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# Invariant Bayesian Inference in Regression Models that is robust against the Jeffreys-Lindley's Paradox

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# Invariant Bayesian Inference in Regression Models that is robust against the Jeffreys-Lindley's Paradox

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## Abstract

We obtain the prior and posterior probability of a nested regression model as the Hausdorff-integral of the prior and posterior on the parameters of an encompassing linear regression model over a lower dimensional set that represents the nested model. The invariant expression of the Hausdorff-integral results from a uniformly converging limit sequence of encompassing full dimensional sets. The uniform convergence avoids the Borel-Kolmogorov paradox. Basing priors and prior probabilities of nested regression models on the prior on the parameters of an encompassing linear regression model reduces the discrepancies between classical and Bayesian inference, like, the Jeffreys-Lindley's paradox. Depending on the parameter of interest in the encompassing linear regression model, the posterior odds ratio is fully robust to the Jeffreys-Lindley's paradox or only allows for a non-informative prior on the parameters of the encompassing model. We illustrate the analysis with examples of linear restrictions, *i.e.* a linear regression model, and non-linear restrictions, *i.e.* a cointegration and an autoregressive moving average model, on the parameters of an encompassing linear regression model.

## 1 Introduction

In Bayesian model comparison based on Bayes factors and prior and posterior odds ratios, prior probabilities for competing models are assigned independently of the priors on the parameters of these models. When one of the models encompasses the others, the prior on its parameters has a specific value at the points in the parameter space that correspond with the competing nested models. Hence, it can occur that this prior has a low value at the location of a competing nested model while that model has a prior probability equal to the prior probability of the encompassing model. This leads to a distinct difference between classical and Bayesian model comparison which is referred to as the Jeffreys-Lindley's paradox, see *e.g.* Lindley (1957), Bernardo and Smith (1994), O'Hagan (1994) and Poirier (1995). Similar differences arise when we compare the functional forms of sampling densities of maximum likelihood estimators and posteriors of the parameters, see Kleibergen and Zivot (2002).

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The priors and prior probabilities are specified independently because the prior on the parameters of an encompassing model does not imply unambiguous probabilities for the lower dimensional sub-sets of its parameter space that constitute the nested models. This is known as the Borel-Kolmogorov paradox, see *e.g.* Kolmogorov (1950), Drèze and Richard (1983), Billingsley (1986) and Wolpert (1995). The paradox results because separate limit sequences can be constructed, that converge to a lower dimensional set and are needed to obtain its probability, which, however, lead to different expressions of the probability of the lower dimensional set. To avoid the Borel-Kolmogorov paradox, we show that it is necessary to use limit sequences that converge uniformly, *i.e.* that converge appropriately for all possible sub-sets of the lower dimensional set. All uniformly converging limit sequences lead to the same expression of the Hausdorff-integral over the lower dimensional set. This expression implies the prior or posterior probability of a nested model that is induced by the prior or posterior on the parameters of an encompassing model. Also the priors and posteriors on the parameters of nested models result from these Hausdorff-integrals.

The discrepancies between classical and Bayesian model comparison reduce when we use priors and prior probabilities for nested regression models that result from a prior on the parameters of an encompassing linear regression (ELR) model. Depending on the specification of the ELR model, the Jeffreys-Lindley's paradox is partially or fully overcome. The Jeffreys-Lindley's paradox is fully overcome when the classical “*t*-values” are the parameters of interest in the ELR model. Just the sensitivity with respect to the prior variance is overcome when we use the parameters of the ELR model as the parameters of interest. Similar results hold for the functional forms of the posteriors and sampling densities of maximum likelihood estimators which are also more in line with one another when we use priors that are induced by the prior on the parameters of an ELR model.

The paper is organized as follows. In the second section, we motivate our analysis and specify the regression models that are nested in the ELR model. In the third section, we discuss the uniformly converging limit sequence by which we avoid the Borel-Kolmogorov paradox and that allows us to obtain the Hausdorff-measure of a lower-dimensional set. In the fourth section, we obtain the prior and prior probability for each nested regression model that is induced by the prior on the parameters of the ELR model. We introduce an example, that we use throughout the paper, to illustrate what this prior probability amounts to. The fifth section extends the results to the posterior and posterior probability. In the sixth section, we discuss the Jeffreys-Lindley's paradox. We use our example to show, that depending on the parameter of interest in the ELR model, that the Jeffreys-Lindley's paradox is partially or fully overcome. These robustness properties imply that the posterior odds ratio can be constructed even in case of a non-informative prior. The seventh section discusses regression models that are conditional on nuisance parameters. The eighth section contains illustrative examples where we focus on the construction of specifications that satisfy the assumption needed to obtain the uniformly converging limit sequence. We show them for regression models that result from linear restrictions, *i.e.* linear regression models, and non-linear restrictions, the cointegration and autoregressive moving average (ARMA) model, on the parameters of an ELR model. Finally, the ninth section concludes.

We use the following notation throughout the paper:  $\text{vec}(A)$  stands for the column vectorization of the  $T \times k$  matrix  $A$  such that  $\text{vec}(A) = (a'_1 \dots a'_k)'$  when  $A = (a_1 \dots a_k)$ .  $M_A = I_T - A(A'A)^{-1}A'$ , with  $I_T$  the  $T \times T$  dimensional identity matrix;  $J(a, (b, c))$  is the Jacobian of the transformation from  $a$  to  $(b, c)$  and  $|_a$  stands for evaluated in  $a$ .

## 2 Motivation

We consider the ELR model,

$$G : y = X\beta + \varepsilon, \quad (1)$$

with  $y$  a  $T \times 1$  vector of observations on the dependent variable,  $X$  a  $T \times k$  matrix that contains the independent explanatory variables,  $\beta$  a  $k \times 1$  vector of parameters and  $\varepsilon$  a  $T \times 1$  vector of disturbances. The support of  $\beta$  is the  $\mathbb{R}^k$ . For expository purposes, we assume that  $\varepsilon \sim N(0, I_T)$ . This distributional assumption can however be generalized which we further discuss in Section 7. We specify a prior on  $\beta$  in model  $G$ ,  $p_G(\beta)$ , that is continuous and continuous differentiable.

We compare and analyze the regression models

$$G_i : y = Xf_i(\varphi_i) + \varepsilon, \quad i = 1, \dots, n, \quad (2)$$

with  $\varphi_i \in \Theta_{G_i}$ ,  $\Theta_{G_i}$  is an open convex set in the  $\mathbb{R}^{m_i}$  and  $f_i$  is a  $k$ -dimensional continuous differentiable function of the  $m_i \times 1$  vector  $\varphi_i$ ,  $m_i \leq k$ . Each model  $G_i$  in (2) is nested in the ELR model  $G$  in (1).

Traditionally, a prior is specified on  $\varphi_i$  in  $G_i$ ,  $p_{G_i}(\varphi_i)$ , without considering that  $G_i$  is nested in  $G$ . This implies that the restrictions that  $G_i$  implies on  $G$  are directly imposed and that the transformation from  $G_i$  to  $G$  is left aside, see *e.g.* Drèze and Richard (1983). The behavior of the posterior of  $\varphi_i$  in  $G_i$  can then be pathological when  $f_i(\varphi_i)$  is a non-linear function while the posterior of  $\beta$  in  $G$  is well-behaved. Examples of such models  $G_i$  are cointegration and simultaneous equations models, see Kleibergen and van Dijk (1994,1998) and Kleibergen and Zivot (2002). Similarly, the prior probability for  $G_i$ ,  $\Pr[G_i]$ , is specified without taking the prior on  $\beta$  in  $G$  into account. It can therefore occur that the prior on  $\beta$  has a relatively low value at the parameter values of  $\beta$  that correspond with  $G_i$  while the prior probability of  $G_i$  is relatively large. To overcome these controversies, we propose a framework which explicitly takes into account that  $G_i$  is nested in  $G$  for the construction of the prior for  $\varphi_i$  and the prior probability of  $G_i$ . We therefore base both the prior for  $\varphi_i$  and the prior probability of  $G_i$  on the prior on  $\beta$  in  $G$ ,  $p_G(\beta)$ . Since the likelihood is a continuous function of the parameters, the results directly extend to the posterior and posterior probabilities.

## 3 Hausdorff-integrals over lower dimensional sets

We represent the nested regression models  $G_i$  in (2) by lower dimensional sets in the parameter space of  $\beta$ , the  $\mathbb{R}^k$ ,

$$S_{G_i} = \{\varphi_i \in \Theta_{G_i} \subset \mathbb{R}^{m_i} | \beta = f_i(\varphi_i)\}, \quad i = 1, \dots, n. \quad (3)$$

The sets  $S_{G_i}$  in (3) are  $m_i$ -dimensional manifolds in the  $\mathbb{R}^k$ . We use the sets  $S_{G_i}$  to induce the prior probability and density of the regression models  $G_i$  from the prior  $p_G(\beta)$  on  $\beta$  in  $G$ .

The prior  $p_G(\beta)$  induces prior probabilities for convex  $k$ -dimensional sets  $S \subset \mathbb{R}^k$ ,

$$\Pr_G [S] = \int_S p_G(\beta) d\beta, \quad (4)$$

where  $d\beta$  is shorthand notation for  $L_k(d\beta)$  because (4) is a Lebesgue-integral. To construct the prior probability of  $G_i$  induced by  $p_G(\beta)$ , we evaluate the integral of  $p_G(\beta)$  over  $S_{G_i}$ . When

$S$  in (4) is a  $m$ -dimensional manifold and  $m$  is less than  $k$ , the Lebesgue-measure of  $S$  in  $\mathbb{R}^k$ ,  $L_k(S)$ , equals zero. Hence, we can not use standard Lebesgue integration to obtain the integral of  $p_G(\beta)$  over  $S$ . Instead of the Lebesgue-measure, we therefore use a measure that is defined for lower dimensional sets: the Hausdorff-measure, see *e.g.* Billingsley (1986) and Rogers (1999).

**Definition 1:** *The Hausdorff-measure of a  $m$ -dimensional set  $S$  in the  $\mathbb{R}^k$ ,  $H_m(S)$ ,  $k \geq m$ , is the infimum of positive numbers  $y$  such that for every  $r > 0$ ,  $S$  can be covered by a countable family of closed sets, each of diameter less than  $r$  and the sum of the  $m$ -th powers of these diameters is less than  $y$ :*

$$H_m(S) = \inf c_m \sum_j \text{diam}(B_j)^m, \quad (5)$$

with  $B_j$ ,  $j = 1, \dots$  the countable number of sets whose union covers  $S$ ,  $\text{diam}(B_j)$  stands for the diameter of  $B_j$ ,  $\text{diam}(B_j) = \sup[|a - b| : a, b \in B_j]$ , and  $c_m$  is a normalizing constant. The Hausdorff-measure is invariant to the specification of  $S$  and can be infinite.

When  $m$  equals  $k$ , the definition of the Hausdorff-measure in (5) gives the volume of  $S$  and the Hausdorff-measure and the Lebesgue-measure coincide if  $c_k$  is specified appropriately. To obtain the Hausdorff-measure of the  $m_i$ -dimensional set  $S_{G_i}$ , where  $m_i$  is less than  $k$ , we use an invertible mapping of  $\beta$ , that spans the  $\mathbb{R}^k$ , into  $\varphi_i$  and an additional  $(k - m_i)$ -dimensional parameter vector  $\lambda_i$  that is such that when  $\lambda_i$  converges to zero  $\beta(\varphi_i, \lambda_i)$  converges uniformly to  $f_i(\varphi_i)$ , *i.e.* for all values of  $\varphi_i$ .

**Assumption 1:** *In model  $G$  from (1), the  $k \times 1$  dimensional vector  $\beta$  is an invertible function of the  $m_i \times 1$  dimensional vector  $\varphi_i$  and the  $(k - m_i) \times 1$  dimensional vector  $\lambda_i$ :*

$$\beta = f_i(\varphi_i) + g_i(\varphi_i, \lambda_i), \quad (6)$$

where  $g_i(\varphi_i, \lambda_i)$  is a continuous differentiable  $k \times 1$  vector function of  $(\varphi_i, \lambda_i)$  which is such that:

- a.  $g_i(\varphi_i, \lambda_i) = 0 \Leftrightarrow \lambda_i = 0$ .
- b. *The set of values of  $\varphi_i$  that lead to a unique value of  $f_i(\varphi_i)$ , or for which  $\frac{\partial f_i}{\partial \varphi_i}$  has full rank, is identical to the set of values of  $\varphi_i$  that lead to a unique value of  $f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)$ , or for which  $(\frac{\partial f_i}{\partial \varphi_i} + \frac{\partial g_i}{\partial \varphi_i} : \frac{\partial g_i}{\partial \lambda_i})$  has full rank, and the latter set does not depend on  $\lambda_i$ , such that  $g_i(\varphi_i, \lambda_i)$  is a strictly monotonic function of  $\lambda_i$  for all values of  $\varphi_i$ .<sup>1</sup>*
- c.  $\left(\frac{\partial g_i(\varphi_i, \lambda_i)}{\partial \lambda_i}\right)' \left(\frac{\partial g_i(\varphi_i, \lambda_i)}{\partial \lambda_i}\right) \equiv A_i$  for all values of  $(\varphi_i, \lambda_i)$ , with  $A_i$  a fixed positive definite symmetric  $(k - m_i) \times (k - m_i)$  matrix that does not depend on  $(\varphi_i, \lambda_i)$ .

