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Enhancing the Performance of EWMA Charts

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Control charts are extensively used in processes and are very helpful in determining the special cause variations so that a timely action may be taken to eliminate them. One of the charting procedures is the Shewhart-type control charts, which are used mainly to detect large shifts. Two alternatives to the Shewhart-type control charts are the cumulative (CUSUM) control charts and the exponentially weighted moving average (EWMA) control charts that are specially designed to detect small and moderately sustained changes in quality. Enhancing the ability of design structures of control charts is always desirable and one may do it in different ways. In this article, we propose two runs rules schemes to be applied on EWMA control charts and evaluate their performance in terms of the Average Run Length (*ARL*). Comparisons of the proposed schemes are made with some existing representative CUSUM and EWMA-type counterparts used for small and moderate shifts, including the classical EWMA, the classical CUSUM, the fast initial response CUSUM and EWMA, the weighted CUSUM, the double CUSUM, the distribution-free CUSUM and the runs rules schemes-based CUSUM. The findings of the study reveal that the proposed schemes are able to perform better than all the other schemes under investigation. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: Average Run Length (*ARL*); control chart; CUSUM; in-control; out-of-control; runs rules

1. Introduction

Production processes contain variations in their output. These variations are classified into two categories named as common cause variation and assignable cause variation. If there are only common cause variations in a process, then there is no need to search for the assignable cause(s) and the process is said to be statistically in-control; otherwise, it is declared to be out-of-control. The common cause variation or 'background noise' is due to small and unavoidable causes, whereas the other variation is always due to some assignable cause(s), and we need to search for that assignable cause(s) in order to remove it. Statistical Process Control (SPC) contains some powerful tools that are helpful in differentiating between the two types of variations. These tools are formally referred as the SPC tool box in the literature and it mainly contains seven tools, often called as the 'magnificent seven' (cf. Montgomery[1]). The most important and commonly used tool of the SPC tool box is the Shewhart control chart which was originated by Walter A. Shewhart in 1920s. A control chart is a trend chart with three additional lines. Two of them are called control limts and are placed (or estimated) at plus or minus three times the standard deviation of the plotted statistic above and below a third line, which is placed at the mean of the statistic and is called the center line (CL). A graphical display of a typical control chart is presented in Figure 1 which has sample numbers (taken over time) on the horizontal axis and some quality characteristic(s) (or some sample statistic of it) on the vertical axis.

From Figure 1 we can see that a control chart has mainly three parameters named as the lower control limit (*LCL*), the *CL* and the upper control limit (*UCL*). *LCL* and *UCL* are selected such that almost all the data points fall between these limits as long as the process is in-control. Assuming that the statistic of interest is normally distributed with known mean or process target value μ_o and known standard deviation σ_o, *CL*, *LCL* and *UCL* for the two-sided standard Shewhart $\overline{X}$ control chart are constructed as:

$$LCL=\mu_o-L\frac{\sigma_o}{\sqrt{n}},\quad CL=\mu_o,\quad UCL=\mu_o+L\frac{\sigma_o}{\sqrt{n}} \tag{1}$$

In (1), n is the sample size and L represents the control limit coefficient which is selected according to the pre-specified false alarm rate or pre-specified in-control average run length (*ARL*). The false alarm rate is the probability of committing a type I error, i.e. probability of concluding that the process is out-of-control when the process is actually in-control, and is denoted by α.

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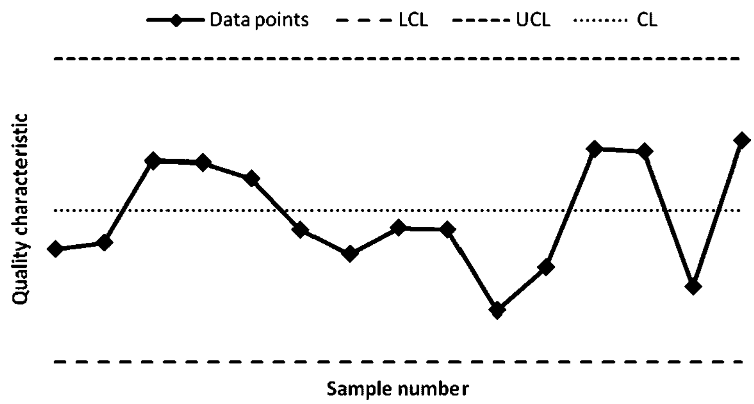

Figure 1. Graphical display of a control chart

ARL is the average number of samples/time periods we have to wait to get an out-of control signal. The in-control *ARL* value is denoted by ARL_0, whereas the out-of-control *ARL* value is denoted by ARL_1.

Shewhart-type control charts make the decision of whether the process is stable based on only the most recent information, unless supplementary rules are used. In contrast the cumulative sum (CUSUM) control charts (cf. Page[2]) and the exponentially weighted moving average (EWMA) control charts (cf. Roberts[3]) are based (in different) ways on past information along with current. Owing to this feature, these charts are more efficient to detect small and moderate shifts. We are mainly concerned with the EWMA control charts in this study; hence, our discussion will be focused on the EWMA control charts, particularly for the location parameter with individual observations.

Referring to Figure 1, we have to examine the plotted data points against the control limits and as long as the plotted points fall inside the control limits we conclude that the process is statistically in-control and otherwise out-of control. This is only one rule of deciding the control situation of a process that is based on a single point falling outside the control limits. However, this rule cannot indicate the out-of-control situation if there is any non-random pattern in the data or some points fall close to the control limits or some points fall in particular zones, which ultimately results into loss of efficiency of control charts for small shifts (cf. Klein[4]). Klein[4] introduced two-runs rules schemes named as two out of two ($\frac{2}{2}$) scheme and two out of three ($\frac{2}{3}$) scheme. An out-of-control signal is given by ($\frac{2}{2}$) scheme if two consecutive points are plotted beyond a specially set control limit which helps fixing the ARL_0 at a pre-specified level. The other scheme ($\frac{2}{3}$) gives an out-of-control signal if two out of three consecutive points are plotted beyond a different specially set control limit obtained on the similar pattern as that of the ($\frac{2}{2}$) scheme. Khoo[5] preceded this idea to more schemes such as $\frac{2}{2}$, $\frac{2}{3}$, $\frac{2}{4}$, $\frac{3}{3}$ and $\frac{3}{4}$ to increase the sensitivity of control charts. The control limits for all these schemes are set differently and are determined on the basis of a pre-specified false alarm rate or ARL_0. Antzoulakos and Rakitzis[6] generalized these rules to the modified *r* out of *m* control chart keeping the ARL_0 value at a pre-specified level. In the literature mentioned, only applications of the runs rules schemes were made with the Shewhart-type control charts. Riaz *et al.* [7] proposed the use of runs rules schemes with the CUSUM charts to enhance their performance. In this article, we introduce the use of these runs rules schemes with the design structure of EWMA control charts.

