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Chapter 6

Entanglement in multiplayer XOR games

The content of this chapter is based on joint work with Harry Buhrman, Troy Lee and Thomas Vidick [BBLV09].

6.1 Introduction

Due to Tsirelson's Theorem, the role of entanglement in two-player XOR games is reasonably well understood. As we have seen in the previous chapters, it implies that the violation ratio is always bounded by the Grothendieck constant. Moreover, it implies that there is a semidefinite program of size polynomial in the number of questions whose optimum value is exactly the entangled bias of a two-player XOR game. This contrasts with the classical setting, where even approximating the bias to within a small constant is NP-hard [Hås01].

Unfortunately, our understanding of entangled games does not extend far beyond the setting of two-player XOR games. Two-player games with larger answer sizes seem to be much harder to get a handle on (see however [BRSW10, JP11, Reg11] for some recent results on the violation ratios achievable in this setting). Even less is known about games involving three players or more. This is in part a reflection of the fact that multipartite entanglement is much less well understood, and seemingly much more diverse than its bipartite counterpart. A simple example of a three-player XOR game that exhibits properties of tripartite entanglement which bipartite entanglement cannot possess is Mermin's game (see Section 1.6.1). In this game classical players can attain bias at most $1/2$, but by sharing the three-qubit GHZ state entangled players can play the game perfectly by performing two-outcome measurements on their local

qubits. That is, the entangled bias equals 1. This kind of separation between entangled and classical biases is impossible in a two-player scenario. Cleve, Høyer, Toner and Watrous [CHTW04, Theorem 8] showed that if the entangled bias of a two-player XOR game equals 1, then the classical bias must be 1 as well. Another important example of a property unique to multipartite entanglement is that of *monogamy* [Ton09], which shows that there is a trade-off in the amount of entanglement between two quantum systems and the amount of entanglement between either one of them and a third system. Monogamy plays a role in many multiplayer nonlocal games [TV06].

Entanglement is not a resource that is easily created or manipulated, and when studying violation ratios of nonlocal games one may ask which types of entanglement are the most useful; in fact this is a question that has preoccupied physicists for the past four decades (see e.g. [MS07] for a survey). In the bipartite setting the most natural measure of entanglement of a quantum state $|\psi\rangle_{AB}$ is its von Neumann entropy $S(\psi) = -\text{Tr}\rho_A \log \rho_A$, where $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$ is the reduced density matrix on Alice's subsystem. However, from the point of view of nonlocality, states with higher entropy are not always the most useful: while in any dimension d the maximally entangled state $|\psi_d\rangle = d^{-1/2} \sum_{i=1}^d |i\rangle|i\rangle$ has the largest entropy, for some games it is known not to be the best resource [AGG05, JP11, VW10, Reg11]. In fact, there is a different "maximally nonlocal" state, the *embezzlement state* [DH03, Oli10] $|\phi_d\rangle$ which is the state proportional to $\sum_{i=1}^d i^{-1/2} |i\rangle|i\rangle$: for any nonlocal game (not necessarily XOR) for which there is an optimal finite-dimensional strategy achieving the violation ratio, and any $\varepsilon > 0$, there is also a d and a strategy using $|\phi_d\rangle$ which achieves the violation ratio up to precision ε . Interestingly, this distinction is not apparent in two-player XOR games, for which it is known that the maximally entangled state is also optimal.

In the case of XOR games with more than two players, little is known about the power of specific states as a resource to produce nonlocal correlations. The most striking recent results in this area are due to Pérez-García et al. [PGWP⁺08], who show that for every positive integer d , there exists a three-player XOR game with violation ratio $\Omega(\sqrt{d})$. Here d refers to the smallest local dimension of the entangled state for any of the three players. An interesting feature of their work is that it makes use of techniques from operator space theory, a field that had not been connected to problems related to nonlocality before. Unfortunately their proof techniques don't give much insight into what *kind* of states can be used to achieve such "unbounded violations".

6.2 Bounded violations for a large class of states

In this chapter we consider multiplayer XOR games in which the players are restricted to sharing specific patterns of entanglement. For this, we introduce two main types of N -partite entanglement. The first is a generalization of GHZ states that we call *Schmidt states*, named so because they admit a sort of tripartite Schmidt decomposition¹: states of the form $|\psi\rangle = \sum_i \alpha_i |i\rangle^{\otimes N}$, for any sequence of positive (normalized) coefficients α_i .² Note that these also contain a natural generalization of the “universal” embezzlement states to more than two parties, and as such one might expect that they are the most highly nonlocal in the context of multiplayer games. The second type of states are formed by what we will refer to as *clique-wise entanglement*. Here, we consider the general setting where the N players are organized in k coalitions of r players each (a given player can take part in any number of coalitions). The members of each of the coalitions are allowed to share a GHZ state of arbitrary dimension, i.e., a state of the form $d^{-1/2} \sum_{i=1}^d |i\rangle \otimes \cdots \otimes |i\rangle$, among themselves. Note that this includes possible collections of EPR pairs shared among two-party coalitions, as these states are simply higher-dimensional two-party GHZ states. Clique-wise entanglement includes states that have been covered extensively in the literature on entanglement, such as GHZ states [GHZ89, Mer90, Zuk93, CB97, BCD01, RW08], which have even been realized experimentally [BPD⁺99, PBD⁺00], and tripartite stabilizer states (see Section 1.6.2), which are of fundamental importance to the theory of quantum error correction [Got97, Nes05, BFG06] and also appear in the context of nonlocal games [GTHB05, TGB06].

We denote by $\beta_S^*(\mathcal{G})$ (resp. $\beta_C^*(\mathcal{G})$) the maximal bias achievable in game \mathcal{G} by players who are restricted to sharing a Schmidt state of arbitrary dimension (resp. arbitrary clique-wise entanglement). We note the following obvious relationships between the biases:

$$\beta(\mathcal{G}) \leq \beta_S^*(\mathcal{G}) \leq \beta^*(\mathcal{G}) \quad \text{and} \quad \beta(\mathcal{G}) \leq \beta_C^*(\mathcal{G}) \leq \beta^*(\mathcal{G}).$$

The main results of this chapter are constant upper bounds on the violation ratios of these quantities.

Concerning Schmidt states we prove the following.

¹For bipartite states, the Schmidt decomposition is simply the singular value decomposition when the state is represented by a matrix.

²The assumption that the α_i s are real and positive is not a restriction, as complex arguments can be introduced to them via a local unitary transformation done by one of the N players.

6.2.1. THEOREM. *Let \mathcal{G} be an N -player XOR game. Then the maximum bias achievable by players sharing a Schmidt state $|\psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle^{\otimes N}$, for an arbitrary dimension d , is at most a constant factor greater than the classical bias. More precisely,*

$$\beta_S^*(\mathcal{G}) \leq 2^{(3N-5)/2} K_G^C \beta(\mathcal{G}),$$

where $K_G^C \lesssim 1.40491$ is the complex Grothendieck constant (see Section 2.3.3).

The exponential dependence on the number of players is necessary in this theorem as Zukowski [Zuk93] gave an explicit sequence of N -player XOR games where players sharing an N -partite GHZ state can achieve a bias that is $2^{-1}(\pi/2)^N$ times larger than the classical bias. The same conclusion follows from Mermin's game, but Zukowski's games give slightly larger separations between the two biases.

