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### Grothendieck inequalities, nonlocal games and optimization

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## Chapter 2

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# Grothendieck inequalities

## 2.1 Introduction

Grothendieck's Inequality is a unifying theme for the chapters in this thesis. Many of the mathematical tools we use to deal with the problems addressed here are variations or extensions of this celebrated inequality. The inequality arose for the first time in Grothendieck's 1953 paper *Résumé de la théorie métrique des produits tensoriels topologiques* [Gro53], nowadays often referred to simply as the *Résumé*. The influence this paper has had until now is difficult to overstate. In particular its main result, Grothendieck's Inequality, has had important applications in huge number of different areas in pure mathematics, theoretical computer science and theoretical physics. A few important examples of such applications are as follows. Tsirelson [Tsi87] showed that the inequality can be interpreted as comparing the classical and quantum biases of a two-player XOR game, which becomes clear after one puts together Tsirelson's Theorem (see Section 1.5) and the form of Grothendieck's Inequality given below. We discuss this application in further detail in Chapters 3 and 6. Alon and Naor [AN06] realized that the inequality gives an upper bound on the ratio of the optima of certain integer optimization problems and their semidefinite relaxations. They showed that this implies the existence of constant-factor approximation algorithms for the problem of computing the cut-norm of a matrix. Their paper kindled a large amount of research on to connections between Grothendieck's Inequality and approximation algorithms based on semidefinite programming. These results are discussed in greater detail in Chapters 4 and 5. Linial and Shraibman [LS09] and Lee, Shraibman and

Schechtman [LSS09] showed that the inequality has connections to communication complexity and Regev and Toner [RT09] adapted techniques used in a proof of the inequality to simulate quantum correlations with classical communication. Pérez-García [PG06] applied the inequality in the context of Banach algebras, a result we discuss in more detail in Chapters 7. Not surprisingly, many equivalent forms of the inequality have been discovered since its first appearance. Arguably its most elementary shape was found by Lindenstrauss and Pełczyński [LP68], which is the shape in which we present the inequality below. In this chapter we present most of the variations of Grothendieck's Inequality that appear in subsequent chapters, though for convenience definitions will often be repeated when they are needed. Many more references regarding applications of Grothendieck's Inequality can be found throughout this thesis. We also refer to the recent extensive surveys of Pisier [Pis11] and Khot and Naor [KN11] for more information on this inequality, variations of it and applications to combinatorial optimization.

## 2.2 Grothendieck's Inequality

To suppress the space needed to state Grothendieck's Inequality and some of the modifications of it that feature in this thesis we introduce the following notation.

**2.2.1. DEFINITION.** For positive integers  $n, r$  and real  $n$ -by- $n$  matrix  $A$ , define

$$\text{SDP}_r(A) = \max \left\{ \sum_{i,j=1}^n A_{ij} x_i \cdot y_j : x_1, \dots, x_n, y_1, \dots, y_n \in S^{r-1} \right\}. \quad (2.1)$$

Define  $\text{SDP}_\infty(A)$  analogously, with a maximum over the unit sphere of  $\ell_2(\mathbb{R})$ .

With regard to the above definition, let us note that since any collection of vectors  $x_1, \dots, x_n, y_1, \dots, y_n$  span a space of dimension at most  $2n$ , we have  $\text{SDP}_\infty(A) = \text{SDP}_{2n}(A)$  for every  $n$ -by- $n$  matrix  $A$ . We also note that the set  $S^0$  consists just of the numbers 1 and  $-1$ . The reason for the abbreviation SDP is a connection to semidefinite programs which will become more explicit in the subsequent chapters. Grothendieck's Inequality can now be stated as follows.

**2.2.2. THEOREM (GROTHENDIECK [GRO53]).** *There exists a real number  $K > 0$  such that for every positive integer  $n$  and any real  $n$ -by- $n$  matrix  $A$ , we have*

$$\text{SDP}_\infty(A) \leq K \text{SDP}_1(A). \quad (2.2)$$

Inequality (2.2) is nowadays known as *Grothendieck's Inequality*. Associated to Grothendieck's Inequality is the smallest number  $K$  for which it holds.

