Affine Markov processes on a general state space
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Chapter 3

Markov processes

In this chapter we consider Markov processes and Feller processes. We develop some general theory for Markov processes where the state space assumes the particular form $\mathcal{X} \times \mathbb{R}^{p-m}$, as given in (3.2). This will be applied in Chapter 4 to obtain the desired characterization for affine processes living on such a state space. After defining Markov transition functions, Markov semigroups and stochastic continuity in Section 3.1, we derive some geometric properties of the state space in Section 3.2. Next we define in Section 3.3 the concept of regularity of a Markov process, similar to the definition of regularity for affine processes in [17, 38, 10], and we translate results of [38] from the affine setting to the more general setting. For this we make use of the symbol of a regular Markov process, a concept that can be found in [32]. We show in Proposition 3.14 that the symbol has a Lévy-Khintchine representation under regularity conditions, which is used in Section 3.4 to characterize Feller processes as solutions of the martingale problem, see Theorem 3.20. This connects Chapter 2 with Chapter 3. For the proof of the theorem, Proposition 2.5 will be crucial.

3.1 Definitions

As in the previous chapter, $E \subset \mathbb{R}^p$ is a closed set and $E_\Delta$ denotes the one-point compactification.

**Definition 3.1.** A function $p_t(x,dz)$ on $[0,\infty) \times E \times \mathcal{B}(E)$ is called a time-homogeneous *Markov transition function* on $E$ if it satisfies the following properties:
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- \( p_t(x, \cdot) \) is a substochastic measure on \( E \) (i.e. a measure with \( p_t(x, E) \leq 1 \)), for all \( t \geq 0, x \in E \);
- \( p_0(x, \cdot) = \delta_x \), for \( x \in E \);
- \( x \mapsto p_t(x, B) \) is Borel-measurable for \( B \in \mathcal{B}(E) \), \( t \geq 0 \);
- the Chapman-Kolmogorov relation holds, i.e.

\[
p_{t+s}(x, B) = \int p_s(y, B) p_t(x, dy), \quad t, s \geq 0, x \in E, B \in \mathcal{B}(E).
\]

Likewise we define a Markov transition function on \( E_\Delta \), with the additional requirement that \( p_t(x, \cdot) \) is a probability measure on \( E_\Delta \).

A Markov transition function \( p_t(x, dz) \) on \( E \) can always be extended to a Markov transition function on \( E_\Delta \) by putting \( p_t(\Delta, \{\Delta\}) = 1 \) and \( p_t(x, \Delta) = 1 - p_t(x, E) \) for \( x \in E \). One verifies that the above properties remain satisfied in this extended setting. Conversely, if \( p_t(x, dz) \) is a Markov transition function on \( E_\Delta \) such that \( p_t(\Delta, \{\Delta\}) = 1 \) and \( p_t(x, \Delta) = 1 - p_t(x, E) \) for \( x \in E \), then \( p_t(x, dz) \) restricted to \( [0, \infty) \times E \times \mathcal{B}(E) \) is a Markov transition function on \( E \). With this in mind, we give the following definition of a time-homogeneous Markov process.

**Definition 3.2.** On some filtered measurable space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})\), let \((\mathbb{P}_x)_{x \in E}\) be a family of probability measures and let \( X \) be an \( E_\Delta \)-valued adapted stochastic process with \( \mathbb{P}_x(X_0 = x) = 1 \) for all \( x \in E \). We call \((X, (\mathbb{P}_x)_{x \in E})\) a time-homogeneous Markov process with Markov transition function \( p_t(x, dz) \) on \( E \) if

\[
\mathbb{P}_x(X_{t+s} \in B | \mathcal{F}_t^X) = p_s(X_t, B), \quad \mathbb{P}_x\text{-a.s., for } x \in E, B \in \mathcal{B}(E_\Delta), \quad (3.1)
\]

where we put \( p_s(\Delta, \{\Delta\}) = 1 \) and \( p_t(x, \Delta) = 1 - p_t(x, E) \) for \( x \in E \).

A Markov transition function \( p_t(x, dz) \) on \( E \) induces a semi-group \((P_t)_{t \geq 0}\) acting on functions \( f \in \mathcal{B}(E) \) by

\[
P_t f(x) = \int f(z)p_t(x, dz), \quad x \in E.
\]

Indeed, the Chapman-Kolmogorov relation yields the semi-group property \( P_{t+s} = P_t P_s \). Note that we can write \( P_t f(x) = \mathbb{E}_x f(X_t) \) for \( f \in \mathcal{B}(E), x \in E \), in view of (3.1), where \((X, (\mathbb{P}_x)_{x \in E})\) is the corresponding Markov process. Here we put \( f(\Delta) = 0 \) by convention. Hence if we let \( 1 \) denote the function in \( \mathcal{B}(E) \) with constant value 1, then we can write

\[
\mathbb{P}_x(X_t \in E) = P_t 1(x) = p_t(x, E) \leq 1.
\]
If $P_t 1 = 1$ for all $t \geq 0$, then we call the Markov process $(X, (P_x)_{x \in E})$ and its associated semi-group $(P_t)_{t \geq 0}$ conservative.

