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Grothendieck inequalities, nonlocal games and optimization

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Appendix A

Some useful linear algebra and analysis

In this section, we provide some basic facts and definitions from linear algebra and analysis which are used in this thesis.

A.1 Vector spaces

Euclidean vector spaces Let n be a positive integer. The vector spaces \mathbb{R}^n and \mathbb{C}^n consist of column vectors of the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where x_1, \dots, x_n are real or complex scalars, respectively. Addition and multiplication by scalars are defined by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}.$$

Transpose The *transpose* of a vector x in a Euclidean vector space, denoted x^T , is defined to be the row-vector (x_1, \dots, x_n) .

Conjugate transpose The *conjugate transpose* of a vector x in a complex Euclidean vector space, denoted x^* , is defined to be the row-vector $(\bar{x}_1, \dots, \bar{x}_n)$.

Normed vector spaces A *norm* on a vector space \mathcal{V} is a function $\| \cdot \| : \mathcal{V} \rightarrow \mathbb{R}$ which satisfies for every $x, y \in \mathcal{V}$ and scalar α ,

1. $\|\alpha x\| = |\alpha| \|x\|$
2. $\|x\| = 0$ if and only $x = 0$
3. $\|x + y\| \leq \|x\| + \|y\|$

The last property is referred to as the *triangle inequality*. A vector space endowed with a norm is a *normed vector space*.

The 2-norm on a Euclidean vector space is defined by

$$\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}.$$

Inner product spaces An *inner product* on a complex vector space \mathcal{V} is a map of the form $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ which satisfies for $x, y, z \in \mathcal{V}$ and scalar α ,

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
3. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0$ if and only $x = 0$

A vector space endowed with an inner product is an *inner product space*.

The *Euclidean inner product* on \mathbb{R}^n is by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. Using the transpose, this can also be denoted as $x^T y$.

The *Euclidean inner product* on \mathbb{C}^n is defined by $\langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$. Using the conjugate transpose, this can also be written as $x^* y$.

Metric spaces For a vector space \mathcal{V} a *metric* is a function $d : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ which satisfies for any $x, y, z \in \mathcal{V}$,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, z) \leq d(x, y) + d(y, z)$

The last property is also referred to as the *triangle inequality*. A vector space endowed with a metric is a *metric space*.

Hilbert spaces Let \mathcal{H} be an inner product space. We can make \mathcal{H} into a normed vector space by endowing it with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. We can make \mathcal{H} a metric space by endowing it with the metric $d(x, y) = \|x - y\|$. A sequence $(x_i)_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there is an integer N such that $d(x_i, x_j) \leq \varepsilon$ for all $i, j > N$. Then, we have that \mathcal{H} is a *Hilbert space* if every Cauchy sequence converges to an element of \mathcal{H} (i.e., if \mathcal{H} is *complete*).

The Euclidean spaces \mathbb{R}^n and \mathbb{C}^n are Hilbert spaces when endowed with the Euclidean inner product. The Hilbert space $L^2([-1, 1])$ consists of the functions $f : [-1, 1] \rightarrow \mathbb{R}$ with finite norm, where the inner product is defined by

$$(f, g) = \int_{-1}^1 f(t)g(t)dt.$$

Cauchy-Schwarz inequality For Hilbert space \mathcal{H} , the *Cauchy-Schwarz inequality* states that for any $x, y \in \mathcal{H}$, we have $|\langle x, y \rangle| \leq \|x\|\|y\|$.

Continuous functions on metric spaces Let \mathcal{X}, \mathcal{Y} be metric spaces. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is *continuous* if for any $\varepsilon > 0$ there is a $\delta > 0$, such that for any $x, y \in \mathcal{X}$ satisfying $d_{\mathcal{X}}(x, y) < \delta$, we have $d_{\mathcal{Y}}(f(x), f(y)) < \varepsilon$.

A.2 Matrices

Transpose The *transpose* of a complex matrix $A \in \mathbb{C}^{n \times m}$ is the complex matrix $A^T \in \mathbb{C}^{m \times n}$ defined by $(A^T)_{ij} = A_{ji}$.