**2.2.3. DEFINITION.** The *Grothendieck constant*  $K_G$  is defined by

$$K_G = \sup \left\{ \frac{\text{SDP}_\infty(A)}{\text{SDP}_1(A)} : n \in \mathbb{N}, A \in \mathbb{R}^{n \times n} \right\}.$$

Despite many efforts the exact value of  $K_G$  is currently not known. However, it is known to be bounded by

$$1.676 \dots \lesssim K_G < \frac{\pi}{2 \ln(1 + \sqrt{2})} = 1.782 \dots$$

The lower bound is due to Davie [Dav84] and Reeds [Ree91], who independently found the same result. The upper bound of  $\pi/(2 \ln(1 + \sqrt{2}))$  due to Krivine [Kri79] was the best known for over thirty years and by many believed to be tight. However, an exciting development took place while this thesis was being written. Using an extension of Krivine's techniques, Braverman, Makarychev, Makarychev and Naor [BMMN11] proved that his upper bound can be improved, disproving Krivine's conjecture that his bound gave the exact value of  $K_G$ . Although they do not give a numerical bound, they prove that  $K_G \leq \pi/(2 \ln(1 + \sqrt{2})) - \varepsilon$  holds for some constant  $\varepsilon > 0$ . A proof of Krivine's upper bound is given in Chapter 5 as part of a more general result regarding a generalization of  $K_G$  based on graphs, which is described below.

Despite the fact that the exact value of  $K_G$  is unknown, Raghavendra and Steurer [RS09] were able to show that  $K_G$  is the UGC hardness threshold for computing the value  $\text{SDP}_1(A)$  for any real matrix  $A$ . Moreover, they show that the exact value of  $K_G$  can be approximated to within an error  $\varepsilon$  in time  $O(\exp(\exp(1/\varepsilon^3)))$  by a linear program.

## 2.3 Generalizations of Grothendieck's Inequality

In this section we define various generalizations of Grothendieck's Inequality that will appear in the subsequent chapters.

### 2.3.1 The rank- $r$ Grothendieck constant

The first generalization we consider relates  $\text{SDP}_r(A)$  for values of  $r$  that may differ from  $\infty$  and 1. This generalization appeared for the first time in the

paper [BBT11], whose content will be presented in Chapter 3. We first define the following generalization of the Grothendieck constant.

**2.3.1. DEFINITION.** For every pair  $q, r \in \mathbb{N} \cup \{\infty\}$  such that  $q \geq r$ , define  $K_G(q \mapsto r)$  by

$$K_G(q \mapsto r) = \sup \left\{ \frac{\text{SDP}_q(A)}{\text{SDP}_r(A)} : n \in \mathbb{N}, A \in \mathbb{R}^{n \times n} \right\}.$$

We refer to  $K_G(\infty \mapsto r)$  as the *rank- $r$  Grothendieck constant*. The reason for the word *rank* is that a matrix  $X \in \mathbb{R}^{n \times n}$  has rank  $r$  if and only if there exist  $r$ -dimensional vectors  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  such that  $X_{ij} = x_i \cdot y_j$ . It follows that  $\text{SDP}_r(A)$  is the maximum of  $\langle A, X \rangle$  over rank- $r$  matrices  $X$ . Based on Definition 2.3.1 we get the following generalization of Grothendieck's Inequality: For every positive integer  $n$  and any real  $n$ -by- $n$  matrix  $A$ , we have

$$\text{SDP}_q(A) \leq K_G(q \mapsto r) \text{SDP}_r(A). \quad (2.3)$$

The constant  $K_G(q \mapsto 1)$  is known as *the Grothendieck constant of order  $q$*  and is usually denoted  $K_G(q)$ . It was studied before by Krivine [Kri77], who proved that  $K_G(2) = \sqrt{2}$  and  $K_G(4) \leq \pi/2$ , and numerically computed upper bounds for other values of  $q$ , including  $K_G(3) < 1.57$ .

Variations of  $K_G(q \mapsto r)$  that will appear in Chapter 4 are based on positive semidefinite matrices and Laplacian matrices.

