



UvA-DARE (Digital Academic Repository)

Grothendieck inequalities, nonlocal games and optimization

Briët, J.

Publication date
2011

[Link to publication](#)

Citation for published version (APA):

Briët, J. (2011). *Grothendieck inequalities, nonlocal games and optimization*. [Thesis, fully internal, Universiteit van Amsterdam]. Institute for Logic, Language and Computation.

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, P.O. Box 19185, 1000 GD Amsterdam, The Netherlands. You will be contacted as soon as possible.

Chapter 7

A problem of Varopoulos: Schatten spaces with the Schur product are Q-algebras

The content of this chapter is based on joint work with Harry Buhrman, Troy Lee and Thomas Vidick [BBLV11].

7.1 Introduction

And now for something completely different. In this chapter, we discuss an old problem posed by Varopoulos [Var75] in the context of Banach algebras. Our contribution to this problem is the solution to a part of it that, when put in conjunction with a series of previous results due to Pietsch and Triebel [PT68], Varopoulos [Var72], Davie [Dav73], Le-Merdy [LM98] and Pérez-García [PG06], leads to its complete resolution.

We begin by giving an informal explanation of what Varopoulos's question is about. Put briefly, the question asks for the existence of isomorphisms between Banach algebras. Roughly speaking, a Banach algebra is a vector space in which one can add the elements and multiply them by scalars as usual, but in addition one can multiply the elements themselves. Two Banach algebras are isomorphic if there is a linear bijection (i.e., a one-to-one correspondence defined by a linear function) between the underlying vector spaces that preserves the multiplication operations. Varopoulos's question concerns two types Banach algebras:

1. algebras in which the vector spaces are formed by sets of matrices and the additional multiplication operation is the entry-wise multiplication

(known as the Schur or Hadamard product),

2. algebras formed by a vector space of complex-valued functions which can be multiplied in the obvious way (the product fg defined by $(fg)(x) = f(x)g(x)$).

Roughly, the problem is to determine whether the first kind (1) are isomorphic to Banach algebras (called Q-algebras) formed by *cosets* of the second kind (2) (details follow below).

Although this problem may appear completely unrelated to the rest of this thesis, Pérez-García et al. [PGWP⁺08] showed that part of it is equivalent to the problem of determining whether Schmidt states (defined in Chapter 6) allow for arbitrarily large violation ratios in multiplayer XOR games: a problem that we solved in the negative in the previous chapter. The negative answer to the Schmidt state problem (Theorem 6.2.1) in conjunction with the results mentioned above, implies a positive answer for the Banach algebra problem.

The main purpose of this chapter is to explain Varopoulos's question in more detail and to explain which part of it was solved by Theorem 6.2.1. We give a proof of our contribution separate from the context of XOR games and explain the relation to the Schmidt state problem found in [PGWP⁺08] afterwards. Last, we briefly sketch why the whole problem is solved when our result is put in conjunction with the results mentioned above.

Before continuing, I want to confess that I am a layman in the subject matter of this chapter. Clearly, the problem about to be discussed was solved in the most part due to more significant partial results and reformulations of others. Nevertheless, the hope is that this presentation may be useful in some way.

In the remainder of this section we gather the mathematical tools needed to explain Varopoulos's problem precisely. More details of the following information can be found in the excellent books by Diestel, Jarchow and Tonge [DJT95], Reed and Simon [RS72], Rudin [Rud86] and Simon [Sim05].

7.1.1 Banach algebras

A *complex algebra* $\mathcal{X} = (\mathcal{V}, *)$ is a vector space \mathcal{V} over \mathbb{C} in which a multiplication $*$ is defined that is distributive and associative. For $A, B \in \mathcal{V}$, we have $A * B \in \mathcal{V}$. If \mathcal{V} has a norm $\| \cdot \|$ defined on it that satisfies for all $A, B \in \mathcal{V}$

$$\|A * B\| \leq \|A\| \|B\|,$$

then \mathcal{X} is called a *normed complex algebra*. If \mathcal{V} is complete with respect to this norm, then \mathcal{X} is called a *Banach algebra*.

A Banach algebra is *commutative* if the multiplication operation is commutative, that is, $A * B = B * A$ for all $A, B \in \mathcal{V}$.