The organization of the rest of the paper is as follows: Section 2 consists of an introduction of the basic structure of EWMA control charts. The definitions and the concepts of the proposed schemes are provided in Section 3. In Section 4, we evaluate the performance of these proposed schemes. Comparisons of the proposed schemes with the existing control charts designed for the small shifts is presented in Section 5. Two illustrative examples are given in Section 6 to explain the procedure of the proposed schemes in practice. Finally, Section 7 concludes the findings of our study.

2. The classical EWMA control charts

The EWMA control chart was introduced by Roberts[3] to particularly address the shifts of smaller and moderate magnitude. The EWMA-type charts perform better than the Shewhart-type charts for small and moderate shifts introduced in the process. The plotting statistic of the EWMA control chart is a weighted combination of the current and past information and is defined as:

$$Z_i = \lambda X_i + (1-\lambda) Z_{i-1} \tag{2}$$

where X_i is the current information (for $i = 1, 2, \ldots$), Z_{i-1} is the past information and λ is the smoothing constant lying between 0 and 1 (i.e. $0 < \lambda \leq 1$). An alternative form of the EWMA statistic given in (2) can be written as:

$$Z_i = \sum_{j=0}^{i-1} \lambda (1-\lambda)^j X_{i-j} + (1-\lambda)^i Z_0 \tag{3}$$

Table I. *ARL* values for the classical EWMA scheme at $ARL_o=500$

δ	$\lambda=0.1$, $L=2.814$	$\lambda=0.25$, $L=2.998$	$\lambda=0.5$, $L=3.071$	$\lambda=0.75$, $L=3.087$
0	500	500	500	500
0.25	106	170	255	321
0.5	31.3	48.2	88.8	140
0.75	15.9	20.1	35.9	62.5
1	10.3	11.1	17.5	30.6
1.5	6.09	5.46	6.53	9.90
2	4.36	3.61	3.63	4.54

The weights $\lambda(1-\lambda)^j$ in (3) decrease exponentially as the sample observations become less recent. λ is the parameter of EWMA-type control charts which makes it a generalized form of the Shewhart-type charts (for $\lambda=1$, the EWMA-type chart becomes equivalent to the Shewhart-type chart). It is actually the sensitivity parameter of the EWMA control chart and for the smaller values of λ, these charts become more sensitive for the smaller shifts. The initial value for the past information, i.e. Z_o, is taken equal to the target mean μ_o. If we do not have information available about the target mean, then it can be estimated by the average of preliminary data. The mean and variance of the EWMA statistic are given as:

$$Mean(Z_i)=\mu_o, \quad Variance(Z_i)=\sigma^2\left\{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})\right\} \tag{4}$$

where σ^2 is the process variance which may have a known value (σ_o^2) or has to be estimated from initial in-control process samples. We continue with the case of a known parameter. Based on the above results, the control structure of an EWMA control chart is given as:

$$LCL=\mu_o-L\sigma\sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})}, \quad CL=\mu_o, \quad UCL=\mu_o+L\sigma\sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})} \tag{5}$$

All the terms used in (5) are defined as earlier. L determines the width of the control limits and its value is chosen according to the choice of the smoothing constant λ and the prefixed ARL_o value. The above-mentioned limits given in (5) are called time-varying limits of the EWMA charts. For large values of i, these limits converge to the constant limits which are given as:

$$LCL=\mu_o-L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \quad CL=\mu_o, \quad UCL=\mu_o+L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{6}$$

Hence, the factor $(1-(1-\lambda)^{2i})$ in (5) tends to 1 as the sample number becomes large and ultimately the time variant limits will become constant. In this article, we will use the time variant limits so that the exact width of the control limits at each sample point is utilized and we will refer it as the classical EWMA control chart in the sequel.

Like the Shewhart $\overline{X}$ charts, the EWMA chart is also based on the normality of the quality characteristic under study but it is not very sensitive to the normality assumption which makes the EWMA chart a robust control chart for quality characteristics which are not normally distributed (cf. Montgomery[1]). Lucas and Saccucci[8] presented a method for studying the run length distribution of the EWMA control charts. They provided a complete *ARL* study of the classical EWMA control chart with different combinations of L and λ and these are given here in Table I for some selective choices of δ, λ and L at $ARL_o=500$, where δ represents the amount of shift in $\sigma/\sqrt{n}$ units throughout this article.

The performance of the classical EWMA control charts can be further improved by different techniques and few attempts have been made in this direction. Lucas and Saccucci[8] proposed the fast initial response (FIR) feature for the EWMA chart. This feature substantially improves the ARL_1 performance but at the cost of a decrease in the ARL_o of the EWMA control chart, which is not recommended by the practitioners. The decision rule for different approaches of EWMA control charts (like the classical EWMA and FIR-based EWMA) is based on one point plotted outside the control limits (i.e. a first sensitizing rule and symbolically we can call it $\frac{1}{1}$ scheme). It is hard to find articles on EWMA-type control charts that exploit extra sensitizing rules (or the runs rules schemes) in order to enhance the performance of their design structure. The application of these extra rules or schemes is generally restricted to the Shewhart-type control charts (cf. References[4–6]).