Theorem 6.2.1 generalizes—with slightly improved constants—a result of Pérez-García et al. who show a constant violation ratio for the case of GHZ states of arbitrary local dimension. The proof of the theorem also uses fairly elementary techniques compared to those used in [PGWP⁺08].

Our second result deals with the case where the players share clique-wise entanglement. Even in this complex setting, we can show that the violation ratio is bounded by a constant depending only on the number of coalitions, and the number of players taking part in each of them, but independent of the dimension of the various states shared among the parties.

6.2.2. THEOREM. *Let \mathcal{G} be an N -player XOR game. Then the maximum bias achievable by players sharing clique-wise entanglement, in which the players are organized in k coalitions of r players each, is greater than the classical bias by at most a constant factor depending only on k and r . More precisely,*

$$\beta_C^*(\mathcal{G}) \leq 2^{k(3r-5)/2} (K_G^C)^k \beta(\mathcal{G}).$$

Stabilizer states were considered in the context of XOR games in [GTHB05, TGB06], where it is shown that they allow for violations that grow exponentially with the number of players sharing them. In view of these results, one might hope to obtain explicit examples of three-player XOR games that exhibit the unbounded violation ratios proved possible by Pérez-García et al. by cleverly grouping some large numbers of players sharing a stabilizer state into three sets that, when treated as three players, can still obtain large violations. Based on a result by Bravyi et al. [BFG06] we obtain the following corollary of Theorem 6.2.2, showing that, unfortunately, such a construction is impossible.

6.2.3. COROLLARY. *Let \mathcal{G} be a 3-player XOR game in which the players are restricted to using a stabilizer state. Then the maximum bias achievable is bounded by a universal constant, independent of the specific state used, or its dimension. More formally, if $|\psi\rangle$ is an arbitrary stabilizer state, then the following inequality holds:*

$$\beta_{|\psi\rangle}^*(\mathcal{G}) \leq 8 (K_G^C)^4 \beta(\mathcal{G}).$$

The above two theorems and corollary provide a perhaps surprising counterpoint to another of [PGWP⁺08]’s results, mentioned above, which shows that some states can achieve much larger gaps. Together, these results indicate a large variation in nonlocality for multipartite states, which is already apparent through their use in XOR games. This contrasts with the bipartite scenario, where all states give at most constant violation ratios, and both the maximally entangled state and the embezzlement states are optimal resources.

In the following section we outline implications of the above results for Banach algebras, for hardness of approximation, and for parallel repetitions.

6.2.1 Implications

Implications for Banach algebras. Theorem 6.2.1 answers an open question of Pérez-García et al. They were particularly interested in this question because they were able to relate the violation ratio with Schmidt states to an old open problem of Varopoulos in Banach algebras [Var75]. Via the reductions given in [Dav73, PGWP⁺08] and in conjunction with the partial answers of LeMerdy [LM98] and Pérez-García [PG06], Theorem 6.2.1 settles Varopoulos’s question completely. We discuss this result in detail in the next chapter, where we explain our contribution separate from the context of nonlocal games, and sketch the connection to Schmidt states made in [PGWP⁺08].

Implications for hardness of approximation. On the one hand, Tsirelson’s characterization of two-player entangled XOR games gives a means to efficiently compute the bias $\beta^*(\mathcal{G})$ to high accuracy via semidefinite programming. On the other hand, approximating the *classical* bias of two-player XOR games within a sufficiently small constant is NP-hard [Hås01]. Hence the natural relaxation that corresponds to allowing the players to share entanglement marks the transition from a hard optimization problem to a tractable one.

As our results show, for multiplayer XOR games the violation ratio can be tightly bounded when the players share specific forms of entanglement,

and it is interesting to ask whether the quantum bias can again be efficiently approximated. It turns out, however, that the situation in this case is quite different. In fact, our results imply the following:

6.2.4. THEOREM. *Unless $P=NP$, for any integer $N \geq 3$ there is no polynomial-time algorithm that approximates the maximum bias of an entangled N -player game in which the players are restricted to sharing either a Schmidt state or clique-wise entanglement to within a factor c for any constant $c > 1$.*

Our results only hold for the specific types of entanglement that we consider, and it could very well be the case that $\beta^*(\mathcal{G})$ can be computed exactly or approximated closely in polynomial-time for general entanglement. The proof of Theorem 6.2.4 follows from a hardness-of-approximation result for Max-E3-Lin2 due to Håstad and Venkatesh [HV04], and we give it in Section 6.7.

Implications for parallel repetition. *Parallel repetition* of a general two-player nonlocal game \mathcal{G} refers to the following situation: The referee samples independently some number ℓ of question pairs $(i^1, j^1), \dots, (i^\ell, j^\ell)$ from the probability distribution π associated to \mathcal{G} , and sends the ℓ questions i^1, \dots, i^ℓ to Alice and j^1, \dots, j^ℓ to Bob. The players are then expected to each return ℓ answers, one corresponding to each of their questions. They win this parallel repetition version of \mathcal{G} if their answers win each of the ℓ instances of \mathcal{G} .

If the maximal winning probability of one round of \mathcal{G} is at most $\omega(\mathcal{G}) < 1$, then it is intuitively clear that an ℓ -fold parallel repetition of it is even more difficult. Determining just how much more difficult parallel repetitions make a game, turns out to be a very non-trivial matter. In general, it is not true that the winning probability simply scales as one would expect, i.e. as $\omega(\mathcal{G})^\ell$, which would be the case if the repetitions of the game are performed sequentially [Fei91, CHTW04, Raz08, BHH⁺08, KR10]. For two-party nonlocal games the celebrated parallel repetition theorem [Raz98, Hol07] states that the winning probability does decrease exponentially in the number of parallel repetitions of the game. Only recently, Kempe and Vidick [KV11] proved that the entangled winning probability of general nonlocal games decreases at all under parallel repetitions. This was shown to hold before for XOR games by Cleve et al. [CSUU08] and unique games by Kempe, Regev and Toner [KRT08].

Closely related to parallel repetition theorems are XOR lemmas. Let us recall that an XOR game can be defined by a probability distribution π on $[n] \times [n]$ and a sign matrix $\Sigma \in \{-1, 1\}^{n \times n}$. The ℓ -fold XOR repetition of an XOR

game $\mathcal{G} = (\pi, \Sigma)$ is again an XOR game, and is defined as $\mathcal{G}^{\otimes \ell} = (\pi^{\otimes \ell}, \Sigma^{\otimes \ell})$. Thus, in this game ℓ question pairs $(i^k, j^k)_{k=1 \dots \ell}$ are picked independently with respect to π , and all i^k are sent to Alice, j^k to Bob. In order to win the game, they should answer signs a and b respectively such that $ab = \sum_{i^1 j^1} \dots \sum_{i^\ell j^\ell}$.

Cleve et al. [CSUU08] show that for any two-player XOR game \mathcal{G} , the game $\mathcal{G}^{\otimes \ell}$ has entangled bias exactly $\beta^*(\mathcal{G})^\ell$. Since the classical and quantum biases are within a constant factor of each other, this also implies that if $\beta^*(\mathcal{G}) < 1$, then the classical bias $\beta(\mathcal{G}^{\otimes \ell})$ must go down exponentially with ℓ (although it does not behave as nicely as the quantum bias with respect to taking XORs). Cleve et al. further use this XOR lemma for the entangled bias to show that the *winning probability* with entanglement of two-player XOR games behaves perfectly under parallel repetition, i.e., as $\omega(\mathcal{G})^\ell$ (such behavior is usually referred to as strong parallel repetition). In fact, quite generally XOR lemmas imply parallel repetition theorems [Ung09].