An important example is the space $\mathcal{C}(K)$ of continuous functions $f : K \rightarrow \mathbb{C}$ on a metric space K , which is a Banach algebra when endowed with the *uniform norm* (also called the supremum norm), defined as $\|f\| = \sup\{|f(x)| : x \in K\}$, and the pointwise multiplication, given by $(f * g)(x) = f(x)g(x)$. Often this Banach algebra is also denoted by $\mathcal{C}(K)$, as we will do here.

7.1.2 Q-algebras

Two complex algebras $\mathcal{X} = (\mathcal{V}, *)$ and $\mathcal{Y} = (\mathcal{W}, \cdot)$ are *isomorphic* if there exists a linear bijective map $\varphi : \mathcal{V} \rightarrow \mathcal{W}$ that preserves the multiplication in the sense that $\varphi(A * B) = \varphi(A) \cdot \varphi(B)$ for all $A \in \mathcal{V}$ and $B \in \mathcal{W}$. Such a map φ is called an isomorphism. If \mathcal{X} and \mathcal{Y} are normed algebras, then they are said to be *isometrically isomorphic* if there exists an isomorphism $\varphi : \mathcal{V} \rightarrow \mathcal{W}$ that is norm-preserving, that is, $\|A\|_{\mathcal{V}} = \|\varphi(A)\|_{\mathcal{W}}$.

A Banach algebra is a *uniform algebra* if it is isometrically isomorphic to a closed subspace of the Banach algebra $\mathcal{C}(K)$ for some space K which is allowed to be a slightly more general space than a metric space (namely a compact Hausdorff space; see for example [Rud86, p. 36]).

Q-algebras are closely related to uniform algebras. Roughly speaking, a Q-algebra is a Banach algebra formed by cosets in a uniform algebra. To define Q-algebras precisely, we need two more definitions, that of an ideal, and that of a quotient algebra.

A subset \mathcal{I} of a commutative complex algebra \mathcal{X} is an *ideal* in \mathcal{X} if \mathcal{I} is a subspace of \mathcal{X} (in the sense of a vector space), and if for every $A \in \mathcal{I}$ and $B \in \mathcal{X}$, we have $A * B \in \mathcal{I}$.

Given an ideal \mathcal{I} , we can associate with each $A \in \mathcal{X}$ the *coset* $\varphi(A) = A + \mathcal{I} = \{A + B : B \in \mathcal{I}\}$. A vector space is obtained out of such cosets by defining

$$\begin{aligned}\varphi(A) + \varphi(B) &= \varphi(A + B) \\ \alpha\varphi(A) &= \varphi(\alpha A)\end{aligned}$$

for every complex scalar α . Moreover, if \mathcal{I} is closed and properly contained

in \mathcal{X} (i.e., $\mathcal{I} \neq \mathcal{X}$), then a commutative Banach algebra is obtained by defining

$$\begin{aligned}\varphi(A)\varphi(B) &= \varphi(A * B) \\ \|\varphi(A)\| &= \inf\{\|B\| : B \in \varphi(A)\}.\end{aligned}$$

The Banach algebra obtained this way is called a *quotient algebra* of \mathcal{X} , and is denoted by \mathcal{X}/\mathcal{I} .

7.1.1. DEFINITION. Let \mathcal{X} be a commutative Banach algebra. Then \mathcal{X} is a *Q-algebra* if there exists a uniform algebra \mathcal{Y} and a closed ideal $\mathcal{I} \subset \mathcal{Y}$ such that \mathcal{X} is isomorphic to the quotient algebra \mathcal{Y}/\mathcal{I} .

The most interesting feature of Q-algebras, discovered by Cole (see [Wer69]), is that they are isometrically isomorphic to a closed (commutative) subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space (where the multiplication is the regular matrix product). In other words, Q-algebras are commutative operator algebras. In general, the converse is false [Var74], but Tonge [Ton78] showed that it is true for every algebra generated by a set of commuting Hilbert-Schmidt operators (endowed with the matrix product). We refer to the Notes and Remarks section of [DJT95, Chapter 18] for more information on the significance and historical developments of Q-algebras.