It is worth noting that from Proposition 2.8 we infer that $(X, (P_x)_{x \in E})$ is a Markov process when $P_x$ is the unique solution of the martingale problem for $(A, \delta_x)$ in $\Omega$ given by (2.8) and $X$ is the coordinate process. Indeed, define $p_t(x, dz)$ by

$$p_t(x, B) = P_x(X_t \in B), \quad t \geq 0, x \in E, B \in B(E) .$$

Then Proposition 2.8 yields measurability of $x \mapsto p_t(x, B)$ and the Markov property (3.1), which on its turn implies the Chapman-Kolmogorov relation. Since $p_t(\Delta, \{\Delta\}) = P_\Delta(X_t = \Delta) = 1$, the restriction of $p_t(x, dz)$ to $[0, \infty) \times E \times B(E)$ is a Markov transition function on $E$, see the remark preceding Definition 3.2. Hence $(X, (P_x)_{x \in E})$ is a Markov process with transition function $p_t(x, dz) = P_x(X_t \in dz)$. In this chapter we are interested in the converse situation, that is, we provide conditions such that a Markov process can be characterized as the solution of a certain martingale problem. This requires certain continuity properties of the process, to begin with stochastic continuity.

**Definition 3.3.** A Markov process $(X, (P_x)_{x \in E})$ is called stochastically continuous if $p_s(x, \cdot) \rightarrow p_t(x, \cdot)$ on $E$, for $s \rightarrow t$, for all $t \geq 0, x \in E$, that is, if

$$\int f(z)p_s(x, dz) \rightarrow \int f(z)p_t(x, dz),$$

as $s \rightarrow t$, for all $f \in C_b(E)$.

**Remark 3.4.** It is important to notice that there exist Markov processes with continuous sample paths in $E_\Delta$, that are not stochastically continuous in the sense of Definition 3.3. For example, consider the ODE

$$\partial_t f(t, x) = f(t, x)^2, \quad f(0, x) = x.$$

with $x \in E := \mathbb{R}_+$. Its unique solution is given by

$$f(t, x) = \begin{cases} 0 & \text{for } x = 0, t \geq 0, \\ (x^{-1} - t)^{-1} & \text{for } x > 0, 0 \leq t < x^{-1}, \\ \Delta & \text{for } x > 0, t \geq x^{-1}. \end{cases}$$

Take $\Omega = \mathbb{R}_+, \mathcal{F} = B(\mathbb{R}_+)$ and define the filtration $(\mathcal{F}_t)_{t \geq 0}$ by $\mathcal{F}_t = B(\mathbb{R}_+)$ for all $t \geq 0$. Define probability measures $(P_x)_{x \in E}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ by $P_x(\{x\}) = 1$ and let the adapted stochastic process $X$ be given by $X_t(\omega) = f(t, \omega)$, so that
\( \mathbb{P}_x(X_t = f(t,x)) = 1 \) for all \( x \in E \). Then \((X, (\mathbb{P}_x)_{x \in E}) \) is a Markov process on \( E = \mathbb{R}_+ \) with Markov transition function \( p_t(x, dz) \) on \( E \) given by

\[
p_t(x, B) = \begin{cases} 
1_B(0) & \text{for } x = 0, \\
1_B((x^{-1} - t)^{-1}), & \text{for } x > 0, 0 \leq t < x^{-1}, \\
0 & \text{for } x > 0, t \geq x^{-1},
\end{cases}
\]

for \( B \in \mathcal{B}(\mathbb{R}_+) \). The sample paths of \( X \) are continuous in \( E_\Delta \), \( \mathbb{P}_x \)-a.s. for all \( x \in E \). However, stochastic continuity does not hold. For example take \( x > 0, t = 1/x \). Then

\[
\int 1_E(z)p_s(x, dz) = 1 \text{ for all } s < t, \text{ which does not converge to } \int 1_E(z)p_t(x, dz) = 0 \text{ for } s \uparrow t.
\]

### 3.2 Properties of the state space

Henceforth and throughout we assume that the state space \( E \subset \mathbb{R}^p \) takes the special form

\[
E = \mathcal{X} \times \mathbb{R}^{p-m}, \tag{3.2}
\]

for some \( 1 \leq m \leq p \), where \( \mathcal{X} \subset \mathbb{R}^m \) is a closed, convex set with non-empty interior and such that

\[
U := \{ u \in \mathbb{C}^m : \sup_{x \in \mathcal{X}} \text{Re } u^\top x < \infty \} \tag{3.3}
\]

has non-empty interior. We decompose \( \{1, \ldots, p\} = I \cup J \) with \( I = \{1, \ldots, m\} \), \( J = \{m+1, \ldots, p\} \) and accordingly a point \( x \in \mathbb{C}^p \) is decomposed as \( x = (x_I, x_J) \in \mathbb{C}^m \times \mathbb{C}^{p-m} \). For \( u \in \mathbb{C}^p \) we let \( f_u \in C(E) \) denote the function \( f_u(x) = e^{u^\top x} \).

Similarly we define \( f_{u_I} \in C(\mathcal{X}) \) and \( f_{u_J} \in C(\mathbb{R}^{p-m}) \), so that we have the decomposition \( f_{u_I, u_J}(x) := f_u(x) = f_{u_I}(x_I)f_{u_J}(x_J) \). Note that \( u \in U \times \text{iR}^{p-m} \) if and only if \( f_u \in C_b(E) \). In particular it holds that \( \text{iR}^m \subset U \). On the other hand, \( \text{iR}^m \cap U^\circ = \emptyset \). This is a consequence of the next lemma, which characterizes the interior of \( U \).