Surprisingly, our results (as well as the previous results by Pérez-García et al. [PGWP⁺08]) imply that there is no such XOR lemma for classical XOR games in the N -player setting for $N > 2$. This can be seen as follows. Suppose that $\beta_S^*(\mathcal{G}) = 1$ and $\beta(\mathcal{G}) < 1$ for some game \mathcal{G} . Then clearly $\beta_S^*(\mathcal{G}^{\otimes \ell}) = 1$, and so by Theorem 6.2.1 it must be the case that the classical bias satisfies

$$\beta(\mathcal{G}) \geq \frac{1}{2^{(3N-5)/2} K_G^C},$$

which is independent of ℓ . Mermin's game is an example of such a game.

6.3 Proof overview and techniques

The main technical contribution of this chapter is the expansion of the connection between violation ratios of two-player XOR games and Grothendieck's Inequality established by Tsirelson's Theorem. We relate violation ratios for N -player XOR games with the patterns of entanglement discussed above and certain multilinear extensions of Grothendieck's inequality. Let us briefly recall that in the case of two players sharing an entangled state $|\psi\rangle$, the easy direction of Tsirelson's Theorem follows from the simple observation that the expected value of the product of the players' answers determined by $\{-1, 1\}$ -valued observables F and G , given by $\langle \psi | F \otimes G | \psi \rangle$, can be written as the inner-product of two complex vectors $(\langle \psi | F \otimes I) \cdot (I \otimes G | \psi \rangle)$. Hence, the optimization over entangled strategies is readily upper-bounded by an optimization

over complex unit vectors, which can in turn be related to the classical bias via Grothendieck's Inequality.

In the multiplayer case, this kind of connection between the entangled and classical biases is not so obvious. The bipartite structure needed to express the expectation as an inner product between vectors is lost already when we consider the case of three players, where this expectation has the form $\langle \psi | F \otimes G \otimes H | \psi \rangle$. The results of this chapter stem from the observation that the function $(F, G, H) \mapsto \langle \psi | F \otimes G \otimes H | \psi \rangle$ is still a multilinear functional, whose exact dependence on the coefficients of F, G and H will depend on the state $|\psi\rangle$. Hence we isolate certain classes of states $|\psi\rangle$ (Schmidt states and clique-wise entanglement, already described above) and study the functionals that arise from them.

Our proofs proceed in two steps. In the first step, we show that, given a class of states $|\psi\rangle$, a maximization over observables F, G, H can be upper-bounded by the maximization of a certain generalized inner product over unit vectors. This step greatly depends on the class of states $|\psi\rangle$ under consideration. In the second step, we bound this last optimization as a function of the classical bias, which is the maximization over products of $\{-1, 1\}$ -valued functions. This step involves a constant-factor loss, as indeed in general the classical bias is smaller than the quantum bias that we started with.

We illustrate those two steps in more detail below by giving an overview of the proof of Theorem 6.2.1 for the case of three-player games in which entangled players share GHZ states of arbitrary local dimension. We use the following definitions introduced in Section 2.3.4. Let us recall that for N -tensor $A : [n]^N \rightarrow \mathbb{R}$, we defined

$$\text{OPT}(A) = \max \left\{ \sum_{I \in [n]^N} A[I] \chi_1(i_1) \cdots \chi_N(i_N) : \chi_1, \dots, \chi_N : [n] \rightarrow \{-1, 1\} \right\}$$

and

$$\text{GIP}(A) = \sup \left\{ \left| \sum_{i_1, \dots, i_N=1}^N A[i_1, \dots, i_N] \langle f_1(i_1), \dots, f_N(i_N) \rangle \right| : \right. \\ \left. d \in \mathbb{N}, f_1, \dots, f_N : [n] \rightarrow B_{\mathbb{C}^d} \right\},$$

where $\langle x_1, \dots, x_N \rangle = \sum_i (x_1)_i \cdots (x_N)_i$ is the generalized inner product. Note that if $A = \pi \circ \Sigma$ for some N -player XOR game $\mathcal{G} = (\pi, \Sigma)$, then $\text{OPT}(A)$ is precisely the classical bias $\beta(\mathcal{G})$ of \mathcal{G} .

6.3.1 First step: relating the entangled bias to the GIP bias

We show that, when $|\psi\rangle$ has a certain structure, one can relate the expected value $\langle\psi|F \otimes G \otimes H|\psi\rangle$ to a certain natural *trilinear functional* over unit vectors. For instance, for the simplest case of GHZ states one obtains the generalized inner-product $\langle x, y, z \rangle$ defined above. Other types of states may lead to more complicated functionals, and hence this step crucially depends on the type of entanglement that the players are allowed to use. Note that, in contrast, the Schmidt decomposition implies that for the case of two-player games the only bilinear functional which arises is essentially a weighted inner product. The many inequivalent classes of multilinear functionals that one can obtain for the case of three or more players are a reflection of the much richer structure of multipartite entanglement.

For the case of GHZ states $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=1}^d |\ell\rangle|\ell\rangle|\ell\rangle$ we show the following:

6.3.1. LEMMA. *Let $\mathcal{G} = (\pi, \Sigma)$ be a 3-player XOR game. Assume that the players are restricted to sharing a GHZ state, and denote the resulting maximum bias by $\beta_Z^*(\mathcal{G})$. Then the following bound holds:*

$$\beta_Z^*(\mathcal{G}) \leq \text{GIP}(\pi \circ \Sigma).$$

PROOF: Fix an optimal strategy of the players based on the shared entangled state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=1}^d |\ell\rangle|\ell\rangle|\ell\rangle$, and let F_i, G_j, H_k be each player's $\{-1, 1\}$ -valued observables in that strategy. Let $A = \pi \circ \Sigma$. The players' bias is given by

$$\begin{aligned} \beta_Z^*(\mathcal{G}) &= \sum_{(i,j,k) \in [n]^3} A[i, j, k] \langle\psi|F_i \otimes G_j \otimes H_k|\psi\rangle \\ &= \frac{1}{d} \sum_{(i,j,k) \in [n]^3} A[i, j, k] \sum_{\ell, m=1}^d \langle\ell|F_i|m\rangle \langle\ell|G_j|m\rangle \langle\ell|H_k|m\rangle \\ &= \frac{1}{d} \sum_{m=1}^d \left(\sum_{(i,j,k) \in [n]^3} A[i, j, k] \sum_{\ell=1}^d \langle\ell|F_i|m\rangle \langle\ell|G_j|m\rangle \langle\ell|H_k|m\rangle \right) \\ &\leq \frac{1}{d} \sum_{m=1}^d \text{GIP}(A) = \text{GIP}(A). \end{aligned}$$

The inequality holds as the inner sum on the third line is a generalized inner product of the m^{th} columns of the $\{-1, 1\}$ -valued observables F_i, G_j, H_k , which are unit vectors since these matrices are unitary. \square