Conjugate transpose The *conjugate transpose* of a complex matrix $A \in \mathbb{C}^{m \times n}$, denoted A^* , is the complex n -by- m matrix defined by $(A^*)_{ij} = A_{ji}^*$.

Trace The *trace function* $\text{Tr} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}$ is defined by $\text{Tr}(A) = A_{11} + \cdots + A_{nn}$.

Trace inner product The *trace inner product* (also known as the Hilbert-Schmidt inner product) is an inner product on the vector space of matrices $\mathbb{C}^{n \times n}$ defined by $\langle A, B \rangle = \text{Tr}(A^*B)$. Endowed with this inner product, $\mathbb{C}^{n \times n}$ forms an n^2 -dimensional Hilbert space.

Rank The *rank* of a matrix is defined to be its largest number of linearly independent columns.

Outer product The *outer product* of two vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ is the matrix $xy^* \in \mathbb{C}^{n \times m}$ given by $(xy^*)_{ij} = x_i \bar{y}_j$.

Hermitian matrix A complex matrix $A \in \mathbb{C}^{n \times n}$ is *Hermitian* if $A^* = A$.

Unitary matrices A complex matrix $U \in \mathbb{C}^{n \times n}$ is *unitary* if it satisfies

$$U^*U = I.$$

Unitary matrices have the property that they preserve inner products between vectors. In fact, this property is equivalent to being unitary. For any pair of vectors $x, y \in \mathbb{C}^n$, we have $\langle Ux, Uy \rangle = \langle x, y \rangle$. It follows that unitary matrices are also norm-preserving: $\|Ux\| = \|x\|$.

Positive semidefinite matrices A complex Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is *positive semidefinite* if one of the following holds.

1. The matrix A has only real nonnegative eigenvalues.
2. There exist a complex n -dimensional vectors z_1, \dots, z_n such that for every $i, j \in \{1, \dots, n\}$, we have $A_{ij} = z_i \cdot z_j$.
3. For any vector $z \in \mathbb{C}^n$, we have $z^*Az \geq 0$.
4. There exists a complex matrix B such that $A = B^*B$.

In fact, Items 1-4 are equivalent (see for example [Bha07]). The factorization given in item 2 is called the *Gram decomposition* of A .

The set of positive semidefinite matrices forms a *convex cone*, meaning that for any n -by- n positive semidefinite matrices A, B and nonnegative scalars $\alpha, \beta \in \mathbb{R}_+$, we have that the matrix $\alpha A + \beta B$ is also positive semidefinite. Sometimes the notation $A \succeq 0$ will be used to denote that A is positive semidefinite.

A positive semidefinite matrix A satisfying $A^2 = A$ is an *orthogonal projector*. An orthogonal projector corresponds to a subspace of \mathbb{C}^n defined by the space spanned by its nonzero eigenvectors.

In the case of real matrices, we have the following analogous characterization of positive semidefinite matrices. A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is *positive semidefinite* if one of the following holds.

1. The matrix A has only real nonnegative eigenvalues.
2. There exist a real n -dimensional vectors z_1, \dots, z_n such that for every $i, j \in \{1, \dots, n\}$, we have $A_{ij} = z_i \cdot z_j$.

3. For any vector $z \in \mathbb{R}^n$, we have $z^T A z \geq 0$.
4. There exists a real matrix B such that $A = B^T B$.

We denote the cone of real n -by- n positive semidefinite matrices by \mathcal{S}_n^+ .

The rank of a positive semidefinite matrix equals the smallest positive integer d such that there exists a Gram decomposition of it in \mathbb{R}^d .

Laplacian matrices Let $G = (V, E)$ be a graph with finite vertex set V and edge set $E \subseteq \binom{V}{2}$. Then, the *Laplacian matrix* of G is the matrix $A : V \times V \rightarrow \mathbb{R}$ (this matrix has rows and columns indexed by the vertices of V) defined by

$$A(u, v) = \begin{cases} \deg(u) & \text{if } v = u \\ -1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(u) = |\{v \in V : \{u, v\} \in E\}|$ denotes the *degree* of vertex u .

