Affine Markov processes on a general state space
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Chapter 2

The martingale problem

This chapter concerns the theory of the martingale problem. Originally developed by Stroock and Varadhan as presented in their classical work [51], this theory characterizes jump-diffusions as solutions of a martingale problem, which has become a fundamental tool for the analysis of stochastic processes. For a full overview we refer to [21].

In this chapter we consider jump-diffusions that assume their values in a certain state space $E \subset \mathbb{R}^p$ and characterize them as solutions of the aforementioned martingale problem. We include the possibility of explosion and killing by a potential, wherefore we follow the framework in [6] (with a few modifications). We revisit the latter paper and deduce an important criterium in Section 2.2 for a solution of the martingale problem, see Proposition 2.5. In Section 2.3 we apply this proposition to derive the Markov property for general jump-diffusions, under well-posedness. Although these results are not new, to the best of our knowledge they are not stated in the literature in the particular form as presented here. We emphasize that Proposition 2.7 is crucial in proving existence and uniqueness for affine processes in Chapter 4. In addition, we give in this section a well-known stochastic representation of a solution to Kolmogorov’s backward equation, extending known results to the general jump-diffusion case.

In Section 2.4 we discuss the positive maximum principle. A consequence of [21 Theorem 4.5.4] is that existence of a solution of the martingale problem is equivalent with the positive maximum principle. Using a technique involving a change of measure, we derive necessary and sufficient conditions for the positive maximum principle in Proposition 2.13 which we use extensively for our analysis of
affine processes in later chapters when we introduce the admissibility conditions. In Section 2.5 we build further on the main result of [6] and deduce sufficient conditions for the martingale property of a stochastic exponential. This will be needed in Chapter 6 to establish the validity of the affine transform formula.

2.1 Set-up and notation

Let $E \subset \mathbb{R}^p$ be the closure of an unbounded open set and $E_{\Delta} = E \cup \{\Delta\}$ the one-point compactification of $E$, where $\Delta \notin \mathbb{R}^p$ corresponds with “the point at infinity”. Every measurable function $f$ on $E$ is extended to $E_{\Delta}$ by setting $f(\Delta) = 0$, except the norm-function, for which we take $|\Delta| = \infty$. Note that the derivatives of $f \in C^2(E)$ are well-defined on $E$, as they are determined by the values of $f$ on $E^\circ$, by the assumption that $E = E^\circ$. Throughout this chapter, $\Omega$ denotes a subset of $D_{E_{\Delta}}[0, \infty)$, the space of càdlàg functions $\omega : [0, \infty) \to E_{\Delta}$. Unless mentioned otherwise, $\Omega$ is equipped with the $\sigma$-algebra $F_X = \sigma(X_s : s \geq 0)$ and the filtration $(F^X_t)$ with $F^X_t = \sigma(X_s : 0 \leq s \leq t)$, generated by the coordinate process $X$ given by $X_t(\omega) = \omega(t)$.

Let us be given measurable functions $b : E \to \mathbb{R}^p$, $c : E \to S^p_+$, $\gamma : E \to \mathbb{R}^+$ and $K$ a transition kernel from $E$ to $\mathbb{R}^p \setminus \{0\}$ with $\text{supp } K(x, dz) \subset E - x$ for all $x \in E$. Assume that

$$
\int (|z|^2 \wedge 1)K(z, dz) \text{ are bounded on compacta of } E,
$$

and let $\chi : \mathbb{R}^p \to \mathbb{R}^p$ denote a truncation function. We are given a linear operator $A : C^2(E) \to M(E)$ by

$$
A f(x) = \nabla f(x)^\top b(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) - \gamma(x)f(x)
+ \int (f(x + z) - f(x) - \nabla f(x)^\top \chi(z))K(x, dz),
$$

and we assume $A(C^2(E)) \subset B(E)$, so that $\int_0^t A f(X_s)ds$ is well-defined pathwise for $f \in C^2(E)$. The example below demonstrates that the assumption $A(C^2(E)) \subset B(E)$ is not redundant.

Example 2.1. Let $E = \mathbb{R}$, take $b = c = \gamma = 0$ and define the transition kernel $K$ by $K(x, dz) = x^4 \mu(dz)$, with $\mu$ a measure with support $\mathbb{N}$ and given by $\mu(\{k\}) = 1/k^2$ for $k \in \mathbb{N}$. Then

$$
\int (|z|^2 \wedge 1)K(x, dz) = x^4 \sum_{k=1}^{\infty} 1/k^2,
$$
whence $x \mapsto \int (|z|^2 \wedge 1)K(x, dz)$ is bounded on compacta. Take $f \in C^2_c(\mathbb{R})$ such that $1_{\{|x| \leq 1\}} \leq f(x) \leq 1_{\{|x| \leq M\}}$ for some $M > 1$. Then for $|x| > M$ with $1 - x \in \mathbb{N}$ it holds that

$$Af(x) = \int f(x + z)K(x, dz) \geq x^4 \int 1_{\{z = 1 - x\}} \mu(dz) = x^4/(1 - x)^2,$$

which can be made arbitrarily large. Hence $A(C^2_c(E)) \not\subset B(E)$.

**Definition 2.2.** A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}^X)$ is called a solution of the *martingale problem* for $\mathcal{A}$ in $\Omega$ if

$$M^f_t = f(X_t) - f(X_0) - \int_0^t Af(X_s)ds \tag{2.3}$$

is a $\mathbb{P}$-martingale with respect to $(\mathcal{F}^X_t)$ for all $f \in C^2_c(E)$. If in addition $\lambda$ is a probability measure on $E_\Delta$ such that $\mathbb{P} \circ X^{-1} = \lambda$, then we say $\mathbb{P}$ is a solution of the martingale problem for $(\mathcal{A}, \lambda)$ and we often write $\mathbb{P} = \mathbb{P}_\lambda$. If $\lambda = \delta_x$, the Dirac-measure at $x$ for some $x \in E_\Delta$, then we write $\mathbb{P}_x$ instead. Likewise, $\mathbb{E}_\lambda$ denotes the expectation with respect to $\mathbb{P}_\lambda$ and $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$. We call the martingale problem for $\mathcal{A}$ *well-posed* in $\Omega$ if for all probability measures $\lambda$ on $E_\Delta$ there exists a unique solution $\mathbb{P}_\lambda$ on $(\Omega, \mathcal{F}^X)$ of the martingale problem for $(\mathcal{A}, \lambda)$.

There is a close relation between the martingale problem and jump-diffusions. For $\Omega = D_E[0, \infty)$ and $\gamma = 0$ this is easy to see. Indeed, if $X$ is a semimartingale on $(D_E[0, \infty), (\mathcal{F}^X_t), \mathbb{P})$ with differential characteristics $(b(X), c(X), K(X, dz))$, then $\mathbb{P}$ is a solution of the martingale problem for $\mathcal{A}$ in $D_E[0, \infty)$ (with $\gamma = 0$), in view of (1.4). For general $\Omega \subset D_{E\Delta}[0, \infty)$ and $\gamma$, we need to extend the definition of jump-diffusions by allowing the possibility of *explosion* and *killing by a potential*. Let $X$ be a process in $D_{E\Delta}[0, \infty)$ with absorbing cemetery state $\Delta$, i.e. if $X_{t^-} = \Delta$ or $X_t = \Delta$, then $X_s = \Delta$ for all $s \geq t$. Define the stopping time

$$S_n = \inf\{t \geq 0 : |X_{t^-}| \geq n \text{ or } |X_t| \geq n\}.$$

Then $X$ is an *exploding* jump-diffusion if for all $n \in \mathbb{N}$ it holds that $X^{S_n}$ is a semimartingale with differential characteristics

$$(b(X_t)1_{[0,S_n]}(t), c(X_t)1_{[0,S_n]}(t), K(X_t, dz)1_{[0,S_n]}(t)),$$

and if in addition we have $T_{expl} := \lim_{n \to \infty} S_n < \infty$ with positive probability as well as $\lim_{t \uparrow T_{expl}} |X_t| = \infty$ for $T_{expl} < \infty$. Let $e$ be an independent exponentially distributed random variable with rate 1 and define the stopping time

$$T_{kill} = \inf\{t \geq 0 : \int_0^t \gamma(X_s)ds \geq e\}.$$
If $X$ is a (possibly exploding) jump-diffusion, then
$$X_t 1_{[0,T_{\text{kill}})}(t) + \Delta 1_{[T_{\text{kill}},\infty)}(t)$$
denotes a killed jump-diffusion, with killing rate $\gamma$. Note that if $T_{\text{kill}} < T_{\text{expl}}$, then a transition from $E$ to $\Delta$ occurs by a jump, while if $T_{\text{kill}} \geq T_{\text{expl}}$, then a transition to $\Delta$ occurs by an explosion. In the latter case the killing is redundant.

With this extended definition in mind, one can show that if $\mathbb{P}$ is the distribution of a jump-diffusion, then $\mathbb{P}$ solves the martingale problem for $\mathcal{A}$ in $D_{E,\Delta}[0,\infty)$. The next section is devoted to the converse of this assertion.

### 2.2 Turning $X$ into a semimartingale

As explained in the preceding section, we relate in this section the solution of a martingale problem with the law of a semimartingale. We follow [6], but slightly adapt their proofs in order to obtain a useful characterization of a solution of the martingale problem. As in [6] we define a couple of ($\mathcal{F}_t^X$)-stopping times. First, we define the ($\mathcal{F}_t^X$)-stopping time
$$T_\Delta = \inf\{t \geq 0 : X_{t-} = \Delta \text{ or } X_t = \Delta\},$$
which can be regarded as the lifetime of $X$ in case $\Delta$ is absorbing. Being a first contact time, $T_\Delta$ is indeed an ($\mathcal{F}_t^X$)-stopping time, in view of [21, Proposition 2.1.5].