Definition 1 implies that the Hausdorff-measure of  $S_{G_i}$  results from a countable number of sets whose union covers  $S_{G_i}$ . Definition 1 does not lead to a straightforward manner of

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<sup>1</sup>We note that this condition refers to the functional relationship  $f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)$ . The spaces where  $\varphi_i$ ,  $\lambda_i$  result from are therefore considered unrestricted,  $\varphi_i \in \mathbb{R}^{m_i}$ ,  $\lambda_i \in \mathbb{R}^{k-m_i}$ , such that  $\Theta_{G_i}$  is not involved and, for example, the intersection of the set of values of  $\varphi_i$  that do not imply a unique value for  $f_i(\varphi_i)$  and  $\Theta_{G_i}$  can even be empty.

constructing the Hausdorff-measure because such a union of sets can be difficult to obtain. Assumption 1 alleviates the construction of the Hausdorff-measure by specifying a functional relationship between  $\beta$  and  $(\varphi_i, \lambda_i)$ . We can use this functional relationship in two different manners to obtain the Hausdorff-measure of  $S_{G_i}$ .

Assumption 1 allows us to project  $\beta$  onto  $f_i(\varphi_i)$  such that all sets  $\beta(\Theta_{G_i}, \Lambda_{i_j})$ , with  $\Lambda_{i_j}$  an open convex set in the  $\mathbb{R}^{k-m_i}$ , are projected onto  $S_{G_i}$ . A Hausdorff-measure of  $S_{G_i}$  can then be constructed by integrating a distance-measure of  $\lambda_i$  with respect to an integrable function of  $\lambda_i$ , like, for example, a density function, see *e.g.* McCulloch and Rossi (1992) and Doster (1998). This leads to the Hausdorff-measure of  $S_{G_i}$  that is marginal with respect to  $\lambda_i$  since it results from integrating over  $\lambda_i$ .

We construct the Hausdorff-measure of  $S_{G_i}$  in the alternative conditional manner that results from conditioning on  $\lambda_i = 0$ . The Hausdorff-measure is then obtained through a sequence of sets in  $\mathbb{R}^k$  that monotonically and uniformly converges to  $S_{G_i}$ . Assumption 1a implies that  $\lambda_i$  reflects the difference between  $\beta$  and  $f_i(\varphi_i)$ . Assumption 1b is a technical condition that ensures that  $\lambda_i$  reflects this difference for all points  $\varphi_i$  where  $\frac{\partial f_i}{\partial \varphi_i}$  has full rank. Assumption 1b is just a translation of the necessary condition of uniform convergence, *i.e.* convergence for all values of  $\varphi_i$  for which  $\frac{\partial f_i}{\partial \varphi_i}$  has full rank. Assumption 1c implies that the difference between  $\beta$  and  $f_i(\varphi_i)$  does not depend on  $\varphi_i$ . The normalizing constant that is used for the Hausdorff-measure of sub-sets of  $S_{G_i}$  is then identical for all sub-sets of  $S_{G_i}$ . This is necessary for an appropriately defined normalizing constant. Assumption 1 is thus a necessary and sufficient condition to obtain an invariant expression of the Hausdorff-measure that results from a limit sequence of sets that converges to the lower dimensional set.

Assumptions 1a-b enable the construction of a limit sequence of sets that converges monotonically and uniformly to a sub-set of  $S_{G_i}$ . To construct the Hausdorff-measure of such a  $m_i$ -dimensional set  $W_{G_i} (\subset S_{G_i})$  in the  $\mathbb{R}^k$ ,

$$W_{G_i} = \{\varphi_i \in \Omega_{G_i} \subset \Theta_{G_i} | \beta = f_i(\varphi_i)\}, \quad i = 1, \dots, n, \quad (7)$$

with  $\Omega_{G_i}$  a convex open  $m_i$ -dimensional sub-set of  $\Theta_{G_i}$ , we use the  $k$ -dimensional set

$$W_{G_i}(\rho) = \{\varphi_i \in \Omega_{G_i}, \lambda_i \in B_{k-m_i}(0, \rho) \subset \mathbb{R}^{k-m_i} | \beta = f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)\}, \quad (8)$$

where  $B_{k-m_i}(0, \rho)$  is a  $(k-m_i)$ -dimensional sphere with radius  $\rho$  and the  $(k-m_i)$ -dimensional vector of zeros as its center. The set  $W_{G_i}(\rho)$  contains  $W_{G_i}$  for all values of  $\rho$  and for a sequence of values of  $\rho$ ,  $\rho_1 > \rho_2 > \dots > \rho_n > 0$ , Assumptions 1a-b imply that,

$$W_{G_i}(\rho_n) \subset W_{G_i}(\rho_{n-1}) \subset \dots \subset W_{G_i}(\rho_2) \subset W_{G_i}(\rho_1) \text{ for all convex sets } \Omega_{G_i} \subset \mathbb{R}^{m_i}. \quad (9)$$

This implies that the convergence of  $W_{G_i}(\rho_n)$  to  $W_{G_i}$  is strictly monotonic and holds for all sets  $\Omega_{G_i}$ ,

$$\lim_{\rho \rightarrow 0} W_{G_i}(\rho) = W_{G_i} \text{ for all convex sets } \Omega_{G_i} \subset \mathbb{R}^{m_i}. \quad (10)$$

To obtain the Hausdorff-measure of  $W_{G_i}$ , we use the Lebesgue-measure of  $W_{G_i}(\rho)$  which we normalize to account for the difference in dimension between the  $m_i$ -dimensional set  $W_{G_i}$  and the  $k$ -dimensional set  $W_{G_i}(\rho)$ . The normalizing constant is the inverse of the Lebesgue-measure of the transformation of the sphere  $B_{k-m_i}(0, \rho)$  by the function  $g_i$ , see the Appendix for a proof:

$$c_i(\rho)^{-1} = |A_i|^{\frac{1}{2}} V_{k-m_i}(\rho), \quad (11)$$

with  $V_{k-m_i}(\rho)$  the volume of a  $(k-m_i)$ -dimensional sphere with radius  $\rho$ . Assumption 1c ensures that the normalizing constant does not depend on  $(\varphi_i, \lambda_i)$  and is therefore the same

for every sub-set  $W_{G_i}$  of  $S_{G_i}$ . The Hausdorff-measure of  $W_{G_i}$  then results as the limit when  $\rho$  converges to zero of the product of the normalizing constant  $c_i(\rho)$  and the Lebesque-measure of  $W_{G_i}(\rho)$  :

$$H_{m_i}(W_{G_i}) = \lim_{\rho \rightarrow 0} c_i(\rho) L_k(W_{G_i}(\rho)). \quad (12)$$

The normalizing constant  $c_i(\rho)$  converges to infinity when  $\rho$  converges to zero and therefore offsets the convergence to zero of the Lebesque-measure of  $W_{G_i}(\rho)$ . The Hausdorff-measure of  $W_{G_i}$  in (12), which is a transformation of  $\Omega_{G_i}$  by  $f_i$ , results from conditioning on a zero value of  $\lambda_i$  and does not result from integrating over  $\lambda_i$ .

**Theorem 1** *When  $m_i$  is less than  $k$  and Assumption 1 holds, the Hausdorff measure  $H_{m_i}(W_{G_i})$  in (12) is equal to*

$$H_{m_i}(W_{G_i}) = \int_{\Omega_{G_i}} \left| \left( \frac{\partial f_i}{\partial \varphi'_i} \right)' M_{\left( \frac{\partial g_i}{\partial \lambda'_i} \right) |_{\lambda_i=0}} \left( \frac{\partial f_i}{\partial \varphi'_i} \right) \right|^{\frac{1}{2}} d\varphi_i, \quad (13)$$

and is invariant with respect to transformations of  $\beta$  and  $(\varphi_i, \lambda_i)$  that satisfy Assumption 1 and control for the transformation of  $\beta$ .

**Proof.** see the Appendix. ■

The definition of the Hausdorff-measure also shows how Hausdorff-integrals of non-negative functions are constructed, see *e.g.* Billingsley (1986) and Rogers (1999).

**Definition 2:** *When  $m_i$  is less than  $k$  and Assumption 1 holds, the Hausdorff-integral of a non-negative function  $q(\beta)$  over the  $m_i$ -dimensional set  $W_{G_i}$  reads*

$$\int_{W_{G_i}} q(\beta) H_{m_i}(d\beta) = \lim_{\rho \rightarrow 0} \left[ c_i(\rho) \int_{W_{G_i}(\rho)} q(\beta) d\beta \right]. \quad (14)$$

**Theorem 2** *When  $m_i$  is less than  $k$  and Assumption 1 holds, the Hausdorff-integral of the non-negative function  $q(\beta)$  over  $W_{G_i}$  from (14) is equal to*

$$\int_{W_{G_i}} q(\beta) H_{m_i}(d\beta) = \frac{1}{|A_i|^{\frac{1}{2}}} \int_{\Omega_{G_i}} q(\beta(\varphi_i, \lambda_i) |_{\lambda_i=0}) |J(\beta, (\varphi_i, \lambda_i)) |_{\lambda_i=0}| d\varphi_i. \quad (15)$$

The Hausdorff-integral in (15) is invariant with respect to transformations of  $\beta$  and  $(\varphi_i, \lambda_i)$  that satisfy Assumption 1 and control for the transformation of  $\beta$ .

**Proof.** see the Appendix. ■

When  $m_i$  equals  $k$ , the Hausdorff-measure and integral are identical to the Lebesque-measure and integral. We use the Hausdorff-integrals to construct prior and posterior probabilities and densities of the models  $G_i$  that are induced by  $p_G(\beta)$ .

The Hausdorff-measure in Theorem 1 gives the measure of a lower dimensional manifold. These lower dimensional manifolds represent different models that are nested in the ELR model. We use the Hausdorff-measure to construct priors and posteriors of the parameters of these different models. When the functional form of a class of density functions, such as posteriors, is given, a Hausdorff-measure can be used to construct the relative distance between all densities within this class and a specific one of them. These kind of analyzes are conducted in, for example, McCulloch and Rossi (1992) and Doster (1998). McCulloch and Rossi (1992) use projection functions to map  $\beta$  onto  $f_i(\varphi_i)$ . Doster (1998) uses Hellinger and



Kullback-Leibler distance metrics for the involved density functions. Doster (1998) shows that the resulting specification of the Hausdorff-measure corresponds with a Jeffreys prior, *i.e.* a prior that is proportional to the square root of the determinant of the information matrix. For a specific choice of the parameter  $\beta$ , the latter also holds for the Hausdorff-measure (13) for some nested regression models. When  $\beta$  is the vector of (conditional) classical  $t$ -values of the parameters of the ELR model, the Hausdorff-measure from Theorem 1 is identical to the Jeffreys prior in these models. An example of such a model is the instrumental variables regression model, see Kleibergen and Zivot (2002).

### Borel-Kolmogorov Paradox

The Borel-Kolmogorov paradox, see *e.g.* Kolmogorov (1950), Drèze and Richard (1983), Billingsley (1986) and Wolpert (1995), implies that the probability of a lower dimensional set is not unambiguously defined. Theorems 1-2 state the Hausdorff-measure and integral for lower dimensional sets. These integrals lead to probabilities that are invariant with respect to their specification when this specification accords with Assumption 1. Assumption 1 gives therefore a manner of specifying probabilities on lower dimensional sets that is robust to the Borel-Kolmogorov paradox. The Hausdorff-measure and integrals in Theorems 1 and 2 avoid the Borel-Kolmogorov paradox because we use a limit sequence in which a sequence of sets converges uniformly to the restricted set. Only in case of such uniform convergence does the limit sequence allways converge to the restricted set. We therefore avoid the issue of non-conglomerability, *i.e.* an ambiguous way of reflecting the restriction, that is one element of the Borel-Kolmogorov paradox, see *e.g.* De Finetti (1972). Assumption 1c allows us to obtain the normalizing constant for the Hausdorff-measure.

