Affine Markov processes on a general state space
Veerman, E.

Citation for published version (APA):
Introduction

Since the rediscovery of Bachelier’s work on modeling stock prices with Brownian motion, many attempts in finance and mathematical finance have been made to capture the movements of the financial market in a mathematical framework. Traditionally, a main instrument for modeling asset prices has been the class of semimartingales, in particular continuous diffusions that arise as solutions to certain stochastic differential equations (SDEs). We mention the models introduced in the last decades of the previous century, such as the Black-Scholes model \([3]\), where the stock price is modeled by a geometric Brownian motion, the Vasicek model \([53]\), where the short interest rate is described as an Ornstein-Uhlenbeck process, the Cox-Ingersoll-Ross model \([9]\), which uses a square root SDE for the interest rate, the affine term structure model by Duffie and Kan \([18]\), which extends the CIR-model to multiple dimensions, and the Heston model \([29]\), which incorporates stochastic volatility into the Black-Scholes model by adding a square root SDE for the variance. All these models share the desired property that they are mathematically tractable. They do not only capture the dynamics of the market (reasonably) well, but also allow for mathematical analysis and the performance of calculations. In particular they make calibration possible.

At the end of the previous century, the above mentioned “classical” models were all subsumed into one framework in the pioneering paper \([19]\) by Duffie, Pan and Singleton. They introduced a class of processes, adopted with the name affine jump-diffusions, that are characterized as solutions of multi-dimensional SDEs, including jumps, where the drift vector, the instantaneous diffusion matrix and the arrival rate of the jumps all depend in an affine way on the current state of the process. Analytic expressions were provided for various transforms of these processes, including Fourier and Laplace transforms. Elaborating on this, in the seminal
paper [17] by Duffie, Filipović and Schachermayer, a fundamental and rigorous mathematical treatment has been given for an even more general class of affine jump-diffusions, called affine processes, which also allows for killing and explosion. These processes are characterized as Markov processes where the logarithm of the characteristic function of the transition function has affine dependence on the initial state of the process. Having a closed form expression for the Fourier transform at hand, Duffie, Filipović and Schachermayer provide examples of financial applications of affine processes, including interest rate term structure modeling, option pricing and risk modeling.

The class of affine processes can be regarded as a complete generalization of and an improvement upon the previously mentioned classical models. Affine processes not only preserve the desired property of mathematical tractability from the old models, but also realize a better fit of the market. The latter is due to their flexibility, since multiple (macro-economical) factors can be included in an affine process. Moreover, as observed in [20], modeling the driving force of stock movements solely by Brownian motion appears to be unsatisfactory, for the tails of the Gaussian distribution are too thin. Incorporating jumps in the model is a possible remedy for this deficiency.

This is not the end of the story though (indeed, it’s the start). A standing assumption in [17] is that the state space in which the affine process takes its values, is of the so-called canonical form $\mathbb{R}_+^m \times \mathbb{R}^{p-m}$, a notion that was introduced in [14]. A consequence of this is that the instantaneous covariance matrix is essentially a diagonal matrix, in other words, infinitesimal small movements of the diffusion part are mutually uncorrelated. For applications this might be problematic, see for instance [50], where an affine model for interest rate and inflation is considered. In that paper it is shown that the mathematical restrictions implied by the shape of the state space, are in contrast with economic principles for the interplay between interest rate and inflation, wherefore they are simply dropped. Though the resulting model is not of the affine type anymore, still the bond prices can be accurately approximated by the closed form expressions for affine processes, as shown by Monte Carlo simulations.

An alternative way to overcome these modeling issues, is to extend the flexibility of affine processes by seeking for other state spaces than the canonical one. It has already been observed in [17] that affine processes are not limited to take values in the canonical state space, but they might also assume their values in a parabolic state space. For the 2-dimensional diffusion case this was further investigated in [27]. Recently the attention has been drawn to matrix-valued affine
processes, which assume their values in the cone of positive semi-definite matrices, see for instance [10] [11] [25] [28] [44]. These processes are typically used for modeling instantaneous stochastic covariance in multi-variate extensions of the Heston model. A mathematical foundation for matrix-valued affine processes has been provided in [10].

This thesis contributes further to the theory of affine processes with a non-canonical state space and complements [17] and [10]. We develop a mathematical foundation for affine processes with a general state space, including the canonical state space and the cone of positive semi-definite matrices, as well as the parabolic state space and the Lorentz cone. The main contributions in these thesis are as follows.

- First, a full characterization is given for affine processes on a rather general (convex) state space. We show the equivalence between an affine process and an affine jump-diffusion by means of the Feller property. The necessary and sufficient conditions on the affine characteristics, called admissibility conditions, are presented in a general form, based on the positive maximum principle.