To handle an explosion of $X$ we introduce the stopping times
$$T'_n = \inf\{t \geq 0 : |X_{t-}| \geq n \text{ or } |X_t| \geq n\}.$$  
A transition to $\Delta$ can occur by either a jump (when the process is killed by a potential) or an explosion. Accordingly, we define

$$T_{\text{jump}} = \begin{cases} T_\Delta, & \text{if } T'_n = T_\Delta \text{ for some } n, \\ \infty, & \text{if } T'_n < T_\Delta \text{ for all } n. \end{cases}$$
$$T_{\text{expl}} = \begin{cases} T_\Delta, & \text{if } T_n < T_\Delta \text{ for all } n, \\ \infty, & \text{if } T'_n = T_\Delta \text{ for some } n, \end{cases}$$
$$T_n = \begin{cases} T'_n, & \text{if } T'_n < T_\Delta, \\ \infty, & \text{if } T'_n = T_\Delta. \end{cases}$$

Note that $T_\Delta = T_{\text{expl}}$ and $T_{\text{jump}} = \infty$ in the case that a transition from $E$ to $\Delta$ occurs by an explosion, while $T_\Delta = T_{\text{jump}}$ and $T_{\text{expl}} = \infty$ in case a transition to $\Delta$
2.2. Turning $X$ into a semimartingale

occurs by a jump (i.e. by killing). Note also that $T_n \uparrow T_{\text{expl}}$ and that $T_{\text{jump}}$ differs from $T_{\text{kill}}$ (as defined in the preceding paragraph), since $T_{\text{jump}}$ assumes the value $\infty$ when an explosion occurs first.

Now we assume that $\Delta$ is an absorbing cemetery state. Therefore, we take $\Omega$ to be equal to

$$\Omega = \{ \omega \in \mathbb{D}_E[0,\infty) : \text{if } \omega(t-) = \Delta \text{ or } \omega(t) = \Delta, \text{ then } \omega(s) = \Delta \text{ for } s \geq t \}. \quad (2.8)$$

We modify $X$ such that it becomes a càdlàg process in $\mathbb{R}^p$. Following [6], we identify a jump to $\Delta$ with a jump to some point $\partial \in \mathbb{R}^p \setminus E$. The existence of such a point can be guaranteed after enlarging the dimension when necessary, see [6, Section 3]. Without loss of generality we may also assume that $\text{dist} (\partial, E) > 0$ (where dist denotes the distance) and that $\chi(\partial - x) = 0$ for all $x \in E$. We put

$$\hat{X}_t = X_t 1_{[0,T_{\text{jump}})}(t) + \partial 1_{[T_{\text{jump}},\infty)}(t). \quad (2.9)$$

Then $\hat{X}$ cannot jump to $\Delta$, but an explosion to $\Delta$ is still possible. Stopping $\hat{X}$ by $T_n$ we obtain a càdlàg process in $\mathbb{R}^p$.

Next we show that $\hat{X}^{T_n}$ is a semimartingale on $(\Omega, (\mathcal{F}_{t+}^X), \mathbb{P})$ and determine its characteristics, in case $\mathbb{P}$ solves the martingale problem for $A$ in $\Omega$. In order to do so, we define a second linear operator $\hat{A} : C^2(\mathbb{R}^p) \to M(E)$ by

$$\hat{A}f(x) = \nabla f(x)^T b(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x) c(x)) + \int (f(x + z) - f(x) - \nabla f(x)^T \chi(z)) K^\partial(x,dz),$$

where $K^\partial(x,dz) = K(x,dz) + \gamma(x) \delta_{\partial - x}(dz)$ (we restrict the domain of $\hat{A}f$ to $E$, as $b(x), c(x), \gamma(x)$ and $K(x,dz)$ are not defined for $x \notin E_\Delta$). Note that

$$\hat{A}f(x) = Af(x) + f(\partial) \gamma(x). \quad (2.10)$$

The assumptions in (2.1) yield that the processes

$$M^{T_n}_t = f(X^{T_n}_t) - f(X^{T_n}_0) - \int_0^{t \wedge T_n} A f(X_s) ds$$

and

$$\hat{M}^{T_n}_t = f(\hat{X}^{T_n}_t) - f(\hat{X}^{T_n}_0) - \int_0^{t \wedge T_n} \hat{A} f(X_s) ds$$

are well-defined for $f \in C^2(\mathbb{R}^p)$ and bounded for $f \in C^2_b(\mathbb{R}^p)$. In addition, we have

$$f(\hat{X}_t) = f(X_t) 1_{\{t < T_{\text{jump}}\}} + f(\partial) 1_{\{T_{\text{jump}} \leq t\}} = f(X_t) + f(\partial) 1_{\{T_{\text{jump}} \leq t\}}.$$
Thus \( N \) have martingale, as it is right-continuous (see [15, Theorem VI.1.3]). For proof.

Note that if \( \Omega \) holds that \( \tilde{X} \), let Proposition 2.3.

Recalling the convention \( b(N) = c(N) = \gamma(N) = K(N, \cdot) = 0 \), we can also write
\[
\tilde{f}(X_t) = \nabla f(\hat{X}_t)^\top b(X_t) + \frac{1}{2} \text{tr} (\nabla^2 f(\hat{X}_t)c(X_t)) \\
+ \int (f(\hat{X}_t + z) - f(\hat{X}_t) - \nabla f(\hat{X}_t)^\top \chi(z)) K^\theta(X_t, dz),
\]
for \( t \geq 0 \). Now Theorem 1.4 translates as follows.

**Proposition 2.3.** Let \( \Omega \) be given by (2.8) and let \( \mathbb{P} \) be a measure on \( (\Omega, \mathcal{F}) \). It holds that \( \hat{X} \) is a semimartingale on \( (\Omega, (\mathcal{F}^X_{t \uparrow}), \mathbb{P}) \) with differential characteristics
\[
(b(X_t)1_{[0,T_n]}(t), c(X_t)1_{[0,T_n]}(t), K^\theta(X_t, dz)1_{[0,T_n]}(t)),
\]
if and only if \( \tilde{M}^{f,T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale for all \( f \in C^2_b(\mathbb{R}^p) \), which in turn holds if and only if \( \tilde{M}^{f,u,T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale for all \( u \in \mathbb{R}^p \).

A direct consequence is the following proposition.

**Proposition 2.4.** Let \( \mathbb{P} \) be a probability measure on \( (\Omega, \mathcal{F}^X) \) with \( \Omega \) given by (2.8). Suppose \( M^{f,u,T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale for all \( u \in \mathbb{R}^p \), some \( n \in \mathbb{N} \). Then it holds that \( N^{T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale and \( \hat{X}^{T_n} \) is a semimartingale on \( (\Omega, (\mathcal{F}^X_{t \uparrow})_{t \geq 0}, \mathbb{P}) \) with differential characteristics (2.13).

**Proof.** Note that if \( M^{f,u,T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale, then it is also an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale, as it is right-continuous (see [15, Theoreme VI.1.3]). For \( u = 0 \) we have \( f_u(x) = 1 \) for \( x \in E \) and \( f_u(\Delta) = 0 \), whence
\[
M^{f_0,T_n} = 1_{\{t \wedge T_n < T\text{jump}\}} - 1_{\{T\text{jump} > 0\}} + \int_0^{t \wedge T_n} \gamma(X_s) ds = -N^{T_n}_t.
\]
Thus \( N^{T_n}_t \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale. It follows from (2.11) that \( \tilde{M}^{f,u,T_n} \) is an \((\mathcal{F}^X_{t \uparrow}, \mathbb{P})\)-martingale for all \( u \in \mathbb{R}^p \). Thus the result follows from Proposition 2.3.

In the following we need a countable collection \( \mathcal{C} \subset C^2_c(E) \), defined as follows. Let \( g_k \) be a sequence in \( C^2_c(E) \) such that \( g_k(x) = 1 \) for \( |x| \leq k \) and \( 0 \leq g_k(x) \leq 1 \)
for all \( x \in E \). Then we define a countable collection \( \mathcal{C} \) of functions \( \eta \) on \( \Omega \) of the form

\[
\eta = (M^f_t - M^f_s)1_F, \text{ some } f \in C^2_c(E), \ s \leq t, \ F \in \mathcal{F}^X_s, \quad (2.14)
\]

with \( f = f_u g_k \) for some \( u \in \mathbb{IQ}^p \), \( k \in \mathbb{N} \), with \( s \leq t \), some \( s, t \in \mathbb{Q}_+ \), and with \( F \in \mathcal{G}_s \subset \mathcal{F}^X_s \) where \( \mathcal{G}_s \) denotes the collection of sets of the form

\[
\bigcap_{r \in A} \{ \omega \in \Omega : \omega_i(r) \in (-\infty, k_i(r)], \forall i = 1, \ldots, p \},
\]

for some finite \( A \subset \mathbb{Q} \cap [0, s] \) and some \( k : A \to \mathbb{Q}^p \). Using this set \( \mathcal{C} \), we are able to give a useful characterization of a solution of the martingale problem, see Proposition 2.5 below. As a corollary we obtain that \( \hat{X}^{T_n} \) is a semimartingale for all \( n \in \mathbb{N} \) if and only if \( \mathbb{P} \) solves the martingale problem, see also [6, Lemma 3.1 and Proposition 3.2].

**Proposition 2.5.** Let \( \Omega \) be given by (2.8), let \( \mathbb{P} \) be a probability measure on \((\Omega, \mathcal{F}^X)\) and let \( \mathcal{C} \) be given as above. Then the following are equivalent:

(i) \( \int \eta \, d\mathbb{P} = 0 \), for all \( \eta \in \mathcal{C} \).

(ii) \( M^{f_u,T_n} \) is a \( \mathbb{P} \)-martingale for all \( u \in \mathbb{IR}^p \), \( n \in \mathbb{N} \).

(iii) \( M^{f,T_n} \) is a \( \mathbb{P} \)-martingale for all \( f \in C^2_c(E) \), \( n \in \mathbb{N} \).

(iv) \( \mathbb{P} \) is a solution of the martingale problem for \( \mathcal{A} \) in \( \Omega \).

**Proof.** Obviously, it holds that (iv) \( \Rightarrow \) (i). Combining Proposition 2.4 with Proposition 2.3 yields (ii) \( \Rightarrow \) (iii). To see (iii) \( \Rightarrow \) (iv), note that for \( f \in C^2_c(E) \) it holds that

\[
M^{f,T_n}_t \overset{L^1}{\to} M^{f,T_n}_{t,\text{expl}} = M^f_t,
\]

as \( n \to \infty \). Hence \( M^f \) is a \( \mathbb{P} \)-martingale for all \( f \in C^2_c(E) \) if we assume (iii).

