\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)\big) \in \Phi$.

So far we have $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_T \notin \Phi$ and, for every $C \in \mathcal{C}_1$, not only $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi_C \in \Phi$ but also $\mathrm{U}\big((\Box\varphi_T \wedge \mathfrak{T}\varphi_C) \to \Box(\Box\varphi_T \wedge \mathfrak{T}\varphi_C)\big) \in \Phi$. The first and theorem $\mathfrak{T}\varphi \leftrightarrow \mathfrak{T}\Box\varphi$ imply $\widehat{\mathrm{U}}\,\mathfrak{T}\Box\varphi_T \notin \Phi$; the second and axiom $\mathfrak{T}\varphi \to \mathfrak{T}\mathfrak{T}\varphi$ imply $\widehat{\mathrm{U}}\,\mathfrak{T}\mathfrak{T}\varphi_C \in \Phi$. These two, the third, and axiom $\big(\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \wedge \neg\widehat{\mathrm{U}}\,\mathfrak{T}\psi \wedge \mathrm{U}((\varphi \wedge \psi) \to \Box(\varphi \wedge \psi))\big) \to \widehat{\mathrm{U}}\,\Box(\varphi \wedge \neg\psi)$ imply $\widehat{\mathrm{U}}\,\Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \in \Phi$. For the final part, take S to be the union of $\{\!\!\{ \Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \}\!\!\}$ for all $C \in \mathcal{C}_1$, that is,

$$S := \bigcup_{C \in \mathcal{C}_1} \{\!\!\{ \Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \}\!\!\}$$

The following two facts about S are crucial for proving $T \notin d(\mathcal{C})$:

(a) $S \cap T = \varnothing$. For its proof, from $T \subseteq \{\!\!\{ \Box\varphi_T \}\!\!\}$ and $\{\!\!\{ \Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \}\!\!\} \subseteq \{\!\!\{ \neg\Box\varphi_T \}\!\!\}$ for all $C \in \mathcal{C}_1$ (for the latter, use axiom $\Box\varphi \to \varphi$), it follows that $T \cap \{\!\!\{ \Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \}\!\!\} = \varnothing$ for all such C. Therefore, $S \cap T = \varnothing$.

(b) For any $C' \in \mathcal{C}$, we have $C' \cap S \neq \varnothing$. For its proof, take any $C' \in \mathcal{C}$. If $C' \in \mathcal{C}_1$ then, as $\{\!\!\{ \Box(\mathfrak{T}\varphi_{C'} \wedge \neg\Box\varphi_T) \}\!\!\} \subseteq \{\!\!\{ \mathfrak{T}\varphi_{C'} \}\!\!\}$ (same as before) and $\{\!\!\{ \Box(\mathfrak{T}\varphi_{C'} \wedge \neg\Box\varphi_T) \}\!\!\} \neq \varnothing$ (by $\widehat{\mathrm{U}}\,\Box(\mathfrak{T}\varphi_C \wedge \neg\Box\varphi_T) \in \Phi$), it follows that $\{\!\!\{ \Box(\mathfrak{T}\varphi_{C'} \wedge \neg\Box\varphi_T) \}\!\!\} \cap \{\!\!\{ \mathfrak{T}\varphi_{C'} \}\!\!\} \neq \varnothing$. But $\{\!\!\{ \mathfrak{T}\varphi_{C'} \}\!\!\} \subseteq C'$ for any $C' \in \mathcal{C}_1$; hence, $S \cap C' \neq \varnothing$. Otherwise, $C' \in \mathcal{C}_2$ and, for a contradiction, assume $S \cap C' = \varnothing$. Then, since $S \cap C'' \neq \varnothing$ for all $C'' \in \mathcal{C}_1$ as we have proved, there is an open in τ, S, such that $S \cap C' = \varnothing$ and $S \cap C'' \neq \varnothing$ for all $C'' \in \mathcal{C}_1$. Thus, C' violates the second condition of acceptance, and hence $C' \notin \mathcal{C}_2$: contradiction. So $S \cap C' \neq \varnothing$ for any $C' \in \mathcal{C}_2$. Therefore, in sum, $C' \in \mathcal{C}$ implies $C' \cap S \neq \varnothing$

Now, since $S \cap T = \varnothing$ and T is not semi-acceptable (so there is no $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ s.t. both $\{\!\!\{ \mathfrak{T}\varphi \}\!\!\} \subseteq T$ and $\widehat{\mathrm{U}}\,\mathfrak{T}\varphi \in \Phi$), we have found an open S in τ with $T \leftarrowtail S$, according to the definition of $\leftarrowtail$. But $S \cap C \neq \varnothing$ for all $C \in \mathcal{C}$, so $S \not\leftarrowtail C$ for all $C \in \mathcal{C}$: no open in $\mathcal{C}$ attacks S. Hence, $T \notin d(\mathcal{C})$.