6.3.2 Second step: relating the GIP bias to the classical bias

The second step in our proofs consists of upper-bounding the multilinear expression resulting from the first step by a similar optimization over real numbers of absolute value less than 1. This step involves a constant-factor loss, and is based on Tonge’s Inequality, Theorem 2.3.10. For our second result, Theorem 6.2.2, we use an inequality proved by Carne [Car80] in the context of Banach lattices, combined with Grothendieck’s inequality. Specialized to the case of real rank-3 tensors, Tonge’s Inequality reads:

6.3.2. THEOREM. *For every positive integer $n \geq 2$ and any 3-tensor $A : [n]^3 \rightarrow \mathbb{R}$, we have*

$$\text{GIP}(A) \leq 4K_G^C \text{OPT}(A). \quad (6.1)$$

Combining Lemma 6.3.1 with Theorem 6.3.2 gives $\beta_Z^*(\mathcal{G}) \leq 4K_G^C \beta(\mathcal{G})$, since putting $A = \pi \circ \Sigma$ makes the maximum on the right-hand side of Eq. (6.1) exactly the classical bias $\beta(\mathcal{G})$. This proves Theorem 6.2.1 for the case of three entangled players sharing GHZ states.

Tonge’s Inequality also plays a role in the proof of [PGWP⁺08] showing a constant violation ratio in the case of GHZ states. It is used there as an intermediate step to show a relationship between different tensor norms that are in turn used to prove the bound. Their technique, however, does not seem to be easily adapted to the case of Schmidt states or clique-wise entanglement.

Outline of the rest of this Chapter In Section 6.4 we introduce a few more notational conventions and definition. In Section 6.5 we prove Theorem 6.2.1, extending the above techniques to the case of Schmidt states. In Section 6.6 we prove Theorem 6.2.2 and Corollary 6.2.3, extending the above techniques to the case of clique-wise entanglement and stabilizer states. In Section 6.8 prove Carne’s Theorem. In Section 6.9 we pose an open question and we give a brief summary of this chapter in Section 6.10

6.4 Notation and definitions

The following definition will be useful in studying the different biases achievable by players who are restricted to sharing a specific type of entanglement.

6.4.1. DEFINITION. Let $\mathcal{G} = (\pi, \Sigma)$ be an N -player XOR game and $|\psi\rangle \in \mathcal{H}^{\otimes N}$ be a fixed entangled state shared by N players. Then the *bias restricted to $|\psi\rangle$* , denoted $\beta_{|\psi\rangle}^*(\mathcal{G})$, is defined as

$$\beta_{|\psi\rangle}^*(\mathcal{G}) = \max_{F_1, \dots, F_N} \mathbb{E}_{I \sim \pi} \left[\sum [I] \langle \psi | F_1(i_1) \otimes \dots \otimes F_N(i_N) | \psi \rangle \right]$$

where the maximum is taken over $F_1, \dots, F_N : [n] \rightarrow \mathcal{O}(\mathcal{H})$.

The following setups are the ones that we will encounter most frequently, and for each we introduce a special notation for the bias. For the case of GHZ states $|\psi\rangle = d^{-1/2} \sum_{i=1}^d |i\rangle_1 \dots |i\rangle_N$ (of arbitrary dimension d) we will denote the maximum bias by $\beta_Z^*(\mathcal{G})$, while for Schmidt states $|\psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle_1 \dots |i\rangle_N$ (with arbitrary dimension d and real positive coefficients α_i satisfying $\sum_{i=1}^d \alpha_i^2 = 1$) we will use the notation $\beta_S^*(\mathcal{G})$. Finally, clique-wise entanglement is any type of entanglement that can be obtained by grouping the N players into k coalitions of r players each (a given player can take part in any number of coalitions), and allowing the members of each of the coalitions to share a GHZ state of arbitrary dimension (recall that collections of EPR pairs shared among a two-party coalition are simply higher dimensional two-party GHZ states). In that case, we denote the maximal bias by $\beta_C^*(\mathcal{G})$. This may depend on the parameters k and r , which are kept implicit so as not to overload the notation, but will always be clear from context.

6.5 Bounded violations for Schmidt states

In this section we prove Theorem 6.2.1. As this chapter is rather heavy on notation, we present the proof of Theorem 6.2.1 in three steps, in order to let the reader get accustomed to the various quantities involved. First, in Section 6.5.1 we analyze the maximum bias $\beta_Z^*(\mathcal{G})$ achievable by strategies that are limited to sharing a GHZ state for games with an arbitrary number of players. In Section 6.5.2, we extend our proof to cover the case where the players are allowed to share a Schmidt state.

6.5.1 Strategies with GHZ states.

We prove the following lemma, which is a (straightforward) generalization of Lemma 6.3.1 proved in Section 6.3:

6.5.1. LEMMA. Let $\mathcal{G} = (\pi, \Sigma)$ be an N -player game. Assume that the players are restricted to sharing a GHZ state (i.e., a state of the form $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=1}^d |\ell\rangle^{\otimes N}$ where d is arbitrary). Then the maximum bias the players can achieve is upper-bounded by

$$\beta_Z^*(\mathcal{G}) \leq \text{GIP}(\pi \circ \Sigma).$$

PROOF: For $i_1, \dots, i_N \in [n]$, let $F_1(i_1), \dots, F_N(i_N)$ be the $\{-1, 1\}$ -valued observables used by the N players to play \mathcal{G} while sharing state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=1}^d |\ell\rangle^{\otimes N}$. Let $A = \pi \circ \Sigma$. The players' bias is then given by

$$\begin{aligned} \beta_Z^*(\mathcal{G}) &= \sum_{I \in [n]^N} A[I] \langle \psi | F_1(i_1) \otimes \dots \otimes F_N(i_N) | \psi \rangle \\ &= \frac{1}{d} \sum_{I \in [n]^N} A[I] \sum_{\ell, m=1}^d \langle \ell | F_1(i_1) | m \rangle \dots \langle \ell | F_N(i_N) | m \rangle \\ &\leq \frac{1}{d} \sum_{m=1}^d \left(\sum_{I \in [n]^N} A[I] \sum_{\ell=1}^d \langle \ell | F_1(i_1) | m \rangle \dots \langle \ell | F_N(i_N) | m \rangle \right) \\ &\leq \frac{1}{d} \sum_{m=1}^d \text{GIP}(A) = \text{GIP}(A). \end{aligned}$$

The inequality holds as the inner sum on the third line is a generalized inner product of the m^{th} columns of the observables $F_1(i_1), \dots, F_N(i_N)$, which are unit vectors since these matrices are unitary. \square

The inequality $\beta_Z^*(\mathcal{G}) \leq 2^{(3N-5)/2} K_G^{\mathbb{C}} \beta(\mathcal{G})$ now follows from Lemma 6.5.1 and Theorem 2.3.10, since for $A = \pi \circ \Sigma$, we have that $\text{OPT}(A)$ is precisely the classical bias of the game $\mathcal{G} = (\pi, \Sigma)$. This proves Theorem 6.2.1 for the special case of GHZ states.

6.5.2 Extension to Schmidt states.

We extend the result of Section 6.5.1 to the case of Schmidt states, thus proving Theorem 6.2.1 in full generality. For this, analogous to Lemma 6.5.1, it is sufficient to show that for a Schmidt state $|\psi\rangle = \sum_{\ell=1}^d \alpha_\ell |\ell\rangle^{\otimes N}$, we have

$$\beta_{|\psi\rangle}^*(\mathcal{G}) \leq \text{GIP}(\pi \circ \Sigma)$$

The theorem then follows by setting $A = \pi \circ \Sigma$ and applying Theorem 2.3.10.