**2.3.2. DEFINITION.** For every pair  $q, r \in \mathbb{N} \cup \{\infty\}$  such that  $q \geq r$ , define  $K_G^{\succ}(q \mapsto r)$  by

$$K_G^{\succ}(q \mapsto r) = \sup \left\{ \frac{\text{SDP}_q(A)}{\text{SDP}_r(A)} : n \in \mathbb{N}, A \in \mathcal{S}_n^+ \right\}.$$

**2.3.3. DEFINITION.** For every pair  $q, r \in \mathbb{N} \cup \{\infty\}$  such that  $q \geq r$ , define  $K_G^L(q \mapsto r)$  by

$$K_G^L(q \mapsto r) = \sup \left\{ \frac{\text{SDP}_q(A)}{\text{SDP}_r(A)} : n \in \mathbb{N}, A \in \mathcal{S}_n^+ \text{ and Laplacian} \right\}.$$

We have the following easy relations between the above constants:

$$K_G^L(q \mapsto r) \leq K_G^{\succ}(q \mapsto r) \leq K_G(q \mapsto r) \leq K_G.$$

The calculations done in Section 1.7.2 to analyze the Goemans-Williamson approximation algorithm for MAX-CUT show that the constant  $K_G^L(\infty \mapsto 1)$  is bounded from above by  $(.878\dots)^{-1} = 1.138\dots$ . Upper bounds on the constants  $K_G^L(q \mapsto 1)$  for  $q = 2, 3$  were computed by Avidor and Zwick [AZ05], who showed that for these values of  $q$ , we have  $K_G^L(q \mapsto 1) < K_G^L(\infty \mapsto 1)$ .

### 2.3.2 The Grothendieck constant of a graph

We also consider a generalization of Grothendieck's Inequality based on graphs. For this, we introduce a variation of the quantity  $\text{SDP}_r(A)$  based on graphs and matrices whose rows and columns are indexed by the vertices of those graphs.

**2.3.4. DEFINITION.** For a graph  $G = (V, E)$ , positive integer  $r$  and matrix  $A : V \times V \rightarrow \mathbb{R}$ , define

$$\text{SDP}_r(G, A) = \max \left\{ \sum_{\{u,v\} \in E} A(u,v) f(u) \cdot f(v) : \forall u \in V, f(u) \in S^{r-1} \right\}.$$

Define  $\text{SDP}_\infty(G, A)$  analogously with a maximum over functions  $f : V \rightarrow S^\infty$  where  $S^\infty$  is the unit sphere of  $\ell_2(\mathbb{R})$ .

Since  $\ell_2(\mathbb{R})$  contains  $\mathbb{R}^n$  as the subspace of its first  $n$  components, we have that  $|V|$ -dimensional unit vectors suffice to achieve the maximum above (note that a collection of  $|V|$  vectors span a space of dimension at most  $|V|$ ). That is,  $\text{SDP}_\infty(G, A) = \text{SDP}_{|V|}(G, A)$ . An important difference between  $\text{SDP}_r(G, A)$  and  $\text{SDP}_r(A)$  defined above is that the latter has a maximum over *two* sequences of unit vectors, while the former has only *one* such sequence.

**2.3.5. DEFINITION.** For a graph  $G = (V, E)$  and pair  $q, r \in \mathbb{N} \cup \{\infty\}$  such that  $q \geq r$ , define  $K(q \mapsto r, G)$  by

$$K(q \mapsto r, G) = \sup \left\{ \frac{\text{SDP}_q(G, A)}{\text{SDP}_r(G, A)} : A : V \times V \rightarrow \mathbb{R} \right\}.$$