**Lemma 3.5.** It holds that \( f_u \in C_0(\mathcal{X}) \) if and only if \( u \in U^\circ \).

**Proof.** First we prove that \( f_u \in C_0(\mathcal{X}) \) for \( u \in \text{Re } U^\circ \). Fix \( u \in \text{Re } U^\circ \). There exists \( \varepsilon > 0 \) such that \( u \pm \varepsilon e_i \in \text{Re } U^\circ \) for all \( i \). Therefore,

\[
\sup_{x \in \mathcal{X}} (u \pm \varepsilon e_i)^\top x < \infty, \text{ for all } i. \tag{3.4}
\]
Let \((x_n)\) be a sequence in \(E\) such that \(|x_n| \to \infty\). Arguing by contradiction, suppose \(\limsup_{n \to \infty} u^\top x_n > -\infty\). Then there exists a subsequence, also denoted by \((x_n)\), such that \(\liminf_{n \to \infty} u^\top x_n > -\infty\), and in addition \(x_{n,i} \to \infty\) or \(x_{n,i} \to -\infty\). Then (3.4) yields \(u^\top x_n \to -\infty\), a contradiction. Hence \(u^\top x_n \to -\infty\) whenever \(u \in \text{Re}\, U^0\) and \(|x_n| \to \infty\) with \(x_n \in E\), in other words, \(f_u \in C_0(\mathcal{X})\) for all \(u \in \text{Re}\, U^0\). For \(u \in U^0\) we have \(|f_u(x)| \leq f_{\text{Re},u}(x)\), so by the previous we have \(f_u \in C_0(\mathcal{X})\) for all \(u \in U^0\).

Now suppose \(u \in U^0\). Translating \(\mathcal{X}\) does not affect \(U\), so we may assume without loss of generality that \(0 \in \mathcal{X}\). There exists \(v \in B(0,1)\) such that \(u + \varepsilon v \notin U\) for all \(\varepsilon > 0\). This yields the existence of a sequence \(x_n \in \mathcal{X}\) such that \((u + v/n)^\top x_n = n\). If \((u^\top x_n) \neq -\infty\), then we are done. If \(u^\top x_n \to -\infty\), then there exists \(N > 0\) such that \(u^\top x_n < -1\) for all \(n \geq N\). Hence \(\bar{x}_n = -x_n/(u^\top x_n) \in \mathcal{X}\), for \(n \geq N\), by convexity of \(\mathcal{X}\) and the fact that \(0 \in \mathcal{X}\) and \(x_n \in \mathcal{X}\). We have \(|\bar{x}_n||v| \geq v^\top \bar{x}_n = -n^2/(u^\top x_n) + n \geq n\), so \(\bar{x}_n \to \Delta\). On the other hand, \(u^\top \bar{x}_n = -1\), which yields that \(f_u \notin C_0(\mathcal{X})\), as we needed to show.

**Corollary 3.6.** If \(u \in U^0\) and \(v \in U\), then \(v + \lambda u \in U^0\) for all \(\lambda \in (0,1]\).

**Proof.** Translating \(\mathcal{X}\) does not affect \(U\), so we may assume without loss of generality that \(0 \in \mathcal{X}\). Let \(|x_n| \to \infty\), with \(x_n \in \mathcal{X}\) and \(\lambda \in (0,1]\) be arbitrary. Since \(0 \in \mathcal{X}\), it holds that \(y_n := \lambda x_n \in \mathcal{X}\) and \(|y_n| \to \infty\). Hence the result follows from Lemma 3.5, as \(f_{v + \lambda u}(x_n) = f_v(x_n)f_u(y_n) \to 0\) for \(n \to \infty\), since \(f_v \in C_0(\mathcal{X})\) and \(f_u \in C_0(\mathcal{X})\).

As we saw in Lemma 3.5, there is equivalence between \(u \in U^0\) and \(f_u \in C_0(\mathcal{X})\). The following lemma shows that a similar equivalence holds between convergence in \(U^0\) and convergence in \(C_0(\mathcal{X})\).

**Lemma 3.7.** It holds that \(u_n \to u\) in \(U^0\) if and only if \(f_{u_n} \to f_u\) in \(C_0(\mathcal{X})\) (i.e. \(\|f_{u_n} - f_u\|_\infty \to 0\)).

**Proof.** The “if”-part is easy. We show the “only if”-part. First assume \(u_n\) and \(u\) are real-valued. Fix \(\varepsilon > 0\) arbitrarily. We first show that there exists \(M > 0\) and \(N > 0\) such that for all \(x \in \mathcal{X}\) with \(|x| \geq M\) it holds that \(|f_u(x)| < \varepsilon\) and \(|f_{u_n}(x)| < \varepsilon\) for all \(n \geq N\). Let \(\delta > 0\) be such that \(v_i := u + \delta e_i \in U^0\) and \(w_i := u - \delta e_i \in U^0\), for all \(i = 1, \ldots, p\). By Lemma 3.5 we have \(f_{v_i}, f_{w_i} \in C_0(\mathcal{X})\) for all \(i\) as well as \(f_u \in C_0(\mathcal{X})\). Since these are finitely many, there exists \(M > 0\)
such that for all $x \in X$ with $|x| > M$ it holds that $|f_u(x)| < \varepsilon$, $|f_{v_i}(x)| < \varepsilon$ and $|f_{w_i}(x)| < \varepsilon$, for all $i$, i.e.

$$u^\top x \leq \log \varepsilon, \quad v_i^\top x \leq \log \varepsilon, \quad w_i^\top x \leq \log \varepsilon,$$

for all $x \in X$. \hfill (3.5)