Recently, Riaz *et al.*[7] proposed the use of these rules or schemes with the CUSUM charts and enhanced their detection ability. Following their proposals we propose, in this article, the application of these runs rules schemes on EWMA control charts in order to enhance the performance of their design structure. The details regarding the proposed schemes for EWMA charts are provided in the next section.

3. Proposed schemes for the EWMA chart

Shewhart control charts are good in detecting the larger disturbances in the process, but it takes too long for Shewhart-type charts to detect a small or moderate shift. To overcome this problem, some sensitizing rules are designed but their implementation inflates the pre-specified false alarm rate. This issue may be resolved by the introduction of the runs rules schemes. The authors in References[4–6] presented runs rules schemes applied on the Shewhart control charts to enhance their performance for small and moderate shifts, keeping the false alarm rate at the pre-specified level. The application of these runs rules schemes is not commonly used with the CUSUM and EWMA control charts. However, Riaz *et al.*[7] presented two-runs rules schemes for the CUSUM charts and have shown that their proposed schemes perform better for small and moderate shifts while they reasonably maintain their efficiency for large shifts as well. Taking inspiration from their work, we propose two new schemes in this article for the design structure of the EWMA control chart named as 'simple $\frac{2}{2}$ EWMA scheme' and 'modified $\frac{2}{3}$ EWMA scheme'. The procedural and conceptual framework of these two proposed schemes is defined as:

Proposed Scheme I (*Simple $\frac{2}{2}$ EWMA Scheme*): According to the simple $\frac{2}{2}$ EWMA scheme, a process is said to be out-of-control if two consecutive points are plotted either below a lower signaling limit (*LSL*) or above an upper signaling limit (*USL*).

Proposed Scheme II (*Modified $\frac{2}{3}$ EWMA Scheme*): According to the modified $\frac{2}{3}$ EWMA scheme, a process is said to be out-of-control if one of the following two conditions is satisfied:

(1) At least two out of three consecutive points fall below an *LSL* and the point above the *LSL* (if any) falls between the *CL* and the *LSL*.
(2) At least two out of three consecutive points fall above a *USL* and the point below the *USL* (if any) falls between *CL* and the *USL*.

The signaling limits *LSL* and *USL* mentioned above in the definitions of our two proposals are specially set limits chosen for the two schemes separately depending upon the desired ARL_O. The control structure for the proposed schemes is given as under:

$$LSL=\mu_O-L_S\sigma\sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})},\quad CL=\mu_O,\quad USL=\mu_O+L_S\sigma\sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2i})} \tag{7}$$

where L_S is the signaling limit coefficient of the proposed schemes and the other terms are as defined in Section 2. It has to be mentioned that the above mentioned signaling limits coefficient L_S is set according to the pre-specified value of ARL_O. Moreover, a signaling limit on either side may be split into two lines (as is done in Reference[7]) to reach at some optimum pair. We opted the choice where the outer split of the line is taken at infinity. However, one may take some different appropriately chosen outer split other than infinity.

The parameters of these two proposed schemes are the central line and two signaling limits as given in (7) (i.e. *CL*, *LSL* and *USL*). The upper and lower signaling limits are symmetric around the central line and will vary according to the pre-specified ARL_O. Using the positive relation between *ARL* and width of the signaling limits (depending upon L_S), we fix ARL_O at the desired level and find the corresponding pair of symmetric signaling limits. Based on these specially set signaling limits, we carry out our ARL_1 study at the desired ARL_O values.

The calculation of *ARL* may be carried out using different approaches such as integral equations, Markov Chains, approximations and Monte Carlo simulations. We have chosen to use Monte Carlo simulations to obtain *ARL* values. The simulation algorithms for the calculation of *ARL* values of the two proposed EWMA schemes are developed in EXCEL using an Add-In feature *MCSim*.

4. Performance evaluation of the proposed schemes

To evaluate the performance of a control chart, there are a number of criteria such as power, *ARL*, Average Time to Signal. We have chosen the most popular one as performance measure for our proposals and that is *ARL*. To investigate the performance of our proposed EWMA schemes, we have considered different in-control and out-of-control situations.

A suitable number of samples (say 100 000) of a fixed size *n* are generated from $N(\mu+\delta\sigma,\sigma)$ where μ and σ are the process mean and standard deviation, respectively, and δ is the amount of shift in the process (note that $\delta=0$ indicates the in-control state, whereas $\delta\neq 0$ refers to the out-of-control state). The EWMA statistics for these samples are then calculated and the conditions of the two proposed EWMA schemes (as listed in Section 3) are applied on them using the signaling limits given in (7) through our simulation algorithm. By executing this process repeatedly, we obtain different run length values which ultimately help computing *ARL* values and other properties as well. It is to be noted that the value of L_S is worked out such that the desired ARL_O value is achieved. For $\delta=0$, the ARL_O values are evaluated with the help of their corresponding L_S and then for $\delta\neq 0$ the ARL_1 values are computed by introducing different shifts in the process.

To evaluate the performance of the two proposals we fix the pre-specified ARL_O values, in this article, at 168, 200 and 500. These choices will suffice to exhibit the behavior of our proposed schemes and will enable us to make valid comparisons with their already existing counterparts. On similar lines, other choices of ARL_O can also be obtained. By fixing the ARL_O values at the above-mentioned levels (using their corresponding L_S), we have obtained the ARL_1 at different values of δ. These ARL_1 values are provided in Tables II–VII for the aforementioned desired ARL_O preferences (along with their corresponding L_S values) at different choices of λ.