PROOF OF THEOREM 6.2.1: For N -player XOR game $\mathcal{G} = (\pi, \Sigma)$, define the tensor $A = \pi \circ \Sigma$. Suppose that the N players share the Schmidt state $|\psi\rangle = \sum_{\ell=1}^d \alpha_\ell |\ell\rangle^{\otimes N}$. For $i_1, \dots, i_N \in [n]$, let $F_1(i_1), \dots, F_N(i_N)$ be a choice of $\{-1, 1\}$ -valued observables used by the players to achieve bias $\beta_{|\psi\rangle}^*(\mathcal{G})$.

We use the following claim, which shows that $|\psi\rangle$ can be expressed as a weighted sum of GHZ-type states.

9. CLAIM. *There exist nonnegative reals v_1, \dots, v_d such that $|\psi\rangle = \sum_{m=1}^d v_m |\phi_m\rangle$, where $|\phi_m\rangle = \sum_{\ell=1}^m |\ell\rangle^{\otimes N}$ for $m = 1, \dots, d$ is a “partial” (un-normalized) GHZ state. Moreover, the v_ℓ satisfy the following equation:*

$$\sum_{m,k=1}^d v_m v_k \cdot \min\{m, k\} = 1 \quad (6.2)$$

PROOF: Renaming the basis vectors as necessary, we can assume that $\alpha_1 \geq \dots \geq \alpha_d$. Let $v_d = \alpha_d$ and $v_m = \alpha_m - \alpha_{m+1}$ for $\ell = 1, \dots, d-1$. Then we have

$$|\psi\rangle = \sum_{m=1}^d v_m |\phi_m\rangle.$$

Moreover, Eq. (6.2) is immediate from the fact that $|\langle\psi|\psi\rangle| = 1$ and $\langle\phi_m|\phi_k\rangle = \min\{m, k\}$ (recall that $|\phi_{m'}\rangle$ itself was not normalized). \blacklozenge

This reformulation of $|\psi\rangle$ reduces the task of showing an upper bound on $\beta_{|\psi\rangle}^*(\mathcal{G})$ to a form similar to what we had before. Namely,

$$\beta_{|\psi\rangle}^*(\mathcal{G}) = \sum_{m,k} v_m v_k \sum_{I \in [n]^N} A[I] \langle\phi_m|F_1(i_1) \otimes \dots \otimes F_N(i_N)|\phi_k\rangle.$$

For fixed m, k , each term of the sum involves unnormalized “partial” GHZ states, which can be handled in the same fashion as Lemma 6.5.1.

10. CLAIM. *For tensor A and states $|\phi_m\rangle$ as defined above, we have*

$$\sum_{I \in [n]^N} A[I] \langle\phi_m|F_1(i_1) \otimes \dots \otimes F_N(i_N)|\phi_k\rangle \leq \min\{m, k\} \text{GIP}(A).$$

PROOF: Writing out $|\phi_m\rangle$ and $|\phi_k\rangle$, we have

$$\begin{aligned} \sum_{I \in [n]^N} A[I] \langle\phi_m|F_1(i_1) \otimes \dots \otimes F_N(i_N)|\phi_k\rangle = \\ \sum_{I \in [n]^N} A[I] \sum_{s=1}^m \sum_{t=1}^k \langle s|F_1(i_1)|t\rangle \dots \langle s|F_N(i_N)|t\rangle. \end{aligned}$$

We will order the double sum over s, t depending on whether m or k is smaller—we want the outer sum to be over the smaller one. Suppose that $m \leq k$. The other case is completely analogous. Then

$$\sum_{I \in [n]^N} A[I] \sum_{s=1}^m \sum_{t=1}^k \langle s | F_1(i_1) | t \rangle \cdots \langle s | F_N(i_N) | t \rangle = \sum_{s=1}^m \left(\sum_{I \in [n]^N} A[I] \sum_{t=1}^k \langle s | F_1(i_1) | t \rangle \cdots \langle s | F_N(i_N) | t \rangle \right). \quad (6.3)$$

For each fixed s , the inner sum is now a generalized inner product of the first k entries of the s^{th} rows of the matrices $F_1(i_1), \dots, F_N(i_N)$. Since the full rows of these matrices have norm at most 1, we obtain complex vectors of norm at most 1 by taking only their first k coordinates. Hence, we have

$$\sum_{I \in [n]^N} A[I] \langle \phi_m | F_1(i_1) \otimes \cdots \otimes F_N(i_N) | \phi_k \rangle \leq \min\{m, k\} \text{GIP}(A).$$

This proves the claim. ◆

We can now finish the proof of the theorem. Combining the above two claims gives

$$\begin{aligned} \beta_{|\psi\rangle}^*(\mathcal{G}) &= \sum_{m,k} v_m v_k \sum_{I \in [n]^N} A[I] \langle \phi_m | F_1(i_1) \otimes \cdots \otimes F_N(i_N) | \phi_k \rangle \\ &\leq \sum_{m,k} v_m v_k \min\{m, k\} \text{GIP}(A) \\ &= \text{GIP}(A). \end{aligned}$$

The first line follows from Claim 10 and the last from Claim 9. □

6.6 Bounded violations for clique-wise entanglement

The proof of Theorem 6.2.2 is based on a result by Carne [Car80] that essentially shows how Grothendieck-type inequalities can be composed in order to prove new inequalities of the same type. This will let us prove bounds on the entangled bias when the players are allowed to share any combination of EPR pairs and GHZ states. We explain Carne's theorem in Section 6.6.1, we explain how it is applied to prove Theorem 6.2.2 in Section 6.6.2 and we end this section with a proof of Corollary 6.2.3.

6.6.1 Carne's Theorem

Carne's Theorem is most easily explained with the use of hypergraphs. A *hypergraph* $H = (V, E)$ consists of a finite set V of vertices and a family E of subsets (called hyper-edges) of V . In a normal graph, the edge set E consists of pairs of vertices, but in a hypergraph, the hyper-edges are allowed to have any size ranging from 1 to $|V|$. For a vertex $u \in V$, we denote by $E(u)$ the set of hyper-edges $e \in E$ that contain u as an element.

Towards understanding Carne's Theorem, let $H = (V, E)$ be a hypergraph. We associate with each hyper-edge $e \in E$ and vertex $u \in e$ a complex Hilbert space $\mathcal{H}(u, e)$. Furthermore, we associate with every edge $e \in E$ a linear functional $\phi_e : \otimes_{u \in e} \mathcal{H}(u, e) \rightarrow \mathbb{C}$. Later, every vertex $u \in V$ will correspond to a player in a $|V|$ -player XOR game and for every hyper edge e containing u , the space $\mathcal{H}(u, e)$ will be u 's local Hilbert space for some state $|\phi_e\rangle \in \otimes_{v \in e} \mathcal{H}(v, e)$ that u shares with the other members of e . The linear functionals ϕ_e will correspond to generalized inner products that arise when the $|\phi_e\rangle$ are GHZ states.