The *rank- $r$  Grothendieck constant of the graph  $G$* , denoted  $K(r, G)$ , is defined by  $K(r, G) = K(\infty \mapsto r, G)$ . This number plays a major role in Chapter 5, where we establish new upper bounds for  $r > 1$ . The constant  $K(G) = K(1, G)$  was considered by Alon, Makarychev, Makarychev and Naor [AMMN06], who called it simply *the Grothendieck constant of the graph  $G$* . They proved that

$$\Omega(\log \omega(G)) \leq K(G) \leq O(\log \vartheta(\overline{G})),$$

where  $\omega(G)$  is the size of the largest clique in  $G$ . This shows in particular that  $K(G)$  depends strongly on the graph, and is not a universal constant like the Grothendieck constant  $K_G$ . Laurent and Varvitsiotis [LV11] showed that for specific graph classes, it is possible to compute  $K(G)$  exactly. In particular, if  $G$  is an  $n$ -cycle, then  $K(G) = n \cos(\pi/n) / (n - 2)$  and if  $G$  has no  $K_5$  minor and is not a forest, then  $K(G)$  equals the maximum value of  $K(C)$  for  $C$  a cycle graph appearing as an induced subgraph in  $G$ . We refer to Chapter 5 for more results on these numbers.

### 2.3.3 The complex Grothendieck constant

Perhaps the most natural generalization of Grothendieck's Inequality is obtained by allowing all quantities involved to take complex values. Let us denote by  $S_{\mathbb{C}}^{r-1}$  the  $r$ -dimensional complex unit sphere.

**2.3.6. DEFINITION.** For positive integers  $n, r$  and complex  $n$ -by- $n$  matrix  $A$ , define

$$\text{SDP}_r^{\mathbb{C}}(A) = \max \left\{ \left| \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle \right| : x_1, \dots, x_n, y_1, \dots, y_n \in S_{\mathbb{C}}^{r-1} \right\}, \quad (2.4)$$

where  $\langle \cdot, \cdot \rangle$  denotes the regular inner product on  $\mathbb{C}^d$ . Define  $\text{SDP}_{\infty}^{\mathbb{C}}$  analogously but with a supremum over the unit sphere of the Hilbert space  $\ell_2(\mathbb{C})$  of complex square-summable sequences.

**2.3.7. DEFINITION.** For every pair  $q, r \in \mathbb{N} \cup \{\infty\}$  such that  $q \geq r$ , define

$$K_G^{\mathbb{C}}(q \mapsto r) = \sup \left\{ \frac{\text{SDP}_q^{\mathbb{C}}(A)}{\text{SDP}_r^{\mathbb{C}}(A)} : n \in \mathbb{N}, A \in \mathbb{C}^{n \times n} \right\}. \quad (2.5)$$

The *complex Grothendieck constant*  $K_G^{\mathbb{C}}$  is defined by  $K_G^{\mathbb{C}} = K_G^{\mathbb{C}}(\infty \mapsto 1)$ . The corresponding variant of Grothendieck's Inequality is then as follows: For any positive integer  $n$  and any matrix  $A \in \mathbb{C}^{n \times n}$ , we have

$$\text{SDP}_{\infty}^{\mathbb{C}}(A) \leq K_G^{\mathbb{C}} \text{SDP}_1(A).$$

The currently best lower and upper bounds on  $K_G^{\mathbb{C}}$ , proved by Davie [Dav84] and Haagerup [Haa87], respectively, are given by  $1.33807 \leq K_G^{\mathbb{C}} \lesssim 1.40491$ . König [Kön91] obtained the numerical bounds  $1.152 \leq K_G^{\mathbb{C}}(2 \mapsto 1) \leq 1.216$ . Another related result of Davie [Dav85] shows that for every positive integer  $n$ , any complex  $n$ -by- $n$  matrix  $A$  and  $r$  the integer part of  $\sqrt{2n-1}$ , we have  $\text{SDP}_{\infty}^{\mathbb{C}}(A) = \text{SDP}_r^{\mathbb{C}}(A)$ ; he noted that similar results hold for the real setting.

### 2.3.4 Tonge's Inequality

Blei [Ble79] and Tonge [Ton78] considered certain multilinear generalizations of Grothendieck's inequality, where the matrix  $A$  is replaced by a higher-order tensor, and the inner product function replaced by a multilinear functional on more than two unit vectors. We use these generalizations in Chapters 6 and 7.

By a (real)  $N$ -tensor we mean a map  $A : [n]^N \rightarrow \mathbb{R}$ , which can be seen as an array of reals whose coordinates are indexed by  $N$ -tuples of integers  $(i_1, \dots, i_N)$  over the set  $[n]$ . The case  $N = 2$  thus gives ordinary real matrices.