Table II. *ARL* values for the proposed scheme I at $ARL_O = 168$

δ	$\lambda = 0.1$, $L_S = 2.145$	$\lambda = 0.25$, $L_S = 2.184$	$\lambda = 0.5$, $L_S = 2.034$	$\lambda = 0.75$, $L_S = 1.83$
0	169.8676	169.4769	169.6763	169.7112
0.25	54.5771	73.4836	94.8246	110.7766
0.5	19.8026	26.6284	37.9883	49.063
0.75	10.5927	12.9547	17.408	23.2892
1	6.9435	7.9456	9.8833	13.0739
1.5	4.1117	4.3476	4.7293	5.5777
2	2.9796	3.0954	3.1231	3.3368

Table III. *ARL* values for the proposed scheme II at $ARL_O = 168$

δ	$\lambda = 0.1$, $L_S = 1.807$	$\lambda = 0.25$, $L_S = 1.936$	$\lambda = 0.5$, $L_S = 1.85$	$\lambda = 0.75$, $L_S = 1.67$
0	167.3173	169.9927	168.5416	170.9464
0.25	34.367	43.3236	55.4569	62.6484
0.5	14.0389	17.8611	23.0836	27.6195
0.75	8.1338	9.4968	11.9229	14.2207
1	5.7064	6.4798	7.6037	8.5804
1.5	3.7755	3.9823	4.2103	4.5232
2	3.2047	3.2708	3.3136	3.4072

Table IV. *ARL* values for the proposed scheme I at $ARL_O = 200$

δ	$\lambda = 0.1$ $L_S = 2.211$	$\lambda = 0.25$ $L_S = 2.24$	$\lambda = 0.5$ $L_S = 2.09$	$\lambda = 0.75$ $L_S = 1.875$
0	200.5694	199.8855	200.8923	201.1229
0.25	60.9801	80.7515	107.709	126.1691
0.5	20.9561	28.6011	42.0774	55.7335
0.75	11.2452	13.7306	19.0848	25.862
1	7.1859	8.1962	10.6258	13.9457
1.5	4.198	4.4825	4.9002	5.7061
2	3.0577	3.1331	3.1899	3.4272

Table V. *ARL* values for the proposed scheme II at $ARL_O = 200$

δ	$\lambda = 0.1$, $L_S = 1.895$	$\lambda = 0.25$, $L_S = 2.008$	$\lambda = 0.5$, $L_S = 1.902$	$\lambda = 0.75$, $L_S = 1.715$
0	201.9206	200.5236	200.5969	200.886
0.25	39.1578	50.0565	63.1482	70.9068
0.5	15.4204	19.5705	25.1144	30.4725
0.75	8.6061	10.1922	12.8642	15.3578
1	5.981	6.7621	7.7437	9.1272
1.5	3.8655	4.1115	4.3324	4.6841
2	3.2523	3.3186	3.3386	3.4238

ARL is one measure of the behavior of the run length distribution. Other measures such as the standard deviation of the run length and some selective percentile points of the run length distribution are also recommended by the authors such as Palm[9], Shmueli and Cohen[10] and Antzoulakos and Rakitzis[6] . Following these authors, we present here the standard deviation of the run lengths denoted by *SDRL* and the *i*th percentiles denoted by P_i ($i = 10$, 25, 50, 75 and 90) at $ARL_O = 500$. Similar results can be easily obtained for other values of ARL_O. These measures along with *ARL* may help studying the behavior of the run length distribution.

The relative standard errors of the results reported in Tables II–XI are also calculated and are found to be around 1%. We have also replicated the results of the classical EWMA chart and found almost the same results as by Lucas and Saccucci[8] which ensures the validity of our simulation algorithm.

Table VI. *ARL* values for the proposed scheme I at $ARL_O = 500$

δ	$\lambda = 0.1$, $L_S = 2.556$	$\lambda = 0.25$, $L_S = 2.554$	$\lambda = 0.5$, $L_S = 2.36$	$\lambda = 0.75$, $L_S = 2.115$
0	501.7558	505.5284	501.2598	502.0725
0.25	103.3109	169.1349	235.1138	280.6187
0.5	29.5748	47.0105	78.0771	108.8792
0.75	14.3216	19.2776	30.8742	45.3405
1	8.9561	10.5964	15.1992	22.1033
1.5	4.9197	5.2578	6.1014	7.7862
2	3.4498	3.5527	3.6815	4.0883

Table VII. *ARL* values for the proposed scheme II at $ARL_O = 500$

δ	$\lambda = 0.1$, $L_S = 2.3$	$\lambda = 0.25$, $L_S = 2.345$	$\lambda = 0.5$, $L_S = 2.202$	$\lambda = 0.75$, $L_S = 1.982$
0	502.883	499.6153	505.3564	501.9698
0.25	66.6864	97.0108	133.7117	155.7078
0.5	21.4251	31.2023	46.3541	57.7739
0.75	11.7427	14.4295	20.6223	26.0312
1	7.5539	8.6761	11.0991	13.8363
1.5	4.4676	4.7066	5.1336	5.7812
2	3.4534	3.549	3.6276	3.7787

Table VIII. *SDRL* values for the proposed scheme I at $ARL_O = 500$

δ	$\lambda = 0.1$, $L_S = 2.556$	$\lambda = 0.25$, $L_S = 2.554$	$\lambda = 0.5$, $L_S = 2.36$	$\lambda = 0.75$, $L_S = 2.115$
0	497.949	504.2443	507.2755	496.1511
0.25	95.3113	163.9435	228.8944	277.5384
0.5	22.8308	42.7233	75.7654	106.8295
0.75	9.2438	15.6482	28.3049	43.8296
1	5.0726	7.3114	12.6517	20.2822
1.5	2.265	2.6916	3.995	6.0512
2	1.3145	1.4353	1.8001	2.4913

Table IX. *SDRL* values for the proposed scheme II at $ARL_O = 500$

δ	$\lambda = 0.1$, $L_S = 2.3$	$\lambda = 0.25$, $L_S = 2.345$	$\lambda = 0.5$, $L_S = 2.202$	$\lambda = 0.75$, $L_S = 1.982$
0	501.9372	500.452	499.948	505.2274
0.25	61.0314	94.0425	132.2482	153.9424
0.5	17.1564	27.1779	43.2896	55.7953
0.75	7.6938	10.9904	18.3501	24.0529
1	4.2428	5.6134	8.5255	11.7163
1.5	1.8184	2.0758	2.7764	3.7044
2	0.8672	0.9694	1.2019	1.5265

Mainly, the findings for the two proposed schemes are:

(i) the two proposed schemes are performing very well at detecting small and moderate shifts while their performance for large shifts is not bad either (cf. Tables II–VII);
(ii) the *SDRL* decreases for both schemes as the value of δ increases (cf. Tables VIII–IX);
(iii) the run length distribution of both schemes is positively skewed (cf. Tables X–XI);
(iv) with an increase in the value of δ the ARL_1 decreases rapidly for both schemes, at a given ARL_O (cf. Tables II–VII);
(v) with a decrease in the value of ARL_O the ARL_1 decreases quickly for both schemes for a given value of δ (cf. Tables II–VII);

Table X. Percentile points for the proposed scheme I at $ARL_O = 500$

λ	P_i	δ 0	0.25	0.5	0.75	1	1.5	2
0.1	P_{10}	55	17	8	5	4	2	2
	P_{25}	150	35	14	8	5	3	2
	P_{50}	347	75	23	12	8	4	3
	P_{75}	696	141	39	19	11	6	4
	P_{90}	1139	228	60	26.1	16	8	5
0.25	P_{10}	54	21	9	5	4	2	2
	P_{25}	146	51	17	9	6	3	2
	P_{50}	352	120	34	15	9	5	3
	P_{75}	703	235	63	25	14	7	4
	P_{90}	1179.1	384	104	39	20	9	5
0.5	P_{10}	56	26	11	5	4	2	2
	P_{25}	144	71	24	11	6	3	2
	P_{50}	345	163	54	23	11	5	3
	P_{75}	688.25	327	108	41	20	8	4
	P_{90}	1165	538	176	69	32	11	6
0.75	P_{10}	51	31	13	6	4	2	2
	P_{25}	145	84	33	15	8	3	2
	P_{50}	347	193	75	32	16	6	3
	P_{75}	697	389	148	62	30	10	5
	P_{90}	1165.1	647	250	102	48	16	7

Table XI. Percentile points for the proposed scheme II at $ARL_O = 500$

λ	P_i	δ 0	0.25	0.5	0.75	1	1.5	2
0.1	P_{10}	50	11	4	4	3	3	3
	P_{25}	141	23	10	6	4	3	3
	P_{50}	350	48	17	10	7	4	3
	P_{75}	699	91	28	15	10	5	4
	P_{90}	1165	145	43	22	13	7	5
0.25	P_{10}	54	13	7	4	3	3	3
	P_{25}	147	30	12	7	5	3	3
	P_{50}	349	67	23	11	7	4	3
	P_{75}	690	133	43	19	11	6	4
	P_{90}	1133	219	67	29	16	7	5
0.5	P_{10}	54	16	7	4	3	3	3
	P_{25}	147.75	40	15	8	5	3	3
	P_{50}	354	92	33	15	9	4	3
	P_{75}	701	186	64	27	15	6	4
	P_{90}	1156.1	304	102	45	22	9	5
0.75	P_{10}	51	18	7	4	3	3	3
	P_{25}	144.75	46	18	9	5	3	3
	P_{50}	349	109	40	19	10	4	3
	P_{75}	695	217	80	36	18	7	4
	P_{90}	1154.1	355	130	57	29	11	6

(vi) the proposed scheme II is performing significantly better than the proposed scheme I for all choices of λ (cf. Tables II–VII);
(vii) performance of the two proposed schemes is generally better for smaller choices of λ (cf. Tables II–VII);
(viii) the proposed scheme II has the ability to perform well even for moderately large values of λ;
(ix) the application of both the schemes is quite simple and easily executable;
(x) performance of the EWMA type charts can further be enhanced by extending the proposed schemes with the help of other runs rules schemes.

5. Comparisons

In this section, we provide a detailed comparison of the proposed schemes with their already existing counterparts meant for detecting small shifts, i.e. EWMA- and CUSUM-type charts. The performance of all the control charting schemes is compared in terms of *ARL*. The control schemes used for the comparison purposes include the classical EWMA, the classical CUSUM, the FIR CUSUM, the FIR EWMA, the weighted CUSUM, the double CUSUM, the distribution-free CUSUM and the runs rules schemes-based CUSUM. The *ARLs* for all these charts/schemes are given in Tables I, XII–XIX which will be used for the comparisons of these schemes with the proposed schemes I and II for the EWMA chart.

Proposed vs the classical EWMA: The classical EWMA is defined by Roberts[3]. *ARL* values for the classical EWMA are given in Table I. The classical EWMA refers to one out of one ($\frac{1}{1}$) scheme. The comparison of the three schemes (i.e. the classical EWMA and the two proposed schemes) shows that both proposed EWMA schemes of Section 3 are performing better than the classical scheme in terms of *ARL* (cf. Tables VI and VII vs Table I). Moreover, scheme II is outperforming the scheme I with a great margin for the small shifts ($0.25 \leq \delta \leq 1.5$). The performance of the two proposed schemes almost coincide for larger values of δ.

Proposed vs the classical CUSUM: The classical CUSUM is defined by Page[2]. The upward and downward deviations are accumulated by two separate statistics C_i^+ and C_i^- plotted against the control limit h. The *ARL* values of the classical CUSUM are given in Table XII at ARL_o 168 and 465. The comparison of the classical CUSUM with the proposed schemes reveals that both schemes are outperforming the classical CUSUM scheme at all the values of δ (cf. Tables II and III vs Table XII). Particularly, comparing the three schemes at $\delta=0.25$ we observe that the proposed scheme II is performing the best with $ARL_1=34.367$ followed by the proposed scheme I with $ARL_1=54.5771$, whereas the classical CUSUM has $ARL_1=74.2$ which mean that the proposed scheme II is giving almost half ARL_1 than the classical CUSUM scheme with ARL_o fixed at 168.