- Second, we determine all possible polyhedral and quadratic state spaces on which an affine process exists, under the presence of a diffusion part, and we work out the admissibility conditions for these.

- Third, we extend the validity of the exponential affine expression for the Fourier-Laplace transform of an affine process beyond its natural domain. In particular we obtain conditions under which an affine process has a finite exponential moment.

In order to understand the topic of this thesis better and as a warming up for the next chapters, we discuss below the main ideas in the theory of affine processes and give a mathematical overview.

**Mathematical overview**

We consider a very simple affine process (namely the square root process, as used in the Cox-Ingersoll-Ross model) and calculate its conditional exponential moments, using heuristic arguments. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) be a filtered probability space and \(X\) a stochastic process that solves the 1-dimensional square root SDE

\[
    dX_t = (aX_t + b)dt + \sqrt{X_t}dW_t, \quad X_0 = x,
\]
with \( a \in \mathbb{R} , b \geq 0 , x \geq 0 \) and \( W \) a 1-dimensional Brownian motion. Observe that \( X \) is an affine process with state space \( \mathbb{R}_+ \). Indeed, the drift \( aX_t + b \) and the instantaneous variance matrix \( X_t \) (the square of the term in front of the Brownian motion) clearly depend in an affine way on \( X_t \), while \( X_t \geq 0 \) for all \( t \), due to the condition \( b \geq 0 \). We aim to calculate the conditional exponential moment

\[
\mathbb{E}(\exp(uX_t)|\mathcal{F}_s), \quad \text{for } s \leq t, \text{ some } u \in \mathbb{C}.
\]

Fix \( t \geq 0 \). As an “ansatz”, we try \( f(s,X_s) \), with \( f : [0,t] \times [0,\infty) \rightarrow \mathbb{C} \) a \( C^{1,2} \)-function given by

\[
f(s,x) = \exp(\phi(t-s) + \psi(t-s)x),
\]

for some \( C^1 \)-functions \( \phi : [0,t] \rightarrow \mathbb{C} , \psi : [0,t] \rightarrow \mathbb{C} \) with \( \phi(0) = 0 , \psi(0) = u \). Note that \( f(t,X_t) = \exp(uX_t) \). Therefore, it suffices to choose \( \phi \) and \( \psi \) in such a way that \( f(s,X_s) \) is a martingale, since in that case we have

\[
\mathbb{E}(\exp(uX_t)|\mathcal{F}_s) = \mathbb{E}(f(t,X_t)|\mathcal{F}_s) = f(s,X_s).
\]

To determine \( \phi \) and \( \psi \), we apply Itô’s formula, which yields (suppressing the arguments of \( \phi \) and \( \psi \))

\[
\partial_s f(s,X_s) = \partial_s f(s,X_s)ds + \partial_x f(s,X_s)dX_s + \frac{1}{2} \partial_{xx} f(s,X_s)d\langle X \rangle_s
\]

\[
= f(s,X_s)(-\dot{\phi} - \dot{\psi}X_s + \psi(aX_s + b) + \frac{1}{2} \psi^2 X_s)ds + f(s,X_s)\psi \sqrt{X_s}dW_s.
\]

Suppose that the stochastic integral is a proper martingale. Then \( f(s,X_s) \) is a martingale under the additional condition that the drift term vanishes, which holds if \( (\phi, \psi) \) satisfies the Riccati equations

\[
\dot{\phi} = b \psi
\]

\[
\dot{\psi} = a \psi + \frac{1}{2} \psi^2.
\]

Thus we have found an “exponential affine” expression for the conditional exponential moments of the square root process. The above derivation can easily be generalized to multi-dimensional diffusions with affine drift and affine instantaneous covariance matrix, or even to jump-diffusions where in addition the jump-rates are affine, see [19].

For turning the previous derivation into a rigorous mathematical proof, some gaps need to be filled, to wit:

- The solutions \( (\phi, \psi) \) to the Riccati equations might explode in finite time, due to the quadratic term \( \frac{1}{2} \psi^2 \). It is obvious that \( \mathbb{E}(\exp(uX_t) \text{ is finite for} \)

\(
\)
\[ u \in \mathbb{C}_- \text{, but a careful analysis is needed to show that explosion of } (\phi, \psi) \text{ for } u \in \mathbb{C}_- \text{ is impossible. In addition, } \mathbb{E}(\exp(uX_t)) = 0 \text{ for some complex } u \in \mathbb{C}_- \text{ is not excluded a priori, which would correspond with an explosion of } \psi. \]

\begin{itemize}
  \item It is assumed that the stochastic integral \( \int_0^t f(s, X_s) \psi(t - s)\sqrt{X_s} dW_s \) is a proper martingale. This is not immediately clear and needs further verification.
\end{itemize}

These issues are taken care of by Duffie, Filipović and Schachermayer in [17] (and generalized to \( \mathbb{R}_+^m \times \mathbb{R}^{p-m} \)-valued jump-diffusions). In addition, they show the reverse statement, in the sense that a Markov process \( X \) necessarily solves a square root SDE, whenever its exponential moments are exponential affine.