7.1.3 Schatten spaces and the Schur product

Varopoulos's question involves Banach algebras formed by Schatten spaces and the Schur product, which we introduce next.

The Spectral Theorem asserts that the Banach space of compact operators on ℓ_2 , which we denote by S_∞ , consists of the operators A that admit a representation of the form

$$A = \sum_{i=1}^{\infty} \lambda_i \langle \cdot, e_i \rangle f_i, \quad (7.1)$$

where $(e_i)_i$ and $(f_i)_i$ are orthonormal bases for ℓ_2 and the sequence $(\lambda_i)_i \subset \mathbb{R}$ satisfies $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\lim_{i \rightarrow \infty} \lambda_i = 0$ (see for example [RS72, Theorem VI.17]). The space S_∞ is endowed with the operator norm $\|A\| = \sup\{|\langle x, Ay \rangle| : \|x\|, \|y\| \leq 1\}$.

For $1 \leq p < \infty$, the *Schatten p -norm* of a compact operator A is given by the ℓ_p -norm $(|\lambda_1|^p + |\lambda_2|^p + \dots)^{1/p}$ of the sequence $(\lambda_i)_i$ appearing in Eq. (7.1). It is a well-known, but nontrivial fact, proved by Schatten and von Neumann [Sch46, SvN46, SvN48], that these functions are indeed norms. The

p -Schatten space $S_p \subseteq S_\infty$ is the normed vector space formed by the set of compact operators that have finite Schatten p -norm, where the norm is the Schatten p -norm. In [SvN48] it was first proved that these spaces are Banach spaces. Much-studied examples of these spaces are the trace class S_1 and the Hilbert-Schmidt operators S_2 .

The *Schur product*, for which we henceforth fix the symbol \circ , is a commutative multiplication for S_∞ defined as the entry-wise product when the elements of S_∞ are represented by matrices using the canonical basis for ℓ_2 . That is, for $A, B \in S_\infty$ such that $A = (A_{ij})_{i,j}$ and $B = (B_{ij})_{i,j}$, we have

$$A \circ B = (A_{ij}B_{ij})_{i,j}.$$

7.2 Varopoulos's question and our part of the answer

Davie [Dav73] and Varopoulos [Var72] proved that the Banach algebra (ℓ_p, \circ) is a Q-algebra for all $1 \leq p \leq \infty$. Here the multiplication \circ is the pointwise multiplication. Note that this notation is consistent with the symbol used for the Schur product (defined in the previous section) when we represent an element $x \in \ell_p$ as a linear combination of the canonical basis vectors. This result implies immediately that the algebra of Hilbert-Schmidt operators (S_2, \circ) (with the Schur product) is a Q-algebra. To see this, notice that a matrix can be seen as a vector by simply appending all of its columns underneath each other. The Hilbert-Schmidt norm (also known as the Frobenius norm) of the matrix then coincides with the ℓ_2 norm of that vector, and the Schur product of two matrices corresponds to the pointwise product of their corresponding vectors. In other words, (S_2, \circ) is isometrically isomorphic to (ℓ_2, \circ) .

Varopoulos's question [Var75] is the natural one following these facts:

Is it true that (S_p, \circ) is a Q-algebra for all $1 \leq p \leq \infty$?

Progress was made by Le-Merdy [LM98] and Pérez-García [PG06], who proved that the property holds *true* for all $2 \leq p \leq 4$ and $1 \leq p \leq 2$, respectively. Mantero and Tonge [MT80] proved that (S_∞, \circ) *fails* to be a so-called 1-summing algebra, which requires only slightly stronger conditions than for being a Q-algebra. Nevertheless, our contribution gives a positive result for the high end of the spectrum.