Suppose that every ϕ_e satisfies a Grothendieck-type inequality, by which we mean that for every $|e|$ -tensor $A : [n]^{|e|} \rightarrow \mathbb{R}$ and functions $f_u : [n] \rightarrow B_{\mathcal{H}(u, e)}$, for each $u \in e$, the inequality

$$\left| \sum_{I \in [n]^{|e|}} A[I] \phi_e \left(\bigotimes_{u \in e} f_u(i_u) \right) \right| \leq C_e \text{OPT}(A), \quad (6.4)$$

holds for some constant C_e independent of A and the f_u . The functionals ϕ_e that we will encounter below are those for which $\phi_e \left(\bigotimes_{u \in e} x_u \right)$ is the generalized inner product between the vectors $x_u \in \mathcal{H}(u, e)$.

Define for every $u \in V$ the Hilbert space $\mathcal{H}_u = \otimes_{e \in E(u)} \mathcal{H}(u, e)$. Carne's Theorem then states that a certain natural combination of the linear functionals ϕ_e in a general multilinear functional Φ defined over the entire Hilbert space $\mathcal{H} = \otimes_{u \in V} \mathcal{H}_u$ also satisfies a Grothendieck-type inequality with a constant equal to the product of the C_e . This combination of the ϕ_e is precisely the type we obtain by allowing the players in each hyper-edge to share a GHZ state. Since a vertex u can be part of many different edges, there can be many functionals ϕ_e that act on the same space \mathcal{H}_u . This is what makes Carne's Theorem non-trivial. We need one last thing, which is the linear *re-arranging map*

$$\sigma : \bigotimes_{u \in V} \left(\bigotimes_{e \in E(u)} \mathcal{H}(u, e) \right) \rightarrow \bigotimes_{e \in E} \left(\bigotimes_{u \in e} \mathcal{H}(u, e) \right),$$

which simply permutes the elements of a vector $x \in \otimes_{u \in V} \mathcal{H}_u$.

6.6.1. THEOREM (CARNE). Define the linear functional $\Phi : \otimes_{u \in V} \mathcal{H}_u \rightarrow \mathbb{C}$ as $\Phi = \left(\otimes_{e \in E} \phi_e \right) \circ \sigma$, where \circ denotes the composition of the two maps. Then, for any $|V|$ -tensor $A : [n]^{|V|} \rightarrow \mathbb{R}$ and set of functions $f_u : [n] \rightarrow B_{\mathcal{H}_u}$, for $u \in V$, we have

$$\sum_{I \in [n]^{|V|}} A[I] \Phi \left(\otimes_{u \in V} f_u(i_u) \right) \leq \left(\prod_{e \in E} C_e \right) \text{OPT}(A), \quad (6.5)$$

where the C_e are as in Eq. (6.4).

If for each $e \in E$, the functional ϕ_e gives the generalized inner product between vectors $x_u \in \mathcal{H}(u, e) = \mathbb{C}^d$, then by Theorem 2.3.10 (Tongue's Inequality), we get that Inequality 6.4 holds with $C_e = 2^{(3|e|-5)/2} K_G^{\mathbb{C}}$.

6.6.2 Bounding the violations achievable by strategies with clique-wise entanglement

Consider an N -player XOR game $\mathcal{G} = (\pi, \Sigma)$. Let the players be organized in k coalitions of r players each³, where each player can take part in any number of coalitions. Each coalition is allowed to share a GHZ state between its members.

To model this setup, we associate a hypergraph $H = (V, E)$ to the coalition structure, with $V = [N]$ and there is a hyperedge for every coalition. For every hyper edge e we introduce a Hilbert space $\mathcal{H}(e) = \otimes_{u \in e} \mathcal{H}(u, e)$, where $\mathcal{H}(u, e)$ is a local space of player u corresponding to edge e . The state of the players in this space is a GHZ state $|\phi_e\rangle = d^{-1/2} \sum_{j=1}^d |j\rangle^{\otimes |e|}$. The global entangled state shared by the players at the start of the game is then

$$|\tilde{\Phi}\rangle = \bigotimes_{e \in E} |\phi_e\rangle \in \bigotimes_{e \in E} \left(\bigotimes_{u \in e} \mathcal{H}(u, e) \right) \quad (6.6)$$

Finally, each player u has an observable $F_u(i_u)$ corresponding to question $i_u \in [N]$. These act on player u 's local space $\mathcal{H}_u = \otimes_{e \in E(u)} \mathcal{H}(u, e)$.

Theorem 6.2.2 states that the maximum bias achievable by a strategy of the form that we have just described is at most a constant times the classical bias of the game. In order to prove it, we first relate the bias with any $\{-1, 1\}$ -valued observables $\{F_u(i_u)\}_{u \in V}$ to an expression similar to the one appearing on the left-hand side of Inequality (6.9) in Carne's Theorem, where ϕ_e will be the linear functional associated with the GHZ state. More precisely, for every

³The organization of these coalitions is independent of the game itself; rather it is used to define the structure of the entanglement that is shared between the players.

$e \in E$, the Hilbert spaces $\mathcal{H}(u, e)$ for $u \in e$ will be \mathbb{C}^d (for some d) and ϕ_e will be such that for any set of vectors $\{x_u\}_{u \in e} \subseteq \mathbb{C}^d$ the value $\phi_e(\bigotimes_{u \in e} x_u)$ equals the generalized inner product of the x_u s. Applying Theorem 6.6.1 will conclude the argument.

PROOF OF THEOREM 6.2.2: Fix observables $\{F_u\}_{u \in V}$ and an entangled state $|\Phi\rangle = \sigma^{-1}(|\tilde{\Phi}\rangle)$, where $|\tilde{\Phi}\rangle$ is described in Eq. (6.6) and σ is the rearrangement map that appears in Carne's Theorem. This map appears because we need to re-arrange the terms of $|\tilde{\Phi}\rangle$ to correspond to the decomposition of space $\bigotimes_{u \in V} \mathcal{H}_u$. (We omit the arguments i_u in the F_u for now to suppress notation and because they do not play a role at this moment.)

We begin by expanding the expectation $\langle \Phi | \bigotimes_{u \in V} F_u | \Phi \rangle$, with the goal of relating it to the map Φ of Theorem 6.6.1. Let $[d]^E$ denote the set of $|E|$ -tuples of the form $(j_e)_{e \in E}$ where each j_e is an integer in $[d]$. Recall that the members of an edge $e \in E$ share a state of the form $|\phi_e\rangle = d^{-1/2} \sum_{j_e=1}^d |j_e\rangle^{\otimes |e|}$. We have

$$\begin{aligned} |\Phi\rangle &= \sigma^{-1} \left(\frac{1}{\sqrt{d^{|E|}}} \bigotimes_{e \in E} \left(\sum_{j_e=1}^d \bigotimes_{u \in e} |j_e\rangle \right) \right) \\ &= \frac{1}{\sqrt{d^{|E|}}} \sum_{J \in [d]^E} \bigotimes_{u \in V} |J_{|E(u)}\rangle \end{aligned}$$

where $J_{|E(u)}$ denotes the tuple $(j_e)_{e \in E(u)}$ and $|J_{|E(u)}\rangle = \bigotimes_{e \in E(u)} |j_e\rangle$ is a state in the Hilbert space \mathcal{H}_u of player u .