We introduce two quantities, reminiscent of the quantities  $\text{SDP}_1(A)$  and  $\text{SDP}_\infty(A)$  appearing on opposing sides of Grothendieck's Inequality.

**2.3.8. DEFINITION.** For positive integers  $n, N$  and  $N$ -tensor  $A : [n]^N \rightarrow \mathbb{R}$ , define

$$\text{OPT}(A) = \max \left\{ \sum_{I \in [n]^N} A[I] \chi_1(i_1) \cdots \chi_N(i_N) : \chi_1, \dots, \chi_N : [n] \rightarrow \{-1, 1\} \right\}. \quad (2.6)$$

It may be helpful to note that for  $N = 2$ , we have  $\text{OPT}(A) = \text{SDP}_1(A)$ .

We introduce a multilinear functional, which replaces the regular inner product appearing in Grothendieck's Inequality. The *generalized inner product* of vectors  $x_1, \dots, x_N \in \mathbb{C}^d$  is defined by

$$\langle x_1, \dots, x_N \rangle = \sum_i (x_1)_i \cdots (x_N)_i,$$

where  $(x_1)_i$  denotes the  $i$ th coordinate of the vector  $x_1$  in the canonical basis. Note that for the case  $N = 2$ ,  $\langle \cdot, \cdot \rangle$  is linear in both arguments, as opposed to conjugate linear in the first and linear in the second. This conflicting notation with the standard inner product will not be an issue later on and will only occur in Chapters 6 and 7 where the cases  $N \geq 3$  are of main interest. Let  $B_{\mathbb{C}^d}$  denote the  $d$ -dimensional complex unit ball.

**2.3.9. DEFINITION.** For positive integers  $n, N$  and  $N$ -tensor  $A : [n]^N \rightarrow \mathbb{R}$ , we define

$$\text{GIP}(A) = \sup \left\{ \left| \sum_{I \in [n]^N} A[i_1, \dots, i_N] \langle f_1(i_1), \dots, f_N(i_N) \rangle \right| : d \in \mathbb{N}, f_1, \dots, f_N : [n] \rightarrow B_{\mathbb{C}^d} \right\}. \quad (2.7)$$

Note carefully that in the definition of  $\text{GIP}(A)$ , the supremum is taken over *complex* vectors, while the tensor  $A$  is assumed to be *real*. For  $N = 2$ , the identity  $\text{GIP}(A) = \text{SDP}_\infty^{\mathbb{C}}(A)$  holds, as there is a 1-1 correspondence between complex unit vectors and their conjugates.

The multilinear generalization of Grothendieck's Inequality given below is a slight variation of a result due to Tonge [Ton78].<sup>1</sup>

**2.3.10. THEOREM (TONGE).** *Let  $n, N \geq 2$  be positive integers. Then, for any  $N$ -tensor  $A : [n]^N \rightarrow \mathbb{R}$ , we have*

$$\text{GIP}(A) \leq 2^{(3N-5)/2} K_G^{\mathbb{C}} \text{OPT}(A). \quad (2.8)$$

In the original inequality proved by Tonge the tensor  $A$  may be complex and the maximization on the right-hand side is over variables in the complex unit disc. The version stated above is tailored specifically to our needs.

The proof of Theorem 2.3.10 that we give here is longer than Tonge's original proof, but more elementary. Both proofs use induction on  $N$ . The base case,  $N = 2$ , is derived from the complex version of Grothendieck's Inequality, which we restate here in its strongest form.

**2.3.11. LEMMA (HAAGERUP).** *For all positive integers  $n, d$ , any complex  $n$ -by- $n$  matrix  $A$  and complex vectors  $x_1, \dots, x_n$  and  $y_1, \dots, y_n \in B_{\mathbb{C}^d}$ , the inequality*

$$\left| \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle \right| \leq K_G^{\mathbb{C}} \max \left\{ \left| \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right| : \sigma_1, \sigma_2 : [n] \rightarrow B_{\mathbb{C}} \right\}, \quad (2.9)$$

where  $K_G^{\mathbb{C}} \lesssim 1.40491$  is independent of  $n$  and  $d$ .