7.2.1. THEOREM. *The Banach algebra (S_∞, \circ) is a Q-algebra.*

The proof of Theorem 7.2.1 relies on an important result of Davie [Dav73, Theorem 3.3], which gives a simple characterization of Q-algebras. We use a slight reformulation of it, as given in [DJT95, Lemma 18.5 and Proposition 18.6]. For tensor $T : [n]^N \rightarrow \mathbb{C}$, define

$$\text{OPT}^{\mathbb{C}}(T) = \sup \left\{ \left| \sum_{I \in [n]^N} T[I] \zeta_1(i_1) \cdots \zeta_N(i_N) \right| : \zeta_1, \dots, \zeta_N : [n] \rightarrow B_{\mathbb{C}} \right\}.$$

7.2.2. THEOREM (DAVIE). *Let $\mathcal{X} = (\mathcal{V}, *)$ be a commutative Banach algebra. Then \mathcal{X} is a Q-algebra if and only if there exists a universal constant $K > 0$, such that for any choice of positive integers n, N , complex tensor $T : [n]^N \rightarrow \mathbb{C}$, and \mathcal{V} -valued sequences $A_1, \dots, A_N : [n] \rightarrow B_{\mathcal{V}}$, the inequality*

$$\left\| \sum_{I \in [n]^N} T[I] A_1(i_1) * \cdots * A_N(i_N) \right\|_{\mathcal{V}} \leq K^N \text{OPT}^{\mathbb{C}}(T), \quad (7.2)$$

holds.

We prove that the Banach algebra $(S_\infty, *)$ satisfies Davie's criterion using the multilinear extension of the complex version of Grothendieck's inequality, due to Blei [Ble79] and Tonge [Ton78], which we encountered in a slightly different form Section 2.3.4.

7.2.3. THEOREM (TONGE). *Let $n, N \geq 2$ and d be positive integers. Then, for any tensor $T : [n]^N \rightarrow \mathbb{C}$ and functions $f_1, \dots, f_N : [n] \rightarrow B_{\mathbb{C}^d}$, we have*

$$\left| \sum_{I \in [n]^N} T[I] \langle f_1(i_1), \dots, f_N(i_N) \rangle \right| \leq 2^{(N-2)/2} K_G^{\mathbb{C}} \text{OPT}^{\mathbb{C}}(T). \quad (7.3)$$

A proof of this theorem can be obtained with some minor modifications of the proof of the variant presented in Section 2.3.4. This theorem was also used by Pérez-García [PG06] to prove that (S_1, \circ) is a Q-algebra.

PROOF OF THEOREM 7.2.1: The case $N = 1$ is trivial and holds for $K = 1$, as for any sequence $A(1), \dots, A(n) \in B_{S_\infty}$, we have

$$\left\| \sum_{i=1}^n T[i] A(i) \right\| = \sup \left\{ \left| \sum_{i=1}^n T[i] \langle u, A(i)v \rangle \right| : u, v \in B_{\ell_2} \right\} = \text{OPT}^{\mathbb{C}}(T).$$

From now on, we fix integers $n, N \geq 2$, tensor $T : [n]^N \rightarrow \mathbb{C}$ and operator-valued maps $A_1, \dots, A_N : [n] \rightarrow B_{S_\infty}$. Define

$$M = \sum_{I \in [n]^N} T[I] A_1(i_1) \circ \dots \circ A_N(i_N).$$

By Theorem 7.2.2 (Davie's criterion) it suffices to show that the inequality

$$\|M\| \leq K^N \|T\|_\infty, \quad (7.4)$$

holds for some constant K independent of n, N, T and A_1, \dots, A_N .

We begin by making four small preliminary steps to show that without loss of generality we may assume that T is real valued and the A_i are finite-dimensional Hermitian matrices. Afterwards we will be able to apply Theorem 7.2.3 in order to prove Eq. (7.4). In the first step we show that without loss of generality, we may assume that the tensor T is real-valued. To this end, define the real-valued tensors T_R and T_C by $T_R[I] = \Re(T[I])$ and $T_C[I] = \Im(T[I])$ for every $I \in [n]^N$. Define

$$\begin{aligned} M_R &= \sum_{I \in [n]^N} T_R[I] A_1(i_1) \circ \dots \circ A_N(i_N) \\ M_C &= \sum_{I \in [n]^N} T_C[I] A_1(i_1) \circ \dots \circ A_N(i_N) \end{aligned}$$

Since $M = M_R + iM_C$, we have $\|M\| \leq 2 \max\{\|M_R\|, \|M_C\|\}$. Proving Eq. (7.4) for real-valued tensors thus suffices.