The Laplacian matrix of a graph is always a positive semidefinite matrix. To see this, let $G = (V, E)$ be some graph and let us define for each edge $\{u, v\}$ in the graph the vector $x_{uv} = e_u - e_v$, where the e_u are the $|V|$ -dimensional canonical unit vectors and the choice of which of the two unit vectors in x_{uv} is subtracted from the other is arbitrary. Then, we have that the matrix

$$A = \sum_{\{u, v\} \in E} x_{uv} x_{uv}^T$$

satisfies

$$\begin{aligned} A(u, v) &= e_u^T \sum_{\{u', v'\} \in E} (e_{u'} - e_{v'}) (e_{u'} - e_{v'})^T e_v \\ &= \begin{cases} \deg(u) & \text{if } v = u \\ -1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, A is the Laplacian matrix of G . This matrix is positive semidefinite because it is a positive linear combination of the rank-1 positive semidefinite matrices $x_{uv} x_{uv}^T$.

A.3 Tensor products

If $\mathcal{X} = \mathbb{C}^{n_1 \times m_1}$ and $\mathcal{Y} = \mathbb{C}^{n_2 \times m_2}$ then the tensor product of the vector spaces \mathcal{X} and \mathcal{Y} is defined as $\mathcal{X} \otimes \mathcal{Y} = \mathbb{C}^{n_1 n_2 \times m_1 m_2}$.

To define the tensor product of complex matrices it is convenient to index the rows and columns of a matrix by sets \mathcal{R} and \mathcal{C} , respectively, and view the matrix as a map from $\mathcal{R} \times \mathcal{C}$ to \mathbb{C} . An n -by- m matrix A is thus viewed as a map $A : \{1, \dots, n\} \times \{1, \dots, m\} \rightarrow \mathbb{C}$ and its (i, j) -entry is written as $A(i, j)$

Let $\mathcal{R}_1, \mathcal{C}_1$ and $\mathcal{R}_2, \mathcal{C}_2$ be sets and let $A : \mathcal{R}_1 \times \mathcal{C}_1 \rightarrow \mathbb{C}$ and $B : \mathcal{R}_2 \times \mathcal{C}_2 \rightarrow \mathbb{C}$ be complex matrices. Then, their *tensor product* is the matrix

$$A \otimes B : (\mathcal{R}_1 \times \mathcal{R}_2) \times (\mathcal{C}_1 \times \mathcal{C}_2) \rightarrow \mathbb{C}$$

is defined by

$$(A \otimes B)((r_1, r_2), (c_1, c_2)) = A(r_1, c_1)B(r_2, c_2).$$

It follows easily that the tensor product satisfies for any matrices A, B, C, D :

$$\begin{aligned} (A \otimes B) \otimes C &= A \otimes (B \otimes C) \\ A \otimes (B + C) &= A \otimes B + A \otimes C \\ (A \otimes B)(C \otimes D) &= (AC) \otimes (BD), \end{aligned}$$

where for the last identity we assumed that A and C have equal size and that B and D have equal size.

We also have for $x_1, y_1 \in \mathbb{C}^n$ and $x_2, y_2 \in \mathbb{C}^m$, the easy identity

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle.$$

A.4 Dirac notation

Dirac notation refers to a notational convention used for the Hilbert space \mathbb{C}^n in the context of quantum information theory. Vectors are usually denoted by a Greek symbol or a non-negative integer wedged between a “|” and a “⟩”. We thus write for example $|\psi\rangle \in \mathbb{C}^n$ or $|1\rangle \in \mathbb{C}^n$. The non-negative integers are reserved for the canonical basis vectors, that is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad |n-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The conjugate transpose of a vector $|\psi\rangle \in \mathbb{C}^n$ is denoted by $\langle\psi|$. Usually the tensor product symbol is omitted when we take the tensor product of two vectors $|\psi\rangle$ and $|\phi\rangle$. So $|\psi\rangle \otimes |\phi\rangle$ is abbreviated to $|\psi\rangle|\phi\rangle$.

Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$ be matrices and let $|\psi\rangle \in \mathbb{C}^n$ and $|\phi\rangle \in \mathbb{C}^m$ be vectors. It follows easily from the properties of the tensor product that

$$\langle\psi|\langle\phi|A \otimes B|\psi\rangle|\phi\rangle = \langle\psi|A|\psi\rangle\langle\phi|B|\phi\rangle.$$