Let us explain the importance of having a closed form expression for the (conditional) exponential moments for financial applications, following [17, Section 13]. A very popular application of affine processes is modeling the term structure of interest rates. In general, the pricing formula based on no-arbitrage reasoning (see for instance [31]) yields that the price \( D_{t,T} \) of a zero-coupon bond at time \( t \) paying one unit of money at maturity time \( T \), is given by

\[ D_{t,T} = \mathbb{E}(\exp(-\int_t^T r_s ds) | \mathcal{F}_t), \]

where \( r \) denotes the short interest rate process and the expectation is taken under the risk-neutral measure. In a short-rate model, one chooses a stochastic process for modeling the short interest rate, from which the dynamics of the bond price can be deduced by the above relation. A desirable feature of a short rate model is that the obtained formula for the bond price can be calculated analytically, rather than by Monte Carlo simulations, so that calibration is possible. One way to achieve this, is by modeling \( r \) as an affine transformation of an affine process, due to the exponential affine expression for the conditional exponential moments. This can be shown as follows.

Let \( r_t = \delta_0 + \delta^\top X_t \), for some \( \delta_0 \in \mathbb{R}, \delta \in \mathbb{R}^p \), with \( X_t \) an affine diffusion on some state space \( E \subset \mathbb{R}^p \), say with affine drift \( b(X_t) \) and affine diffusion matrix \( c(X_t) \). Define \( Y_t = -\int_0^t r_s ds \). Then it is easy to see that \( Z_t = (X_t, Y_t) \) is an affine diffusion with state space \( E \times \mathbb{R} \) and drift and diffusion matrix given by

\[
\begin{pmatrix}
  b(X_t) \\
  -\delta_0 - \delta^\top X_t
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  c(X_t) & 0 \\
  0 & 0
\end{pmatrix}.
\]
Now the assertion follows from the observation that
\[ \mathbb{E}(\exp(-\int_t^T r_s ds) | \mathcal{F}_t) = \exp(-Y_t)\mathbb{E}(\exp(Y_T) | \mathcal{F}_t) \]
\[ = \exp(-Y_t)\mathbb{E}(\exp(u^T Z_T) | \mathcal{F}_t), \]
with \( u \in \mathbb{R}^{p+1} \) given by \( u_i = 0 \) for \( i \leq p \) and \( u_{p+1} = 1 \).

A second application can be found in option pricing. Consider the price of a European put-option that gives the buyer the right (but not the obligation) to sell a certain stock at time \( T \) for a fixed price \( K > 0 \). Suppose the price of the stock at time \( T \) is given by \( f(X_T) \) for some positive function \( f \) and \( X \) a stochastic process. Then the pricing formula based on no-arbitrage yields that the fair price of the option at time 0 equals (with \( r_t \) the short rate process)
\[ K\mathbb{E}(\exp(-\int_0^T r_t dt) 1_{\{f(X_T) \leq K\}}) - \mathbb{E}(\exp(-\int_0^T r_t dt) f(X_T) 1_{\{f(X_T) \leq K\}}). \]

Let the price of the stock be modeled as \( k_0 \exp(k^T X_t) \) and the short rate as \( r(X_t) = \delta_0 + \delta^T X_t \), for some \( p \)-dimensional affine process \( X \), with \( k_0, \delta_0 \in \mathbb{R} \) and \( k, \delta \in \mathbb{R}^p \). Then both expectations on the right-hand side of the above display are of the form
\[ \mathbb{E}(\exp(-\int_0^T r(X_t) dt) e^{a^T X_T} 1_{\{b^T X_T \leq c\}}), \]
for some \( a, b \in \mathbb{R}^p, c \in \mathbb{R} \). This expectation can be regarded as the distribution function of \( b^T X_T \) at \( c \) under the measure
\[ \exp(-\int_0^T r(X_t) dt) e^{a^T X_T} d\mathbb{P}. \]
Hence to determine the expectation it is enough to compute the Fourier transform of the above measure, in view of Plancherel’s Theorem, see e.g. [5]. The Fourier transform can be seen to be of the form
\[ \mathbb{E}(\exp(-\int_0^T r(X_t) dt) e^{u^T X_T}), \]
for some \( u \in \mathbb{C}^p \). As in the case of the zero-coupon bond price (corresponding with \( u = 0 \)), this expectation is of the exponential affine form, which can explicitly be determined by solving a system of Riccati equations.

