Grothendieck inequalities, nonlocal games and optimization
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Appendix A

Some useful linear algebra and analysis

In this section, we provide some basic facts and definitions from linear algebra and analysis which are used in this thesis.

A.1 Vector spaces

**Euclidean vector spaces** Let $n$ be a positive integer. The vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ consist of column vectors of the form

$$
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix},
$$

where $x_1, \ldots, x_n$ are real or complex scalars, respectively. Addition and multiplication by scalars are defined by

$$
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} + \begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix} = \begin{pmatrix}
x_1 + y_1 \\
\vdots \\
x_n + y_n
\end{pmatrix}, \quad \alpha \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} = \begin{pmatrix}
\alpha x_1 \\
\vdots \\
\alpha x_n
\end{pmatrix}.
$$

**Transpose** The *transpose* of a vector $x$ in a Euclidean vector space, denoted $x^T$, is defined to be the row-vector $(x_1, \ldots, x_n)$.

**Conjugate transpose** The *conjugate transpose* of a vector $x$ in a complex Euclidean vector space, denoted $x^*$, is defined to be the row-vector $(\bar{x}_1, \ldots, \bar{x}_n)$.
Normed vector spaces  A norm on a vector space \( V \) is a function \( \| \cdot \| : V \to \mathbb{R} \) which satisfies for every \( x, y \in V \) and scalar \( \alpha \),

1. \( \| \alpha x \| = |\alpha| \| x \| \)
2. \( \| x \| = 0 \) if and only if \( x = 0 \)
3. \( \| x + y \| \leq \| x \| + \| y \| \)

The last property is referred to as the triangle inequality. A vector space endowed with a norm is a normed vector space.

The 2-norm on a Euclidean vector space is defined by \( \| x \|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2} \).

Inner product spaces  An inner product on a complex vector space \( V \) is a map of the form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) which satisfies for \( x, y, z \in V \) and scalar \( \alpha \),

1. \( \langle x, y \rangle = \overline{\langle y, x \rangle} \)
2. \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \)
3. \( \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \)
4. \( \langle x, x \rangle \geq 0 \)
5. \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \)

A vector space endowed with an inner product is an inner product space.

The Euclidean inner product on \( \mathbb{R}^n \) is by \( x \cdot y = x_1y_1 + \cdots + x_ny_n \). Using the transpose, this can also be denoted as \( x^T y \).

The Euclidean inner product on \( \mathbb{C}^n \) is defined by \( \langle x, y \rangle = \bar{x}_1y_1 + \cdots + \bar{x}_n y_n \). Using the conjugate transpose, this can also be written as \( x^* y \).

Metric spaces  For a vector space \( V \) a metric is a function \( d : V \times V \to \mathbb{R} \) which satisfies for any \( x, y, z \in V \),

1. \( d(x, y) \geq 0 \)
2. \( d(x, y) = 0 \) if and only if \( x = y \)
3. \( d(x, z) \leq d(x, y) + d(y, z) \)

The last property is also referred to as the triangle inequality. A vector space endowed with a metric is a metric space.
Hilbert spaces  Let $\mathcal{H}$ be an inner product space. We can make $\mathcal{H}$ into a normed vector space by endowing it with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. We can make $\mathcal{H}$ a metric space by endowing it with the metric $d(x, y) = \|x - y\|$. A sequence $(x_i)_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Cauchy sequence if for any $\epsilon > 0$ there is an integer $N$ such that $d(x_i, x_j) \leq \epsilon$ for all $i, j > N$. Then, we have that $\mathcal{H}$ is a Hilbert space if every Cauchy sequence converges to an element of $\mathcal{H}$ (i.e., if $\mathcal{H}$ is complete).

The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces when endowed with the Euclidean inner product. The Hilbert space $L^2([-1, 1])$ consists of the functions $f: [-1, 1] \to \mathbb{R}$ with finite norm, where the inner product is defined by

$$(f, g) = \int_{-1}^{1} f(t)g(t)dt.$$

Cauchy-Schwarz inequality  For Hilbert space $\mathcal{H}$, the Cauchy-Schwarz inequality states that for any $x, y \in \mathcal{H}$, we have $|\langle x, y \rangle| \leq \|x\|\|y\|$.

Continuous functions on metric spaces  Let $\mathcal{X}, \mathcal{Y}$ be metric spaces. A function $f: \mathcal{X} \to \mathcal{Y}$ is continuous if for any $\epsilon > 0$ there is a $\delta > 0$, such that for any $x, y \in \mathcal{X}$ satisfying $d_{\mathcal{X}}(x, y) < \delta$, we have $d_{\mathcal{Y}}(f(x), f(y)) < \epsilon$.