We now show (i) \( \Rightarrow \) (ii). Let \( f = f_u g_k \) for some \( u \in \mathbb{IQ}^p \), \( k \in \mathbb{N} \). We first show that \( M^f \) is a \( \mathbb{P} \)-martingale. Let \( s \leq t, \ s, t \in \mathbb{Q}_+ \). Note that \( \mathcal{G}_s \) is a \( \pi \)-system containing \( \Omega \) (by the convention that \( \Omega \) equals the empty intersection) and generating \( \mathcal{F}^X_s \). Since \( \mathbb{E}(M^f_t - M^f_s)1_F = 0 \) for all \( F \in \mathcal{G}_s \) and \( M^f \) is bounded, it follows from \([34, \text{Lemma 1.17}]\) that \( \mathbb{E}(M^f_t - M^f_s)1_F = 0 \) for all \( F \in \mathcal{F}^X_s \). Hence

\[
\mathbb{E}(M^f_t | \mathcal{F}^X_s) = M^f_s, \quad \mathbb{P} \text{-a.s. for } s \leq t, \ s, t \in \mathbb{Q}_+.
\]

Since \( M^f \) is bounded and right-continuous, the equality in the above display can be extended for \( s, t \in \mathbb{R}_+ \), by taking limits. This yields that \( M^f \) is a \( \mathbb{P} \)-martingale for all \( f \) of the form \( f = f_u g_k \), with \( u \in \mathbb{IQ}^p \), \( k \in \mathbb{N} \).
Next we show $M_{f^u,T_n}$ is a $\mathbb{P}$-martingale for all $u \in \mathbb{Q}^p$, $n \in \mathbb{N}$. For $k \geq n$ it holds that
\[
|M_{f^u,T_n}^k - M_{f^u,T_n}| \leq |f_u(X_{T_n}^k) - f_u(X_{T_n}^n)| + |f_u(X_0) - f_u(X_n)|
\]
\[
+ \int_0^{t \wedge T_n} \int |f_u(X_s + z) - f_u g_k(X_s + z)| K(X_s, dz) ds
\]
\[
\leq |f_u(X_{T_n}^k) - f_u g_k(X_{T_n}^n)| + |f_u(X_0) - f_u g_k(X_0)|
\]
\[
+ \int_0^{t \wedge T_n} \int 1_{|z| \geq k-n} K(X_s, dz) ds.
\]
The first two terms on the right-hand side of the second inequality are uniformly bounded, whence they tend to 0 in $L^1$ as $k \to \infty$. The last term also tends to 0 in $L^1$, in view of (2.1). Thus $M_{f^u g_k, T_n} \xrightarrow{L^1} M_{f^u, T_n}$ as $k \to \infty$, which implies that $M_{f^u, T_n}$ is a martingale for $u \in \mathbb{Q}^p$, as $M_{f^u g_k, T_n}$ is a martingale.

It remains to show that $M_{f^u, T_n}$ is a $\mathbb{P}$-martingale for all $u \in \mathbb{Q}^p$, $n \in \mathbb{N}$. Note that
\[
\mathcal{A}f_u(x) = f_u(x) \left( u^T b(x) + \frac{1}{2} u^T c(x) u - \gamma(x) \right)
\]
\[
+ \int (e^{u^T z} - 1 - u^T \chi(z)) K(x, dz) \quad (2.15)
\]
Take $u \in \mathbb{Q}^p$ and let $u_k \to u$ for $k \to \infty$, with $u_k \in \mathbb{Q}^p$. Then $\mathcal{A}f_{u_k}(x)$ is uniformly bounded in $k$ and $x$ for $k \in \mathbb{N}$ and $|x| \leq n$ in view of (2.15) and (2.1). It follows that
\[
|M_{f^u,T_n} - M_{f^{u_k},T_n}| \leq |f_u(X_{T_n}^k) - f_{u_k}(X_{T_n}^n)| + |f_u(X_0) - f_{u_k}(X_0)|
\]
\[
+ \int_0^{t \wedge T_n} |\mathcal{A}f_u(X_s) - \mathcal{A}f_{u_k}(X_s)| ds \overset{L^1}{\to} 0, \text{ as } k \to \infty.
\]
Hence $M_{f^u, T_n}$ is a martingale for all $n \in \mathbb{N}$, $u \in \mathbb{Q}^p$, which concludes the proof.

**Corollary 2.6.** Let $\Omega$ be given by (2.5). It holds that $\mathbb{P}$ is a solution of the martingale problem for $\mathcal{A}$ if and only if $X_{T_n}$ is a semimartingale on $(\Omega, (\mathcal{F}_{t+})_{t \geq 0}, \mathbb{P})$ with differential characteristics (2.13), for all $n \in \mathbb{N}$.

**Proof.** The “only if”-part follows by combining Proposition 2.5 and Proposition 2.4. The “if”-part follows from Proposition 2.3 and Proposition 2.5. Indeed, if $\hat{X}_{T_n}$ is a semimartingale under $\mathbb{P}$ for all $n \in \mathbb{N}$, then $\hat{M}_{f^u,T_n}$ is a $\mathbb{P}$-martingale for all $n \in \mathbb{N}$, $f \in C^2_b(\mathbb{R}^p)$, by Proposition 2.3. Each $f \in C^2_b(\mathbb{R}^p)$ can be extended to $C^2_b(E)$ such that $f(\theta) = 0$, so that $M_{f^u,T_n}$ is a $\mathbb{P}$-martingale for all $n \in \mathbb{N}$, $f \in C^2_b(E)$, in view of (2.11). Thus $\mathbb{P}$ solves the martingale problem for $\mathcal{A}$ by Proposition 2.5. \qed
2.3 The Markov property

In this section we show the Markov property of $X$, under well-posedness of the martingale problem for $A$ in $\Omega$, when $\Omega$ is given by (2.8). We generalize [34, Theorems 21.10 and 21.11] from real-valued diffusions to general jump-diffusions with the aid of Proposition [2.5]. The next proposition is a key step, which will also be needed in a subsequent chapter. Below, $\theta_t : \Omega \to \Omega$ denotes the “shift”-operator, defined by $\theta_t(\omega) = \omega(t + \cdot)$, for $t \geq 0$.

**Proposition 2.7.** Let $\Omega$ be given by (2.8), $P$ be a solution of the martingale problem for $A$ in $\Omega$ and let $T \geq 0$ be arbitrary. Then for $P$-almost all $\omega \in \Omega$ it holds that $P(X \circ \theta_T \in \cdot | F^X_T)(\omega)$ is a solution of the martingale problem for $(A, \delta_{X_T(\omega)})$.

**Proof.** We first prove the existence of a regular version of $P(X \circ \theta_T \in \cdot | F^X_T)$. By [34, Lemma 3.1] we have that $\sigma(X_T) = \bigvee_{t \geq 0} \sigma(X_T^t) = \bigvee_{0 \leq t \leq T} \sigma(X_t) = F^X_T$, where $X_T$ denotes the stopped process $X^T_s = X_{T \wedge s}$. Moreover, $(\Omega, F^X)$ is a Borel space. To see this, let $Y$ denote the coordinate process in $D(\Delta, E \Delta \Delta [0, \infty))$. Then it holds that $F^Y$ is equal to the Borel $\sigma$-algebra $B(D(\Delta, E \Delta \Delta [0, \infty))$ generated by the Skorohod topology, see [34, Theorem A2.2] ($E$ is a separable, completely metrizable space, whence the same holds for its one-point compactification $E_\Delta$). Since $X = Y|_\Omega$, we have $F^X = \{ B \cap \Omega : B \in F^Y \}$ and it follows that $F^X$ is a Borel $\sigma$-algebra with respect to the subspace topology, see [34, Lemma 1.6]. Hence [34, Theorem 6.3] yields the existence of a probability kernel $\mu$ from $\Omega$ to $\Omega$ such that

$$\mu(X^T(\omega), \cdot) = P(X \circ \theta_T \in \cdot | F^X_T)(\omega),$$

for $P$-almost all $\omega$. We now show that $\nu(\omega, \cdot) := \mu(X^T(\omega), \cdot)$ is a solution of the martingale problem for $(A, \delta_{X_T(\omega)})$ for $P$-almost all $\omega$. First observe that

$$\nu(\omega, X_0 = x) = P((X \circ \theta_T)_0 = x | F^X_T)(\omega) = P(X_T = x | F^X_T)(\omega) = 1_{\{X_T(\omega) = x\}},$$

for all $x \in E_\Delta$, $P$-almost all $\omega$. Hence

$$\nu(\omega, X_0 = X_T(\omega)) = 1,$$

for $P$-almost all $\omega$. It remains to show that $\nu(\omega, \cdot)$ solves the martingale problem for $A$. 
2. The martingale problem

Let \( C \) be the countable collection of functions \( \eta \) on \( \Omega \) of the form (2.14) as defined in the paragraph preceding Proposition 2.5. By Proposition 2.5 we have \( P \) is a solution of the martingale problem for \( A \) in \( \Omega \) if and only if \( \int \eta \, dP = 0 \) for all \( \eta \in C \). Since \( C \) is countable, it suffices to show that for all \( \eta \in C \) it holds that

\[
E(\eta \circ \theta_T | F^X_T)(\omega) = \int \eta(\omega') \nu(\omega, d\omega') = 0, \quad \text{for } \mathbb{P}\text{-almost all } \omega.
\]

Let \( \eta \) be of the form (2.14). It holds that \( M^f_t \circ \theta_T = M^f_{t+T} - M^f_T \). Note that

\[
\{ A \subset \Omega : \theta_T^{-1}(A) \in F^X_{T+s} \}
\]

is a \( \sigma \)-algebra that includes \( \sigma(X_t) \) for all \( t \leq s \), whence it includes \( \bigvee_{0 \leq t \leq s} \sigma(X_t) = F^X_s \). In other words, \( \theta_T^{-1}(F^X_s) \subset F^X_{T+s} \). It follows that

\[
E(\eta \circ \theta_T | F^X_T)(\omega) = \mathbb{E}(E((M^f_t - M^f_s)1_F \circ \theta_T | F^X_T)
= \mathbb{E}(E((M^f_t + T - M^f_{s+T})1_{\theta_T^{-1}(F)} | F^X_{T+s} \cap F^X_T) = 0,
\]

\( \mathbb{P}\)-almost surely, since \( M^f \) is a \( \mathbb{P}\)-martingale, as \( \mathbb{P} \) is a solution of the martingale problem for \( A \).