The traditional example of the Borel-Kolmogorov paradox is one where we have two random variables,  $\varphi$  and  $\lambda$ , with joint density  $p(\varphi, \lambda)$  and we want to condition on a zero value of  $\lambda$ , see *e.g.* Drèze and Richard (1983) and Wolpert (1995). We can, for example, use either  $\lambda$  or  $\mu = \frac{\lambda}{\varphi}$  to construct a limit sequence of sets that converges to the zero value of  $\lambda$ . When we use  $\lambda$  in the limit sequence,  $p(\varphi|\lambda)|_{\lambda=0}$  is the density on the restricted set while  $|\varphi|p(\varphi|\lambda)|_{\lambda=0}$  is the density on the restricted set when we use  $\mu$ . The difference between these densities reflects the Borel-Kolmogorov paradox and shows De Finetti's (1972) issue of non-conglomerability, *i.e.* the restriction can be represented in a non-denumerably infinite number of ways. The difference between the densities arises because the limit sequence does not converge uniformly when we use  $\mu$  to reflect the restriction. If  $\varphi$  equals zero, the limit sequence that involves  $\mu$  is not defined and hence its convergence is not uniform over all values of  $\varphi$ . Assumption 1b implies uniform convergence and the specification that involves  $\mu$  does thus not satisfy Assumption 1b. The specification that involves  $\mu$  does also not satisfy Assumption 1c. The specification that involves  $\lambda$  satisfies Assumption 1.

We conclude from Assumption 1 that the Borel-Kolmogorov paradox is avoided when we only use limit sequences that converge uniformly. A Bayesian specifies the restricted set, *i.e.* the nested models  $G_i$ , a priori and limit sequences that converge uniformly are obvious since they always converge to the sets that reflect the models  $G_i$ . This also holds for the traditional example discussed previously. The arisal of the Borel-Kolmogorov paradox when we use other limit sequences further emphasizes this point.

## 4 Prior density and prior probability

We construct the prior probability of  $G_i$ ,  $i = 1, \dots, n$ , that is induced by  $p_G(\beta)$ . In order to obtain these probabilities we assume that the set of models  $G_i$ ,  $i = 1, \dots, n$ , is complete.

**Assumption 2:** *The true model is an element of  $\{G_i, i = 1, \dots, n\}$  such that the joint prior probability of the regression models  $G_i$ ,  $i = 1, \dots, n$ , is equal to one.*

Assumption 2 shows that we consider the models  $G_i$ ,  $i = 1, \dots, N$ , as mutually exclusive events, unless they result from functions  $f_i(\varphi_i)$  that are invertible transformations of one another, even when one of them equals  $G$  and encompasses all the other models. Hence, all sets  $S_{G_i}$  constitute a mutually exclusive event, the model  $G_i$ , although they are lower dimensional sets in the  $\mathbb{R}^k$ . The probabilities for these events result from the Hausdorff-integral over  $S_{G_i}$  with respect to the prior  $p_G(\beta)$  after an appropriate normalization for the completeness of the set of models  $G_i$ ,  $i = 1, \dots, n$ . The Hausdorff-integrals result from Theorem 2.

**Theorem 3** *When Assumptions 1 and 2 hold, the invariant prior probability for model  $G_i$ ,  $i = 1, \dots, n$ , that is induced by  $p_G(\beta)$  reads*

$$Pr_G [G_i] = \frac{Q_{G_i}}{Q} \quad i = 1, \dots, n, \quad (16)$$

with

$$Q_{G_i} = \int_{S_{G_i}} p_G(\beta) H_{m_i}(d\beta), \quad (17)$$

and

$$Q = \sum_{j=1}^w \int_{\cup_{i=1}^{n_j} S_{i_j}} p_G(\beta) H_{m_j}(d\beta), \quad (18)$$

with  $w$  the number of sets  $S_{G_i}$  that have a different function  $f_i$ ,  $w \leq n$ ,  $n_j$  is the number of sets that have the identical function  $f_j$  (or an invertible transformation thereof),  $m_j$  is the dimension of  $S_{G_j}$  and  $S_{i_j}$ ,  $i_j = 1, \dots, n_j$  are the sets with the same function  $f_j$  involved.

**Proof.** follows directly from Theorem 2. The specification of  $Q$  ensures the completeness that results from Assumption 2. ■

When  $m_i$  equals  $k$ , the Hausdorff-integral is identical to the Lebesque-integral and

$$Q_{G_i} = \int_{S_{G_i}} p_G(\beta) d\beta. \quad (19)$$

If  $m_i$  is less than  $k$ , we obtain from Theorem 2 that

$$Q_{G_i} = \frac{\left[ \frac{\partial Pr_G [\beta(\{\Theta_{G_i}, (-\infty, \lambda_i)\})]}{\partial \lambda_i} \Big|_{\lambda_i=0} \right]}{\left| \frac{\partial \beta(0, \lambda_i)}{\partial \lambda_i} \Big|_{\lambda_i=0} \right|} = \frac{p_G(\lambda_i) \Big|_{\lambda_i=0}}{|A_i|^{\frac{1}{2}}} \left[ \int_{\Theta_{G_i}} p_G(\varphi_i | \lambda_i) \Big|_{\lambda_i=0} d\varphi_i \right], \quad (20)$$

where we have used that

$$\begin{aligned} p_G(\varphi_i, \lambda_i) &= p_G(\beta(\varphi_i, \lambda_i)) |J(\beta, (\varphi_i, \lambda_i))| \\ &= p_G(\varphi_i | \lambda_i) p_G(\lambda_i). \end{aligned} \quad (21)$$

The resulting specification of  $Q$  is then

$$Q = \sum_{j=1}^{w-1} \frac{p_G(\lambda_j) \Big|_{\lambda_j=0}}{|A_j|^{\frac{1}{2}}} \left[ \int_{\cup_{i_j=1}^{n_j} \Theta_{G_{i_j}}} p_G(\varphi_j | \lambda_j) \Big|_{\lambda_j=0} d\varphi_j \right] + \int_{\cup_{i=1}^{n_k} S_{G_{i_w}}} p_G(\beta) d\beta,$$

with  $n_k$  the number of sets of dimension  $k$ . Because of Theorem 2, the prior probability (16) is invariant with respect to the specification of  $\beta$ ,  $(\varphi_i, \lambda_i)$  that satisfy Assumption 1.

### Example Model

For expository purposes, we consider an example with  $n = 2$ . We use this example throughout and explicitly indicate whenever we use it. We use Theorem 3 to obtain the prior probabilities of respectively a nested (non-linear) regression model,

$$G_1 : y = Xf_1(\varphi_1) + \varepsilon, \quad (22)$$

with  $\varphi_1 \in \Theta_{G_1} \subset \mathbb{R}^{m_1}$ , such that

$$S_{G_1} = \{\varphi_1 \in \Theta_{G_1} \subset \mathbb{R}^{m_1} | \beta = f_1(\varphi_1)\}, \quad (23)$$

where  $m_1$  is less than  $k$  and  $f_1(\varphi_1)$  continuous and continuous differentiable, and an ELR model,

$$G_2 : y = X\beta + \varepsilon, \quad (24)$$

with  $\beta \in \mathbb{R}^k$  such that  $S_{G_2} = \{\beta \in \mathbb{R}^k\}$ . Hence, the model set under consideration includes (22) and the encompassing model in (24), that is identical to (1).

The vital element of the applicability of Theorem 3 is the existence of a function  $g_1(\varphi_1, \lambda_1)$  which is such that  $\beta$  and  $(\varphi_1, \lambda_1)$  satisfy the conditions from Assumption 1. It depends on  $f_1(\varphi_1)$  whether  $g_1(\varphi_1, \lambda_1)$  is straightforward to obtain. We therefore give examples of its specification for some commonly used regression models in Section 8. Alongside Assumption 1, we also make Assumption 2. Because  $\int_{S_{G_2}} p_G(\beta) d\beta = 1$ , we obtain the probabilities induced by  $p_G(\beta)$  for  $S_{G_1}$  and  $S_{G_2}$  from Theorem 3,

$$\Pr_G [S_{G_1}] = \frac{Q_{G_1}}{1+Q_{G_1}}, \quad \Pr_G [S_{G_2}] = 1 - \Pr_G [S_{G_1}], \quad (25)$$

with

$$Q_{G_1} = \frac{p_G(\lambda_1)|_{\lambda_1=0}}{|A_1|^{\frac{1}{2}}} \left[ \int_{\Theta_{G_1}} p_G(\varphi_1 | \lambda_1) |_{\lambda_1=0} d\varphi_1 \right]. \quad (26)$$

These prior probabilities imply the prior odds ratio (PROR):

$$\begin{aligned} \text{PROR}[G_1, G_2] &= \frac{\Pr_G[G_1]}{\Pr_G[G_2]} \\ &= Q_{G_1}. \end{aligned} \quad (27)$$

The prior probability from Theorem 3 also implies a prior density of  $\varphi_i$  on  $\Theta_{G_i}$ .

**Theorem 4** *When Assumption 1 holds, the prior probabilities (16) induce the prior densities*

$$\begin{aligned} p_{G_i}(\varphi_i) &= \lim_{\rho \rightarrow 0} \frac{\Pr_G[G_i(\varphi_i, \rho)]}{L_{m_i}[B_{m_i}(\varphi_i, \rho)]} & i = 1, \dots, n, \\ &= \frac{p_G(\varphi_i | \lambda_i) |_{\lambda_i=0}}{\int_{\Theta_{G_i}} p_G(u | \lambda_i) |_{\lambda_i=0} du}, \end{aligned} \quad (28)$$

on  $\Theta_{G_i}$ , where  $B_{m_i}(\varphi_i, \rho)$  is a  $m_i$ -dimensional sphere with radius  $\rho$  centered at  $\varphi_i \in \Theta_{G_i}$  and  $\Pr_G[G_i(\varphi_i, \rho)]$  is the prior probability for  $G_i$  when  $\varphi_i$  only results from  $B_{m_i}(\varphi_i, \rho)$ . The prior density (28) is invariant with respect to transformations of  $\beta$ ,  $(\varphi_i, \lambda_i)$  that satisfy the conditions from Assumption 1 and control for the transformation of  $\beta$ .

**Proof.** see the Appendix. ■

### Example Model

Theorem 4 implies the prior on  $\varphi_1$  in  $G_1$  :

$$p_{G_1}(\varphi_1) = \frac{p_G(\varphi_1|\lambda_1)|_{\lambda_1=0}}{\int_{\Theta_{G_1}} p_G(u|\lambda_1)|_{\lambda_1=0} du}, \quad (29)$$

and on  $\beta$  in  $G_2$  :

$$p_{G_2}(\beta) = p_G(\beta). \quad (30)$$

Theorem 4 implies that these priors are invariant with respect to the specification of  $(\varphi_1, \lambda_1)$  and  $\beta$  that satisfy Assumption 1.

Theorems 3-4 show how we conduct Bayesian inference in regression models that are non-linear in the parameters in a manner that is consistent with the Bayesian inference in ELR models. The latter analysis is well-developed and Theorems 3-4 show how we extend this analysis to regression models that are non-linear in the parameters. For example, sufficient statistics exist for the parameter  $\beta$  in  $G$  and we therefore know how the prior influences the posterior, see *e.g.* Box and Tiao (1992) and Chao and Phillips (1998). By specifying the prior on  $\varphi_i$  in  $G_i$  according to Theorem 4, this property also holds for the prior and posterior of  $\varphi_i$  in  $G_i$ . We discuss it for the posterior in the next section.

## 5 Posterior density and posterior probability

The posterior for  $\beta$  in  $G$  from (1) is obtained by updating the prior with the likelihood:

$$p_G(\beta|D) = \frac{p_G(\beta)\mathcal{L}(D|\beta)}{\int_{\mathbb{R}^k} p_G(u)\mathcal{L}(D|u)du}, \quad (31)$$

where  $\mathcal{L}(D|\beta)$  is the likelihood function, which in our case of standard normal disturbances corresponds with

$$\mathcal{L}(D|\beta) = (2\pi)^{-\frac{1}{2}T} \exp \left[ -\frac{1}{2} (y - X\beta)' (y - X\beta) \right]. \quad (32)$$

However, any other likelihood that is a continuous and continuous differentiable function of  $\beta$  can be used as well. Because the posterior in (31) is a proper density function, and therefore non-negative, we can, analogous to Theorem 3, construct posterior probabilities by usage of Theorem 2.