For all $n \in \mathbb{N}$ we can write

$$u_n = u + \sum_{i=1}^{p} \lambda_i^n e_i - \sum_{i=1}^{p} \mu_i^n e_i,$$

with $\lambda_i^n, \mu_i^n \geq 0$. Since $u_n \to u$, there exists $N > 0$ such that both $\sum_{i=1}^{p} \lambda_i^n$ and $\sum_{i=1}^{p} \mu_i^n$ are bounded by $\delta/2$ for $n \geq N$. Fix $n \geq N$ and define $[0, 1]$-valued numbers

$$t_i = \lambda_i^n / \delta, \quad s_i = \mu_i^n / \delta, \quad t_0 = 1 - \sum_{i=1}^{p} (t_i + s_i).$$

Then we can write

$$u_n = t_0 u + \sum_{i=1}^{p} t_i v_i + \sum_{i=1}^{p} s_i w_i,$$

so that it follows that for all $x \in X$ with $|x| > M$ we have $u_n^\top x \leq \log \varepsilon$ in view of (3.5). Hence $|f_{u_n}(x)| \leq \varepsilon$ for $|x| > M$ and $n \geq N$.

Next we write

$$f_{u_n}(x) - f_u(x) = (e^{(u_n-u)^\top x} - 1) e^{a^\top x}.$$

For $x \in X$ with $|x| > M$, the left-hand side is bounded by $2\varepsilon$ for $n \geq N$. For $|x| \leq M$ we can bound the right-hand side by

$$\exp(|(u_n-u)^\top x|) \exp(|u^\top x|) \leq \exp(M|u_n-u|) \exp(M|u|),$$

which can be made arbitrarily small by choosing $n$ large enough. It follows that $f_{u_n}(x) \to f_u(x)$ uniformly in $x$.

Now assume $u_n$ and $u$ are complex-valued. Let $u_n = a_n + ib_n$, $u = a + ib$, with $a_n \to a$ in $\text{Re}U^\circ$ and $b_n \to b$ in $\mathbb{R}^p$. We can write

$$f_{u_n}(x) - f_u(x) = (e^{a_n^\top x} - e^{a^\top x}) e^{ib_n x} + e^{(a+ib)^\top x} (e^{i(b_n-b)^\top x} - 1),$$

and bound it by

$$|e^{a_n^\top x} - e^{a^\top x}| + C|b_n - b|,$$

with $C = \exp(\sup_{x \in X} a^\top x) < \infty$. Both terms tend to zero uniformly in $x$, which yields the result.
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Corollary 3.8. For all compact $K \subset U^o$ it holds that $\sup_{u \in K} \| f_u \|_{\infty} < \infty$.

Proof. By Lemma 3.7 it holds that $u \mapsto \| f_u \|_{\infty}$ is continuous on $U^o$. This immediately gives the result.

We now investigate continuity properties of $u \mapsto \int f_u(x) \mu(dx)$, where $\mu$ is some finite measure on $E$. An immediate consequence of Lemma 3.7 is that $u \mapsto \int f_u(x) \mu(dx)$ is continuous on $U^o \times i\mathbb{R}^{p-m}$. However, continuity on $U \times i\mathbb{R}^{p-m}$ fails in general, as is demonstrated in the next example.

Example 3.9. Consider $E = \mathcal{X} = \{ x \in \mathbb{R}^2 : x_2 \geq x_1^2 \}$. Then one verifies that $\text{Re} \ U = (\mathbb{R} \times (-\infty,0)) \cup \{(0,0)\}$. Define the finite measure $\mu$ on $E$ with support $\{(k,k^2) : k \in \mathbb{N}\}$ by $\mu(\{(k,k^2)\}) = 1/k^2$, for $k \in \mathbb{N}$.

Take $u_n = (n^{-1}, -n^{-4}) \in \text{Re} \ U$, which converges to $u_0 = 0 \in \text{Re} \ U$. Then it holds that

$$\int f_{u_0}(x) \mu(dx) = \sum_{k=1}^{\infty} 1/k^2 = \frac{\pi^2}{6},$$

while

$$\int f_{u_n}(x) \mu(dx) = \sum_{k=1}^{\infty} 1/k^2 \exp(k/n - k^2/n^4) \geq \frac{1}{n^4} \exp(n-1) \to \infty,$$

for $n \to \infty$. Hence $\int f_{u_n}(x) \mu(dx)$ does not converge to $\int f_{u_0}(x) \mu(dx)$.

Although continuity fails on the whole of $U \times i\mathbb{R}^{p-m}$, it does hold on certain compacta of $U \times i\mathbb{R}^{p-m}$, as we show in the following lemma.

Lemma 3.10. Let $u \in U$, $K \subset U^o$ be compact and define the compact set $C = \{ \lambda u + (1 - \lambda) v : \lambda \in [0,1], v \in K \} \subset U$. Then for all finite measures $\mu$ on $E$ it holds that $z \mapsto \int f_z(x) \mu(dx)$ is continuous on $C \times i\mathbb{R}^{p-m}$.

Proof. Let $z_n \to z$ with $z_n, z \in C \times i\mathbb{R}^{p-m}$. Then $z_{n,I} = \lambda_n u + (1 - \lambda_n)v_n$, for some $\lambda_n \in [0,1], v_n \in K$. It follows that

$$\| f_{z_n} \|_{\infty} \leq \| f_u \|_{\infty} \| f_{v_n} \|_{\infty},$$

which is uniformly bounded in $n \in \mathbb{N}$, in view of Corollary 3.8. Dominated convergence yields

$$\int f_{z_n}(x) \mu(dx) \to \int f_z(x) \mu(dx),$$

for all finite measures $\mu$. This gives the result.
The next continuity property will be needed later on.