Note that the maximization on the right-hand side of Eq. (2.9) is over sequences  $\sigma_1(1), \dots, \sigma_1(n)$  and  $\sigma_2(1), \dots, \sigma_2(n)$  of scalars in the *complex* unit disc.

The inductive step relies on a slight modification of an Inequality of Littlewood (Lemma 2.3.12 below) [Lit30] (see also [Pie72, page 43] and [Sza76]).

**2.3.12. LEMMA (LITTLEWOOD).** *For all positive integers  $n, d$  and any complex  $n$ -by- $d$  matrix  $B$ , we have*

$$\sum_{i=1}^n \left( \sum_{j=1}^d |B_{ij}|^2 \right)^{1/2} \leq 2^{3/2} \max \left\{ \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi(i) \zeta(j) \right| : \chi(i), \zeta(j) \in \{-1, 1\} \right\}. \quad (2.10)$$

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<sup>1</sup>A weaker version of Tonge's result was proved earlier by Blei [Ble79] (though it was published shortly after Tonge's paper was).

We first prove Theorem 2.3.10. Afterwards, we prove Lemma 2.3.12

PROOF OF THEOREM 2.3.10: By induction on  $N$ . For the base case,  $N = 2$ , we use Lemma 2.3.11 and relate the right-hand side of Inequality (2.9) to  $\text{OPT}(A)$  using the following version of [MST99, Proposition 15]:

**3. CLAIM.** *For any real  $n$ -by- $n$  matrix  $A$  and sequences of scalars  $\sigma_1(1), \dots, \sigma_1(n)$  and  $\sigma_2(1), \dots, \sigma_2(n)$  in the complex unit disc  $B_{\mathbb{C}}$ , we have*

$$\left| \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right| \leq \max \left\{ \sum_{i,j=1}^n A_{ij} \Re(\sigma'_1(i) \sigma'_2(j)) : \sigma'_1, \sigma'_2 : [n] \rightarrow B_{\mathbb{C}} \right\}. \quad (2.11)$$

PROOF: Using polar coordinates, the complex number  $\sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j)$  can be written as  $re^{i\phi}$  for some non-negative real number  $r$  and angle  $\phi$ . This gives

$$\begin{aligned} \left| \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right| &= \left| e^{-i\phi} \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right| \\ &= \Re \left( e^{-i\phi} \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right) \\ &= \sum_{i,j=1}^n A_{ij} \Re \left( e^{-i\phi} \sigma_1(i) \sigma_2(j) \right), \end{aligned}$$

where the second identity follows because the number between brackets is real and nonnegative (it is  $r$ ), and the third identity follows because  $A$  is real. The result follows by defining  $\sigma'_1 = e^{-i\phi/2} \sigma_1$  and  $\sigma'_2 = e^{-i\phi/2} \sigma_2$ .  $\square$

We can write the real part  $\Re(\sigma_1 \sigma_2)$  of the product of two complex numbers  $\sigma_1, \sigma_2$  as the inner product between real vectors  $a = (\Re(\sigma_1), \Im(\sigma_1))^T$  and  $b = (\Re(\sigma_2), -\Im(\sigma_2))^T$ . Using this, Lemma 2.3.11 and Claim 3, we get that for every sequence of unit vectors  $x_1, \dots, x_n$  and  $y_1, \dots, y_n \in B_{\mathbb{C}^d}$ ,

$$\begin{aligned} \left| \sum_{i,j=1}^n A_{ij} \langle x_i, y_j \rangle \right| &\leq K_G^{\mathbb{C}} \max_{\sigma_1, \sigma_2 : [n] \rightarrow B_{\mathbb{C}}} \left| \sum_{i,j=1}^n A_{ij} \sigma_1(i) \sigma_2(j) \right| \\ &\leq K_G^{\mathbb{C}} \max_{\sigma_1, \sigma_2 : [n] \rightarrow B_{\mathbb{C}}} \sum_{i,j=1}^n A_{ij} \Re(\sigma_1(i) \sigma_2(j)) \\ &\leq K_G^{\mathbb{C}} \max_{a, b : [n] \rightarrow B_{\mathbb{R}^2}} \sum_{i,j=1}^n A_{ij} a(i) \cdot b(j) \end{aligned}$$