Therefore, regardless of whether T is semi-acceptable or not, we have proved that $T \notin \mathcal{C}$ implies that $T \notin d(\mathcal{C})$.

By the fact that $\mathcal{C}_1$ is not attacked and $\mathcal{C}_2 \subseteq d(\mathcal{C}_1)$, which we have proved in our proof of $\mathcal{C} \subseteq \mathsf{LFP}$, it follows that $\mathcal{C} \subseteq d(\mathcal{C})$. Together with $d(\mathcal{C}) \subseteq \mathcal{C}$, it follows that $\mathcal{C} = d(\mathcal{C})$. By the fact that $\mathcal{C} \subseteq \mathsf{LFP}$ and LFP is the least fixed point, it follows that $\mathcal{C} = \mathsf{LFP}$.

Thus, in both cases $T \notin \mathcal{C}$ implies $T \notin \mathsf{LFP}$. This completes the proof. □

Proposition 18 (Truth lemma) *For any $\varphi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ and any $\Gamma \in W$,*

$$\Gamma \in \{\!|\varphi|\!\}_{\mathcal{M}^{\Phi}} \textit{ if and only if } \Gamma \in [\![\varphi]\!]_{\mathcal{M}^{\Phi}}$$

Proof The proof proceeds by induction, with the cases for atomic propositions and Boolean connectives being routine, and those for and $\Box$ and U relying on Proposition 17. Here we focus on the case for $\mathfrak{T}$.

From left to right, suppose $\Gamma \in \{\!|\mathfrak{T}\varphi|\!\}$. Then, $\mathfrak{T}\varphi \in \Gamma$ so, by Proposition 17, $\widehat{\mathrm{U}}\mathfrak{T}\varphi \in \Phi$ which, by Item (iii) of Fact 2, implies $\{\!|\mathfrak{T}\varphi|\!\} \in \tau$. Now, let $T = \{\!|\mathfrak{T}\varphi|\!\}$. Then, (i) from $\{\!|\mathfrak{T}\varphi|\!\} \subseteq T$ and $\widehat{\mathrm{U}}\mathfrak{T}\varphi \in \Phi$, it follows that $T \in \mathcal{C}_1$ which, by Lemma 5, implies $T \in \mathsf{LFP}$; (ii) $\Gamma \in T$, as $\Gamma \in \{\!|\mathfrak{T}\varphi|\!\}$; (iii) from axiom $\mathfrak{T}\varphi \rightarrow \varphi$ it follows that $T \subseteq \{\!|\varphi|\!\}$ which, by inductive hypothesis $\{\!|\varphi|\!\} = [\![\varphi]\!]$, implies $T \subseteq [\![\varphi]\!]$. Hence, by $\mathfrak{T}$'s truth condition, $\Gamma \in [\![\mathfrak{T}\varphi]\!]$.

From right to left, suppose $\Gamma \in [\![\mathfrak{T}\varphi]\!]$. Then, by $\mathfrak{T}$'s truth condition, there is $T \in \mathsf{LFP}$ with $\Gamma \in T$ and $T \subseteq [\![\varphi]\!]$.

The inductive hypothesis implies $[\![\varphi]\!] = \{\!|\varphi|\!\}$. By Item (iv) of Fact 2 and $T \subseteq \{\!|\varphi|\!\}$, we have $T \subseteq \{\!|\Box\varphi|\!\}$. So we have proved that $\Gamma \in T$ and $T \subseteq \{\!|\Box\varphi|\!\}$.