**Lemma 3.11.** Let \((X,(\mathbb{P}_x)_{x \in E})\) be a stochastically continuous Markov process with state space \(E\) and let the compact set \(C \subset \mathbb{R}^m\) be given as in Lemma 3.10. Then for fixed \(x \in E\), it holds that \((t,u) \mapsto P_t f_u(x)\) is jointly continuous on \(\mathbb{R}_+ \times C \times i\mathbb{R}^{p-m}\).

**Proof.** Let \((t_n,u_n) \to (t,u)\) in \(\mathbb{R}_+ \times C \times i\mathbb{R}^{p-m}\). As in the proof of Lemma 3.10 it holds that \(\|f_{u_n}\|_\infty\) is uniformly bounded in \(n\). We have to show that \(P_{t_n}f_{u_n}(x) \to P_t f_u(x)\). By stochastic continuity we have \(p_{t_n}(x,dz) \xrightarrow{w} p_t(x,dz)\). Note that if \(p_t(x,E) = 0\), then
\[
|P_{t_n}f_{u_n}(x)| \leq \|f_{u_n}\|_\infty p_{t_n}(x,E) \to 0 = P_t f_u(x).
\]
which yields the result. Therefore, suppose \(p_t(x,E) > 0\). Then we have \(p_{t_n}(x,E) > 0\) for \(n\) large enough. Define probability measures on \(E\) by
\[
\tilde{p}_n(dz) = \frac{p_{t_n}(x,dz)}{p_{t_n}(x,E)}, \quad \tilde{p}(dz) = \frac{p_t(x,dz)}{p_t(x,E)}.
\]
Then \(\tilde{p}_n \xrightarrow{w} \tilde{p}\). By Skorohod’s Representation Theorem \[34\] Theorem 4.30 there exist \(E\)-valued random variables \(Y_n, Y\) defined on a common probability space \((\Omega, \mathcal{F}, P)\) such that \(P \circ Y_n^{-1} = \tilde{p}_n\), \(P \circ Y^{-1} = \tilde{p}\) and \(Y_n \to Y\), \(P\)-a.s. By dominated convergence we have
\[
\int f_{u_n}(z)\tilde{p}_n(dz) = \int f_{u_n}(Y_n(\omega))P(d\omega) \\
\to \int f_u(Y(\omega))P(d\omega) = \int f_u(z)\tilde{p}(dz).
\]
This gives the result. 

\(\square\)

### 3.3 The symbol of a regular Markov process

In this section we consider regular Markov processes. Intuitively speaking, such a process behaves locally like a Lévy process. Recall that if \((X,(\mathbb{P}_x)_{x \in E})\) is a Lévy process with Lévy triplet \((b,c,K)\) with respect to a truncation function \(\chi\), then its Fourier-Laplace transform can be written as
\[
P_t f_u(x) = f_u(x) \exp(t\Psi(u)),
\]
for \(u \in U \times i\mathbb{R}^{p-m}\), where \(\Psi\) is of the Lévy-Khintchine form
\[
\Psi(u) = u^\top b + \frac{1}{2} u^\top cu + \int (e^{u^\top z} - 1 - u^\top \chi(z))K(dz).
\]
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Hence, we call a Markov processes regular if this holds “locally”, that is, if

\[ P_t f_u(x) \approx f_u(x) \exp(t \Psi(u, x)), \]

for some \( \Psi(u, x) \), \( t \geq 0 \) small enough. We make this precise in the next definition.

**Definition 3.12.** We call \((X, (P_x)_{x \in E})\) regular if the right-hand derivative

\[ \partial^+_t |_{t=0} P_t f_u(x) \]

exists for all \( x \in E, u \in U^o \times i\mathbb{R}^{p-m}, \) and is continuous in \( u \) for all \( x \in E \). In that case we define the symbol of the process \( X \) by

\[ \Psi(u, x) = \frac{\partial^+_t |_{t=0} P_t f_u(x)}{f_u(x)}, \quad (3.6) \]

for \( u \in U^o \times i\mathbb{R}^{p-m}, x \in E \). We call \( X \) strongly regular if it is regular and in addition if for all \( x \in E, y \in U^o, \) compact \( K \subset \mathbb{R}^{p-m}, \) there exists \( T > 0 \) such that

\[ \sup_{z \in K, t \in [0, T]} \left| \frac{1}{t} (P_t f_{y,iz}(x) - f_{y,iz}(x)) \right| < \infty. \]

Note that our definition of regularity slightly differs from the definition in [17], as the parameter \( u \) is restricted to \( U^o \times i\mathbb{R}^{p-m}. \) Moreover, one usually defines the symbol with a minus-sign on the right-hand side of (3.6), see [32]. Similar as for Lévy processes, the symbol \( \Psi \) of a regular process is also of the Lévy-Khintchine form. This is proved in Proposition 3.14 below, for which we need the following lemma. The proofs are based on the proof of [38, Theorem 4.3]. Recall that \( \mu_n \) converges vaguely to \( \mu \), denoted by \( \mu_n \xrightarrow{v} \mu \), if

\[ \int f(z) \mu_n(dz) \to \int f(z) \mu(dz), \quad \text{for all } f \in C_0(E). \]

**Lemma 3.13.** Let \( \mu_n \) be a sequence of finite measures on \( E \) and suppose

\[ \int f_u(z) \mu_n(dz) \to \Phi(u), \]

for all \( u \in U^o \times i\mathbb{R}^{p-m}, \) with \( \Phi : U^o \times i\mathbb{R}^{p-m} \to \mathbb{C} \) a continuous function. Then there exists a finite measure \( \mu \) such that

- \( (i) \) \( f_{\theta,0}(z) \mu_n(dz) \xrightarrow{u_\theta} f_{\theta,0}(z) \mu(dz) \) for all \( \theta \in \text{Re } U^o, \)
- \( (ii) \) \( \mu_n \xrightarrow{v} \mu, \)
- \( (iii) \) \( \Phi(u) = \int f_u(z) \mu(dz) \) for \( u \in U^o \times i\mathbb{R}^{p-m}, \)
(iv) If in addition \( \mu_n(E) \rightarrow \mu(E) \), then \( \mu_n \overset{w}{\rightarrow} \mu \).