In the second step we show that it suffices to consider the case where the operators $A_1(i_1), \dots, A_N(i_N) \in B_{S_\infty}$ are finite-dimensional matrices (in the canonical basis for ℓ_2). Recall that norm of M is given by

$$\|M\| = \sup\{|\langle u, Mv \rangle| : u, v \in B_{\ell_2}\}.$$

For any $u \in \ell_2$ with $\|u\| \leq 1$ and any $\varepsilon > 0$ there exists a $D \in \mathbb{N}$ such that the vector $u' = \sum_{\ell=1}^D u_\ell e_\ell$ has norm at least $1 - \varepsilon$. Hence, for any $u, v \in B_{\ell_2}$ and $\varepsilon > 0$ there exist $D \in \mathbb{N}$ and $u', v' \in B_{\ell_2}$ supported only on e_1, \dots, e_D such that

$$|\langle u, Mv \rangle| \leq |\langle u', Mv' \rangle| + (2\varepsilon(1 - \varepsilon) + \varepsilon^2) |\langle u, Mv \rangle|.$$

It follows that for some $D \in \mathbb{N}$ and vectors $u', v' \in B_{\ell_2}$ supported only on e_1, \dots, e_D , we have

$$\|M\| \leq 2|\langle u', Mv' \rangle|. \quad (7.5)$$

Define for every $k = 1, \dots, N$ and $i_k = 1, \dots, n$ the D -by- D complex matrix $A'_k(i_k) = (\langle e_\ell, A_k(i_k)e_m \rangle)_{\ell, m=1}^D$. Note that $\|A'_k(i_k)\| \leq \|A_k(i_k)\| \leq 1$. Expanding the definition of M then gives

$$\begin{aligned} \langle u', Mv' \rangle &= \left\langle u', \sum_{I \in [n]^N} T[I] A_1(i_1) \circ \dots \circ A_N(i_N) v' \right\rangle = \\ &= \sum_{I \in [n]^N} T[I] \langle u', A_1(i_1) \circ \dots \circ A_N(i_N) v' \rangle = \\ &= \sum_{I \in [n]^N} T[I] \langle u', A'_1(i_1) \circ \dots \circ A'_N(i_N) v' \rangle. \end{aligned} \quad (7.6)$$

Define the complex number $\Theta = \langle u', Mv' \rangle$. Eq. (7.5) shows that to prove the theorem, it suffices to show that the inequality

$$|\Theta| \leq K^N \|T\|_\infty, \quad (7.7)$$

holds for some constant K , and Eq. (7.6) shows that we can write Θ using the matrix-valued maps A'_1, \dots, A'_N .

In the third step we absorb the complex part of the number Θ into the matrix-valued map A'_1 . Let us write Θ in polar coordinates as $|\Theta|e^{i\phi}$ for some $\phi \in [0, 2\pi]$. Define $A''_1(i_1) = e^{-i\phi} A'_1(i_1)$. Then by Eq. (7.6), we have

$$\sum_{I \in [n]^N} T[I] \langle u', A''_1(i_1) \circ A'_2(i_2) \circ \dots \circ A'_N(i_N) v' \rangle = |\Theta|. \quad (7.8)$$

In the fourth step we symmetrize the situation by making the matrices Hermitian. To this end, define the map $\rho : \mathbb{C}^{D \times D} \rightarrow \mathbb{C}^{2D \times 2D}$ by

$$\rho(A) = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

Define matrix-valued maps $B_1, \dots, B_N : [n] \rightarrow \mathbb{C}^{2D \times 2D}$ by

$$\begin{aligned} B_1(i_1) &= \rho(A''_1(i_1)) \\ B_2(i_2) &= \rho(A'_2(i_2)) \\ &\vdots \\ B_N(i_N) &= \rho(A'_N(i_N)). \end{aligned}$$

Note that $\|B_k(i_k)\| \leq 1$ for all $k = 1, \dots, N$ and $i_k = 1, \dots, n$, since the map ρ leaves the norm unchanged. Define the matrices

$$\begin{aligned} M' &= \sum_{I \in [n]^N} T[I] A''_1(i_1) \circ A'_2(i_2) \circ \dots \circ A'_N(i_N) \\ M'' &= \sum_{I \in [n]^N} T[I] B_1(i_1) \circ B_2(i_2) \circ \dots \circ B_N(i_N). \end{aligned}$$

Since the tensor T is real-valued we have $M'' = \rho(M')$.