Table XII. *ARL* Values for the classical CUSUM scheme with $k=0.5$

δ	0	0.25	0.5	0.75	1	1.5	2	2.5	3
$h=4$	168	74.2	26.6	13.3	8.38	4.75	3.34	2.62	2.19
$h=5$	465	139	38.0	17.0	10.4	5.75	4.01	3.11	2.57

Table XIII. *ARLs* for the FIR CUSUM scheme with $k=0.5$

δ	0	0.25	0.5	0.75	1	1.5	2
$h=4$, $C_o=1$	163	71.1	24.4	11.6	7.04	3.85	2.7
$h=4$, $C_o=2$	149	62.7	20.1	8.97	5.29	2.86	2.01

Table XIV. *ARL* values for the FIR EWMA scheme

δ	% Head Start	$\lambda=0.1$, $L=2.814$	$\lambda=0.25$, $L=2.998$	$\lambda=0.5$, $L=3.071$	$\lambda=0.75$, $L=3.087$
0	25	487	491	497	498
	50	468	483	487	496
0.5	25	28.3	46.5	87.8	140
	50	24.2	43.6	86.1	139
1	25	8.75	10.1	16.9	30.2
	50	6.87	8.79	15.9	29.7
2	25	3.57	3.11	3.29	4.33
	50	2.72	2.5	2.87	4.09

Table XV. *ARL* values for the symmetric two-sided weighted CUSUM scheme at $ARL_o=500$

$k=0.5$		δ			
γ	h	0.5	1	1.5	2
0.7	3.16	86.30	15.90	6.08	3.52
0.8	3.46	70.20	13.30	5.66	**3.50**
0.9	3.97	54.40	**11.40**	**5.50**	3.60
1.0	5.09	**39.00**	10.50	5.81	4.02

Table XVI. *ARLs* for the double CUSUM with $k=3.3$, $h'=6.8$ and $k'=0.3$ at $ARL_o=500$					
δ	0	0.5	1	1.5	2
$h=2.6$	507	27.1	9.85	5.55	3.57

Table XVII. *ARL* values for different distribution-free CUSUM schemes with nominal $ARL_o=200$

		δ		
j_{max}	Chart	0	0.5	1
5	B1	178.43	25.31	12.54
	B2	173.78	18.37	7.94
	B3	201.86	27.38	9.14
30	B1	202.92	18.89	6.60
	B2	194.44	18.68	6.43
	B3	197.79	19.20	6.45
40	B1	195.04	22.40	5.66
	B2	198.98	20.52	5.70
	B3	201.87	21.36	5.77
50	B1	190.88	16.96	6.59
	B2	199.35	18.73	6.84
	B3	202.79	17.51	6.50

Table XVIII. *ARL* values for the runs rules-based CUSUM scheme I at $ARL_o=200$

Limits		δ					
WL	AL	0.25	0.5	0.75	1	1.5	2
3.9	4.24	82.9524	28.6972	13.8006	8.9038	4.9088	3.4918
3.8	4.29	84.537	28.736	14.065	8.6606	5.002	3.5234
3.7	4.4	82.1152	28.4956	13.8216	8.8124	4.9928	3.561
3.6	4.77	84.1504	28.716	13.9612	8.9812	5.2276	3.7252
3.57	∞	79.4742	28.9396	14.2622	9.213	5.5104	4.076

Table XIX. *ARL* values for the runs rules-based CUSUM scheme II at $ARL_o=200$

Limits		δ					
WL	AL	0.25	0.5	0.75	1	1.5	2
3.9	4.23	82.975	28.206	13.7702	8.869	4.9782	3.4346
3.8	4.28	81.4026	28.3564	13.9032	8.6994	4.966	3.5028
3.7	4.6	82.7322	28.8236	14.0778	8.8744	5.1404	3.687
3.64	∞	81.5176	29.1382	14.2656	9.1142	5.4572	4.1198

Proposed vs the FIR CUSUM: The FIR CUSUM presented by Lucas and Crosier[11] gives a head start to the CUSUM statistic rather than setting it equal to zero. The *ARLs* of FIR CUSUM with two different values for the head start (C_o) are given in Table XIII. Comparing the performance of FIR CUSUM with the proposed schemes, we can see that the proposed scheme II is performing better than the FIR CUSUM although that there is a problem with the FIR CUSUM that it has ARL_o value less than 168 (the desired level). Moreover, we see that if the value of C_o increases then the value of ARL_o decreases, which is not recommended in case of sensitive processes (cf. Bonetti and Waeckerlin[12]). The proposed schemes are not only fixing the ARL_o at the pre-specified level (so that valid comparison may be made) but also performing better in terms of ARL_1s, i.e. the proposed schemes are minimizing the ARL_1 (with a fixed ARL_o) without a decrease in ARL_o and without the need of any head start value (cf. Tables II and III vs Table XIII).

Proposed vs the FIR EWMA: Lucas and Saccucci[8] proposed the application of the FIR feature with the EWMA control chart (especially with the small values of λ). The *ARL* values of the EWMA control chart with FIR feature are provided in Table XIV. Comparing the FIR EWMA with the proposed schemes, we observe that the proposed schemes are not only having smaller ARL_1s

but they also fix the ARL_o value at the desired level which is not the case with the FIR EWMA (cf. Tables VI & VII vs Table XIV). The other comments made in favor of the proposed schemes vs the FIR CUSUM are also valid here with the same spirit and strength.

Proposed vs the weighted CUSUM: Yashchin[13] proposed a class of weighted CUSUM charts that generalize the classical CUSUM charts by giving weights to the past information and can be viewed as the EWMA version of the CUSUM charts. The *ARLs* for the weighted CUSUM are given in Table XV where the weights given to the past information are represented by γ. Comparing the weighted CUSUM with the proposed schemes, we notice that the proposed schemes are performing better than the weighted CUSUM for all the values of δ which shows the uniform superiority of the proposed schemes over the weighted CUSUM (cf. Tables VI & VII vs Table XV).

Proposed vs the double CUSUM: Waldmann[14] has shown that the simultaneous use of two classical CUSUMs improves the *ARL* performance of the CUSUM chart. This simultaneous use of the two CUSUM charts is being given the name of double CUSUM. The *ARL* performance of the double CUSUM is given in Table XVI in which parameters of the first CUSUM are h and k and parameters of the second CUSUM are h' and k'. Comparison of the double CUSUM with the proposed schemes shows that the double CUSUM performs better than the proposed scheme I for $\delta=0.5$, but the proposed scheme II performs better than both the proposed scheme I and the double CUSUM. For all other values of δ, the proposed scheme II is performing the best followed by the proposed scheme I (cf. Tables VI & VII vs Table XVI).