**Proof.** Let \( \theta \in \text{Re}U^\circ \) be arbitrary. By assumption, the characteristic functions of the exponentially tilted measures \( f_{\theta,0}(z)\mu_n(dz) \) converge to \( u \mapsto \Phi(\theta + u_I, u_J) \), which is continuous in 0 as \( u \mapsto \Phi(u) \) is continuous on \( U^\circ \times i\mathbb{R}^{p-m} \). By Lévy’s continuity theorem we have

\[
f_{\theta,0}(z)\mu_n(dz) \overset{w}{\rightarrow} \mu_\theta(dz)
\]

for some measure \( \mu_\theta(dz) \) with characteristic function \( u \mapsto \Phi(\theta + u_I, u_J) \). Let \( \mu_\theta \) be a vague accumulation point of \( \mu_n \), which exists by Helly’s selection theorem (see [4, Corollary 5.7.6]). Then it holds that

\[
f_{\theta,0}(z)\mu_n(dz) \overset{w}{\rightarrow} f_{\theta,0}(z)\mu(dz),
\]

along a subsequence. Since weak convergence implies vague convergence, we infer from the uniqueness of the vague limit that

\[
\mu_\theta(dz) = f_{\theta,0}(z)\mu(dz),
\]

by combining (3.7) and (3.8). This yields the first assertion. Moreover, we conclude that \( \mu_n \) has only one vague accumulation point \( \mu \), which yields the second assertion. Since \( \mu_\theta(dz) \) has characteristic function \( u \mapsto \Phi(\theta + u_I, u_J) \), it follows from (3.9) that

\[
\Phi(\theta + u_I, u_J) = \int f_u(z)\mu_\theta(dz) = \int f_{\theta+u_I,u_J}(z)\mu(dz),
\]

for all \( u \in i\mathbb{R}^p \). This gives the third assertion, as \( \theta \in \text{Re}U^\circ \) was chosen arbitrarily. The last assertion is a consequence of [41, II.6.8] (but notice the difference in terminology).

**Proposition 3.14.** Let \((X, (\mathbb{P}_x)_{x \in E})\) be a regular Markov process with symbol \( \Psi \). Then there exist functions \( b : E \rightarrow \mathbb{R}^p \), \( c : E \rightarrow S^p_+ \), \( \gamma : E \rightarrow \mathbb{R}_+ \), a transition kernel \( K \) from \( E \) to \( \mathbb{R}^p \setminus \{0\} \) with \( \text{supp} K(x,dz) \subseteq E -x \) and

\[
\int (|z|^2 \wedge 1)K(x,dz) < \infty,
\]

such that

\[
\Psi(u,x) = u^\top b(x) + \frac{1}{2} u^\top c(x)u - \gamma(x) + \int (e^{u^\top z} - 1 - u^\top \chi(z))K(x,dz),
\]

for \( u \in U^\circ \times i\mathbb{R}^{p-m}, x \in E \), where \( \chi \) denotes a continuous truncation function.
3.3. The symbol of a regular Markov process

Proof. Fix \( x \in E \) and let \( t_n \downarrow 0 \). We define measures \( K_n(x, dz) \) on \( \mathbb{R}^p \setminus \{0\} \) with \( \text{supp} \, K_n(x, dz) \subset E - x \), by

\[
K_n(x, B) = \frac{1}{t_n} \int 1_B(z - x)p_{t_n}(x, dz), \quad \text{for } B \in \mathcal{B}(\mathbb{R}^p \setminus \{0\}).
\]

Note that (3.6) can be rewritten as

\[
\Psi(u, x) = \lim_{n \to \infty} \left( \int (e^{u^\top z} - 1)K_n(x, dz)) + \frac{1}{t_n}(p_{t_n}(x, E) - 1) \right),
\]

for \( u \in U^o \times i\mathbb{R}^{p-m} \). Let \( \tilde{\mu}_n(x, dz) \) be the compound Poisson distribution with Lévy measure \( K_n(x, dz) \) and define the infinitely divisible substochastic measure \( \mu_n(x, dz) \) by

\[
\mu_n(x, dz) = \exp\left(\frac{1}{t_n}(p_{t_n}(x, E) - 1)\right)\tilde{\mu}_n(x, dz)
\]

Then we have

\[
\exp(\Psi(u, x)) = \lim_{n \to \infty} \exp \left( \int (e^{u^\top z} - 1)K_n(x, dz)) \right) \exp \left( \frac{1}{t_n}(p_{t_n}(x, E) - 1) \right)
\]

for \( u \in U^o \times i\mathbb{R}^{p-m} \). By Lemma 3.13 there exists a finite measure \( \mu(x, dz) \) that is the vague limit of \( \mu_n(x, dz) \). Moreover, for all \( \theta \in \text{Re } U^o \) we have

\[
f_{\theta,0}(z)\mu_n(x, dz) \xrightarrow{w} f_{\theta,0}(z)\mu(x, dz).
\]