The base case now follows from Krivine's [Kri79] bound  $K_G(2) \leq \sqrt{2}$  on the Grothendieck constant of order 2 (see Section 2.3.1). This implies that

$$\sum_{i,j=1}^n A_{ij} a(i) \cdot b(j) \leq \sqrt{2} \max \left\{ \sum_{i,j=1}^n A_{ij} \chi_1(i) \chi_2(j) : \chi_1, \chi_2 : [n] \rightarrow \{-1, 1\} \right\}$$

holds for any sequence of vectors  $a(1), \dots, a(2)$  and  $b(1), \dots, b(2) \in B_{\mathbb{R}^2}$ , and hence proves the base case.

We continue with the induction step. Suppose that Inequality (2.8) holds for some  $N \geq 2$ . Let  $A : [n]^{N+1} \rightarrow \mathbb{R}$  be a real  $(N+1)$ -tensor. Define the complex  $n$ -by- $d$  matrix  $B$  by

$$B_{ij} = \sum_{i_1, \dots, i_N=1}^n A[i_1, \dots, i_N, i] f_1(i_1)_j \cdots f_N(i_N)_j,$$

where  $f_1(i_1)_j$  stands for the  $j^{\text{th}}$  coordinate of the  $d$ -dimensional complex vector  $f_1(i_1)$ . Then, we can write the left-hand side of Inequality (2.8) as

$$\sum_{i_1, \dots, i_N, i=1}^n A[i_1, \dots, i_N, i] \langle f_1(i_1), \dots, f_{N+1}(i) \rangle = \sum_{i=1}^n \sum_{j=1}^d B_{ij} f_{N+1}(i)_j.$$

By the triangle inequality, the Cauchy-Schwarz inequality and Inequality (2.10) from Lemma 2.3.12, we can bound the absolute value of this quantity by

$$\begin{aligned} \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} f_{N+1}(i)_j \right| &\leq \sum_{i=1}^n \left| \sum_{j=1}^d B_{ij} f_{N+1}(i)_j \right| \\ &\leq \sum_{i=1}^n \left( \sum_{j=1}^d |B_{ij}|^2 \right)^{1/2} \\ &\leq 2^{3/2} \max \left\{ \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi(i) \xi(j) \right| \right\}, \end{aligned} \quad (2.12)$$

where the maximum is taken over  $\chi : [n] \rightarrow \{-1, 1\}$  and  $\xi : [d] \rightarrow \{-1, 1\}$ . Let  $\chi$  and  $\xi$  be the functions with which this maximum is achieved.

Expanding the definition of the matrix  $B$  gives

$$\begin{aligned} &2^{3/2} \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi(i) \xi(j) \right| \\ &= 2^{3/2} \left| \sum_{i=1}^n \sum_{j=1}^d \left( \sum_{i_1, \dots, i_N=1}^n A[i_1, \dots, i_N, i] f_1(i_1)_j \cdots f_N(i_N)_j \right) \chi(i) \xi(j) \right| \end{aligned} \quad (2.13)$$

Define the real  $N$ -tensor  $A' : [n]^N \rightarrow \mathbb{R}$  defined by

$$A'[i_1, \dots, i_N] = \sum_{i=1}^n A[i_1, \dots, i_N, i] \chi(i).$$

Then, we can write the right-hand side of Eq. (2.13) as

$$2^{3/2} \left| \sum_{j=1}^d \sum_{I \in [n]^N} A'[I] f_1(i_1)_j \cdots f_N(i_N)_j \xi(j) \right| = 2^{3/2} \left| \sum_{I \in [n]^N} A'[I] \langle f_1(i_1)_j, \dots, f_N(i_N) \circ \xi \rangle \right|,$$

where by  $f_N(i_N) \circ \xi$  we mean the entry-wise product of the  $d$ -dimensional complex vectors  $f_N(i_N)$  and  $(\xi(1), \dots, \xi(d))$ . By the induction hypothesis, the last quantity is bounded from above by  $2^{3/2} 2^{(3N-5)/2} K_G^{\mathbb{C}} \text{OPT}(A')$ . Since  $\text{OPT}(A)$  involves a re-maximization over  $\chi$  that appears in the definition of  $A'$ , we have  $\text{OPT}(A') \leq \text{OPT}(A)$ . This completes the proof.  $\square$