The following proposition is similar to [21, Theorem 4.4.6] and [31, Theorems 21.10 and 21.11], and is proved along the same lines. Here we write \( \mathcal{P}(\Omega) \) for the class of probability measures on \( (\Omega, \mathcal{F}) \), with \( \mathcal{F} \) the \( \sigma \)-algebra from the context.

**Proposition 2.8.** Let \( \Omega \) be given by (2.8). Suppose for all \( x \in E_\Delta \) there exists a unique solution \( \mathbb{P}_x \) of the martingale problem for \( (A, \delta_x) \) in \( \Omega \). Then \( x \in E_\Delta \mapsto \mathbb{P}_x(B) \) is measurable for all Borel sets \( B \in \mathcal{B}(E_\Delta) \). Moreover, for all probability measures \( \lambda \) on \( E_\Delta \) it holds that \( \mathbb{P}_\lambda = \int_{E_\Delta} \mathbb{P}_x \lambda(dx) \) is the unique solution of the martingale problem for \( (A, \lambda) \) and the martingale problem is well-posed in \( \Omega \). In addition, the Markov property holds, i.e.

\[
\mathbb{P}_\lambda(X \circ \theta_t \in \cdot | F^X_t) = \mathbb{P}_x, \quad \mathbb{P}_\lambda \text{-a.s.,}
\]

for all \( t \geq 0 \).

**Proof.** Since \( E_\Delta \) is a separable, completely metrizable space, it holds that \( \mathcal{P}(E_\Delta) \) and \( \mathcal{P}(D_{E_\Delta}[0, \infty)) \) are complete, separable metric spaces with respect to the Prohorov metric, see [21, Theorems 3.5.6 and 3.1.7]. We first show that the same holds for \( \mathcal{P}(\Omega) \). Let \( Y \) denote the coordinate process in \( D_{E_\Delta}[0, \infty) \) and let \( T_\Delta \) be given by (2.4) with \( Y \) instead of \( X \). We can write

\[
\Omega = \bigcap_{q, r \in \mathbb{Q}^+} \bigcap_{t \in \mathbb{Q} \cap [q+r, \infty)} \{ Y_t = \Delta \} \cup \{ T_\Delta \notin B(q, r) \}.
\]


whence $\Omega \in \mathcal{F}^Y$. The projection $\pi: \mathcal{P}(D_{E\Delta}[0,\infty)) \to [0,1]: P \mapsto P(\Omega)$ is continuous. Therefore, $\pi^{-1}(\{1\})$ is a closed subset of the complete separable metric space $\mathcal{P}(D_{E\Delta}[0,\infty))$, whence it is itself complete and separable. Since $\mathcal{F}^X = \{B \cap \Omega : B \in \mathcal{F}^Y\}$, it follows that

$$\mathcal{P}(\Omega) = \{P|_{\mathcal{F}^X} : P \in \pi^{-1}(\{1\})\}$$

is complete and separable with respect to the Prohorov metric.

Let $\mathcal{C}$ be the countable set of functions on $\Omega$ as in Proposition 2.7, so that $P \in \mathcal{P}(\Omega)$ is a solution of the martingale problem for $\mathcal{A}$ if and only if $\int \eta dP = 0$ for all $\eta \in \mathcal{C}$. Define $G: \mathcal{P}(\Omega) \to \mathcal{P}(E\Delta)$ by $G(P) = P \circ X_0^{-1}$. Then $G$ is continuous and hence measurable. Moreover, $D := \{\delta_x \in \mathcal{P}(E\Delta) : x \in E\Delta\}$ is measurable, as it is closed. Also,

$$\{P \in \mathcal{P}(\Omega) : \int \eta dP = 0\}$$

is measurable, as the projection $P \mapsto \int \eta dP$ is continuous. Write $\mathcal{M}$ for the collection of all solutions of the martingale problem for $\mathcal{A}$ with degenerate initial condition, which is equal to $\{\mathbb{P}_x : x \in E\Delta\}$ by the well-posedness. Then it follows that

$$\mathcal{M} = \bigcap_{\eta \in \mathcal{C}} \{P \in \mathcal{P}(\Omega) : \int \eta dP = 0\} \cap G^{-1}(\{\delta_x : x \in E\Delta\})$$

is measurable. Moreover, $G|_{\mathcal{M}}$ is one-to-one and onto on $D$. Kuratowski’s Theorem states that a one-to-one Borel measurable map of a Borel subset of a complete, separable metric space onto a Borel subset of a complete, separable metric space has a Borel measurable inverse, see [45, Theorem 3.9]. Hence $\delta_x \mapsto \mathbb{P}_x$ is Borel measurable. Since $x \mapsto \delta_x$ and the projections $\mathbb{P}_x \mapsto \mathbb{P}_x(B)$ are also measurable for $B \in \mathcal{B}(E\Delta)$, it follows that $x \mapsto \mathbb{P}_x(B)$ is measurable.

Proposition 2.7 together with the uniqueness of the solution for $(\mathcal{A},\delta_x)$ yields that

$$\mathbb{P}(X \circ \theta_t \in \cdot |\mathcal{F}^X_t) = \mathbb{P}_{X_t}, \quad \mathbb{P}\text{-a.s.},$$

for all $t \geq 0$ and all solutions $\mathbb{P}$ of the martingale problem for $\mathcal{A}$, which is the Markov property. In particular

$$\mathbb{P}(X \in \cdot |\mathcal{F}^X_0) = \mathbb{P}_{X_0}, \quad \mathbb{P}\text{-a.s.},$$

so that

$$\mathbb{P}(B) = \int \mathbb{P}(B|\mathcal{F}^X_0) d\mathbb{P} = \int \mathbb{P}_{X_0}(B) d\mathbb{P} = \int \mathbb{P}_x(B)(\mathbb{P} \circ X_0^{-1})(dx),$$
whence $\mathbb{P}$ is uniquely determined by its initial condition $\mathbb{P} \circ X_0^{-1}$. Now let $\lambda$ be a probability measure on $E_\Delta$ and define $\mathbb{P}_\lambda = \int_{E_\Delta} \mathbb{P}_x \lambda(dx)$. Then $\int \eta d\mathbb{P} = 0$ for all $\eta \in \mathcal{C}$ and for all $\mathbb{P} = \mathbb{P}_\lambda$, whence $\mathbb{P}_\lambda$ is a solution of the martingale problem. Its initial condition equals $\lambda$, since

$$
\mathbb{P}_\lambda(X_0 \in B) = \int_{E_\Delta} \mathbb{P}_x(X_0 \in B)\lambda(dx) = \int_{E_\Delta} 1_{\{x \in B\}}\lambda(dx) = \lambda(B),
$$

for all $B \in \mathcal{B}(E_\Delta)$. This yields that the martingale problem for $\mathcal{A}$ is well-posed and concludes the proof.

A weaker version of the Markov property can be obtained under existence of a solution to a certain differential equation. Below, for a function $f \in C^{1,2}_{\mathbb{R}^+ \times E}$, we write $A f(t,x)$ instead of $A f(t,\cdot)(x)$. The following proposition gives a well-known stochastic representation for a solution to Kolmogorov’s backward equation. See also [34, Chapter 24].

**Proposition 2.9.** Let $\Omega$ be given by (2.8). Suppose for all $x \in E$ there exists a solution $\mathbb{P}_x$ of the martingale problem for $(\mathcal{A}, \delta_x)$. Fix some $T > 0$ and $f : E \to \mathbb{R}$ and suppose there exists a solution $u \in C^{1,2}_0([0,T] \times E)$ of Kolmogorov’s backward equation

$$
\partial_t u + A u = 0, \quad u(T,\cdot) = f. \tag{2.16}
$$

Then $u(t,X_t)$ is a $\mathbb{P}_x$-martingale for all $x \in E$ and necessarily $u$ takes the form

$$
u(t,x) = \mathbb{E}_x f(X_{T-t}).
$$

Consequently, we have $\mathbb{E}_{X_s} f(X_{t-s}) = \mathbb{E}_x(f(X_t)|\mathcal{F}_s^X), \mathbb{P}_x$-a.s., for $0 \leq s \leq t \leq T$ and $x \in E$.

**Proof.** Fix $x \in E$ arbitrarily. Let $\tilde{X}$ be given by (2.9) and $T_n$ by (2.7). Corollary 2.6 gives that $\tilde{X}^{T_n}$ is a semimartingale under $\mathbb{P}_x$ with differential characteristics given by (2.13). Hence $(t \land T_n, \tilde{X}_t^{T_n})$ is a semimartingale in $\mathbb{R}^{p+1}$ under $\mathbb{P}_x$ with differential characteristics

$$(\tilde{b}(t,X_t)1_{[0,T_n]}(t), \tilde{c}(t,X_t)1_{[0,T_n]}(t), \tilde{K}^\partial(X_t,ds,dz)1_{[0,T_n]}(t))$$

with

$$
\tilde{b}(t,x) = \begin{pmatrix} 1 \\ b(x) \end{pmatrix}, \quad \tilde{c}(t,x) = \begin{pmatrix} 0 & 0 \\ 0 & c(x) \end{pmatrix}, \quad \tilde{K}^\partial(x,ds,dz) = \delta_0(ds) \times K^\partial(x,dz).
$$
Extend $u$ to a function in $C^{1,2}_0([0,T] \times \mathbb{R}^p)$ such that $u(t, \partial) = 0$ for all $t$, so that $u(t, \hat{X}_t) = u(t, X_t)$. It follows from Theorem 1.2 that
\[
\int_{0}^{t \wedge T_n} (\partial_s u(s, X_s) + Au(s, X_s)) ds
\]
is a $\mathbb{P}_x$-martingale on $[0,T]$, for all $n \in \mathbb{N}$. In view of (2.16), the above expression equals $u(t \wedge T_n, X_T) - u(0, X_0)$. Since $u$ is vanishing at infinity, we can extend it to a bounded continuous function on $[0,T] \times E\Delta$, by putting $u(t, \Delta) = 0$ for all $t \in [0,T]$. Therefore,
\[
u(t \wedge T_n, X_T) \to u(t \wedge T_{expl}, X_T) = u(t, X_t),
\]
pathwise, as $n \to \infty$. Since $u$ is bounded, it follows that $u(t, X_t)$ is a $\mathbb{P}_x$-martingale on $[0,T]$. Since $x$ was chosen arbitrarily, we infer that $u(t, X_t)$ is a $\mathbb{P}_x$-martingale for all $x \in E$. In particular
\[
\mathbb{E}_x f(X_T) = \mathbb{E}_x u(T, X_T) = u(0, x), \quad \text{for all } x \in E.
\]
Now let $s \in [0,T]$ be arbitrary and define $v \in C^{1,2}_0([0,T-s] \times E)$ by $v(t, x) = u(s + t, x)$. Then $v$ solves the equation
\[
\partial_t v + Av = 0, \quad v(T-s, \cdot) = f,
\]
so by the above we have $u(s, x) = v(0, x) = \mathbb{E}_x f(X_{T-s})$. This yields the result. \qed