Since the class of infinitely divisible measures is closed under exponentially tilting and weak convergence, it follows that \( f_{\theta,0}(z)\mu(x, dz) \) is infinitely divisible and hence the same holds for \( \mu(x, dz) \). In addition, we have by Lemma 3.13 that

\[
\exp(\Psi(u, x)) = \int f_u(z)\mu(x, dz),
\]

(3.12)

for \( u \in U^o \times i\mathbb{R}^{p-m} \). In particular \( \mu(x, E) > 0 \). Since \( \mu(x, E) \) is the vague limit of a sequence of substochastic measures, we also have \( \mu(x, E) \leq 1 \). Therefore, \( \gamma(x) := -\log \mu(x, E) \geq 0 \). By the Lévy-Khintchine formula, there exist \( b(x) \in \mathbb{R}^p \), \( c(x) \in S_+^p \) and a measure \( K(x, dz) \) on \( \mathcal{B}(\mathbb{R}^p \setminus \{0\}) \) satisfying (3.10) such that

\[
\int f_u(z)\mu(x, dz) = \exp \left( -\gamma(x) + u^\top b(x) + \frac{1}{2}u^\top c(x)u 
+ \int (e^{u^\top z} - 1 - u^\top \chi(z))K(x, dz) \right),
\]
for \( u \in \mathbb{R}^p \) and even for \( u \in U \times \mathbb{R}^{p-m} \) in view of [48, Theorem 27.17]. Here \( \chi \) denotes a truncation function, which we choose to be continuous. Combining the above display with (3.12) gives
\[
\Psi(u,x) + k(x,u)\pi = \gamma(x) + u^\top b(x) + \frac{1}{2} u^\top c(x) u + \int (e^{u^\top z} - 1 - u^\top \chi(z)) K(x,dz),
\]
for some \( k(x,u) \in 2\mathbb{Z} \), possibly depending on \( x \) and \( u \). Since \( \Psi(u,x) \) as well as the right-hand side of the above display are continuous in \( u \) on \( U^o \times \mathbb{R}^{p-m} \) by Lemma 3.7 it holds that \( k(x,u) \) does not depend on \( u \). In addition, choosing \( u \in \text{Re} U^o \times \{0\} \) gives \( k(x,u) = 0 \), as all other terms are real-valued. Thus we have derived that
\[
\Psi(u,x) = \gamma(x) + u^\top b(x) + \frac{1}{2} u^\top c(x) u + \int (e^{u^\top z} - 1 - u^\top \chi(z)) K(x,dz).
\]
Since \( \chi \) is chosen to be continuous, [48, Theorem 8.7] yields
\[
\int f(z) K_n(x,dz) \to \int f(z) K(x,dz), \text{ as } n \to \infty,
\]
for \( f \in C_b(\mathbb{R}^p) \) vanishing on a neighborhood of 0. Note that if \( f(z) = 0 \) for \( z \in E-x \), then the left-hand side of the above display is zero, since supp \( K_n(x,dz) \subset E-x \). This yields that the support of \( K(x,dz) \) is contained in \( E-x \) and we have proved the result.

\[ \square \]

### 3.4 Feller processes

In this section we recall the definition of a Feller process and provide conditions under which a (regular) Feller process solves a certain martingale problem.

**Definition 3.15.** A Markov process \((X,(\mathbb{P}_x)_{x \in E})\) is called a **Feller process** and the corresponding semigroup \((P_t)_{t \geq 0}\) is called a **Feller semi-group** if

(i) \( P_tC_0(E) \subset C_0(E) \) for all \( t \geq 0 \);

(ii) \( P_tf(x) \to f(x) \) as \( t \downarrow 0 \), uniformly in \( x \in E \), for \( f \in C_0(E) \).

With every Feller semi-group one can associate its **infinitesimal generator**, which is the linear operator \( \mathcal{A} \) on \( C_0(E) \) (i.e. linear operator from a subspace \( \mathcal{D}(\mathcal{A}) \subset C_0(E) \) to \( C_0(E) \)) defined by
\[
\mathcal{A}f = \lim_{t \downarrow 0} \frac{P_tf-f}{t}, \tag{3.13}
\]
where the limit is taken with respect to the supremum norm. The domain $\mathcal{D}(A) \subset C_0(E)$ of $A$ is the subspace of those $f \in C_0(E)$ for which this limit exists.

Note that for a Feller process $(X, (\mathbb{P}_x)_{x \in E})$ we may take $\Omega$ to be of the form $(2.8)$ and $X$ the coordinate process, in view of [34, Theorem 19.15] and the ensuing paragraph. In that case we call $(X, (\mathbb{P}_x)_{x \in E})$ a canonical Feller process. For a stochastically continuous Markov process to be Feller, it suffices to have $P_t f \in C_0(E)$ for $f$ in a certain subclass of $C_0(E)$. This is the content of Proposition 3.17 below, the proof of which uses the following lemma, taken from [38, Theorem 3.5].

**Lemma 3.16.** Define the class of functions $\mathcal{H}$ by
\[
\mathcal{H} = \{ x \mapsto f_y(x_I) \int f_{iz}(x_J)g(z)dz : y \in U^0, g \in C_c(\mathbb{R}^{p-m}) \}. \tag{3.14}
\]
It holds that the linear span of $\mathcal{H}$ is dense in $C_0(E)$ with respect to the supremum norm.