**Theorem 5** *When Assumptions 1 and 2 hold, the invariant posterior probability for model  $G_i$ ,  $i = 1, \dots, n$ , that is induced by  $p_G(\beta|D)$  (31) reads*

$$Pr_G [G_i|D] = \frac{Q_{G_i|D}}{Q_D} \quad i = 1, \dots, n, \quad (33)$$

with

$$Q_{G_i|D} = \int_{S_{G_i}} p_G(\beta|D) H_{m_i}(d\beta), \quad (34)$$

and

$$Q_D = \sum_{j=1}^w \int_{\cup_{i=1}^{n_j} S_{i_j}} p_G(\beta|D) H_{m_j}(d\beta). \quad (35)$$

**Proof.** results directly from the proofs of Theorem 2. ■

When  $m_i$  equals  $k$ , the Hausdorff-integral is identical to the Lebesgue-integral and

$$Q_{G_i|D} = \int_{S_{G_i}} p_G(\beta|D) d\beta. \quad (36)$$

If  $m_i$  is less than  $k$ , we use Theorem 2 to obtain that

$$Q_{G_i|D} = \frac{p_G(\lambda_i|D)|_{\lambda_i=0}}{|A_i|^{\frac{1}{2}}} \left[ \int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0} d\varphi_i \right], \quad (37)$$

where

$$\begin{aligned} p_G(\varphi_i, \lambda_i|D) &= p_G(\beta(\varphi_i, \lambda_i)|D) |J(\beta, (\varphi_i, \lambda_i))| \\ &= p_G(\varphi_i|\lambda_i, D) p_G(\lambda_i|D). \end{aligned} \quad (38)$$

The accompanying specification of  $Q_D$  is given by

$$Q_D = \sum_{j=1}^{w-1} \frac{p_G(\lambda_j|D)|_{\lambda_j=0} \left[ \int_{\bigcup_{i_j=1}^{n_j} \Theta_{G_{i_j}}} p_G(\varphi_j|\lambda_j, D)|_{\lambda_j=0} d\varphi_j \right]}{|A_j|^{\frac{1}{2}}} + \int_{\bigcup_{i=1}^{n_k} S_{G_{i_w}}} p_G(\beta|D) d\beta. \quad (39)$$

We refer to Theorem 3 for further clarification of the different symbols. Theorem 2 shows that the posterior probabilities are invariant to the specification of  $\beta$ ,  $(\varphi_i, \lambda_i)$  that satisfy Assumption 1.

Analogous to the result in Theorem 4, the posterior probabilities in (33) also imply a posterior density for  $\varphi_i$  on  $\Theta_{G_i}$ .

**Theorem 6** *When Assumption 1 holds, the posterior probabilities (33) induce the posterior densities*

$$p_{G_i}(\varphi_i|D) = \frac{p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{\int_{\Theta_{G_i}} p_G(u|\lambda_i, D)|_{\lambda_i=0} du} \quad i = 1, \dots, n, \quad (40)$$

on  $\Theta_{G_i}$ , and these posterior densities are invariant with respect to the specification of  $\beta$ ,  $(\varphi_i, \lambda_i)$  that satisfy the conditions from Assumption 1.

**Proof.** results directly from the proof of Theorem 4. ■

Naturally, the posterior densities (40) also result when we update the prior  $p_{G_i}(\varphi_i)$  with the likelihood:

$$p_{G_i}(\varphi_i|D) = \frac{p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)}}{\int_{\Theta_{G_i}} p_{G_i}(\psi_i) \mathcal{L}(D|u)|_{u=f_i(\psi_i)} d\psi_i} \quad i = 1, \dots, n. \quad (41)$$

Similarly, the posterior probabilities (33) result from the equality between the posterior odds ratio (POR) and the prior odds ratio (PROR) times the Bayes factor (BF):

$$\text{POR}(G_i, G_j) = \text{PROR}(G_i, G_j) \times \text{BF}(G_i, G_j), \quad (42)$$

where

$$\text{POR}(G_i, G_j) = \frac{\text{Pr}_G[G_i|D]}{\text{Pr}_G[G_j|D]}, \quad \text{PROR}(G_i, G_j) = \frac{\text{Pr}_G[G_i]}{\text{Pr}_G[G_j]}, \quad \text{BF}(G_i, G_j) = \frac{p_{G_i}(D)}{p_{G_j}(D)}, \quad (43)$$

and  $p_{G_i}(D)$  is the marginal data density,

$$\begin{aligned} p_{G_i}(D) &= \int_{\Theta_{G_i}} p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)} d\varphi_i \\ &= c_\beta \times \frac{p_G(\lambda_i|D)|_{\lambda_i=0}}{p_G(\lambda_i)|_{\lambda_i=0}} \times \frac{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0} d\varphi_i}{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i)|_{\lambda_i=0} d\varphi_i}, \end{aligned} \quad (44)$$

with  $c_\beta^{-1} = \int_{\mathbb{R}^k} p_G(\beta) \mathcal{L}(D|\beta) d\beta$ . For a proof of (44) we refer to the Appendix, see also Verdinelli and Wasserman (1995).

The specification of the prior  $p_{G_i}(\varphi_i)$  in (28) satisfies the conditions for the Bayes factor to equal the Savage-Dickey density ratio, see *e.g.* Dickey (1971) and Verdinelli and Wasserman (1995). The Bayes factor is therefore equal to the ratio of the posterior heights divided by the prior heights:

$$\text{BF}(G_i, G_j) = \frac{\left[ \frac{p_G(\lambda_i|D)|_{\lambda_i=0}}{p_G(\lambda_i)|_{\lambda_i=0}} \right] \left[ \frac{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0} d\varphi_i}{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i)|_{\lambda_i=0} d\varphi_i} \right]}{\left[ \frac{p_G(\lambda_j|D)|_{\lambda_j=0}}{p_G(\lambda_j)|_{\lambda_j=0}} \right] \left[ \frac{\int_{\Theta_{G_j}} p_G(\varphi_j|\lambda_j, D)|_{\lambda_j=0} d\varphi_j}{\int_{\Theta_{G_j}} p_G(\varphi_j|\lambda_j)|_{\lambda_j=0} d\varphi_j} \right]}. \quad (45)$$

Substituting this expression for the Bayes factor in (42) results in the posterior odds ratio that accords with the one that results directly from the posterior probabilities (33), *i.e.*

$$\text{POR}(G_i, G_j) = \frac{Q_{G_i|D}}{Q_{G_j|D}}. \quad (46)$$

### Example Model

The Bayes factor for comparing  $G_1$  with  $G_2$  becomes

$$\text{BF}(G_1, G_2) = \left[ \frac{p_G(\lambda_1|D)|_{\lambda_1=0}}{p_G(\lambda_1)|_{\lambda_1=0}} \right] \left[ \frac{\int_{\Theta_{G_1}} p_G(\varphi_1|\lambda_1, D)|_{\lambda_1=0} d\varphi_1}{\int_{\Theta_{G_1}} p_G(\varphi_1|\lambda_1)|_{\lambda_1=0} d\varphi_1} \right] \quad (47)$$

and the posterior odds ratio for comparing  $G_1$  and  $G_2$  reads

$$\begin{aligned} \text{POR}(G_1, G_2) &= \frac{Q_{G_1|D}}{|A_1|^{\frac{1}{2}}} \left[ \int_{\Theta_{G_1}} p_G(\varphi_1|\lambda_1, D)|_{\lambda_1=0} d\varphi_1 \right]. \end{aligned} \quad (48)$$

The first part in the Bayes factor (47) is the Savage-Dickey density ratio, see Dickey (1971) and Verdinelli and Wasserman (1995). The second part arises because the integrals of the conditional densities  $p_G(\varphi_1|\lambda_1, D)|_{\lambda_1=0}$  and  $p_G(\varphi_1|\lambda_1)|_{\lambda_1=0}$  over  $\Theta_{G_1}$  do not have to be equal to one. When  $\Theta_{G_1} = \mathbb{R}^{m_1}$ , the integrals of both conditional densities are equal to one and the Bayes factor simplifies to the usual expression of the Savage-Dickey density ratio.

## 6 Jeffreys-Lindley's Paradox

### Example model

To discuss the Jeffreys-Lindley's paradox, we further simplify our example. Model G,

$$G: y = \iota_T \beta + \varepsilon, \quad (49)$$

with  $y$  a  $T \times 1$  vector of observations on the dependent variable and  $\iota_T$  a  $T \times 1$  vector of ones, now contains only one parameter so  $\beta$  is a scalar and its support is  $\mathbb{R}$ . We specify a normal prior on  $\beta$  with mean  $b$  and variance  $\tau^2$ ,

$$p_G(\beta) = (2\pi\tau^2)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2\tau^2} (\beta - b)^2 \right]. \quad (50)$$

When we combine the prior with the likelihood, we obtain the posterior,

$$p_G(\beta|D) = (2\pi)^{-\frac{1}{2}} \left[ \frac{1}{\tau^2} + T \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \frac{1}{\tau^2} + T \right) (\beta - \tilde{b})^2 \right], \quad (51)$$

with

$$\tilde{b} = \left(\frac{1}{\tau^2} + T\right)^{-1} \left(\frac{1}{\tau^2} b + T\hat{b}\right) \quad (52)$$

and  $\hat{b} = \frac{\sum_{t=1}^T y_t}{T}$ .  $G_1$  is the model that results when  $\beta$  equals zero,

$$G_1: y = \varepsilon, \quad (53)$$

and  $G_2$  is identical to  $G$ . The restriction imposed on  $\beta$  to obtain  $G_1$  is such that it satisfies Assumption 1 for  $\lambda$  identical to  $\beta$ . We use this specification to reflect the difference between  $G_1$  and  $G_2$ . The Bayes factor  $\text{BF}(G_1, G_2)$  for comparing  $G_1$  and  $G_2$  then equals the Savage-Dickey density ratio (47),

$$\begin{aligned} \text{BF}(G_1, G_2) &= \frac{p_G(\lambda|D)|_{\lambda=0}}{p_G(\lambda)|_{\lambda=0}} = \frac{p_G(\beta|D)|_{\beta=0}}{p_G(\beta)|_{\beta=0}} \\ &= [1 + \tau^2 T]^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( T\tilde{b}^2 + \frac{1}{\tau^2} (\tilde{b}^2 - b^2) \right) \right]. \end{aligned} \quad (54)$$

We distinguish two instances of the Jeffreys-Lindley's paradox that imply a degeneracy of the Bayes factor and a distinct difference between classical and Bayesian model comparison, see *e.g.* Lindley (1957), Bernardo and Smith (1994), O'Hagan (1994) and Poirier (1995):

- a. When the prior variance  $\tau^2$  converges to infinity,  $\text{BF}(G_1, G_2)$  goes to infinity:

$$\lim_{\tau^2 \rightarrow \infty} \text{BF}(G_1, G_2) = \infty. \quad (55)$$

- b. When the number of observations  $T$  converges to infinity,  $\text{BF}(G_1, G_2)$  goes to zero unless  $\tilde{\beta}$  is equal to zero:

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{BF}(G_1, G_2) &= 0 && \tilde{\beta} \neq 0 \\ &= \infty && \tilde{\beta} = 0. \end{aligned} \quad (56)$$

We separately discuss these two instances of the Jeffreys-Lindley's paradox and analyze whether the induced probability approach overcomes either one of these two instances.

- a. The prior odds ratio to compare  $G_1$  and  $G_2$  that results from Theorem 3 and (27) is:

$$\begin{aligned} \text{PROR}(G_1, G_2) &= p_G(\lambda(\beta))|_{\lambda=0} |J(\beta, \lambda)|_{\lambda=0}| \\ &= (2\pi\tau^2)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2\tau^2} b^2 \right], \end{aligned} \quad (57)$$

since  $A_1 = 1$ . When we multiply the prior odds ratio with the Bayes factor in (54), or make use of Theorem 5, we obtain the posterior odds ratio

$$\begin{aligned} \text{POR}(G_1, G_2) &= p_G(\lambda(\beta))|_{\lambda=0} |J(\beta, \lambda)|_{\lambda=0}| \\ &= (2\pi)^{-\frac{1}{2}} \left[ \frac{1}{\tau^2} + T \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \frac{1}{\tau^2} + T \right) \tilde{b}^2 \right]. \end{aligned} \quad (58)$$

The posterior odds ratio in (58) does not converge to infinity when the prior variance  $\tau^2$  becomes infinite. Instead, using the expression for  $\tilde{b}$  in (52), it can be shown that

$$\lim_{\tau^2 \rightarrow \infty} \text{POR}(G_1, G_2) = (2\pi)^{-\frac{1}{2}} T^{\frac{1}{2}} \exp \left[ -\frac{1}{2} T \hat{b}^2 \right], \quad (59)$$

where the right-hand side of (59) is a finite non-zero constant for finite  $T$ . Hence, the posterior odds ratio is well-defined in case of an infinite prior variance. Examples of priors with an

infinite variance are non-informative priors. The Bayes factor is infinite when we use such a non-informative prior as shown by (55). Similarly, the prior odds ratio in (57) is equal to zero in case of a non-informative prior, which is obtained by using (57) and letting  $\tau^2$  converge to infinity. Theorem 5, however, still gives a well-defined expression for the posterior odds ratio in case of a non-informative prior. The convergence to zero of the prior odds ratio and the convergence to infinity of the Bayes factor therefore cancel each other out in the posterior odds ratio. We can also use Theorem 5 to obtain the prior that leads to the same posterior odds ratio as the limit expression in (59). This non-informative prior is,

$$p_G(\beta) \propto 1. \quad (60)$$

When the prior variance converges to infinity, the Bayes factor becomes infinite because of the zero value of the prior in the denominator of the Savage-Dickey density ratio in (54). The finite value of the posterior odds ratio therefore shows that the prior odds ratio offsets the zero value in the denominator of the Bayes factor and thus corrects the Bayes factor for the plausibility of the competing models reflected in the prior.