By Lemma 5, $\mathsf{LFP} = \mathcal{C}_1 \cup \mathcal{C}_2$; thus, $T \in \mathcal{C}_1 \cup \mathcal{C}_2$. Suppose $T \in \mathcal{C}_1$; then there is $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $\{\!|\mathfrak{T}\psi|\!\} \subseteq T$ and $\widehat{\mathrm{U}}\mathfrak{T}\psi \in \Phi$. Thus, $\{\!|\mathfrak{T}\psi|\!\} \subseteq T \subseteq \{\!|\Box\varphi|\!\}$, so $\mathrm{U}(\mathfrak{T}\psi \rightarrow \Box\varphi) \in \Phi$. Now, take any $\Delta \in \{\!|\mathfrak{T}\psi|\!\}$. The fact that $\mathrm{U}(\mathfrak{T}\psi \rightarrow \Box\varphi) \in \Delta$, together with theorem $\mathrm{U}(\varphi \rightarrow \psi) \rightarrow (\mathfrak{T}\varphi \rightarrow \mathfrak{T}\psi)$ (Proposition 7), implies $\mathfrak{T}\mathfrak{T}\psi \rightarrow \mathfrak{T}\Box\varphi \in \Delta$. Moreover: $\Delta \in \{\!|\mathfrak{T}\psi|\!\}$ implies $\Delta \in \{\!|\mathfrak{T}\mathfrak{T}\psi|\!\}$, so $\Delta \in \{\!|\mathfrak{T}\Box\varphi|\!\}$, that is, $\mathfrak{T}\Box\varphi \in \Delta$. The latter, together with theorem $\mathfrak{T}\varphi \leftrightarrow \mathfrak{T}\Box\varphi$ and axiom $\mathfrak{T}\varphi \rightarrow \mathrm{U}(\Box\varphi \rightarrow \mathfrak{T}\varphi)$, imply $\mathrm{U}(\Box\varphi \rightarrow \mathfrak{T}\varphi) \in \Delta$, and thus $\mathrm{U}(\Box\varphi \rightarrow \mathfrak{T}\varphi) \in \Phi$. Hence, $\{\!|\Box\varphi|\!\} \subseteq \{\!|\mathfrak{T}\varphi|\!\}$ and thus, since $\Gamma \in T$ and $T \subseteq \{\!|\Box\varphi|\!\}$, we have $\Gamma \in \{\!|\mathfrak{T}\varphi|\!\}$. Otherwise, $T \in \mathcal{C}_2$, and hence for any $\psi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $T \subseteq \{\!|\Box\psi|\!\}$ there is $\xi \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ with $\{\!|\mathfrak{T}\xi|\!\} \subseteq \{\!|\Box\psi|\!\}$ and $\widehat{\mathrm{U}}\mathfrak{T}\xi \in \Phi$. Thus, since φ satisfies that $T \subseteq \{\!|\Box\varphi|\!\}$, there is $\eta \in \mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$ such that $\{\!|\mathfrak{T}\eta|\!\} \subseteq \{\!|\Box\varphi|\!\}$ and $\widehat{\mathrm{U}}\mathfrak{T}\eta \in \Phi$. From here we can repeat the argument used in the case of $T \in \mathcal{C}_1$ in order to get $\Gamma \in \{\!|\mathfrak{T}\varphi|\!\}$ again. Thus, in both cases, $\Gamma \in \{\!|\mathfrak{T}\varphi|\!\}$, which completes the proof. □

Lemma 6 $\mathcal{M}^{\Phi}$ *is Alexandroff.*

Proof Whether $\mathcal{M}^{\Phi}$ is Alexandroff has nothing to do with $\leftarrow\!\!\prec$; thus, we can apply Prop. 5.6.15 in Özgün (2017), which states that if $\tau = \{\bigcup_{\Gamma \in U} \leq[\Gamma] \mid U \subseteq W\}$ then $\mathcal{M}^{\Phi}$ is Alexandroff. But Item (i) of Fact 2 and the definition of $\mathcal{E}_0$ imply the required condition; then, $\mathcal{M}^{\Phi}$ is Alexandroff. □

Since $\mathcal{M}^{\Phi}$ is Alexandroff, Proposition 16 tells us it has a modally equivalent topological argumentation model. Hence, the $\mathsf{L}_{\Box,\mathrm{U},\mathfrak{T}}$-consistent set of $\mathcal{L}_{\Box,\mathrm{U},\mathfrak{T}}$-formulas

Φ_0 is satisfiable in a topological argumentation model which satisfies the following condition: for every $T, T_1, T_1' \in \tau$: if $T_1 \leftarrow\!\!\prec T$ and $T_1' \subseteq T_1$, then $T_1' \leftarrow\!\!\prec T$ (see Lemma 3.2).

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