Since observables are Hermitian, the expected value $\langle \Phi | \bigotimes_{u \in V} F_u | \Phi \rangle$ equals

$$\begin{aligned} \langle \Phi | \bigotimes_{u \in V} F_u | \Phi \rangle &= \frac{1}{2 \cdot d^{|E|}} \sum_{J', J \in [d]^E} \left(\prod_{u \in V} \langle J'_{|E(u)} | F_u | J_{|E(u)} \rangle + \prod_{u \in V} \langle J_{|E(u)} | F_u | J'_{|E(u)} \rangle \right) \\ &= \frac{1}{2 \cdot d^{|E|}} \sum_{J', J \in [d]^E} \left(\prod_{u \in V} \langle J'_{|E(u)} | F_u | J_{|E(u)} \rangle + \prod_{v \in V} \langle J'_{|E(v)} | F_v^* | J_{|E(v)} \rangle \right) \\ &= \frac{1}{d^{|E|}} \sum_{J' \in [d]^E} \left(\sum_{J \in [d]^E} \Re \left(\prod_{u \in V} \langle J'_{|E(u)} | F_u | J_{|E(u)} \rangle \right) \right). \end{aligned} \quad (6.7)$$

Note that, since the expression on the left-hand side is real, the one on the right-hand side is too, and we can safely ignore the \Re symbol on the right. Since the F_u are unitary matrices, their columns are unit vectors. This implies that there exist unit vectors $x_u \in \bigotimes_{e \in E(u)} \mathcal{H}(u, e)$ (depending on J') such that the expression between the brackets in equation (6.7) is of the form

$$\sum_{J' \in [d]^E} \prod_{u \in V} (x_u)_{J_{|E(u)}}$$

where $(x_u)_{J_{|E(u)}}$ denotes $J_{|E(u)}$ -coordinate of the vector x_u , when written in the basis defined by the vectors $|J_{|E(u)}\rangle$.

11. CLAIM. For $\Phi = \left(\bigotimes_{e \in E} \phi_e \right) \circ \sigma$ where each ϕ_e corresponds to the generalized inner product function on $\bigotimes_{u \in e} \mathcal{H}(u, e)$, we have

$$\sum_{J \in [d]^E} \prod_{u \in V} (x_u)_{J_{|E(u)}} = \Phi \left(\bigotimes_{u \in V} x_u \right).$$

PROOF: Since Φ is linear, it suffices to prove the claim for vectors of the form $x_u = \bigotimes_{e \in E(u)} x_{u,e}$, where each $x_{u,e} \in \mathcal{H}(u, e)$. In this case, we have

$$\begin{aligned} \left(\bigotimes_{e \in E} \psi_e \right) \circ \sigma \left(\bigotimes_{u \in V} \left(\bigotimes_{e \in E(u)} x_{u,e} \right) \right) &= \bigotimes_{e \in E} \left(\psi_e \left(\bigotimes_{u \in e} x_{u,e} \right) \right) \\ &= \prod_{e \in E} \left(\sum_{j_e=1}^d \left(\prod_{u \in e} (x_{u,e})_{j_e} \right) \right) \\ &= \sum_{J \in [d]^E} \prod_{e \in E} \left(\prod_{u \in e} (x_{u,e})_{j_e} \right) \\ &= \sum_{J \in [d]^E} \prod_{u \in V} \left(\prod_{e \in E(u)} (x_{u,e})_{j_e} \right), \end{aligned}$$

where the last product is $\prod_{e \in E(u)} (x_{u,e})_{j_e} = (x_u)_{J_{|E(u)}}$. \square

Let $F_u(i_u)$ be the observable used by player u on question i_u , so that the bias achieved by this strategy in the game $\mathcal{G} = (\pi, \Sigma)$ is

$$\sum_{I \in [n]^V} A[I] \langle \Phi | \bigotimes_{u \in V} F_u(i_u) | \Phi \rangle$$

where $A = \pi \circ \Sigma$. We can bound this expression by

$$\begin{aligned} & \left| \sum_{I \in [n]^V} A[I] \left(\frac{1}{d^{|E|}} \sum_{J' \in [d]^E} \sum_{J \in [d]^E} \prod_{u \in V} [F_u(i_u)]_{J_{|E(u)}, J'_{|E(u)}} \right) \right| \\ & \leq \frac{1}{d^{|E|}} \sum_{J' \in [d]^E} \left| \sum_{I \in [n]^V} A[I] \cdot \sum_{J \in [d]^E} \prod_{u \in V} [F_u(i_u)]_{J_{|E(u)}, J'_{|E(u)}} \right| \\ & \leq \max_{J' \in [d]^E} \left| \sum_{I \in [n]^V} A[I] \cdot \sum_{J \in [d]^E} \prod_{u \in V} [F_u(i_u)]_{J_{|E(u)}, J'_{|E(u)}} \right| \\ & \leq \max_{f_u: [n] \rightarrow \mathcal{B}\mathcal{H}_u: u \in V} \left| \sum_{I \in [n]^V} A[I] \Phi \left(\bigotimes_{u \in V} f_u(i_u) \right) \right|, \quad (6.8) \end{aligned}$$

where the first equality is (6.7), and the last inequality follows from Claim 11. The result then follows directly from Theorem 6.6.1 combined with the bound in Theorem 2.3.10, giving the last part of the theorem. \square

We end this section with a proof of Corollary 6.2.3.

PROOF OF COROLLARY 6.2.3: Theorem 5 in [BFG06] states that, if $|\Psi\rangle$ is any stabilizer state shared in an arbitrary way among three parties, then there exist unitary matrices U_1 , U_2 and U_3 on the Hilbert spaces \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 , respectively, such that $U_1 \otimes U_2 \otimes U_3 |\Psi\rangle$ is a state of the form $|\Phi\rangle$ considered above. In other words, $|\Psi\rangle$ is local-unitarily equivalent to a number of EPR pairs shared between each of the three pairs of players, together with a GHZ state shared in common. By defining local observables $U_1 F_1(i_1) U_1^*$, etc, it is not difficult to see that for any three-player XOR game \mathcal{G} , the bias $\beta_{|\Psi\rangle}^*(\mathcal{G})$ is at most the bias attainable with clique-wise entanglement shared among the three players.

It now suffices to consider the hypergraph H with vertex set $V = \{1, 2, 3\}$, and edge set $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. In the notation of Theorem 6.2.2, this hypergraph has $k = 4$ and $r \leq 3$, which gives the bound $2^8(K_G^C)^4$. However, a careful examination of the proof of Theorem 6.2.2 easily reveals that the inequality holds with the smaller constant $8(K_G^C)^4$. \square

6.7 Hardness of approximation of the entangled bias

Khot and Naor [KN08] observed that the hardness-of-approximation results for Max-E3-Lin2 of Håstad and Venkatesh [HV04] can be extended to:

6.7.1. THEOREM (HÅSTAD-VENKATESH-KHOT-NAOR). *Unless $P=NP$, there is no polynomial-time algorithm that approximates the classical bias of a three-party XOR game to within a multiplicative factor c for any constant $c > 1$.*

The inapproximability results in [HV04] only hold for *symmetric* strategies, in which the players all share the same strategy. However, Khot and Naor show that the inapproximability result holds even when restricted to games $\mathcal{G} = (\pi, \Sigma)$ that are invariant under permutations of the three players (i.e. for $A = \pi \circ \Sigma$ we have $A[i, j, k] = A[i, k, j] = A[j, i, k] = A[j, k, i] = A[k, i, j] = A[k, j, i]$) and are such that the same question is never asked to two players simultaneously (i.e. $A[i, j, j] = A[j, i, j] = A[j, j, i] = 0$). In this case Lemma 2.1 in [KN08] shows that the optimum with respect to symmetric strategies is

within a factor 10 of the general optimum. Combining this result with Theorems 6.2.1 and 6.2.2 immediately proves Theorem 6.2.4. Indeed, Theorem 6.2.1 (resp. Theorem 6.2.2) shows that, as long as the players are restricted to using an arbitrary Schmidt state (resp. clique-wise entanglement), the quantum bias is at most a constant times the classical bias. Hence any constant-factor approximation to the quantum bias would give a constant approximation to the classical bias, which is ruled out by the hardness result from [HV04].