A.2 Matrices

Transpose  The transpose of a complex matrix $A \in \mathbb{C}^{n \times m}$ is the complex matrix $A^T \in \mathbb{C}^{m \times n}$ defined by $(A^T)_{ij} = A_{ji}$.

Conjugate transpose  The conjugate transpose of a complex matrix $A \in \mathbb{C}^{m \times n}$, denoted $A^*$, is the complex $n$-by-$m$ matrix defined by $(A^*)_{ij} = (A^*)_{ji}$.

Trace  The trace function $\text{Tr}: \mathbb{C}^{n \times n} \to \mathbb{C}$ is defined by $\text{Tr}(A) = A_{11} + \cdots + A_{nn}$.

Trace inner product  The trace inner product (also known as the Hilbert-Schmidt inner product) is an inner product on the vector space of matrices $\mathbb{C}^{n \times n}$ defined by $\langle A, B \rangle = \text{Tr}(A^*B)$. Endowed with this inner product, $\mathbb{C}^{n \times n}$ forms an $n^2$-dimensional Hilbert space.

Rank  The rank of a matrix is defined to be its largest number of linearly independent columns.
Outer product  The *outer product* of two vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ is the matrix $xy^* \in \mathbb{C}^{n \times m}$ given by $(xy^*)_{ij} = x_i \overline{y}_j$.

Hermitian matrix  A complex matrix $A \in \mathbb{C}^{n \times n}$ is *Hermitian* if $A^* = A$.

Unitary matrices  A complex matrix $U \in \mathbb{C}^{n \times n}$ is *unitary* if it satisfies

$$U^* U = I.$$  

Unitary matrices have the property that they preserve inner products between vectors. In fact, this property is equivalent to being unitary. For any pair of vectors $x, y \in \mathbb{C}^n$, we have $\langle Ux, Uy \rangle = \langle x, y \rangle$. It follows that unitary matrices are also norm-preserving: $\|Ux\| = \|x\|$.

Positive semidefinite matrices  A complex Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is *positive semidefinite* if one of the following holds.

1. The matrix $A$ has only real nonnegative eigenvalues.
2. There exist a complex $n$-dimensional vectors $z_1, \ldots, z_n$ such that for every $i, j \in \{1, \ldots, n\}$, we have $A_{ij} = z_i \cdot z_j$.
3. For any vector $z \in \mathbb{C}^n$, we have $z^* Az \geq 0$.
4. There exists a complex matrix $B$ such that $A = B^* B$.

In fact, Items 1-4 are equivalent (see for example [Bha07]). The factorization given in item 2 is called the *Gram decomposition* of $A$.

The set of positive semidefinite matrices forms a *convex cone*, meaning that for any $n$-by-$n$ positive semidefinite matrices $A, B$ and nonnegative scalars $\alpha, \beta \in \mathbb{R}_+$, we have that the matrix $\alpha A + \beta B$ is also positive semidefinite. Sometimes the notation $A \succeq 0$ will be used to denote that $A$ is positive semidefinite.

A positive semidefinite matrix $A$ satisfying $A^2 = A$ is an *orthogonal projector*. An orthogonal projector corresponds to a subspace of $\mathbb{C}^n$ defined by the space spanned by its nonzero eigenvectors.

In the case of real matrices, we have the following analogous characterization of positive semidefinite matrices. A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is *positive semidefinite* if one of the following holds.

1. The matrix $A$ has only real nonnegative eigenvalues.
2. There exist a real $n$-dimensional vectors $z_1, \ldots, z_n$ such that for every $i, j \in \{1, \ldots, n\}$, we have $A_{ij} = z_i \cdot z_j$.
3. For any vector \( z \in \mathbb{R}^n \), we have \( z^T A z \geq 0 \).

4. There exists a real matrix \( B \) such that \( A = B^T B \).

We denote the cone of real \( n \)-by-\( n \) positive semidefinite matrices by \( S_n^+ \).

The rank of a positive semidefinite matrix equals the smallest positive integer \( d \) such that there exists a Gram decomposition of it in \( \mathbb{R}^d \).

Laplacian matrices

Let \( G = (V, E) \) be a graph with finite vertex set \( V \) and edge set \( E \subseteq \binom{V}{2} \). Then, the Laplacian matrix of \( G \) is the matrix \( A : V \times V \to \mathbb{R} \) (this matrix has rows and columns indexed by the vertices of \( V \)) defined by

\[
A(u, v) = \begin{cases} 
\deg(u) & \text{if } v = u \\
-1 & \text{if } \{u, v\} \in E \\
0 & \text{otherwise},
\end{cases}
\]

where \( \deg(u) = |\{v \in V : \{u, v\} \in E\}| \) denotes the degree of vertex \( u \).