We now return to affine processes with a general state space \( E \). For ease of exposition, we consider the continuous diffusion case. Suppose \( X \) is an affine diffusion with state space \( E \), given as the solution to the multi-dimensional SDE
\[ dX_t = b(X_t) dt + c(X_t)^{1/2} dW_t, \quad X_0 = x \in E, \]
with affine drift \( b(X_t) \) and affine diffusion matrix \( c(X_t) \). The following observations are crucial.

- In order to have existence of the square root, it is necessary that \( c(x) \) is positive semi-definite for all \( x \in E \). This imposes restrictions on both the diffusion matrix and the state space \( E \).

- The process \( X \) is not allowed to leave the state space \( E \) (a notion called \textit{stochastic invariance} of \( E \)). Therefore, conditions on the behavior of the drift and diffusion matrix on the boundary of the state space have to be imposed. In particular, the drift \( b(x) \) should point inwards and the diffusion matrix \( c(x) \) should vanish parallel to the boundary, for all \( x \in \partial E \). The necessary and sufficient conditions on \( b \) and \( c \) (and also on the jumps for general jump-diffusions) are called \textit{admissibility conditions}.

- Uniqueness of the solution to the SDE is not immediate. Since \( x \mapsto c(x)^{1/2} \) is in general not Lipschitz-continuous for \( x \) on the boundary, standard uniqueness results for SDEs fail. In addition, we note that [55, Theorem 1] is only applicable for affine diffusions on the canonical state space \( \mathbb{R}_+^m \times \mathbb{R}^{p-m} \), but not for general state spaces.

- The behavior of the solutions \((\phi,\psi)\) to the Riccati equations is crucial for deriving uniqueness. When the exponential affine expression is established for the characteristic function, uniqueness of the affine diffusion follows from uniqueness of \((\phi,\psi)\).

The main challenge in developing a theory for affine processes on an arbitrary state space, compared to the canonical state space \( \mathbb{R}_+^m \times \mathbb{R}^{p-m} \), is that the admissibility conditions are much more delicate. This is due to singularities on and curvedness of the boundary, resulting in an additional “Stratonovich term” when deriving the boundary conditions required for stochastic invariance. As a consequence, it is much harder for general state spaces to analyze the solutions \((\phi,\psi)\) to the Riccati equations directly by means of these admissibility conditions. To circumvent this difficulty, we use an indirect approach and rely on probabilistic methods instead. For our analysis we extensively make use of stochastic calculus and the theory on semimartingales.

The contents of the thesis are as follows. In the first three chapters we recall and extend theory for general stochastic processes, which we use in the remaining chapters for the analysis of affine processes. In Chapter 1 we treat general semimartingales, while in Chapter 2 we consider jump-diffusions and analyze them by
means of the martingale problem. Using the positive maximum principle, we are able to derive necessary and sufficient conditions for the existence of general jump-diffusions, see Proposition 2.15. This proposition will be used for proving existence of affine processes in Chapter 4. In addition, we infer a result on the martingale property of a stochastic exponential, see Proposition 2.23, which will be used in Chapter 6 for establishing the validity of the affine transform formula. Chapter 3 is devoted to general Markov and Feller processes that live on a state space $E$ of the form $E = \mathcal{X} \times \mathbb{R}^{p-m}$, where $\mathcal{X} \subset \mathbb{R}^m$ is a closed convex set satisfying certain properties. We characterize regular Feller processes as the solution of a martingale problem in Theorem 3.20.

Next, we establish existence and uniqueness of affine processes living on an arbitrary state space $E$ of the aforementioned form $\mathcal{X} \times \mathbb{R}^{p-m}$ in Chapter 4. The main result of this chapter is Theorem 4.4. Here, the admissibility conditions are given in a general form, which are explicitly worked out for polyhedral and quadratic state spaces in Chapter 5; see Theorems 5.12, 5.17 and 5.22. In this chapter we also specify the form of a general polyhedral state space and characterize all quadratic state spaces. It turns out that the parabolic state space and the Lorentz cone are the only possibilities for the latter. Finally, in Chapter 6 we aim to extend the validity of the exponential affine expression for exponential moments. The main results here are Theorems 6.4 and 6.7. With the aid of the first theorem, we provide tractable conditions under which the Fourier-Laplace function does not vanish in Theorem 6.5.