Proposed vs the distribution-free CUSUM: Chatterjee and Qiu[15] proposed a class of distribution-free CUSUM charts. The three non-parametric control charts named as B1, B2 and B3 depend upon the variable T_i, which is defined as:

$$T_i=\left\{\begin{matrix} 0 & \text{if } C_i=0 \\ j & \text{if } C_i\neq 0,\ C_{i-1}\neq 0,\ldots,C_{i-j+1}\neq 0;\ j=1,2,\ldots,n \end{matrix}\right\}$$

where T_i is the number of samples since the last time the statistic C_i was zero. The *ARL* performance of these non-parametric charts is given in Table XVII. For $\delta=0.5$, the best *ARL* performance is at $j_{\max}=50$ by chart B1. In this case the $ARL_1=16.96$, whereas the ARL_1 for the proposed schemes I and II is 20.9561 and 15.4204, respectively, which shows superiority of the proposed scheme II. For $\delta=1$ the distribution-free charts perform slightly better for $j_{\max}=40$, but for all other values of $j_{\max}$, the proposed scheme II is again performing better. This proves the dominance of the proposed scheme II as compared with the distribution-free CUSUM charts in general (cf. Tables IV & V vs Table XVII).

Proposed vs the runs rules based CUSUM: Riaz *et al.*[7] proposed two runs rules schemes, namely CUSUM Scheme I and CUSUM Scheme II, on the CUSUM charts and computed the *ARL* values for the two schemes which are given in Tables XVIII and XIX where *WL* and *AL* represent the warning and action limits. Comparing the proposed EWMA schemes with these runs rules based CUSUM schemes I and II we see that the two proposed EWMA schemes are performing better than the CUSUM schemes of Riaz *et al.*[7] (cf. Tables IV and V vs. Tables XVIII and XIX).

Moreover, for an overall comparison of the proposed schemes with their existing counterparts mentioned and compared above, we have made some graphs showing *ARL* curves of different schemes.

It is evident from the analysis of Figures 2–4 that the *ARL* curves of the two proposed EWMA schemes exhibit dominance in general as compared with all the other schemes covered in this study. Particularly, the *ARL* curve of the proposed scheme II is on the lower side compared with all other schemes. This shows the best *ARL* performance of the proposed scheme II compared with all others. For the small shifts, the gap between the *ARL* curves of the proposed schemes with those of the other schemes is large whereas this gap reduces as the size of the shift increases. This implies that the proposals of the study (particularly proposed scheme II) are generally more beneficial for smaller shifts.

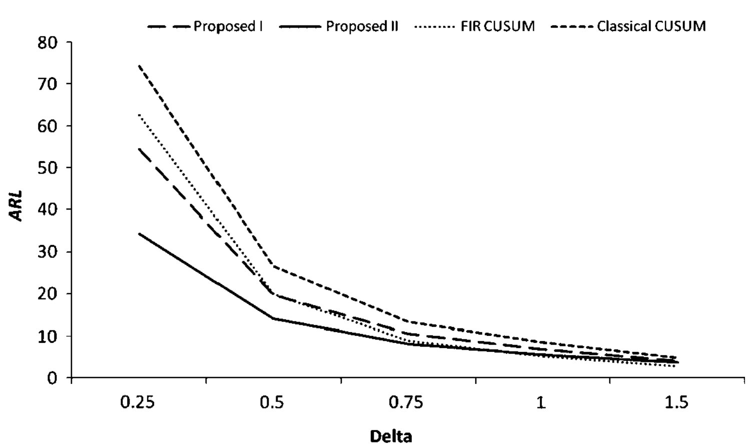

Figure 2. *ARL* curves for the proposed schemes I and II, the classical CUSUM and the FIR CUSUM at $ARL_o=168$

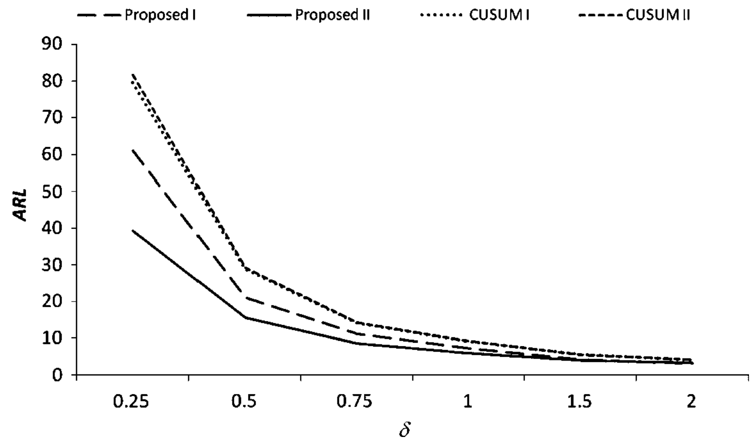

Figure 3. *ARL* curves for the proposed schemes I and II, the runs rules scheme I for CUSUM and the runs rules scheme II for CUSUM at $ARL_O = 200$

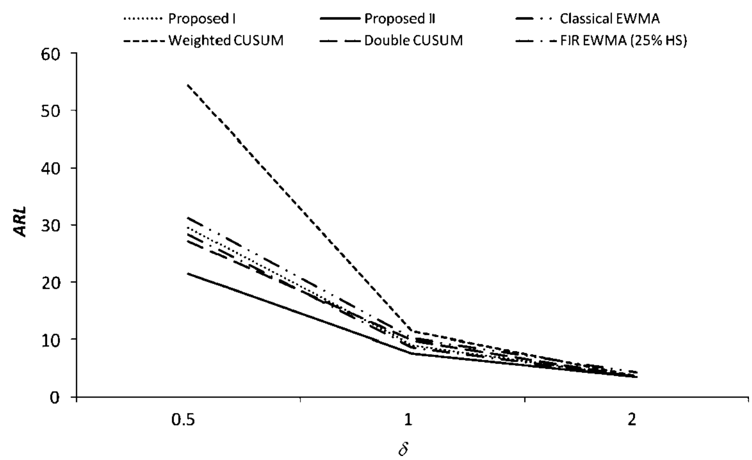

Figure 4. *ARL* curves for the proposed schemes I and II, the classical EWMA, the weighted CUSUM and the double CUSUM at $ARL_O = 500$

6. Illustrative examples

Antzoulakos and Rakitzis[6] provided an example to illustrate the proposed runs rules schemes. On similar lines, we also present an illustrative example to show how the proposed schemes can be applied in real situations. For this purpose, two data sets are generated which will be later referred as data set 1 and data set 2.