The Laplacian matrix of a graph is always a positive semidefinite matrix.

To see this, let \( G = (V, E) \) be some graph and let us define for each edge \( \{u, v\} \) in the graph the vector \( x_{uv} = e_u - e_v \), where the \( e_u \) are the \( |V| \)-dimensional canonical unit vectors and the choice of which of the two unit vectors in \( x_{uv} \) is subtracted from the other is arbitrary. Then, we have that the matrix

\[
A = \sum_{\{u,v\} \in E} x_{uv} x_{uv}^T
\]

satisfies

\[
A(u, v) = e_u^T \sum_{\{u',v'\} \in E} (e_{u'} - e_v)(e_{u'} - e_{v'})^T e_v
\]

\[
= \begin{cases} 
\deg(u) & \text{if } v = u \\
-1 & \text{if } \{u, v\} \in E \\
0 & \text{otherwise}.
\end{cases}
\]

Hence, \( A \) is the Laplacian matrix of \( G \). This matrix is positive semidefinite because it is a positive linear combination of the rank-1 positive semidefinite matrices \( x_{uv} x_{uv}^T \).

A.3 Tensor products

If \( \mathcal{X} = \mathbb{C}^{n_1 \times m_1} \) and \( \mathcal{Y} = \mathbb{C}^{n_2 \times m_2} \) then the tensor product of the vector spaces \( \mathcal{X} \) and \( \mathcal{Y} \) is defined as \( \mathcal{X} \otimes \mathcal{Y} = \mathbb{C}^{n_1 n_2 \times m_1 m_2} \).
To define the tensor product of complex matrices it is convenient to index the rows and columns of a matrix by sets $\mathcal{R}$ and $\mathcal{C}$, respectively, and view the matrix as a map from $\mathcal{R} \times \mathcal{C}$ to $\mathbb{C}$. An $n$-by-$m$ matrix $A$ is thus viewed as a map $A : \{1, \ldots, n\} \times \{1, \ldots, m\} \to \mathbb{C}$ and its $(i, j)$-entry is written as $A(i, j)$.

Let $\mathcal{R}_1, \mathcal{C}_1$ and $\mathcal{R}_2, \mathcal{C}_2$ be sets and let $A : \mathcal{R}_1 \times \mathcal{C}_1 \to \mathbb{C}$ and $B : \mathcal{R}_2 \times \mathcal{C}_2 \to \mathbb{C}$ be complex matrices. Then, their tensor product is the matrix

$$A \otimes B : (\mathcal{R}_1 \times \mathcal{R}_2) \times (\mathcal{C}_1 \times \mathcal{C}_2) \to \mathbb{C}$$

is defined by

$$(A \otimes B)((r_1, r_2), (c_1, c_2)) = A(r_1, c_1)B(r_2, c_2).$$

It follows easily that the tensor product satisfies for any matrices $A, B, C, D$:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

where for the last identity we assumed that $A$ and $C$ have equal size and that $B$ and $D$ have equal size.

We also have for $x_1, y_1 \in \mathbb{C}^n$ and $x_2, y_2 \in \mathbb{C}^m$, the easy identity

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle.$$

### A.4 Dirac notation

Dirac notation refers to a notational convention used for the Hilbert space $\mathbb{C}^n$ in the context of quantum information theory. Vectors are usually denoted by a Greek symbol or a non-negative integer wedged between a “$|$” and a “$\rangle$”. We thus write for example $|\psi \rangle \in \mathbb{C}^n$ or $|1\rangle \in \mathbb{C}^n$. The non-negative integers are reserved for the canonical basis vectors, that is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad |n-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The conjugate transpose of a vector $|\psi \rangle \in \mathbb{C}^n$ is denoted by $\langle \psi |$. Usually the tensor product symbol is omitted when we take the tensor product of two vectors $|\psi \rangle$ and $|\phi \rangle$. So $|\psi \rangle \otimes |\phi \rangle$ is abbreviated to $|\psi \rangle |\phi \rangle$. 

Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$ be matrices and let $|\psi\rangle \in \mathbb{C}^n$ and $|\phi\rangle \in \mathbb{C}^m$ be vectors. It follows easily from the properties of the tensor product that

$$\langle \psi | (|\phi\rangle \otimes |\psi\rangle) \langle \phi | A \otimes B \langle \psi | \rangle$$

$$= \langle \psi | A \langle \psi | \rangle \langle \phi | B \langle \phi | \rangle.$$