We now prove Inequality (2.10) of Lemma 2.3.12. We derive it from Khintchine's Inequality (see for example [MS86]), which states that there exists a constant  $\kappa$  such that for any any finite sequence of complex scalars  $\sigma(1), \dots, \sigma(n)$  the inequality

$$\left( \sum_{i=1}^n |\sigma(i)|^2 \right)^{1/2} \leq \kappa \int_{t=0}^1 \left| \sum_{i=1}^n \sigma(i) r_i(t) \right| dt, \quad (2.14)$$

where  $r_i(t) = \text{sign}(\sin(2^i \pi t))$  denotes the  $i^{\text{th}}$  Rademacher function. The best value of  $\kappa$  is due to Szarek [Sza76] (see also [LO94]), who proved that  $\kappa \leq \sqrt{2}$ .

PROOF OF LEMMA 2.3.12: By Inequality (2.14), we have

$$\begin{aligned} \sum_{i=1}^n \left( \sum_{j=1}^d |B_{ij}|^2 \right)^{1/2} &\leq \sqrt{2} \int_{t=0}^1 \sum_{i=1}^n \left| \sum_{j=1}^d B_{ij} r_j(t) \right| dt \\ &\leq \sqrt{2} \max \left\{ \sum_{i=1}^n \left| \sum_{j=1}^d B_{ij} \xi(j) \right| \right\}, \end{aligned} \quad (2.15)$$

where the above maximum is over maps  $\xi : [d] \rightarrow \{-1, 1\}$ . Let  $\xi$  achieve this maximum. Define  $\chi : [n] \rightarrow B_{\mathbb{C}}$  by

$$\chi(i) = \frac{\left( \sum_{j=1}^d B_{ij} \xi(j) \right)^*}{\left| \sum_{j=1}^d B_{ij} \xi(j) \right|}$$

Then, we have that the maximum on the right-hand side of Eq. (2.15) equals

$$\sum_{i=1}^n \chi(i) \sum_{j=1}^d B_{ij} \zeta(j) = \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi(i) \zeta(j).$$

Now, we make  $\chi$  real-valued, at the cost of a factor at most 2. By the triangle inequality, we have

$$\sum_{i=1}^n \chi(i) \sum_{j=1}^d B_{ij} \zeta(j) \leq \left| \sum_{i=1}^n \Re(\chi(i)) \sum_{j=1}^d B_{ij} \zeta(j) \right| + \left| \sum_{i=1}^n \Im(\chi(i)) \sum_{j=1}^d B_{ij} \zeta(j) \right| \quad (2.16)$$

Set  $\chi'$  to be either the real or imaginary part of  $\chi$ , whichever gives the largest value on the right-hand side of Eq. (2.16). Then,  $\chi' : [n] \rightarrow [-1, 1]$  and we have

$$\begin{aligned} \sum_{i=1}^n \left( \sum_{j=1}^d |B_{ij}|^2 \right)^{1/2} &\leq 2^{3/2} \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi'(i) \zeta(j) \right| \\ &\leq 2^{3/2} \max \left\{ \left| \sum_{i=1}^n \sum_{j=1}^d B_{ij} \chi''(i) \zeta(j) \right| \right\}, \end{aligned}$$

where the maximum is taken over  $\chi'' : [n] \rightarrow \{-1, 1\}$  and  $\zeta : [d] \rightarrow \{-1, 1\}$ . Here, the second inequality follows because  $\chi'$  can be written as a convex combination of functions  $\chi'' : [n] \rightarrow \{-1, 1\}$  and by the triangle inequality. This completes the proof.  $\square$