**Remark 2.10.** The assertion in Proposition 2.9 remains valid for non-exploding $X$ if we relax the condition that $u$ is vanishing at infinity to boundedness of $u$. For exploding $X$ however, it cannot be dispensed with. For instance, take $u(t, x) = 1$ for all $t \in [0,T]$, $x \in E$ and suppose $\gamma = 0$, so that $\mathbb{P}_x(T_{\text{jump}} = \infty) = 1$ in view of Proposition 2.4. Then $u$ solves Kolmogorov’s backward equation (2.16) with $f = 1$, but
\[
\mathbb{E}_x f(X_T) = \mathbb{E}_x 1_{\{X_t \neq \Delta\}} = \mathbb{P}_x(t < T_{\text{expl}}),
\]
which is not equal to $u(T-t, x) = 1$ in case $\mathbb{P}_x(t < T_{\text{expl}}) < 1$. For other examples where the stochastic representation fails, we refer to [30].

### 2.4 The positive maximum principle

In this section we assume $\mathcal{A}(C_c^2(E)) \subset C_b(E)$ and discuss the positive maximum principle of the operator $\mathcal{A}$ and its implications on the parameters $b, c, \gamma, K$. This
enables us to find sufficient and necessary conditions for existence of a solution of the martingale problem for \( A \).

We say that \( A \) satisfies the **positive maximum principle**, if \( Af(x_0) \leq 0 \) for all \( f \in C^2_c(E), x_0 \in E \), such that \( f(x_0) = \sup_{x \in E} f(x) \geq 0 \). The positive maximum principle is satisfied under existence of a solution of the martingale problem for \((A, \delta_x)\) for all \( x \in E \), which is a consequence of the next lemma.

**Lemma 2.11.** Let \( \Omega \subset D_{E_\Delta}[0, \infty), x_0 \in E \) and \( P_{x_0} \) a probability measure on \((\Omega, \mathcal{F}^X)\) such that \( P_{x_0}(X_0 = x_0) = 1 \). Suppose \( A(C^2_c(E)) \subset C_b(E) \) and \( M^{f, T} \) is a \( P_{x_0}\)-martingale for some \( f \in C^2_c(E) \) and a strictly positive stopping time \( T \). Then we have \( Af(x_0) = \partial_+^t \big|_{t=0} P_{x_0} f(X^T_t) \). Consequently, \( Af(x_0) \leq 0 \) if \( f(x_0) = \sup_{x \in E} f(x) \geq 0 \).

**Proof.** Since \( Af(X_s) \) is bounded, we can apply Fubini to derive that

\[
\int_0^t P_{x_0}(Af(X_s)1_{[0,T]}(s))ds = P_{x_0} f(X^T_t) - f(x_0).
\]

Moreover, \( s \mapsto P_{x_0}(Af(X_s)1_{[0,T]}(s)) \) is right-continuous in 0, since \( X_s \) is right-continuous, \( Af \) is continuous and \( Af(X_s) \) is bounded. Therefore,

\[
Af(x_0) = \partial_+^t \big|_{t=0} \int_0^t P_{x_0}(Af(X_s)1_{[0,T]}(s))ds = \partial_+^t \big|_{t=0} P_{x_0} f(X^T_t) = \lim_{t\downarrow 0}(P_{x_0} f(X^T_t) - f(x_0))/t,
\]

as \( T > 0 \). Since \( f(x_0) \geq 0 \), we have \( f(x_0) \geq f(x) \) for all \( x \in E_\Delta \), whence \( f(X^T_t) - f(x_0) \leq 0 \) for \( t \geq 0 \). This yields the result. \( \square \)

A sufficient condition for \( A(C^2_c(E)) \subset C_b(E) \) is derived in the following lemma. Recall that a transition kernel \( \mu(x, dz) \) is called **weakly continuous** if it holds that \( x \mapsto \int f(z) \mu(x, dz) \) is continuous for any bounded continuous function \( f \).

**Lemma 2.12.** Under the assumption

\[
b(\cdot), c(\cdot), \gamma(\cdot), \chi(\cdot) \text{ are continuous, } (|z|^2 \wedge 1)K(\cdot, dz) \text{ is weakly continuous, } \tag{2.17}
\]

it holds that \( A(C^2_c(E)) \subset C_b(E) \).

**Proof.** Recall that the assumption \( A(C^2_c(E)) \subset B(E) \) was made a priori. Let \( f \in C^2_c(E) \). It suffices to show that

\[
x \mapsto \int (f(x + z) - f(x) - \nabla f(x) \chi(z))K(x, dz)
\]
is continuous. Extend $f$ to a function in $C^2_b(\mathbb{R}^p)$, so that $f(x + \chi(z))$ is well-defined for all $x \in E$, $z \in \mathbb{R}^p$. Write $g(x, z) = f(x + \chi(z)) - f(x) - \nabla f(x) \top \chi(z)$ for $x \in E$, $z \in \mathbb{R}^p$ and

$$
\int g(x, z)K(x, dz) - \int g(y, z)K(y, dz) = \int (g(x, z) - g(y, z))K(x, dz) + \int g(y, z)(K(x, dz) - K(y, dz)),
$$

for $y \in E$. The second term on the right-hand side tends to zero for $x \to y$, by weak continuity of $(|z|^2 \wedge 1)K(\cdot, dz)$ and the fact that $z \mapsto g(y, z)$ is continuous and bounded by a constant times $|z|^2 \wedge 1$. The first term on the right-hand side equals (where $f_{ij}$ is short-hand notation for $\partial_i \partial_j f$)

$$
\int \left( \sum_{i,j} \int_0^1 \int_0^1 (f_{ij}(x + st\chi(z)) - f_{ij}(y + st\chi(z)))s \chi_i(z)\chi_j(z) \, ds \, dt \right) K(x, dz).
$$

Since $f$ has compact support, $f$ and also $f_{ij}$ are uniformly continuous. Therefore, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for $|x - y| \leq \delta$ the above display is bounded by $\varepsilon \int (|z|^2 \wedge 1)K(x, dz)$ (as $|\chi(z)|^2$ is bounded by a constant times $|z|^2 \wedge 1$), which tends to $\varepsilon \int (|z|^2 \wedge 1)K(y, dz)$ for $x \to y$ by weak continuity of $(|z|^2 \wedge 1)K(\cdot, dz)$. Since $\varepsilon > 0$ can be made arbitrarily small, it follows that $\int (g(x, z) - g(y, z))K(x, dz) \to 0$ for $x \to y$.

It remains to show that $x \mapsto \int (f(x + z) - f(x + \chi(z)))K(x, dz)$ is continuous. Write $h(x, z) = f(x + z) - f(x + \chi(z))$. Since $\chi(z) = z$ in a neighborhood of zero, we have $h(x, z) = 0$ for $|z| \leq \varepsilon$, for some $\varepsilon$. Therefore, we can write

$$
\int h(x, z)K(x, dz) - \int h(y, z)K(y, dz) = \int_{\{|z| > \varepsilon\}} (h(x, z) - h(y, z))K(x, dz) + \int_{\{|z| > \varepsilon\}} h(y, z)(K(x, dz) - K(y, dz)).
$$

The second term on the right-hand side tends to zero by the same arguments as above. The integrant of the first term on the right-hand side can be bounded by a constant times $|x - y|$, uniformly in $z$, as $f$ is bounded. Hence the integral tends to zero as $x \to y$, since $x \mapsto \int 1_{\{|z| > \varepsilon\}}K(x, dz)$ is bounded on compacts by weak continuity. This concludes the proof.\hfill $\square$

In the proposition below we give necessary conditions for existence of a solution of the martingale problem for $(A, \delta_x)$. The approach is as follows. By turning $X$ into a semimartingale as done in Section 2.2 we are able to apply a change of measure, which yields existence of a solution of the martingale problem for a whole
class of operators, given that we have a solution of the original martingale problem. In conjunction with the positive maximum principle, this enforces restrictions on the parameters.

**Proposition 2.13.** Let $\Omega$ be given by (2.8) and assume (2.17). Suppose there exists a solution $\mathbb{P}_{x_0}$ of the martingale problem for $(\mathcal{A}, \delta_{x_0})$ with $x_0 \in E$. Then for $f \in C^2_c(E)$ with $f(x_0) = \sup_{x \in E} f(x) \geq 0$ it holds that

(i) $\nabla f(x_0)^T c(x_0) = 0$,

(ii) $\int \nabla f(x_0)^T \chi(z) K(x_0, dz)$ is well-defined and finite,

(iii) $\nabla f(x_0)^T b(x_0) - \int \nabla f(x_0)^T \chi(z) K(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0) c(x_0)) \leq 0$.