**b.** When  $T$  goes to infinity, the posterior odds ratio in (58) converges to

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{POR}(G_1, G_2) &= \lim_{T \rightarrow \infty} (2\pi)^{-\frac{1}{2}} \left[ \frac{1}{\tau^2} + T \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \frac{1}{\tau^2} + T \right) \tilde{b}^2 \right] \\ &= 0 && \tilde{b} \neq 0 \\ &= \infty && \tilde{b} = 0. \end{aligned} \quad (61)$$

The convergence behavior of the posterior odds ratio is in this case identical to the convergence behavior of the Bayes factor. The posterior odds ratio equals the product of the prior odds ratio and the Bayes factor. Since the prior odds ratio remains fixed, the posterior odds ratio has the same convergence behavior as the Bayes factor.

Another way in which to analyze the limit behavior of the Bayes factor and the posterior odds ratio, when  $T$  goes to infinity, is to express them using a classical “ $t$ -value”,

$$\tilde{z} = \tilde{b} \sqrt{\frac{1}{\tau^2} + T}. \quad (62)$$

The expressions of the Bayes factor and the posterior odds ratio then become respectively

$$\text{BF}(G_1, G_2) = [1 + \tau^2 T]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \tilde{z}^2 - \frac{1}{\tau^2} b^2 \right) \right] \quad (63)$$

and

$$\text{POR}(G_1, G_2) = (2\pi)^{-\frac{1}{2}} \left[ \frac{1}{\tau^2} + T \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \tilde{z}^2 \right]. \quad (64)$$

When the classical  $t$ -value,  $\tilde{z}$ , remains fixed, both the Bayes factor in (63) and the posterior odds ratio in (64) become infinite when  $T$  converges to infinity, see *e.g.* Berger (1985). Classical statistical analysis has in this case a non-zero probability of rejecting  $G_1$  against  $G_2$  since  $\tilde{z}$  remains finite. A Bayesian that uses the posterior odds ratio from (64) or the Bayes factor from (63), however, always chooses  $G_1$ .

The classical  $t$ -value  $\tilde{z}$  is not the sufficient statistic for the posterior of  $\beta$  in (51). To analyze the limit behavior of the Bayes factor in terms of a statistic that is not a sufficient statistic can be considered as rather peculiar. The classical  $t$ -statistic is the sufficient statistic of the posterior of  $\zeta$  with

$$\zeta = \beta \sqrt{\frac{1}{\tau^2} + T}. \quad (65)$$



Expressed in  $\zeta$ , model G from (49) reads

$$\text{G: } y = \nu_T \frac{1}{\sqrt{\frac{1}{\tau^2} + T}} \zeta + \varepsilon. \quad (66)$$

Our parameter of interest is  $\beta$  which is identical to  $\frac{1}{\sqrt{\frac{1}{\tau^2} + T}} \zeta$ . When expressed using the parameter of interest  $\beta$ ,  $G_1$  corresponds with  $\beta = 0$ . The specification of  $\beta$  as a function of  $\zeta$  that is in line with Assumption 1 reads

$$\beta = g(\zeta), \quad (67)$$

with  $g(\zeta) = \frac{1}{\sqrt{\frac{1}{\tau^2} + T}} \zeta$ . In (67),  $G_1$  is obtained when  $\zeta$  equals zero.

We construct the posterior odds ratio and Bayes factor for comparing  $G_1$  and  $G_2$  using (67). The prior on  $\beta$  (50) implies the prior on  $\zeta$ ,<sup>2</sup>

$$p_G(\zeta) = (2\pi)^{-\frac{1}{2}} (1 + \tau^2 T)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (1 + \tau^2 T)^{-1} (\zeta - z)^2 \right], \quad (68)$$

with  $z = b \sqrt{\frac{1}{\tau^2} + T}$ . Similarly, the posterior of  $\zeta$  that is implied by the posterior of  $\beta$  in (51) reads

$$p_G(\zeta|D) = (2\pi)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (\zeta - \tilde{z})^2 \right]. \quad (69)$$

The Bayes factor results from the Savage-Dickey density ratio,

$$\begin{aligned} \text{BF}(G_1, G_2) &= \frac{p_G(\lambda(\zeta)|D)|_{\lambda=0} |J(\zeta, \lambda)|_{\lambda=0}}{p_G(\lambda(\zeta))|_{\lambda=0} |J(\zeta, \lambda)|_{\lambda=0}} \\ &= \left[ 1 + \tau^2 T \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( T \tilde{b}^2 + \frac{1}{\tau^2} (\tilde{b}^2 - b^2) \right) \right], \end{aligned} \quad (70)$$

which is identical to the expression in (54). The posterior odds ratio that results from the prior odds ratio implied by Theorem 3 and the Bayes factor, or directly from Theorem 5 using (69) and (67), becomes,

$$\begin{aligned} \text{POR}(G_1, G_2) &= |J(\beta, \zeta)|_{\zeta=0}^{-1} p_G(\zeta|D)|_{\zeta=0} \\ &= |J(\beta, \zeta)|_{\zeta=0}^{-1} (2\pi)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \tilde{z}^2 \right] \\ &= (2\pi)^{-\frac{1}{2}} \left( \frac{1}{\tau^2} + T \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \tilde{z}^2 \right], \end{aligned} \quad (71)$$

since  $A_1 = J(\beta, \zeta)' J(\beta, \zeta)$  such that  $|A_1|^{\frac{1}{2}} = |J(\beta, \zeta)|$ . The posterior odds ratio in (71) is identical to the posterior odds ratio in (64). This results from the invariance of the Hausdorff-integral stated in Theorem 2. The reason why this invariance occurs is that  $\beta$  is the parameter of interest and we express  $G_1$  in terms of  $\beta$  in both cases.

Instead of  $\beta$ , we can also use  $\zeta$  as our parameter of interest. We then use (66) as the basic specification of G instead of (49). The posterior of  $\zeta$  in (69) shows that  $\tilde{z}$  is a sufficient statistic for this posterior. A representation of  $\zeta$  that satisfies Assumption 1 is

$$\zeta = h(\theta), \quad (72)$$

with  $h(\theta) = \theta$ .  $G_1$  is obtained when  $\theta$  equals zero. We can also express the posterior odds ratio and Bayes factor using specification (72). The Bayes factor that results from the Savage-Dickey density ratio remains identical to (70). The posterior odds ratio that results from

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<sup>2</sup>We note that the prior on  $\zeta$  depends on the data, since  $T = \nu_T' \nu_T$ , and therefore violates the likelihood principle.

Theorem 5, that equals the posterior from (69) evaluated in  $\zeta = 0$  and which can also be obtained from combining the prior odds ratio from Theorem 3 with the Bayes factor, does, however, change

$$\begin{aligned} \text{POR}(G_1, G_2) &= p_G(\zeta|D)|_{\zeta=0} \\ &= (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\tilde{z}^2\right]. \end{aligned} \tag{73}$$

The difference between the posterior odds ratios (71) and (73) shows that the invariance of the Hausdorff-integral is only with respect to specifications that involve the same parameter of interest. Hence, the posterior odds ratios differ depending on whether  $\beta$  or  $\zeta$  is the parameter of interest. When we use  $\zeta$  as our parameter of interest, the posterior odds ratio (73) remains constant when  $T$  converges to infinity and  $\tilde{z}$  remains fixed. It is also only negligibly affected by an increase in the prior variance  $\tau^2$ . Hence, when  $\zeta$  is our parameter of interest neither of the two instances of the Jeffreys-Lindley's paradox occurs.

The posterior odds ratio that we used to compare  $G_1$  and  $G_2$  when  $\beta$  is the parameter of interest, namely (58) (or (64) or (71)), is based on a prior and posterior which are identical to the prior and posterior underlying the posterior odds ratio for  $\zeta$  in (73). The posterior odds ratios therefore only differ because of the different choice of the parameter of interest or, put differently, the basic specification of  $G$ . A Bayesian considers the parameters of a model, from which a set of observations is generated, as a realization from a prior distribution. Hence, for a Bayesian the parameters of the model serve naturally as the parameters of interest. When we use classical  $t$ -values as parameters of interest, the resulting Bayesian analysis shares many features with classical statistical tests, like, for example, the robustness to the Jeffreys-Lindley's paradox. So if one wants to conduct Bayesian inference that is closely related to classical statistical analysis, one should use the classical  $t$ -values as the parameters of interest. We note, however, that there are several problems attached to such an approach. For example, the priors on the classical  $t$ -values violate the likelihood principle and the prior odds ratio therefore depends on the involved data-set. Hence, the prior odds ratio changes when the sample size increases. Another problem concerns extensions to a multiple parameter setting where the prior is then on the vector of classical  $t$ -values. It is not clear what such a prior implies for each individual parameter.

The Bayes factor is popular for model comparison because it is equal to the posterior odds ratio when the prior odds ratio is equal to one, which implies equal prior probabilities for the competing models, see *e.g.* Kass and Raftery (1995). This specification of the posterior odds ratio is affected by the Jeffreys-Lindley's paradox. Loosely speaking the Jeffreys-Lindley's paradox implies that the Bayes factor converges to infinity when the prior variance or the number of observations go to infinity, see *e.g.* Lindley (1957), Bernardo and Smith (1994), O'Hagan (1994) and Poirier (1995). The above example shows that the induced probability approach leads to posterior odds ratios that are partially or fully robust to the Jeffreys-Lindley's paradox. The results from the example extend in a straightforward manner to the general setting of comparing  $n$  regression models. It depends on the choice of the parameter of interest whether the posterior odds ratios that result from the posterior probabilities from Theorem 5 are partially or fully robust to the Jeffreys-Lindley's paradox. When we use the parameters of the encompassing model  $G$ ,  $\beta$ , as the parameters of interest, the posterior odds ratios that result from Theorem 5 are robust against the element of the Jeffreys-Lindley's paradox that is concerned with the prior variance. When we use the classical  $t$ -values of the parameters of the encompassing model as the parameters of interest, the posterior odds ratios are robust against both elements of the Jeffreys-Lindley's paradox.

The robustness against the prior variance element of the Jeffreys-Lindley's paradox implies

that we can obtain the posterior odds ratio using Theorem 5 in case of an improper prior, like, for example, a non-informative prior. A non-informative prior implies that the prior odds ratio that results from Theorem 3 is equal to zero. The Jeffreys-Lindley's paradox implies that the Bayes factor is infinite when we use a non-informative prior. Theorem 5 can be used whenever the posterior is proper and implies that the zero value of the prior odds ratio cancels out the infinite value of the Bayes factor in case of a non-informative prior. To obtain a proper posterior, it is not necessary that the prior integrates to one.

The posterior odds ratios that result from Theorem 5 are related to the posterior information criterium of Phillips and Ploberger, see *e.g.* Phillips and Ploberger (1994,1996) and Phillips (1996). Although the posterior information criterium is intended for time-series only, an identical expression of the posterior odds ratio from Theorem 5 results, apart from the  $2\pi$  terms, when the classical  $t$ -values are our parameters of interest. The  $|A_i|^{-\frac{1}{2}} |J(\beta, (\varphi_i, \lambda_i))|$  term in the Jacobian in Theorem 5 is then identical to the Jeffreys' prior which is the penalty term in the posterior information criterium.