Define the vector $w = (v' \oplus u')/\sqrt{2}$ and note that $\|w\| \leq 1$. We have

$$\begin{aligned}
 \langle w, M''w \rangle &= \frac{1}{2}[(u')^*, (v')^*] \begin{bmatrix} 0 & M' \\ (M')^* & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \\
 &= \Re(\langle u', M'v' \rangle) \\
 &= \Re \left(\sum_{I \in [n]^N} T[I] \langle u', A_1''(i_1) \circ \cdots \circ A_N'(i_N)v' \rangle \right) \\
 &= |\Theta|,
 \end{aligned} \tag{7.9}$$

where the last identity follows from Eq. (7.8), which shows that the term between brackets on the third line is the real number $|\Theta|$.

Next, we absorb the complex parts of the vector w into the matrix-valued map B_1 . Using polar coordinates we can write

$$w = \sum_{\ell=1}^{2D} w_\ell e^{i\psi_\ell} e_\ell$$

for some moduli $w_\ell \in \mathbb{R}_+$ and arguments $\psi_\ell \in [0, 2\pi]$. Let $U \in \mathbb{C}^{D \times D}$ be the diagonal unitary matrix given by $U = \text{diag}(e^{i\psi_1}, \dots, e^{i\psi_D})$. Define the non-negative real vector $w' = U^*w = \sum_{\ell=1}^{2D} w_\ell e_\ell$ and define the matrix-valued map B'_1 by $B'_1(i_1) = U^*B_1(i_1)U$. Note that $\|B'_1(i_1)\| \leq \|B_1(i_1)\| \leq 1$.

Then, by Eq. (7.9) and by expanding the definition of M'' we have

$$\sum_{I \in [n]^N} T[I] \langle w', B'_1(i_1) \circ B_2(i_2) \circ \cdots \circ B_N(i_N)w' \rangle = \langle w, M''w \rangle = |\Theta|. \tag{7.10}$$

We can now make a connection to Theorem 7.2.3 using the following two claims.

12. CLAIM. *There exist real numbers $\mu_1, \dots, \mu_{2D} \geq 0$ such that*

$$0 \leq \sum_{\ell, m=1}^{2D} \mu_\ell \mu_m \min\{\ell, m\} \leq 1 \tag{7.11}$$

and for $1_\ell = e_1 + \cdots + e_\ell$,

$$|\Theta| = \sum_{\ell, m=1}^{2D} \mu_\ell \mu_m \theta_{\ell, m}, \tag{7.12}$$

where

$$\theta_{\ell, m} = \sum_{I \in [n]^N} T[I] \langle 1_\ell, B'_1(i_1) \circ B_2(i_2) \circ \cdots \circ B_N(i_N)1_m \rangle.$$

PROOF: By relabeling the basis vectors e_1, \dots, e_{2D} appropriately, we may assume that the coefficients of the above vector w' satisfy $w_1 \geq w_2 \geq \dots \geq w_{2D}$. Setting $\mu_\ell = (w_\ell - w_{\ell-1})$ for $\ell = 1, \dots, 2D - 1$ and $\mu_{2D} = w_{2D}$ gives

$$w' = \sum_{\ell=1}^{2D} \mu_\ell 1_\ell,$$

since $\langle w', e_k \rangle = \mu_k + \mu_{k+1} + \dots + \mu_{2D} = w_k$. Eq. (7.11) follows from the fact that $0 \leq \langle w', w' \rangle \leq 1$ and $\langle 1_\ell, 1_m \rangle = \min\{\ell, m\}$, and Eq. (7.12) follows by expanding w' in Eq. (7.10). \square

13. CLAIM. For every $1 \leq \ell, m \leq 2D$, we have

$$|\theta_{\ell, m}| \leq C_N \min\{\ell, m\} \|T\|_\infty, \quad (7.13)$$

where $C_N = 2^{(N-2)/2} K_G$.