**Proof.** Let $\tilde{X}$ be given by (2.9) and $T_n$ by (2.7). Choose $n > |x_0|$, so that $T_n > 0$, fix an arbitrary deterministic time $S > 0$ and write $T := T_n \wedge S$. Corollary 2.6 together with Proposition 1.5 yields that $\tilde{X}^T$ is a semimartingale with differential characteristics given by

$$(b(X_t) 1_{[0,T]}(t), c(X_t) 1_{[0,T]}(t), K^0(X_t, dz) 1_{[0,T]}(t)).$$

Let $\lambda \in \mathbb{R}^p$ and $0 < \varepsilon < 1$ be arbitrary. Take a non-negative $\phi_\varepsilon \in C_b(\mathbb{R}^p)$ such that $\phi_\varepsilon \uparrow 1_{\mathbb{R}^p \setminus \{0\}}$ pointwise for $\varepsilon \downarrow 0$, and $\phi_\varepsilon(z) = 0$ for $|z| \leq \varepsilon$. Define $w_\varepsilon(z) = (\varepsilon - 1)\phi_\varepsilon(z)$ and

$$Z = \lambda \cdot \tilde{X}^{T,c} + w_\varepsilon \ast (\mu^Q - \nu^{\tilde{X}}).$$

Then $\mathcal{E}(Z)$ is a uniformly integrable martingale by Proposition 1.10 since

$$\int_0^T \left( \frac{1}{2} \lambda^T c(X_s) \lambda + \int ((w_\varepsilon(z) + 1) \log(w_\varepsilon(z) + 1) - w_\varepsilon(z)) K^0(X_s, dz) \right) ds$$

$$\leq \int_0^T \left( \frac{1}{2} \lambda^T c(X_s) \lambda + \int_{|z| > \varepsilon} (\varepsilon \log \varepsilon - \varepsilon + 1) K^0(X_s, dz) \right) ds$$

has finite expectation as it is bounded. The inequality in the above display follows from the observation that $0 < \varepsilon \leq w_\varepsilon + 1 \leq 1$ and the fact that $x \mapsto x \log x - x + 1$ is decreasing for $x \in (0, 1]$.

Define $Q = \mathcal{E}(Z)_\infty \cdot \mathbb{P}$, which is a probability measure on $\mathcal{F}^X$ equivalent to $\mathbb{P}$. For $x \in E$ we write

$$\tilde{b}(x) = b(x) + c(x) \lambda + \int \chi(z) w_\varepsilon(z) K(x, dz)$$

$$\tilde{\gamma}(x) = (w_\varepsilon(\partial - x) + 1) \gamma(x)$$

$$\tilde{K}(x, dz) = (w_\varepsilon(z) + 1) K(x, dz),$$
and \( \tilde{K}^\theta(x, dz) = \tilde{K}(x, dz) + \tilde{\gamma}(x)\delta_{0-x}(dz) \). By Proposition 1.9 it holds that \( \tilde{X}^T \) is a semimartingale under \( \mathbb{Q} \) with differential characteristics given by
\[
(\tilde{b}(X_t)1_{[0,T]}(t), c(X_t)1_{[0,T]}(t), \tilde{K}^\theta(X_t, dz)1_{[0,T]}(t)),
\]

since \( \chi(\partial - x) = 0 \) and \( (w_\varepsilon(z) + 1)K^\theta(x, dz) = \tilde{K}^\theta(x, dz) \). Define the linear operator \( \tilde{A} : C^2(E) \to M(E) \) by
\[
\tilde{A}f(x) = \nabla f(x)\top \tilde{b}(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) - \tilde{\gamma}(x)f(x)
+ \int (f(x + z) - f(x) - \nabla f(x)\top \chi(z))\tilde{K}(x, dz).
\]

We show that \( \tilde{A}(C^2_c(E)) \subset C_b(E) \). Let \( f \in C^2_c(E) \) and suppose \( f(x) = 0 \) for \( |x| \geq M \), for some \( M > 0 \). Note that for \( |x| \geq M \) it holds that
\[
Af(x) = \int f(x + z)K(x, dz) \quad \text{and} \quad \tilde{A}f(x) = \int f(x + z)(w_\varepsilon(z) + 1)K(x, dz),
\]

whence \( \tilde{A}f(x) \) is bounded by a constant times \( |Af(x)| \) for \( |x| \geq M \). This yields that \( \tilde{A}f \) is bounded, as \( Af \) is bounded by assumption. The continuity conditions in (2.17) for the parameters \( b, c, \gamma, K \) yield that (2.17) also holds for the parameters \( \tilde{b}, \tilde{\gamma}, \tilde{K} \). Therefore, \( \tilde{A}(C^2_c(E)) \subset C_b(E) \) in view of Lemma 2.12.

Fix \( f \in C^2_c(E) \) and suppose \( f \) assumes a non-negative maximum at \( x_0 \in E \). We extend \( f \) to a function in \( C^2(\mathbb{R}^p) \) such that \( f(\partial) = 0 \). Note that then
\[
\int (f(x + z) - f(x) - \nabla f(x)\top \chi(z))\tilde{K}^\theta(x, dz)
= \int (f(x + z) - f(x) - \nabla f(x)\top \chi(z))\tilde{K}(x, dz) - \tilde{\gamma}(x)f(x),
\]

and \( f(\tilde{X}) = f(X) \), so that Theorem 1.4 yields that
\[
f(X_t^T) - f(X_0) - \int_0^{t\wedge T} \tilde{A}f(X_s)ds
\]
is a \( \mathbb{Q} \)-martingale. Therefore, Lemma 2.11 gives \( \tilde{A}f(x_0) \leq 0 \). Since we can write
\[
\tilde{A}f(x) = Af(x) + \nabla f(x)\top c(x)\lambda
+ (1 - \varepsilon) \left( \phi_\varepsilon(\partial - x)\gamma(x)f(x) + \int (f(x) - f(x + z))\phi_\varepsilon(z)K(x, dz) \right),
\]
we infer that
\[
Af(x_0) + \nabla f(x_0)\top c(x_0)\lambda + (1 - \varepsilon) \left( \phi_\varepsilon(\partial - x)\gamma(x_0)f(x_0)
+ \int (f(x_0) - f(x_0 + z))\phi_\varepsilon(z)K(x_0, dz) \right) \leq 0,
\]
(2.18)
for all $\lambda \in \mathbb{R}^p$ and $0 < \varepsilon < 1$. This implies that $\nabla f(x_0)^\top c(x_0) = 0$, which is the first assertion. It follows that

$$\mathcal{A} f(x_0) + (1-\varepsilon) \left( \phi_\varepsilon(\partial - x) \gamma(x_0) f(x_0) + \int (f(x_0) - f(x_0 + z)) \phi_\varepsilon(z) K(x_0, dz) \right) \leq 0,$$

for all $0 < \varepsilon < 1$. Letting $\varepsilon \downarrow 0$ and applying the Monotone Convergence Theorem gives

$$\mathcal{A} f(x_0) + \gamma(x_0) f(x_0) + \int (f(x_0) - f(x_0 + z)) K(x_0, dz) \leq 0.$$

Therefore, $\int (f(x_0) - f(x_0 + z)) K(x_0, dz) < \infty$, which gives the second assertion.

The left-hand side of the inequality in the above display equals

$$\nabla f(x_0)^\top b(x_0) - \int \nabla f(x_0)^\top \chi(z) K(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0) c(x_0)),$$

which yields the third assertion. \hfill \Box

The above proposition together with \cite[Theorem 4.5.4]{21} gives sufficient and necessary conditions for the existence of a solution of the martingale problem for $\mathcal{A}$ for all initial conditions, which is the content of Proposition \cite{2.15} below. The following lemma is needed to transfer a solution of the martingale problem from $D_{E_\Delta}[0, \infty)$ to $\Omega$ given by \cite{2.8} and vice versa.

**Lemma 2.14.** Let $\Omega$ be given by \cite{2.8} and let $X$ denote the coordinate process in $D_{E_\Delta}[0, \infty)$. If $\mathbb{P}$ is a solution of the martingale problem for $\mathcal{A}$ in $D_{E_\Delta}[0, \infty)$, then $\mathbb{P} \circ (X^{T_\Delta})^{-1}$ is a solution of the martingale problem for $\mathcal{A}$ in $\Omega$. Conversely, if $\mathbb{P}$ is a solution of the martingale problem for $\mathcal{A}$ in $\Omega$, then $\mathbb{P}(\cdot \cap \Omega)$ is a solution in $D_{E_\Delta}[0, \infty)$.

**Proof.** Note that $X'(\omega) := X^{T_\Delta}(\omega) \in \Omega$ for all $\omega \in D_{E_\Delta}[0, \infty)$ and that $X'$ is $\mathcal{F}^X$-measurable. For all $f \in C^\infty_c(E)$ it holds that (recall $\mathcal{A} f(\Delta) = 0$)

$$f(X'_t) - f(X'_0) - \int_0^t \mathcal{A} f(X'_s) ds = f(X^{T_\Delta}_t) - f(X^{T_\Delta}_0) - \int_0^{t \wedge T_\Delta} \mathcal{A} f(X_s) ds = (M^f)^{T_\Delta}_t,$$

where $M^f$ is given by \cite{2.3}. Since $M^f$ is a right-continuous $\mathbb{P}_x$-martingale on $(\mathcal{F}^X_t)$ for $f \in C^\infty_c(E)$, it holds that $(M^f)^{T_\Delta}$ is a martingale on $(\mathcal{F}^X_{T_\Delta})$. Hence $\mathbb{P} \circ (X')^{-1}$ is a solution of the martingale problem for $\mathcal{A}$ on $(\Omega, \mathcal{F}^{X'})$, which yields the first assertion.

Now let $Y$ denote the coordinate process in $\Omega$ and $\mathbb{P}$ be a solution in $\Omega$. To show that $\mathbb{P}(\cdot \cap \Omega)$ is a solution in $D_{E_\Delta}[0, \infty)$, it suffices to show that $A \cap \Omega \in \mathcal{F}^Y$ for all $A \in \mathcal{F}^X$. Since $Y = X|_{\Omega}$ it holds that $\sigma(Y_t) = \{B \cap \Omega : B \in \sigma(X_t)\}$. This yields the result. \hfill \Box
Proposition 2.15. Let $\Omega$ be given by (2.8), assume (2.17) and $A(C^2_c(E)) \subset C_0(E)$. Then for all probability measures $\lambda$ on $E_\Delta$ there exists a solution $\mathbb{P}_\lambda$ of the martingale problem for $(A, \lambda)$ in $\Omega$ if and only if for all $f \in C^2_c(E)$, $x_0 \in E$ with $f(x_0) = \sup_{x \in E} f(x) \geq 0$ it holds that

(i) $\nabla f(x_0)^\top c(x_0) = 0$,

(ii) $\int \nabla f(x_0)^\top \chi(z)K(x_0, dz)$ is well-defined and finite,

(iii) $\nabla f(x_0)^\top b(x_0) - \int \nabla f(x_0)^\top \chi(z)K(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0)c(x_0)) \leq 0$.