## 7 Nuisance Parameters

For expository purposes, thus far, we have only discussed regression models that contain no nuisance parameters. When model G in (1) is a linear regression model conditional on a realization of a  $l \times 1$  vector of nuisance parameters  $\eta$ , we specify it as

$$G : P_y(\eta)y = P_X(\eta)X\beta + \varepsilon, \quad (74)$$

and model  $G_i$  as

$$G_i : P_y(\eta)y = P_X(\eta)Xf_i(\varphi_i) + \varepsilon, \quad i = 1, \dots, n, \quad (75)$$

where the  $T \times T$  matrices  $P_y(\eta)$  and  $P_X(\eta)$  are observable given a realization of the nuisance parameter vector  $\eta$ . The matrices  $P_y(\eta)$  and  $P_X(\eta)$  incorporate the nuisance parameters such that the disturbances  $\varepsilon : T \times 1$  have a pre-defined distribution that does not depend on nuisance parameters. We specify a joint prior on  $(\beta, \eta)$ ,

$$p_G(\beta, \eta) = p_G(\beta|\eta)p_G(\eta). \quad (76)$$

**Theorem 7** *When Assumptions 1 and 2 hold and model G in (74) is a linear regression model given a realization of the nuisance parameter vector  $\eta$ , the expressions of the prior and posterior probabilities in Theorem 3 and 5 induced by  $p_G(\beta, \eta)$  and  $p_G(\beta, \eta|D)$  remain unaltered when we replace  $Q_{G_i}$  and  $Q_{G_i|D}$  by*

$$\begin{aligned} Q_{G_i} &= \int_{\Theta_\eta} \left[ \int_{\Theta_{G_i}} p_G(\beta|\eta) H_{m_i}(d\beta) \right] p_G(\eta) d\eta, & i = 1, \dots, n, \\ Q_{G_i|D} &= \int_{\Theta_\eta} \left[ \int_{\Theta_{G_i}} p_G(\beta|\eta, D) H_{m_i}(d\beta) \right] p_G(\eta|D) d\eta, & i = 1, \dots, n, \end{aligned} \quad (77)$$

where  $\Theta_\eta$  is the parameter region of  $\eta$ . Similarly, the joint prior and posterior densities of  $(\varphi_i, \eta)$  defined on  $\Theta_{G_i} \times \Theta_\eta$  that result from Theorems 4 and 5 read

$$\begin{aligned} p_{G_i}(\varphi_i, \eta) &= \frac{p_G(\varphi_i, \lambda_i|\eta)|_{\lambda_i=0} p_G(\eta)}{\int_{\Theta_{G_i}} \left[ \int_{\Theta_{G_i}} p_G(\varphi_i, \lambda_i|\eta)|_{\lambda_i=0} d\varphi_i \right] p_G(\eta) d\eta}, & i = 1, \dots, n, \\ p_{G_i}(\varphi_i, \eta|D) &= \frac{p_G(\varphi_i, \lambda_i|\eta, D)|_{\lambda_i=0} p_G(\eta|D)}{\int_{\Theta_{G_i}} \left[ \int_{\Theta_{G_i}} p_G(\varphi_i, \lambda_i|\eta, D)|_{\lambda_i=0} d\varphi_i \right] p_G(\eta|D) d\eta}, & i = 1, \dots, n. \end{aligned} \quad (78)$$

The probabilities that result from (77) and the densities (78) are invariant with respect to transformations of  $\beta$ ,  $(\varphi_i, \lambda_i)$  that satisfy Assumption 1 and control for the transformation of  $\beta$ .

**Proof.** results directly from Theorem 2. ■

Theorem 7 shows that we can extend the invariant probabilities and densities to restrictions on linear regression models that condition on nuisance parameters. These restrictions should, however, be such that they do not involve the nuisance parameters. This explains why we refer to these parameters as nuisance parameters. Assumption 1 should also not involve the nuisance parameters in any of its elements.

For many regression models a function  $g_i(\varphi_i, \lambda_i)$  can be constructed such that the conditions for Theorem 7 are satisfied. Amongst these models are not only linear regression models but also models that are non-linear in the parameters, like, for example, cointegration, instrumental variables and ARMA models. Hence, for all these models prior/posterior probabilities and densities result through Theorem 7 from a prior specified on the parameters of an ELR model. In the next section, we briefly discuss a few examples of these models and focus on the specification of  $g_i(\varphi_i, \lambda_i)$ .

The prior/posterior probabilities and densities stated in Theorem 7 are invariant with respect to transformations that satisfy the conditions from Assumption 1. They are not invariant to transformations that involve the nuisance parameter  $\eta$ . Invariance to these kind of transformations can be achieved by an appropriate specification of the prior  $p_G(\beta, \eta)$  in (76).

## 8 Examples

We discuss some examples of regression models that result from a restriction on the parameters of an ELR model. For reasons of brevity, we only discuss the construction of the specification that satisfies Assumption 1. The explicit expressions of the priors, prior probabilities etc. are left aside but are straightforward to construct given the specification that satisfies Assumption 1. The first example concerns linear restrictions that lead to a nested linear regression model. The second and third example are concerned with non-linear restrictions that lead to a cointegration model and an ARMA(1,1) model. Similar results hold for other regression models that are obtained from non-linear restrictions on the parameters of an ELR model, for example, for the instrumental variables regression models, see Kleibergen and Zivot (2002), and the simultaneous equation model, see Kleibergen (1997) and Kleibergen and van Dijk (1998).

### 8.1 Linear regression model

Our first example considers linear restrictions on the parameters of a linear regression model, see also Tiao *et. al.* (1977),

$$G : y = (X \ Z)\beta + u, \quad (79)$$

where  $y : T \times 1$ ,  $X : T \times m$ ,  $Z : T \times (k - m)$ ,  $m$  is less than  $k$ ,  $\beta : k \times 1$ ,  $\beta \in \mathbb{R}^k$  and  $u \sim N(0, \sigma^2 I_T)$ . Our linear regression model of interest  $G_1$ ,

$$G_1 : y = X\varphi + u, \quad (80)$$

with  $\varphi : m \times 1$ ,  $\varphi \in \mathbb{R}^m$ , is nested in the encompassing model G from (79). We therefore specify  $S_{G_1}$  as

$$S_{G_1} = \left\{ \varphi \in \mathbb{R}^m \mid \beta = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \right\}. \quad (81)$$

The model with which we compare  $G_1$ ,  $G_2$ , is identical to G, such that  $S_{G_2}$  reads

$$S_{G_2} = \{ \beta \in \mathbb{R}^k \}. \quad (82)$$

Because  $\sigma^2 \in \mathbb{R}^+$  is a nuisance parameter, we respecify  $G_1$  and  $G_2$  using the notation introduced in Section 7,

$$\begin{aligned} G_1 : P(\sigma)y &= P(\sigma)X\varphi + \varepsilon, \\ G_2 : P(\sigma)y &= P(\sigma)(X \ Z)\beta + \varepsilon, \end{aligned} \quad (83)$$

where  $P(\sigma) = \sigma^{-1}I_T$  and  $\varepsilon \sim N(0, I_T)$ .

When we specify  $g_i(\varphi_i, \lambda_i)$  as  $g(\varphi, \lambda) = (0 \ I_{k-m})'\lambda$  with  $\lambda : (k - m) \times 1$ , Assumption 1 holds and the specification of  $\beta$  becomes

$$\beta = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \varphi + \begin{pmatrix} 0 \\ I_{k-m} \end{pmatrix} \lambda. \quad (84)$$

Specification (84) satisfies Assumption 1 since it is an invertible relationship and (a.)  $\beta = (I_m \ 0)'\varphi \Leftrightarrow \lambda = 0$ , (b.) all values of  $\varphi$  lead to a unique value of  $\beta$  both when  $\lambda = 0$  and when  $\lambda \neq 0$ , (c.)  $\frac{\partial g(\varphi, \lambda)}{\partial \lambda'} = (0 \ I_{k-m})'$  and does not depend on  $\varphi$  and  $\lambda$ . A prior specified on  $(\beta, \sigma^2)$  in model G therefore implies invariant prior/posterior probabilities for  $G_1$  and  $G_2$  and densities for  $\varphi$  when we apply Theorem 7.

Since  $\frac{\partial \beta}{\partial \varphi' \partial \lambda'}$  does not depend on  $\varphi$  and  $\lambda$ , the priors and posteriors that result from Theorem 7 are identical to the priors and posteriors for  $\varphi$  that typically result when we specify them directly on  $\varphi$ . Hence, inducing the priors and posteriors on the parameters of linear models from the priors and posteriors on the parameters of ELR models does not lead to a notable difference for the resulting priors and posteriors compared to specifying them directly. This property of linear models has mistakenly been thought to hold for regression models that are non-linear in the parameters as well. However, for these models priors that are specified directly on the parameters do not correspond with the priors that are in general used on the parameters of an ELR model. Inducing the prior and posterior probability from the prior and posterior on the parameters of an ELR model does lead to a notable change of the prior and posterior probability in linear models.

## 8.2 Cointegration model

Cointegration implies a non-linear restriction on the parameters of a linear regression model. The restriction that cointegration implies is that the long run multiplier of a vector autoregressive model has a reduced rank value, see *e.g.* Engle and Granger (1987) and Johansen (1991). For a vector autoregressive model of order 1, cointegration with  $r$  cointegrating vectors implies that we can specify it as

$$G_r : \Delta y_t = \alpha_r' \beta_r' y_{t-1} + u_t, \quad r = 0, \dots, k-1, \quad (85)$$

where  $y_t, u_t : k \times 1$ ;  $\Delta y_t = y_t - y_{t-1}$ ,  $\alpha_r', \beta_r' : k \times r$ , and  $u_t, t = 1, \dots, T$ , are independently and identically normal distributed with mean zero and  $k \times k$  covariance matrix  $\Omega$ . In case  $r$

equals zero,  $\alpha'_0\beta'_0$  is a  $k \times k$  matrix of zeros. When we do not impose any normalization on  $\alpha_r$  and  $\beta_r$ , the elements of  $\alpha_r$  and  $\beta_r$  are non-identified since  $\alpha'_r\beta'_r = \alpha'^*_r\beta'^*_r$ ,  $\alpha^*_r = A\alpha_r$  and  $\beta^*_r = \beta_r A^{-1}$  for any non-singular  $r \times r$  matrix  $A$ . We therefore need to normalize either  $\alpha_r$  and  $\beta_r$ . A straightforward normalization is to use  $\beta_r = (I_r - \beta'_{2,r})'$  with  $\beta_{2,r} : (k-r) \times r$ . Because of the invariance of the Hausdorff-integrals to transformations, the chosen normalization has no consequences for the prior and posterior (probabilities) that result from the induced probability approach. Hence, the prior and posterior using one specific normalization are a transformation of the prior and posterior of another normalization, see Kleibergen and Paap (2002) for a proof. The prior and posterior probabilities are the same for all possible normalizations.

We represent the cointegration models  $G_r$  from (85) in matrix notation

$$G_r : Y = X\beta_r\alpha_r + U, \quad r = 0, \dots, k-1, \quad (86)$$

where  $Y = (\Delta y_1 \dots \Delta y_T)'$ ,  $X = (y_0 \dots y_{T-1})'$ ,  $U = (u_1 \dots u_T)'$ . The cointegration models  $G_r$  in (86) are nested in the multi-variate ELR model

$$G : Y = X\Pi + U, \quad (87)$$

with  $\Pi : k \times k$ . We specify the cointegration models  $G_r$  in (86) and the ELR model  $G$  from (87) in line with Theorem 7 as

$$\begin{aligned} G_r : P(\Omega)\text{vec}(Y) &= P(\Omega)(I_k \otimes X)\text{vec}(\beta_r\alpha_r) + \text{vec}(\varepsilon), & r = 0, \dots, k-1, \\ G : P(\Omega)\text{vec}(Y) &= P(\Omega)(I_k \otimes X)\text{vec}(\Pi) + \text{vec}(\varepsilon), \end{aligned} \quad (88)$$

where  $P(\Omega) = (\Omega^{-\frac{1}{2}} \otimes I_T)$ ,  $\varepsilon = U\Omega^{-\frac{1}{2}}$ ,  $\text{vec}(\varepsilon) \sim N(0, I_{kT})$ . Equation (88) shows that  $G_r$ ,  $r = 0, \dots, k-1$ , is represented by the lower dimensional sets

$$S_{G_r} = \left\{ \alpha_r \in \mathbb{R}^{k,r}, \beta_{2,r} \in \mathbb{R}^{(k-r),r} \mid \Pi = \begin{pmatrix} I_r \\ -\beta_{2,r} \end{pmatrix} \alpha_r \right\}, \quad r = 0, \dots, k-1, \quad (89)$$

so  $S_{G_0}$  only consists of the  $k \times k$  matrix of zeros.

The unrestricted full rank model  $G_k$  is identical to  $G$  such that  $S_{G_k}$  reads

$$S_{G_k} = \{ \Pi \in \mathbb{R}^{k,k} \}. \quad (90)$$

The sets  $S_{G_r}$ ,  $r = 0, \dots, k$  are such that  $S_{G_0} \subset S_{G_1} \subset \dots \subset S_{G_k}$ .