PROOF: Expanding the vectors 1_ℓ in the canonical basis gives

$$\begin{aligned} \langle 1_\ell, B'_1(i_1) \circ B_2(i_2) \circ \dots \circ B_N(i_N) 1_m \rangle = \\ \sum_{s=1}^{\ell} \sum_{t=1}^m \langle e_s, B'_1(i_1) \circ B_2(i_2) \circ \dots \circ B_N(i_N) e_t \rangle. \end{aligned} \quad (7.14)$$

Note that each term in the double sum on the right-hand side of Eq. (7.14) is simply the product of (s, t) -entries of the matrices $B'_1(i_1), B_2(i_2), \dots, B_N(i_N)$.

Suppose that $\ell \leq m$. Since the matrices $B'_1(i_1), B_2(i_2), \dots, B_N(i_N)$ have norm at most 1, their rows belong to $B_{\ell_2^m}$ (where ℓ_2^m is the set of length- m 2-summarable sequences). Hence, the inner sum on the right-hand side of Eq. (7.14),

$$\begin{aligned} \sum_{t=1}^m \langle e_s, B'_1(i_1) \circ B_2(i_2) \circ \dots \circ B_N(i_N) e_t \rangle = \\ \sum_{t=1}^m \langle e_s, B'_1(i_1) e_t \rangle \langle e_s, B_2(i_2) e_t \rangle \cdots \langle e_s, B_N(i_N) e_t \rangle, \end{aligned}$$

is the generalized inner product of a set of N vectors in $B_{\ell_2^m}$. The result for the case $\ell \leq m$ now follows from the triangle inequality and Theorem 7.2.3, as

$$\begin{aligned} |\theta_{\ell, m}| &= \left| \sum_{I \in [n]^N} T[I] \langle 1_\ell, B'_1(i_1) \circ B_2(i_2) \circ \dots \circ B_N(i_N) 1_m \rangle \right| \leq \\ &\sum_{s=1}^{\ell} \left| \sum_{I \in [n]^N} T[I] \sum_{t=1}^m \langle e_s, B'_1(i_1) e_t \rangle \langle e_s, B_2(i_2) e_t \rangle \cdots \langle e_s, B_N(i_N) e_t \rangle \right| \leq \\ &\ell 2^{(N-2)/2} K_G \|T\|_\infty. \end{aligned}$$

The case $\ell \geq m$ is proved in the same manner. ◆

Putting Claim 12 and Claim 13 together gives

$$\begin{aligned}
|\Theta| &= \sum_{\ell, m=1}^{2D} \mu_\ell \mu_m \theta_{\ell, m} \\
&\leq \sum_{\ell, m=1}^{2D} \mu_\ell \mu_m |\theta_{\ell, m}| \\
&\leq C_N \|T\|_\infty \sum_{\ell, m=1}^{2D} \mu_\ell \mu_m \min\{\ell, m\} \\
&\leq C_N \|T\|_\infty.
\end{aligned}$$

We conclude that Eq. (7.7) (Davie's criterion) holds for $K \leq 4$. □

7.2.1 The connection to the Schmidt states

For completeness, we now sketch the connection made in [PGWP⁺08] to the problem of determining whether Schmidt states, which are states of the form $|\psi\rangle = \sum_{\ell=1}^d \alpha_\ell |\ell\rangle^{\otimes N}$ for arbitrary real nonnegative coefficients α_i , allow for arbitrary large violation ratios in N -player XOR games.

The starting point is the last line of Eq. (7.6) in the proof above, which is of the form:

$$\sum_{I \in [n]^N} T[I] \langle u, A_1(i_1) \circ \cdots \circ A_N(i_N) v \rangle,$$

for some d -dimensional vectors u, v and matrices $A_\ell(i_\ell)$. Theorem 7.2.1 was proved by showing that the absolute value of this quantity is bounded from above by $K^N \text{OPT}^{\text{C}}(T)$ for some universal constant K .