6.8 Proof of Carne's Theorem

In this section we prove Theorem 6.6.1, which we restate here for convenience.

6.8.1. THEOREM (CARNE). *Define the linear functional $\Phi : \bigotimes_{u \in V} \mathcal{H}_u \rightarrow \mathbb{C}$ as $\Phi = \left(\bigotimes_{e \in E} \phi_e \right) \circ \sigma$, where \circ denotes the composition of the two maps. Then, for any $|V|$ -tensor $A : [n]^{|V|} \rightarrow \mathbb{R}$ and set of functions $f_u : [n] \rightarrow B_{\mathcal{H}_u}$, for $u \in V$, we have*

$$\sum_{I \in [n]^{|V|}} A[I] \Phi \left(\bigotimes_{u \in V} f_u(i_u) \right) \leq \left(\prod_{e \in E} C_e \right) \text{OPT}(A), \quad (6.9)$$

where the C_e are as in Eq. (6.4).

PROOF OF THEOREM 6.8.1: The proof is by induction on the number of edges $|E|$. If the edge set is empty, then there is nothing to prove. Let e_0 be any edge in the hypergraph H , and consider the graph $H_0 = (V, E \setminus \{e_0\})$. To re-write the expression, first assume that each vector $f_u(i_u) \in \mathcal{H}_u = \bigotimes_{e \in E(u)} \mathcal{H}(u, e)$ has the following tensor structure:

$$f_u(i_u) = f_u^0(i_u) \otimes f_u^1(i_u),$$

where $f_u^0(i_x) \in \bigotimes_{e \in E \setminus \{e_0\}} \mathcal{H}(u, e)$ and $f_u^1(i_u) \in \mathcal{H}(u, e_0)$.

Define $\Phi_{H_0} = \left(\bigotimes_{e \in E \setminus \{e_0\}} \phi_e \right) \circ \sigma_{H_0}$, where σ_{H_0} is the re-arranging map for H_0 . With this notation we have

$$\begin{aligned} \Phi \left(\bigotimes_{u \in V} f_u(i_u) \right) &= \Phi \left(\bigotimes_{u \in V} f_u^0(i_u) \otimes f_u^1(i_u) \right) \\ &= \Phi_{H_0} \left(\bigotimes_{u \in V} f_u^0(i_u) \right) \cdot \phi_{e_0} \left(\bigotimes_{u \in e_0} f_u^1(i_u) \right) \end{aligned}$$

Define the tensor $B[I] = A[I] \cdot \phi_{e_0} \left(\bigotimes_{u \in e_0} f_u^1(i_u) \right)$. Applying the induction hypothesis to $B[I]$ and the graph H_0 (note that the $\phi_{e_0}(\dots)$ term is simply a number, dependent on I) gives

$$\sum_{I \in [n]^V} B[I] \cdot \Phi_{H_0} \left(\bigotimes_{u \in V} f_u^0(i_u) \right) \leq \left(\prod_{e \in E \setminus \{e_0\}} C_e \right) \text{OPT}(B) \quad (6.10)$$

By definition,

$$\begin{aligned} \text{OPT}(B) &= \max \left\{ \sum_I B[I] \prod_{u \in V} \chi_u(i_u) : \chi_u : [n] \rightarrow \{-1, 1\} \right\} \\ &= \max \left\{ \sum_I A[I] \left(\prod_{u \in V} \chi_u(i_u) \right) \phi_{e_0} \left(\bigotimes_{u \in e_0} f_u^1(i_u) \right) : \chi_u : [n] \rightarrow \{-1, 1\} \right\}. \end{aligned}$$

Fix χ_u that achieve this maximum, and define the tensor $C[I] = A[I] \prod_{u \in V} \chi_u(i_u)$. By hypothesis, the function ϕ_e enjoys a Grothendieck-type inequality, hence the expression above can be bounded by

$$\text{OPT}(B) = \sum_I C[I] \cdot \phi_{e_0} \left(\bigotimes_{x \in e_0} f_x^1(i_x) \right) \leq C_{e_0} \text{OPT}(C) \quad (6.11)$$

To conclude, we can relate $\text{OPT}(C)$ to $\text{OPT}(A)$ in the following way:

$$\begin{aligned} \text{OPT}(C) &= \max \left\{ \sum_I C[I] \prod_{u \in V} \chi'_u(i_u) : \chi'_u : [n] \rightarrow \{-1, 1\} \right\} \\ &= \max \left\{ \sum_I A[I] \prod_{u \in V} \chi_u(i_u) \chi'_u(i_u) : \chi'_u : [n] \rightarrow \{-1, 1\} \right\} \\ &= \max \left\{ \sum_I A[I] \prod_{u \in V} \chi''_u(i_u) : \chi''_u : [n] \rightarrow \{-1, 1\} \right\} \\ &= \text{OPT}(A). \end{aligned}$$

Combining Eqs. (6.10) and (6.11) gives the result in the case where all $f_u(i_u)$ have the tensor structure we described earlier. If not, since Φ is linear, writing their Schmidt decomposition will result in a weighted sum of expressions involving only unit vectors of this form. The weighted sum can be bounded by its maximum component, for which we can apply the reasoning above. \square

6.9 Open questions

We proved that Schmidt states admit only constant-factor violation ratios. We also proved that clique-wise entanglement admits violation ratios bounded from above by a factor depending only on the number of coalitions. Clique-wise entanglement consists of combinations of GHZ states, which are special cases of Schmidt states. Unfortunately, we were not able to unify these results in the sense that we defined clique-wise entanglement as combinations of Schmidt states. A natural open question thus is: Do the bounds on the violation ratios for clique-wise entanglement still hold when we allow the players in each coalition to share general Schmidt states, instead of restricting them to sharing GHZ states?

6.10 Summary

In this chapter, we considered the problem of *upper bounding* the largest possible violation ratio for XOR games that involve possibly many players and where entangled players are restricted to using one of two types of entanglement: Schmidt states, or clique-wise entanglement. We proved that when the players use these types of entanglement, their advantage over classical players is at most a *constant* factor, depending only on the number of players. The case of Schmidt states settled an open problem of [PGWP⁺08] and by a reduction given in that paper, a much older problem of [Var75] (see Chapter 7). The case of clique-wise entanglement shows that, perhaps surprisingly, entanglement consisting of arbitrary combinations of EPR pairs and GHZ states shared among the players is *insufficient* to reproduce the violation ratios proved possible in [PGWP⁺08]. A theorem of [BFG06] implies that the same holds for stabilizer states.