Because cointegration imposes a non-linear restriction on the parameters of a linear regression model, the specification of a function  $g_i(\varphi_i, \lambda_i)$  that makes Assumption 1 hold is rather difficult to obtain. In Kleibergen and Paap (2002) a specification of  $\Pi$  that, results from a singular value decomposition and, satisfies Assumption 1 is given:

$$\begin{aligned} \Pi &= \beta_r\alpha_r + \beta_{r,\perp}\lambda_r\alpha_{r,\perp}, & r = 1, \dots, k-1, \\ &\Leftrightarrow & \\ \text{vec}(\Pi) &= \text{vec}(\beta_r\alpha_r) + (\alpha'_{r,\perp} \otimes \beta_{r,\perp})\text{vec}(\lambda_r), & r = 1, \dots, k-1, \end{aligned} \quad (91)$$

where  $\lambda_r : (k-r) \times (k-r)$ ;  $\beta_{r,\perp}$ ,  $\alpha'_{r,\perp} : k \times (k-r)$  and  $\beta'_{r,\perp}\beta_r \equiv 0$ ,  $\beta'_{r,\perp}\beta_{r,\perp} \equiv I_{k-r}$ ,  $\alpha_r\alpha'_{r,\perp} \equiv 0$ ,  $\alpha_{r,\perp}\alpha'_{r,\perp} \equiv I_{k-r}$ , such that

$$g_r(\varphi_r, \lambda_r) = (\alpha'_{r,\perp} \otimes \beta_{r,\perp})\text{vec}(\lambda_r), \quad r = 1, \dots, k-1, \quad (92)$$

with  $\varphi_r = (\alpha_r, \beta_{2,r})$ . When  $r$  equals 0,  $\lambda_0 = \Pi$  since  $\beta_0\alpha_0$  is a  $k \times k$  matrix of zeros. Kleibergen and Paap (2002) show that an invertible relationship between  $\Pi$  and  $(\alpha_r, \beta_{2,r}, \lambda_r)$  exists.

Furthermore, with respect to the conditions of Assumption 1: (a.)  $\Pi = \beta_r \alpha_r$  is equivalent to  $\lambda_r = 0$ . (b.)  $(\alpha_r, \beta_{2,r})$  implies a unique value of  $\beta_r \alpha_r$  when  $\alpha_r$  has full rank and identically  $(\alpha_r, \beta_{2,r}, \lambda_r)$  implies a unique value of  $\beta_r \alpha_r + \beta_{r,\perp} \lambda_r \alpha_{r,\perp}$  when  $\alpha_r$  has full rank. (c.)  $\frac{\partial g_r(\varphi_r, \lambda_r)}{\partial \text{vec}(\lambda_r)'} = (\alpha'_{r,\perp} \otimes \beta_{r,\perp})$  such that  $\left(\frac{\partial g_r(\varphi_r, \lambda_r)}{\partial \text{vec}(\lambda_r)'}\right)' \left(\frac{\partial g_r(\varphi_r, \lambda_r)}{\partial \text{vec}(\lambda_r)'}\right) = I_{(k-r)^2}$  and does not depend on  $\varphi_r$  and  $\lambda_r$ . Hence, all conditions of Assumption 1 are satisfied. Theorem 7 therefore applies and a prior specified on  $(\Pi, \Omega)$  in  $G$  implies a prior probability for  $G_r$ ,  $r = 0, \dots, k$ , and a prior for  $(\alpha_r, \beta_{2,r}, \Omega)$  in  $G_r$  that are invariant with respect to the specification of  $\Pi$  and  $(\alpha_r, \beta_{2,r}, \lambda_r)$  that satisfy Assumption 1. For more details about the resulting Bayesian analysis of the cointegration model and the expressions of the priors and posteriors, we refer to Kleibergen and Paap (2002).

### 8.3 ARMA(1,1)

Another model that results from a non-linear restriction on the parameters of an ELR model is the ARMA(1,1) model, see *e.g.* Box *et. al.* (1994),

$$G_1 : y_t = \rho y_{t-1} - \alpha u_{t-1} + u_t, \quad t = 1, \dots, T, \quad (93)$$

where the disturbances  $u_t$  are independently and identically distributed,  $u_t \sim N(0, \sigma^2)$ . When we recurrently substitute  $u_{t-1}$  in (93), we obtain

$$G_1 : y_t = (\rho - \alpha) \sum_{j=1}^T \alpha^{j-1} y_{t-j} + u_t, \quad t = 1, \dots, T. \quad (94)$$

We specify (94) as a regression model that is non-linear in the parameters,

$$G_1 : y = X f(\alpha, \rho) + u, \quad (95)$$

where  $y = (y_1 \dots y_T)'$ ,  $X = (x_1 \dots x_T)'$ ,  $x_i = (y_{i-1} \dots y_0 \ 0 \dots 0)'$  :  $T \times 1$ ,  $i = 1, \dots, T$ ;  $u = (u_1 \dots u_T)'$ , and

$$f(\alpha, \rho) = (\rho - \alpha) \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{T-1} \end{pmatrix} : T \times 1. \quad (96)$$

$G_1$  in (95) is nested in the ELR model

$$G : y = X \beta + u, \quad (97)$$

with  $\beta : T \times 1$ . We specify both  $G_1$  and  $G$  in the notation of Theorem 7,

$$\begin{aligned} G_1 : P(\sigma) y &= P(\sigma) X f(\alpha, \rho) + \varepsilon, \\ G : P(\sigma) y &= P(\sigma) X \beta + \varepsilon, \end{aligned} \quad (98)$$

with  $P(\sigma) = \sigma^{-1} I_T$  and  $\varepsilon \sim N(0, I_T)$ .

The ARMA(1,1) model imposes a non-linear restriction on the parameters of  $G$ ,  $\beta = f(\alpha, \rho)$ . This implies that it is not straightforward to obtain a specification of  $g_i(\varphi_i, \lambda_i)$  that

satisfies Assumption 1. A (unrestricted) specification of  $\beta$  that gives such a function  $g_i(\varphi_i, \lambda_i)$  is

$$\beta = (\rho - \alpha) \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{T-1} \end{pmatrix} + \begin{pmatrix} 0 \\ I_{T-2} \end{pmatrix} \lambda, \quad (99)$$

with  $\lambda : (T - 2) \times 1$  and  $g(\varphi, \lambda) = (0 \ I_{T-2})' \lambda$  with  $\varphi = (\alpha, \rho)$ . Equation (99) satisfies the conditions from Assumption 1 since:  $\beta$  has an invertible relationship with  $(\alpha, \rho, \lambda)$ , (a.)  $\beta = f(\alpha, \rho) \Leftrightarrow \lambda = 0$ , (b.)  $(\rho, \alpha)$  implies a unique value of  $f(\alpha, \rho)$  when  $\rho - \alpha \neq 0$ , identically  $(\rho, \alpha, \lambda)$  implies a unique value of  $f(\alpha, \rho) + (0 \ I_{T-2})' \lambda$  when  $\rho - \alpha \neq 0$ , (c.)  $\frac{\partial g(\varphi, \lambda)}{\partial \lambda} = (0 \ I_{T-2})'$  and independent of  $(\alpha, \rho, \lambda)$ . Theorem 7 therefore applies and a prior that is specified on  $(\beta, \sigma^2)$  in G induces a prior probability for  $G_1$  and a prior on  $(\alpha, \rho, \sigma^2)$  that are invariant with respect to the specification of  $\beta$  and  $(\alpha, \rho, \lambda)$  that satisfy Assumption 1. For more details on the resulting Bayesian analysis of the ARMA(1,1) model, we refer to Kleibergen and Hoek (1999).

## 9 Conclusions

We obtain expressions for prior/posterior probabilities and densities of the parameters of nested regression models that are induced by the prior/posterior on the parameters of an encompassing linear regression model. The resulting probabilities and densities are invariant with respect to specifications that satisfy a necessary set of assumptions. Hence, by specifying a prior and a likelihood for the parameters of an encompassing linear regression model, we obtain a complete Bayesian analysis, that includes both prior/posterior probabilities and densities, for all of its nested regression models that allow for a specification that satisfies the set of assumptions. The resulting Bayesian analyzes of these nested regression models are consistent with one another.

The Bayes factor in the resulting analysis corresponds with the Savage-Dickey density ratio and equals the ratio of the posterior and prior height in the hypothesized parameter point. When we multiply the Bayes factor with the prior odds ratio, we obtain the posterior odds ratio. Because the prior and prior probability result from the same prior on the parameters of the encompassing linear regression model, the posterior odds ratio is such that the prior odds ratio corrects the Bayes factor for the plausibility of the competing models reflected in the prior. The posterior odds ratio is therefore robust to increases in the prior variance which is an element of the Jeffreys-Lindley's paradox.

Applications of the above results are especially important for regression models that result from non-linear restrictions on the parameters of encompassing linear regression models. In these models, the resulting analysis leads to priors and posteriors that are different from the ones that are used traditionally. The traditional Bayesian analysis leads to anomalies in these models, like, for example, in simultaneous equation models, see Kleibergen (1997) and Kleibergen and van Dijk (1998); cointegration models, see Kleibergen and van Dijk (1994) and Martin and Martin (2000); and fractional cointegration models, see Martin (2001). When we deduce the priors and posteriors of the parameters in these models from priors and posteriors on the parameters of encompassing linear regression models, these anomalies disappear, see *e.g.* Kleibergen (1997), Kleibergen and van Dijk (1998) and Kleibergen and van Paap (2002).

## Appendix



**Proof of Equation (11).**

The normalizing constant is specified as the transformation over  $g_i$  of the  $(k-m_i)$ -dimensional sphere with center zero and radius  $\rho$  :

$$\begin{aligned}
c_i(\rho)^{-1} &= L_{k-m_i}(g_i(0, B_i(0, \rho))) \\
&= \int_{B_{k-m_i}(0, \rho)} \left| \frac{\partial g_i}{\partial \lambda_i} \right| L_{k-m_i}(d\lambda_i) \\
&= \int_{B_{k-m_i}(0, \rho)} \left| \left( \frac{\partial g_i}{\partial \lambda_i} \right)' \left( \frac{\partial g_i}{\partial \lambda_i} \right) \right|^{\frac{1}{2}} d\lambda_i \\
&= \int_{B_{k-m_i}(0, \rho)} |A_i|^{\frac{1}{2}} d\lambda_i \\
&= |A_i|^{\frac{1}{2}} \int_{B_{k-m_i}(0, \rho)} d\lambda_i \\
&= |A_i|^{\frac{1}{2}} V_{k-m_i}(\rho),
\end{aligned}$$

with  $V_{k-m_i}(\rho)$  the volume of the  $(k-m_i)$ -dimensional sphere with radius  $\rho$ . Because  $|A_i|$  does not depend on  $\varphi_i$  and  $\lambda_i$ , the value of  $\varphi_i$  in  $g_i(\varphi_i, \lambda_i)$  and the center of the sphere  $B_i(0, \rho)$  do not affect the expression of  $c_i(\rho)$  and we have therefore used vectors of zeros for convenience.

**Proof of Theorem 1.**

Before we obtain the specification of the Hausdorff-measure, we note the structure that Assumption 1 imposes on the Jacobian of the transformation from  $\beta$  to  $(\varphi_i, \lambda_i)$  :

$$J(\beta, (\varphi_i, \lambda_i)) = \begin{pmatrix} \frac{\partial f_i}{\partial \varphi_i} + \frac{\partial g_i}{\partial \varphi_i} & \frac{\partial g_i}{\partial \lambda_i} \end{pmatrix}.$$

Because  $g_i(\varphi_i, \lambda_i)$  is a strictly monotonic function of  $\lambda_i$  and  $g_i(\varphi_i, \lambda_i) = 0 \Leftrightarrow \lambda_i = 0$ ,  $\frac{\partial g_i}{\partial \varphi_i}|_{\lambda_i=0} = 0$ . Hence, the Jacobian in  $\lambda_i = 0$  reads

$$J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0} = \begin{pmatrix} \frac{\partial f_i}{\partial \varphi_i} & \frac{\partial g_i}{\partial \lambda_i}|_{\lambda_i=0} \end{pmatrix},$$

and

$$\begin{aligned}
|J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0}| &= \left| \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right)' \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right) \right|^{\frac{1}{2}} \left| \left( \frac{\partial f_i}{\partial \varphi_i} \right)' M \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right) \left( \frac{\partial f_i}{\partial \varphi_i} \right) \right|^{\frac{1}{2}} \\
&= \left| \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right)' M \left( \frac{\partial f_i}{\partial \varphi_i} \right) \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right) \right|^{\frac{1}{2}} \left| \left( \frac{\partial f_i}{\partial \varphi_i} \right)' \left( \frac{\partial f_i}{\partial \varphi_i} \right) \right|^{\frac{1}{2}},
\end{aligned}$$

where  $\left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right)' \left( \frac{\partial g_i}{\partial \lambda_i} \Big|_{\lambda_i=0} \right) = \left( \frac{\partial g_i}{\partial \lambda_i} \right)' \left( \frac{\partial g_i}{\partial \lambda_i} \right) = A_i$ .