Renaming the basis vectors e_1, \dots, e_d for \mathbb{C}^d as $|1\rangle, \dots, |d\rangle$ gives $u = \sum_{\ell=1}^d \alpha_\ell |\ell\rangle$ and $v = \sum_{\ell=1}^d \beta_\ell |\ell\rangle$. The crucial observation is now that for Schmidt states

$$|\psi\rangle = \sum_{\ell=1}^d \alpha_\ell |\ell\rangle^{\otimes N} \quad \text{and} \quad |\phi\rangle = \sum_{\ell=1}^d \beta_\ell |\ell\rangle^{\otimes N},$$

we have

$$\begin{aligned} \sum_{I \in [n]^N} T[I] \langle u, A_1(i_1) \circ \cdots \circ A_N(i_N) v \rangle &= \\ \sum_{I \in [n]^N} T[I] \sum_{\ell, m=1}^d \alpha_\ell^* \beta_m \langle \ell | A_1(i_1) \circ \cdots \circ A_N(i_N) | m \rangle &= \\ \sum_{I \in [n]^N} T[I] \langle \psi | A_1(i_1) \otimes \cdots \otimes A_N(i_N) | \phi \rangle. \end{aligned}$$

The form of the last quantity above and the fact that $|\psi\rangle$ and $|\phi\rangle$ are Schmidt states already give a strong indication that it cannot be far from the entangled bias of some N -player XOR game where the players are restricted to sharing a Schmidt state. As shown in [PGWP⁺08], this is indeed the case. The fact that this bias is at most K^N times larger than the classical bias (for some universal constant K), as stated in Theorem 6.2.1, then implies the required bound, as the classical bias of N -player XOR game (π, Σ) equals $\text{OPT}(\pi \circ \Sigma) \leq \text{OPT}^{\text{C}}(\pi \circ \Sigma)$.

7.3 The intermediate cases

It turns out that once the cases $p = 1$ and $p = \infty$ of Varopoulos's question are answered in the positive, the same results for intermediate ones $1 < p < \infty$ are obtained essentially for free. The reason for this comes from a pair of very useful results of Pietsch and Triebel [PT68] and Varopoulos [Var72], which give that the Banach algebras (S_p, \circ) can be characterized as algebras "between" (S_1, \circ) and (S_∞, \circ) . What is meant by "between" is that there is a way to obtain the spaces S_p for $1 < p < \infty$ by taking certain combinations of S_1 and S_∞ . This method is known as the *complex interpolation method*; we refer to Berg and Löfström [BL76] for a detailed account.

We give rough a description of what the complex interpolation method entails in the current setting. Consider the space \mathcal{F} of functions $f : \mathbb{C} \rightarrow S_\infty$ that are analytic in the open strip $\{0 < \Re(\zeta) < 1 : \zeta \in \mathbb{C}\}$ and continuous on the closed strip $\{0 \leq \Re(\zeta) \leq 1 : \zeta \in \mathbb{C}\}$ (additionally, the functions in \mathcal{F} have to approach 0 sufficiently rapidly when their argument moves away from the real line; see [BL76] for details). We endow \mathcal{F} with the norm

$$\|f\|_{\mathcal{F}} = \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{S_\infty}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{S_1} \right\}.$$

For $0 < \theta < 1$, the *interpolation space* $(S_\infty, S_1)_{[\theta]}$ is defined as the subset of elements $A \in S_\infty$ such that $A = f(\theta)$ for some $f \in \mathcal{F}$. The norm on this space

is defined by

$$\|A\|_{[\theta]} = \inf \{ \|f(\theta)\|_{\mathcal{F}} : f(\theta) = A, f \in \mathcal{F} \}.$$

Surprisingly, the p -Schatten spaces for the intermediate values $1 < p < \infty$ can be *characterized* in this way.

7.3.1. LEMMA (PIETSCH AND TRIEBEL). *For any $1 < p < \infty$, we have*

$$(S_{\infty}, S_1)_{[1/p]} = S_p.$$

Varopoulos [Var72] proved that the property of being a Q-algebra is inherited under the complex interpolation method. Specialized to the current setting, his result says the following.

7.3.2. LEMMA (VAROPOULOS). *If (S_1, \circ) and (S_{∞}, \circ) are Q-algebras, then for any value $0 < p < 1$, we have that $((S_{\infty}, S_1)_{[1/p]}, \circ)$ is a Q-algebra.*

Combining the above two lemmas with the result of Pérez-García [PG06] showing that (S_1, \circ) is a Q-algebra and Theorem 7.2.1 thus gives the following corollary, showing that Varopoulos's question is now completely answered.

7.3.3. COROLLARY. *For any $1 \leq p \leq \infty$, the Banach algebra (S_p, \circ) is a Q-algebra.*