Data set 1 contains 50 observations out of which the first 20 are generated from $N(0, 1)$ referring to an in-control situation and the remaining 30 observations are generated from $N(0.25, 1)$ referring to a small shift in the process mean. The EWMA statistics for these 50 observations are calculated with $\lambda = 0.1$ and the three schemes (i.e. the two proposed schemes and the classical scheme) are applied to the data set. The graphical display of the control chart with all the three schemes applied to the data set 1 is given in Figure 5.

From Figure 5 we can see that first 20 points are plotted around the central line, whereas an upward shift in the points can be seen afterwards. Both the classical scheme and the proposed scheme I are giving two out-of control signals. The classical scheme is signaling at point # 47 and 50, whereas the proposed scheme I is signaling at point # 47 and 48. The proposed scheme II is giving six out-of-control signals and these are at points # 42, 43, 47, 48, 49 and 50. This clearly indicated that the proposed scheme II is not only signaling earlier than the classical scheme but also is giving more number of signals.

Data set 2 contains 30 observations out of which the first 20 are generated from $N(0, 1)$ referring to an in-control situation and the remaining 10 observations are generated from $N(1, 1)$ referring to a moderate shift in the process mean. Again the EWMA statistic for these 30 observations is calculated using $\lambda = 0.1$ and the graphical display of the three schemes is given in Figure 6.

From Figure 6 we observe that the classical scheme is giving six out-of-control signals, the proposed scheme I is giving seven signals and the proposed scheme II is giving eight signals. The classical scheme is giving the first signal at point # 26, whereas the proposed scheme I and II are giving first signals at points # 25 and # 24, respectively. Hence, the proposed scheme II show the same superiority in signaling earlier and in terms of number of signals for the data set 2 as well.

The above example clearly indicates that the proposed scheme II is giving the advantage in terms of run length as well the number of signals for both small and moderate shifts. The outcomes of these two illustrative examples are completely in accordance with the findings of Section 5.

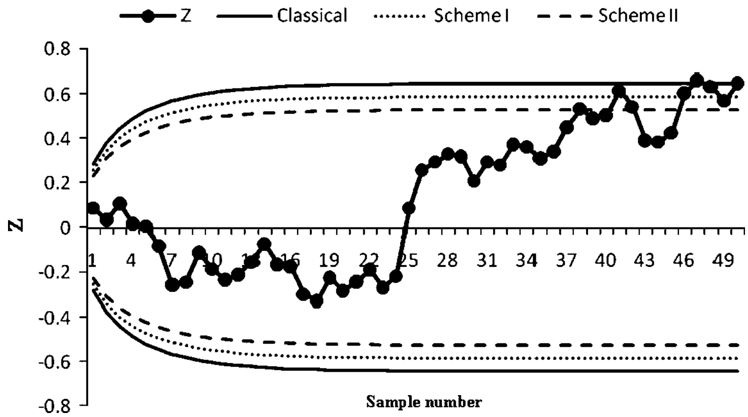

Figure 5. EWMA chart of the classical scheme and the proposed schemes I and II for the data set 1

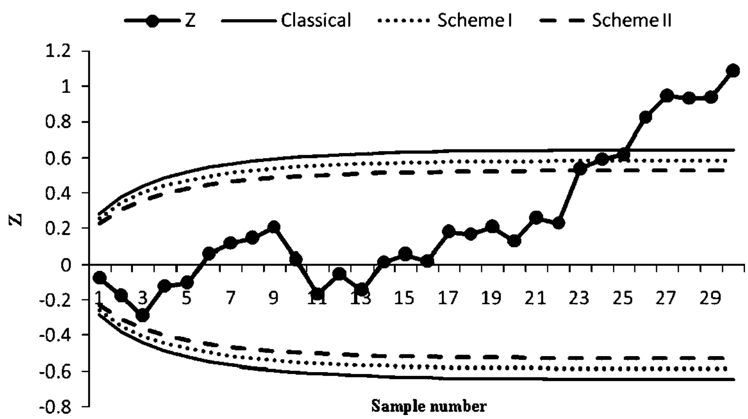

Figure 6. EWMA chart of the classical scheme and the proposed schemes I and II for the data set II

7. Summary and conclusions

Every process shows some variations in its output. These variations need to be reduced so that quality of the output of a process may be improved. Control charts are used to monitor the current situation of a process and to differentiate whether the variations in the process are due to common causes or assignable causes. Shewhart-type location control charts are considered effective for detecting large shifts in the process location, whereas CUSUM and EWMA control charts are efficient for small and moderate shifts. The efficiency of these charts can further be enhanced by using different runs rules schemes with their design structures. We have proposed, in this article, two runs rules schemes to be applied with the EWMA charts for the location parameter. The proposed scheme I gives an out-of control signal if two consecutive points are plotted either below an *LSL* or above a *USL*, whereas the proposed scheme II gives an out-of-control signal if at least two out of three consecutive points are plotted either below an *LSL* (keeping the in-control point (if any) between this limit and the CL) or above a upper signaling limit (keeping the in-control point (if any) between the CL and this limit). The performance of the two proposed schemes is investigated in terms of *ARL* and is compared with some existing schemes used for the same purposes. The proposed schemes (particularly scheme II) are found to be performing really well for small and moderate shifts while reasonably maintaining their performance for large shifts as well.

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