The Hausdorff-measure  $H_{m_i}(W_{G_i})$  is obtained by considering that  $g_i(\varphi_i, \lambda_i)$  is a strictly monotonic function of  $\lambda_i$ . We use the sequence of sets  $W_{G_i}(\rho)$  centered at  $\lambda_i = 0$ ,

$$W_{G_i}(\rho) = \left\{ \varphi_i \in \Omega_{G_i} \subset \mathbb{R}^{m_i}, \lambda_i \in B_{k-m_i}(0, \rho) \subset \mathbb{R}^{k-m_i} \mid \beta = f(\varphi_i) + g_i(\varphi_i, \lambda_i) \right\},$$

where  $B_{k-m_i}(0, \rho)$  is a  $(k-m_i)$ -dimensional sphere with radius  $\rho$  centered at 0. We use a limiting sequence of  $W_{G_i}(\rho)$  that is obtained by letting  $\rho$  converge to zero,

$$\lim_{\rho \rightarrow 0} W_{G_i}(\rho) = W_{G_i}.$$

This results because  $g_i(\varphi_i, \lambda_i)$  is a strictly monotonic function of  $\lambda$ .

Because  $\left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right)' \left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right) = A_i$  and  $\frac{\partial g_i}{\partial \varphi_i'}\bigg|_{\lambda_i=0} = 0$ , the Lebesgue-measure of  $W_{G_i}(\rho)$ ,  $L_k(W_{G_i}(\rho))$ , is for small values of  $\rho$  equal to:

$$\begin{aligned} L_k(W_{G_i}(\rho)) &= \int_{\Omega_{G_i}} \int_{B_{G_i}(0,\rho)} |J(\beta, (\varphi_i, \lambda_i))| d\lambda_i d\varphi_i \\ &\approx \int_{\Omega_{G_i}} \left| \left(\frac{\partial f_i}{\partial \varphi_i'}\right)' M_{\left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right)} \left(\frac{\partial f_i}{\partial \varphi_i'}\right) \right|^{\frac{1}{2}} \left[ \int_{B_{k-m_i}(0,\rho)} |A_i|^{\frac{1}{2}} d\lambda_i \right] d\varphi_i \\ &\approx \left\{ \int_{\Omega_{G_i}} \left| \left(\frac{\partial f_i}{\partial \varphi_i'}\right)' M_{\left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right)} \left(\frac{\partial f_i}{\partial \varphi_i'}\right) \right|^{\frac{1}{2}} d\varphi_i \right\} |A_i|^{\frac{1}{2}} V_{k-m_i}(\rho). \end{aligned}$$

The Hausdorff-measure equals the limit of  $c_i(\rho)$  times  $L_k(W_{G_i}(\rho))$  when  $\rho$  converges to zero:

$$\begin{aligned} H_{m_i}(W_{G_i}) &= \lim_{\rho \rightarrow 0} [c_i(\rho) L_k(W_{G_i}(\rho))] \\ &= \lim_{\rho \rightarrow 0} \frac{1}{|A_i|^{\frac{1}{2}} V_{k-m_i}(\rho)} \left\{ \int_{\Omega_{G_i}} \left| \left(\frac{\partial f_i}{\partial \varphi_i'}\right)' M_{\left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right)} \left(\frac{\partial f_i}{\partial \varphi_i'}\right) \right|^{\frac{1}{2}} d\varphi_i \right\} |A_i|^{\frac{1}{2}} V_{k-m_i}(\rho) \\ &= \int_{\Omega_{G_i}} \left| \left(\frac{\partial f_i}{\partial \varphi_i'}\right)' M_{\left(\frac{\partial g_i}{\partial \lambda_i'}\bigg|_{\lambda_i=0}\right)} \left(\frac{\partial f_i}{\partial \varphi_i'}\right) \right|^{\frac{1}{2}} d\varphi_i. \end{aligned}$$

To show the invariance of the Hausdorff-measure, we consider an invertible function  $h : \mathbb{R}^k \rightarrow \mathbb{R}^k$ ,  $\mu = h(\beta)$ . Because of Assumption 1, we can specify  $\beta$  as

$$\beta = f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)$$

and  $\mu$  can therefore be specified as

$$\mu = l_i(\psi_i) + r_i(\psi_i, \theta_i),$$

with  $l_i(\psi_i) = h(f_i(\varphi_i))$  and  $r_i(\psi_i, \theta_i) = h(f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)) - h(f_i(\varphi_i))$ . Because of Assumption 1b, that  $g_i(\varphi_i, \lambda_i)$  is a strictly monotonic function of  $\lambda_i$ ,  $h$  has to be strict monotonic. This implies that  $\left(\frac{\partial h}{\partial \beta'}\right)' \left(\frac{\partial h}{\partial \beta'}\right)$  is a positive definite symmetric matrix for all values of  $\beta$  and that  $\theta_i$  is an invertible function of  $\lambda_i$  only. Because of Assumption 1c,  $\frac{\partial r_i}{\partial \theta_i'} = \frac{\partial \mu}{\partial \beta'} \frac{\partial \beta}{\partial \lambda_i'} \frac{\partial \lambda_i}{\partial \theta_i'} = \frac{\partial h}{\partial \beta'} \frac{\partial g_i}{\partial \lambda_i'} \frac{\partial \lambda_i}{\partial \theta_i'}$  should be such that

$$\begin{aligned} \left(\frac{\partial r_i}{\partial \theta_i'}\right)' \left(\frac{\partial r_i}{\partial \theta_i'}\right) &= B_i \Leftrightarrow \\ \left(\frac{\partial h}{\partial \beta'} \left(\frac{\partial g_i}{\partial \lambda_i'}\right) \left(\frac{\partial \lambda_i}{\partial \theta_i'}\right)\right)' \left(\frac{\partial h}{\partial \beta'} \left(\frac{\partial g_i}{\partial \lambda_i'}\right) \left(\frac{\partial \lambda_i}{\partial \theta_i'}\right)\right) &= B_i \Leftrightarrow \\ \left(\frac{\partial \lambda_i}{\partial \theta_i'}\right)' \left(\frac{\partial g_i}{\partial \lambda_i'}\right)' \left(\frac{\partial h}{\partial \beta'}\right)' \left(\frac{\partial h}{\partial \beta'}\right) \left(\frac{\partial g_i}{\partial \lambda_i'}\right) \left(\frac{\partial \lambda_i}{\partial \theta_i'}\right) &= B_i, \end{aligned}$$

with  $B_i$  independent of  $\psi_i$ . Since  $\theta_i$  is an invertible function of  $\lambda_i$  only and  $\left(\frac{\partial g_i}{\partial \lambda_i'}\right)' \left(\frac{\partial g_i}{\partial \lambda_i'}\right) = A_i$ , with  $A_i$  independent of  $(\varphi_i, \lambda_i)$ ,  $\left(\frac{\partial h}{\partial \beta'}\right)' \left(\frac{\partial h}{\partial \beta'}\right)$  should be equal to some fixed positive definite symmetric matrix that is independent of  $\beta$ . Unlike  $g_i$ ,  $h$  is an invertible function such that the only specification of  $h$  that satisfies all conditions is an invertible linear function. Hence every specification  $\mu = l_i(\psi_i) + r_i(\psi_i, \theta_i)$  that satisfies Assumption 1 is such that (1.)  $\mu$  is an invertible linear function of  $\beta$  and (2.)  $\theta_i$  is an invertible function of  $\lambda_i$  only and  $\psi_i$  is an invertible function of  $\varphi_i$  only. It is straightforward to show that these transformations lead to an identical Hausdorff-measure.

**Proof of Theorem 2.**

For small values of  $\rho$ , the expression of  $\int_{W_{G_i}(\rho)} q(\beta) d\beta$  reads:

$$\begin{aligned} \int_{W_{G_i}(\rho)} q(\beta) d\beta &= \int_{\Omega_{G_i}} \int_{B_{k-m_i}(0,\rho)} q(\beta(\varphi_i, \lambda_i)) |J(\beta, (\varphi_i, \lambda_i))| d\lambda_i d\varphi_i \\ &\approx \int_{\Omega_{G_i}} \left[ \int_{B_{k-m_i}(0,\rho)} q(\beta(\varphi_i, \lambda_i)|_{\lambda_i=0}) |J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0} d\lambda_i \right] d\varphi_i \\ &\approx \left\{ \int_{\Omega_{G_i}} q(\beta(\varphi_i, \lambda_i)|_{\lambda_i=0}) |J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0} d\varphi_i \right\} V_{k-m_i}(\rho). \end{aligned}$$

The Hausdorff-integral is then obtained by,

$$\begin{aligned} \int_{W_{G_i}} q(\beta) H_{m_i}(d\beta) &= \lim_{\rho \rightarrow 0} \left[ c_i(\rho) \int_{W_{G_i}(\rho)} q(\beta) d\beta \right] \\ &= \lim_{\rho \rightarrow 0} \left[ \frac{\left\{ \int_{\Theta_{G_i}} q(\beta(\varphi_i, \lambda_i)|_{\lambda_i=0}) |J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0} d\varphi_i \right\} V_{k-m_i}(\rho)}{|A_i|^{\frac{1}{2}} V_{k-m_i}(\rho)} \right] \\ &= \frac{1}{|A_i|^{\frac{1}{2}}} \left\{ \int_{\Theta_{G_i}} q(\beta(\varphi_i, \lambda_i)|_{\lambda_i=0}) |J(\beta, (\varphi_i, \lambda_i))|_{\lambda_i=0} d\varphi_i \right\}. \end{aligned}$$

The proof of the invariance of the Hausdorff-integral to specifications of  $\beta, (\varphi_i, \lambda_i)$  that satisfy Assumption 1 is analogous to the proof for Theorem 1.

**Proof of Theorem 4.**

Equation (28) gives the definition of a density function. The invariance of it follows from the proof of Theorem 1. We have shown in this proof that when

$$\beta = f_i(\varphi_i) + g_i(\varphi_i, \lambda_i)$$

and

$$\mu = l_i(\psi_i) + r_i(\psi_i, \theta_i),$$

are two specifications that satisfy assumption 1 that  $\psi_i$  is an invertible function of  $\varphi_i$  only and  $\theta_i$  is an invertible function of  $\lambda_i$  only. Hence, we can independently transform  $\varphi_i$  to  $\psi_i$  and  $\lambda_i$  to  $\theta_i$ . This does not affect the specification of the prior from Theorem 4.

### Proof of equation (44)

$$\begin{aligned}
p_{G_i}(D) &= \int_{\Theta_{G_i}} p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)} d\varphi_i \\
&= p_G(\lambda_i|D)|_{\lambda_i=0} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)}}{p_G(\lambda_i|D)|_{\lambda_i=0}} \right] d\varphi_i \right\} \\
&= p_G(\lambda_i|D)|_{\lambda_i=0} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{p_G(\lambda_i|D)|_{\lambda_i=0} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}} \right] d\varphi_i \right\} \\
&= p_G(\lambda_i|D)|_{\lambda_i=0} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{p_G(\varphi_i, \lambda_i|D)|_{\lambda_i=0}} \right] d\varphi_i \right\} \\
&= p_G(\lambda_i|D)|_{\lambda_i=0} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_{G_i}(\varphi_i) \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{\frac{p_G(\varphi_i, \lambda_i)|_{\lambda_i=0} \mathcal{L}(D|\beta)|_{\beta=f_i(\varphi_i)}}{c_\beta}} \right] d\varphi_i \right\} \\
&= c_\beta \times p_G(\lambda_i|D)|_{\lambda_i=0} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_{G_i}(\varphi_i) p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{p_G(\varphi_i, \lambda_i)|_{\lambda_i=0}} \right] d\varphi_i \right\} \\
&= c_\beta \times \frac{p_G(\lambda_i|D)|_{\lambda_i=0}}{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i)|_{\lambda_i=0} d\varphi_i} \left\{ \int_{\Theta_{G_i}} \left[ \frac{p_G(\varphi_i|\lambda_i)|_{\lambda_i=0} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0}}{p_G(\lambda_i)|_{\lambda_i=0} p_G(\varphi_i|\lambda_i)|_{\lambda_i=0}} \right] d\varphi_i \right\} \\
&= c_\beta \times \frac{p_G(\lambda_i|D)|_{\lambda_i=0}}{p_G(\lambda_i)|_{\lambda_i=0}} \times \frac{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i, D)|_{\lambda_i=0} d\varphi_i}{\int_{\Theta_{G_i}} p_G(\varphi_i|\lambda_i)|_{\lambda_i=0} d\varphi_i}
\end{aligned}$$

where

$$\begin{aligned}
c_\beta^{-1} &= \int_{\mathbb{R}^k} p_G(\beta) \mathcal{L}(D|\beta) d\beta \\
&= \int_{\mathbb{R}^{m_i}} \int_{\mathbb{R}^{k-m_i}} p_G(\varphi_i, \lambda_i) \mathcal{L}(D|\beta(\lambda_i, \varphi_i)) d\lambda_i d\varphi_i, \\
p_{G_i}(\varphi_i) &= \frac{p_G(\varphi_i|\lambda_i)|_{\lambda_i=0}}{\int_{\Theta_{G_i}} p_G(u|\lambda_i)|_{\lambda_i=0} du}.
\end{aligned}$$

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