Proof. The “only if”-part follows from Proposition 2.13. The “if”-part follows from [21, Theorem 4.5.4] together with Lemma 2.14. Indeed, by Lemma 2.12, $A$ is a linear operator on $C_0(E)$ whose domain is dense in $C^0(E)$, while the above conditions imply that $A$ satisfies the positive maximum principle. To see the latter, note that if $f \in C^2_c(E)$, $x_0 \in E$ with $f(x_0) = \sup_{x \in E} f(x) \geq 0$, the above conditions enable us to write

$$Af(x_0) = \nabla f(x_0)^\top b(x_0) - \int \nabla f(x_0)^\top \chi(z)K(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0)c(x_0))$$

$$- \gamma(x_0)f(x_0) + \int (f(x_0 + z) - f(x_0))K(x, dz),$$

which is non-positive as $\gamma(x_0) \geq 0$, $f(x_0) \geq 0$ and $f(x_0 + z) - f(x_0) \leq 0$ for all $z$.

As a corollary, we derive necessary conditions for existence of a solution of the martingale problem in case $E$ is a closed, convex set. Recall that a closed, convex set $E$ can be written as a countable intersection of halfspaces, i.e.

$$E = \bigcap_{i=1}^{\infty} \{x \in \mathbb{R}^p : \zeta_i + \eta_i^\top x \geq 0\},$$

(2.19)

for some $\zeta_i \in \mathbb{R}$, $\eta_i \in \mathbb{R}^p$. It is worth noting that the conditions derived in the next proposition are similar to the stochastic invariance conditions as obtained in [52], see also [43, 13] for the diffusion case.

Proposition 2.16. Let $\Omega$ be given by (2.8), assume (2.17) and let $E$ be a closed, convex set given by (2.19). Write $\partial E_i = \{x \in E : \zeta_i + \eta_i^\top x = 0\}$. If for all $x \in E$ there exists a solution $\mathbb{P}_x$ of the martingale problem for $(A, \delta_x)$ in $\Omega$, then necessarily for all $i \in \mathbb{N}$ and $x \in \partial E_i$ it holds that

(i) $\eta_i^\top c(x) = 0$, 
Proposition 2.13. Let \( \gamma \) be a solution of the martingale problem for \( A \) in \( D_E[0, \infty) \) if and only if \( X \) is a semimartingale under \( \mathbb{P} \) with differential characteristics

\[
(b(X_t), c(X_t), K(X_t, dz)).
\]

(2.20)

Let \( \mathbb{P} \) be a solution of the martingale problem for \( A \) in \( D_E[0, \infty) \). In this section we derive conditions which assure that the stochastic exponential

\[
Z = \mathcal{E}(h(X_\cdot))^\mathbb{P} X^c + w(X_\cdot, z) * \mathbb{P}(\mu^X - \nu^X))
\]

(2.21)

is a (well-defined) \( \mathbb{P} \)-martingale, for some measurable functions \( h : E \to \mathbb{R}^p \), \( w : E \times \mathbb{R}^p \to (-1, \infty) \) with \( h(X_\cdot) \in L^2_{\text{loc}}(X^c, \mathbb{P}) \) and \( w(X_\cdot, z) \in G_{\text{loc}}(\mu^X, \mathbb{P}) \).

For the moment assume that \( Z \) is indeed a martingale. Then \( \mathbb{Q}_t = Z_t \cdot \mathbb{P} \) defines a consistent sequence of probability measure \( \mathbb{Q}_t \) on \( \mathcal{F}_t^X \), whence there exists a probability measure \( \mathbb{Q} \) on \( \mathcal{F}^X \) such that \( \mathbb{Q}|_{\mathcal{F}_t} = \mathbb{Q}_t \), see [31, Lemma 18.18] (right-continuity nor completeness of the filtration is needed here; in fact the assertion fails in general when the filtration is complete, see the discussion in [37, Section 3.5]). We have that \( \mathbb{Q} \) is locally equivalent to \( \mathbb{P} \), so Proposition 1.9 yields \( X \) is a semimartingale under \( \mathbb{Q} \) with differential characteristics \((\tilde{b}(X), c(X), \tilde{K}(X, dz))\),

\[
(2.22)
\]

where

\[
\tilde{b}(x) = b(x) + c(x)h(x) + \int \chi(z)w(x, z)K(x, dz)
\]

\[
\tilde{K}(x, dz) = (w(x, z) + 1)K(x, dz).
\]

Hence \( \mathbb{Q} \) is a solution of the martingale problem for \( A \) given by

\[
\tilde{A}f(x) = \nabla f(x)^\top \tilde{b}(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) + \int (f(x + z) - f(x) - \nabla f(x)^\top \chi(z)) \tilde{K}(x, dz).
\]

(2.23)
We have proved the following result.

**Proposition 2.17.** Let $P$ be a solution of the martingale problem for $(\mathcal{A}, \lambda)$ in $D_E[0, \infty)$, some $\lambda \in \mathcal{P}(E_\Delta)$. If $Z$ given by (2.21) is a $P$-martingale, then there exists a solution $Q$ of the martingale problem for $(\tilde{\mathcal{A}}, \lambda)$ in $D_E[0, \infty)$ that satisfies

$$Q|_{\mathcal{F}_t} = Z_t \cdot P|_{\mathcal{F}_t}.$$  

Remarkably, if the martingale problem for $\tilde{\mathcal{A}}$ is well-posed in $D_E[0, \infty)$, then also the converse of the assertion in the above proposition holds, under some additional boundedness conditions. This gives us sufficient conditions for a stochastic exponential to be a proper martingale. In the following, we let $\tilde{\mathcal{A}}$ be given by (2.23) with $\tilde{b}, \tilde{K}$ given by (2.22), and we assume $\tilde{b}$ and $\tilde{K}$ satisfy (2.1) and $\tilde{\mathcal{A}}(C^2_c(E)) \subset B(E)$. The next theorem can be considered as a particular case of [6, Theorem 2.4]. For clarity we include the proof, which is more transparent in this simpler case.

**Theorem 2.18.** Let $P$ be a solution of the martingale problem for $(\mathcal{A}, \lambda)$ in $D_E[0, \infty)$, some $\lambda \in \mathcal{P}(E)$. Suppose that the martingale problem for $\tilde{\mathcal{A}}$ is well-posed in $D_E[0, \infty)$ and that the following functions are bounded on compacta

$$x \mapsto h(x)^\top c(x)h(x),$$

$$x \mapsto \int ((w(x, z) + 1) \log(w(x, z) + 1) - w(x, z))K(x, dz).$$

Then $Z$ given by (2.21) is a $P$-martingale.

**Proof.** We first show that $h(X_-) \in L^2_{\text{loc}}(X^c, P)$ and $w(X_-, z) \in G_{\text{loc}}(\mu^X, P)$, so that $Z$ is well-defined. Since $x \mapsto h(x)^\top c(x)h(x)$ is bounded on compacta, we have that $\int_0^t h(X_s)^\top c(X_s)h(X_s)ds$ is locally integrable, whence $h(X_-) \in L^2_{\text{loc}}(X^c, P)$. To see that $w(X_-, z) \in G_{\text{loc}}(\mu^X, P)$, we show $\int_0^t \int (1 - \sqrt{w(X_s, z) + 1})^2 K(X_s, dz)ds$ is locally integrable. We have the inequality

$$(1 - \sqrt{x + 1})^2 = \int_0^x (1 - (1 + y)^{-1/2})dy$$

$$\leq \int_0^x (1 - (1 + y)^{-1})dy = x - \log(x + 1), \text{ for } x > -1,$$

so substituting $x = -z/(z + 1) > -1$ for $z > -1$ and multiplying by $z + 1 > 0$ gives the inequality

$$(1 - \sqrt{z + 1})^2 \leq (z + 1) \log(z + 1) - z, \text{ for } z > -1.$$
Together with (2.24) this yields $w(X_-, z) \in G_{\text{loc}}(\mu^X, \mathbb{P})$.

Since $Z$ is a positive local martingale, it is a supermartingale. Hence to show $Z$ is a proper martingale, it suffices to show it has constant expectation equal to 1. Let $T_n = \inf\{t \geq 0 : |X_t| \geq n \text{ or } X_t \geq n\}$ and fix $T \geq 0$ arbitrarily. Then it holds that

$$Z^{T_n \wedge T} = \mathcal{E}(h(X_-)1_{[0, T_n \wedge T]} \cdot X^c + w(X_-, z)1_{[0, T_n \wedge T]} * (\mu^X - \nu^X))$$

is a uniformly integrable $\mathbb{P}$-martingale for all $n$ by Proposition 1.10 with $Z^{T_n \wedge T} = Z_{T_n \wedge T}$. Define measures $Q^n$ on $\mathcal{F}^X$ by

$$Q^n := Z_{T_n \wedge T} \cdot \mathbb{P}.$$ 

By Proposition 1.5 it holds that $X^{T_n \wedge T}$ is a semimartingale under $\mathbb{P}$ with differential characteristics

$$(b(X_t)1_{[0, T_n \wedge T]}, c(X_t)1_{[0, T_n \wedge T]}, K(X_t, dz)1_{[0, T_n \wedge T]}).$$

Proposition 1.9 yields that $X^{T_n \wedge T}$ is a semimartingale under $Q^n$ with differential characteristics

$$(\tilde{b}(X_t)1_{[0, T_n \wedge T]}, \tilde{c}(X_t)1_{[0, T_n \wedge T]}, K(X_t, dz)1_{[0, T_n \wedge T]}).$$

Therefore, Theorem 1.4 gives that

$$f(X_{t \wedge T}^{T_n \wedge T}) - f(X_0) - \int_0^{t \wedge T_n \wedge T} \tilde{A}f(X_s)ds$$

is a $Q^n$-martingale for all $f \in C^2_\mathbb{C}(E)$. Hence $Q^n \circ X^{T_n \wedge T}$ is a solution of the stopped martingale problem for $\tilde{A}$ in the sense of [21, Chapter 4, Section 6]. Let $Q$ be the unique solution of the martingale problem for $(\tilde{A}, \lambda)$ in $D_E[0, \infty)$. Then $Q \circ X^{T_n \wedge T}$ is also a solution of the stopped martingale problem. From [21, Theorem 4.6.1] we infer that well-posedness of the martingale problem for $\tilde{A}$ implies uniqueness of the stopped martingale problem, whence $Q^n \circ X^{T_n \wedge T} = Q \circ X^{T_n \wedge T}$, i.e.

$$Q^n|_{\mathcal{F}^X_{T_n \wedge T}} = Q|_{\mathcal{F}^X_{T_n \wedge T}}.$$ 

It follows from monotone convergence and the fact that $\{T \leq T_n\} \in \mathcal{F}^X_{T_n \wedge T}$ (see [34, Lemma 7.1]), that

$$\mathbb{E}_P Z_T = \lim_{n \to \infty} \mathbb{E}_P Z_{T_n \wedge T}1_{[0, T_n]}(T) = \lim_{n \to \infty} \mathbb{E}_P Z_{T_n \wedge T}1_{[0, T_n]}(T) = \lim_{n \to \infty} Q^n(T \leq T_n) = 1.$$ 

Since $T \geq 0$ was chosen arbitrarily, we have $\mathbb{E}_P Z_T = 1$ for all $T \geq 0$, as we needed to show.
We would rather reverse the roles of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ in Theorem 2.18 and assume well-posedness for $\mathcal{A}$ instead of $\tilde{\mathcal{A}}$. In order to do so, write $\tilde{h}(x) = -h(x)$, $\tilde{w}(x, z) = 1/(w(x, z) + 1) - 1$, so that

$$b(x) = \tilde{b}(x) + c(x)\tilde{h}(x) + \int \chi(z)\tilde{w}(x, z)\tilde{K}(x, dz)$$

$$K(x, dz) = (\tilde{w}(x, z) + 1)\tilde{K}(x, dz).$$

Then one verifies that the functions in (2.24) associated to $\tilde{\mathcal{A}}$ translate to

$$x \mapsto h(x)^{\top}c(x)h(x),$$

$$x \mapsto \int (w(x, z) - \log(w(x, z) + 1))K(x, dz).$$

(2.25)

**Corollary 2.19.** Suppose the martingale problem for $\mathcal{A}$ is well-posed in $D_E[0, \infty)$ and let $\mathbb{P}$ be a solution of the martingale problem for $(\mathcal{A}, \lambda)$ in $D_E[0, \infty)$, some $\lambda \in \mathcal{P}(E)$. In addition, suppose there exists a solution $\mathbb{Q}$ of the martingale problem for $(\tilde{\mathcal{A}}, \lambda)$ in $D_E[0, \infty)$ and assume the functions in (2.25) are bounded on compacta. Then $Z$ given by (2.21) is a $\mathbb{P}$-martingale.

**Proof.** Reversing the roles of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ in Theorem 2.18 yields that

$$\tilde{Z} = \mathcal{E}(-h(X_{-}) \cdot \mathbb{Q}\tilde{X}^c - w(X_{-}, z)/(w(X_{-}, z) + 1) \ast \mathbb{Q}(\mu^X - \tilde{\nu}^X))$$

is a $\mathbb{Q}$-martingale, where $\tilde{X}^c$ denotes the continuous martingale part of $X$ under $\mathbb{Q}$ and $\tilde{\nu}^X$ the compensator of $\mu^X$ under $\mathbb{Q}$. We infer from Proposition 2.17 with the roles of $\mathbb{P}$ and $\mathbb{Q}$ reversed that

$$\mathbb{P}|_{\mathcal{F}_t^X} = \tilde{Z}_t \cdot \mathbb{Q}|_{\mathcal{F}_t^X}.$$ 

Since $\tilde{Z}$ is strictly positive, this yields

$$\mathbb{Q}|_{\mathcal{F}_t^X} = \tilde{Z}_t^{-1} \cdot \mathbb{P}|_{\mathcal{F}_t^X},$$

so that $\tilde{Z}^{-1}$ is a $\mathbb{P}$-martingale. Proposition 1.12 gives $\tilde{Z}^{-1} = Z$. \qed

Under boundedness conditions and well-posedness of the martingale problem for $\mathcal{A}$ in $D_E[0, \infty)$, we combine the previous corollary with Proposition 2.17 and derive that the stochastic exponential $Z$ is a martingale if and only if existence holds of a solution for the martingale problem for $\tilde{\mathcal{A}}$ in $D_E[0, \infty)$. Similar results for the diffusion case can be found in [54], while [42] consider affine stochastic exponentials.
Corollary 2.20. Suppose the martingale problem for $A$ is well-posed in $D_E[0, \infty)$, let $P$ be a solution of the martingale problem for $(A, \lambda)$ in $D_E[0, \infty)$, some $\lambda \in P(E)$ and assume the functions in (2.25) are bounded on compacta. It holds that $Z$ given by (2.21) is a $P$-martingale if and only if there exists a solution $Q$ of the martingale problem for $(A, \lambda)$ in $D_E[0, \infty)$, which necessarily is unique as it satisfies

$$Q|_{\mathcal{F}_t^\infty} = Z_t \cdot P|_{\mathcal{F}_t^\infty}, \text{ for all } t \geq 0.$$ 

In the following propositions, we derive sufficient conditions on the parameters for existence of a solution for the martingale problem for $\tilde{A}$ in $D_E[0, \infty)$, by combining Proposition 2.15 with Proposition 1.6. This gives us sufficient conditions for the martingale property of the stochastic exponential $Z$, in view of the above corollary, see Proposition 2.23.

Proposition 2.21. Let $\Omega$ be given by (2.8). Assume (2.17) both for the parameters of $A$ and of $\tilde{A}$ and in addition suppose $A(C^2_t(E)) \subset C_0(E)$. If for all $x \in E$ there exists a solution of the martingale problem for $(A, \delta_x)$ in $\Omega$, then for all probability measures $\lambda$ on $E_\Delta$ there exists a solution of the martingale problem for $(\tilde{A}, \lambda)$ in $\Omega$.

Proof. Let $f \in C^2_t(E)$ attain a positive maximum at $x_0 \in E$. We check the conditions of Proposition 2.15 for $\tilde{A}$. The first and the second condition are satisfied in view of Proposition 2.13. By the same proposition we have

$$\nabla f(x_0)^\top b(x_0) - \int \nabla f(x_0)^\top \chi(z) \tilde{K}(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0) c(x_0)) = 0,$$

which yields the third requirement.

Proposition 2.22. Let $\Omega$ be given by (2.8) and $X$ resp. $Y$ denote the coordinate process in $D_E[0, \infty)$ resp. $\Omega$. Suppose $P$ is a solution of the martingale problem for $(A, \lambda)$ in $\Omega$, some $\lambda \in P(E)$. Write $b'(x) = b(x) + \int (z - \chi(z)) K(x, dz)$. If

$$|b'(x)|^2 + |c(x)|^2 + \int |z|^2 K(x, dz) \leq C(1 + |x|^2), \text{ for all } x \in E, \text{ some } C > 0,$$

and $C_0 := \int |x|^2 \lambda(dx) < \infty$, then $P|_{\mathcal{F}_X}$ is a solution of the martingale problem for $(A, \lambda)$ in $D_E[0, \infty)$.

Proof. Let $T_n$ be given by (2.7) with $Y$ instead of $X$. For all $n \in \mathbb{N}$ it holds that $Y^{T_n}$ is a semimartingale under $P$ with differential characteristics

$$(b(Y)1_{[0,T_n]}, c(Y)1_{[0,T_n]}, K(Y, dz)1_{[0,T_n]}),$$
relative to the truncation function $\chi$, by Corollary 2.6. By Proposition 1.3 it is a special semimartingale and its differential characteristics relative to the “truncation function” $z$ are

$$(b'(Y)1_{[0,T_n]}, c(Y)1_{[0,T_n]}, K(Y,dz)1_{[0,T_n]}).$$

Condition 1.6 is satisfied, whence Proposition 1.6 yields

$$\mathbb{E}\|Y_{T_n}\|_t^2 \leq (4C_0 + C(t))e^{C(t)t},$$

for all $n \in \mathbb{N}$, where $C(t)$ is a constant, independent of $n$. Letting $n \to \infty$ gives

$$\mathbb{E}\sup_{s \leq t \wedge T_{\text{exp}}} |Y_s|^2 < \infty,$$

for all $t \geq 0$, so in particular $\mathbb{P}(T_{\text{exp}} = \infty) = 1$, i.e. $\mathbb{P}(Y \in D_E) = 1$. Hence $\mathbb{P}|_{\mathcal{F}_X}$ is a solution of the martingale problem for $(A,\lambda)$ in $D_E[0,\infty)$. □

**Proposition 2.23.** Suppose the martingale problem for $A$ is well-posed in $D_E[0,\infty)$ and in addition assume the following conditions.

(i) We have (2.17) both for the parameters of $A$ and of $\tilde{A}$.

(ii) It holds that $\tilde{A}(C^2(E)) \subset C_0(E)$.

(iii) The functions in (2.25) are bounded on compacta.

(iv) There exists $C > 0$ such that for all $x \in E$ we have

$$|\tilde{b}'(x)|^2 + |c(x)| + \int |z|^2 \tilde{K}(x,dz) \leq C(1 + |x|^2),$$

where we write $\tilde{b}'(x) = \tilde{b}(x) + \int(z - \chi(z))\tilde{K}(x,dz)$.

Then the martingale problem for $\tilde{A}$ is well-posed in $D_E[0,\infty)$. In addition for all $\lambda \in \mathcal{P}(E)$ it holds that $Z$ given by (2.21) (with $\mathbb{P} = \mathbb{P}_\lambda$) is a $\mathbb{P}_\lambda$-martingale and

$$Q_{\lambda}|_{\mathcal{F}_X} = Z_t \cdot \mathbb{P}_\lambda|_{\mathcal{F}_X}, \text{ for all } t \geq 0,$$

where $\mathbb{P}_\lambda$ resp. $Q_{\lambda}$ denotes the solution of the martingale problem for $(A,\lambda)$ resp. $(\tilde{A},\lambda)$.

**Proof.** By Proposition 2.21 there exists a solution $Q_x$ of the martingale problem for $(A,\delta_x)$ in $\Omega$ given by (2.8), for all $x \in E$. Proposition 2.22 gives that $Q_x|_{\mathcal{F}_X}$ is a solution of the martingale problem for $(\tilde{A},\delta_x)$ in $D_E[0,\infty)$, which is unique in view of Corollary 2.20. The martingale problem for $\tilde{A}$ is well-posed in $D_E[0,\infty)$ by Proposition 2.8 and we have (2.26) for all $\lambda \in \mathcal{P}(E)$ by Corollary 2.20. □