**Proof.** Lemma 3.5 yields that $x_I \mapsto f_y(x_I) \in C_0(\mathcal{X})$ for all $u \in U^0$, while $x_I \mapsto \int f_{iz}(x_J)g(z)dz \in C_0(\mathbb{R}^{p-m})$ for all $g \in C_c(\mathbb{R}^{p-m})$ by the Riemann-Lebesgue Lemma. Hence $\mathcal{H} \subset C_0(E)$. Note that the functions in $\mathcal{H}$ can be written as $x \mapsto f_y(x_I)\hat{g}(x_J)$, where $\hat{g}$ denotes the Fourier transform of $g$. Therefore, the product of two elements
\[
x \mapsto f_{y_1}(x_I)\hat{g}_1(x_J) \quad \text{and} \quad x \mapsto f_{y_2}(x_I)\hat{g}_2(x_J)
\]
can be written as
\[
x \mapsto f_{y_1+y_2}(x_I)\hat{g}_1 \ast \hat{g}_2(x_J),
\]
where $g_1 \ast g_2$ denotes the convolution. Since $U^0$ is closed under addition and $C_c(\mathbb{R}^{p-m})$ is closed under convolution, it follows that $\mathcal{H}$ is closed under multiplication, i.e. $\mathcal{H}$ is a subalgebra of $C_0(E)$. In addition, $\mathcal{H}$ vanishes nowhere (i.e. there is no $x_0 \in E$ such that $h(x_0) = 0$ for all $h \in \mathcal{H}$) and $\mathcal{H}$ separates points (i.e. for all $x, y \in E$ with $x \neq y$ there exists $h \in \mathcal{H}$ with $h(x) \neq h(y)$). Since $E$ is closed, it is locally compact, whence the result follows from a locally compact version of the Stone-Weierstrass Theorem (see [49, Corollary 7.3.9]). \hfill $\square$

**Proposition 3.17.** Let $(X, (\mathbb{P}_x)_{x \in E})$ be a Markov process and let $\mathcal{H}$ be given by (3.14). Then $X$ is a Feller process if

(i) $P_t(\mathcal{H}) \subset C_0(E)$ for all $t \geq 0$;

(ii) $X$ is stochastically continuous.
Proof. We check the properties stated in Definition 3.15. Let \( f \in C_0(E) \). According to Lemma 3.16 there exists a sequence \( f_n \) in the linear span of \( \mathcal{H} \) such that \( \| f_n - f \|_\infty \to 0 \), as \( n \to \infty \). By the contraction property of the semigroup \( P_t \) we have \( \| P_t f_n - P_t f \|_\infty \leq \| f_n - f \| \to 0 \). Since \( P_t f_n \in C_0(E) \) and \( C_0(E) \) is a complete space with respect to the supremum norm, it follows that \( P_t f \in C_0(E) \), which gives the first property. Stochastic continuity immediately yields

\[
P_t f(x) = \int f(z)p_t(x, dz) \to \int f(z)p_0(x, dz) = f(x), \quad \text{for } f \in C_b(E),
\]

so in particular for \( f \in C_0(E) \). The uniform convergence follows from Lemma 19.6], whence we have shown the second property. 

In Theorem 3.20 below we provide conditions such that the generator of a regular Feller process \( (X, (\mathbb{P}_x)_{x \in E}) \) assumes the form (2.2) for \( f \in C^2_0(E) \) and we show that \( \mathbb{P}_x \) is a solution of the corresponding martingale problem. The special form of the generator is a consequence of the Lévy-Khintchine form of the symbol \( \Psi \), while we use Proposition 2.5 to deduce that \( \mathbb{P}_x \) solves the martingale problem.

Lemma 3.18. Let \( f \in C^{k,0}(\mathbb{R}^p \times \mathbb{R}^q) \) and define \( g : \mathbb{R}^p \to \mathbb{C} \) by \( g(x) = \int_C f(x, y)dy \), for some compact set \( C \subset \mathbb{R}^q \). Then it holds that \( g \in C^k(\mathbb{R}^p) \) and

\[
\partial_x g(x) = \int_C \partial_x f(x, y)dy.
\]

Proof. It is sufficient to prove this for \( k = p = 1 \). By Fubini we can write

\[
g(x) = g(0) + \int_C \int_0^x \partial_t f(t, y)dydt = g(0) + \int_0^x \int_C \partial_t f(t, y)dydt,
\]

since \( (t, y) \mapsto \partial_t f(t, y) \) is continuous, whence bounded on \([0, x] \times C\). Dominated convergence yields that \( t \mapsto \int_C \partial_t f(t, y)dy \) is continuous, so that we infer from the fundamental theorem of calculus that \( g \) is continuously differentiable with

\[
\partial_x g(x) = \int_C \partial_x f(x, y)dy.
\]

Lemma 3.19. Let \( K \) be a transition kernel from \( E \) to \( \mathbb{R}^p \setminus \{0\} \) with \( \text{supp} K(x, dz) \subset E - x \), and satisfying (3.10). Let \( f \in C_0^2(E \times \mathbb{R}^q) \) and define \( g : E \to \mathbb{C} \) by \( g(x) = \int_C f(x, y)dy \), for some compact set \( C \subset \mathbb{R}^q \). Then it holds that \( g \in C^2(E) \) and

\[
\int (g(x + z) - g(x) - \nabla g(x)^\top \chi(z))K(x, dz) = \\
\int_C \int (f(x + z, y) - f(x, y) - \nabla f(x, y)^\top \chi(z))K(x, dz)dy.
\]
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Proof. By Lemma 3.18 we have \( g \in C^2(E) \) and \( \nabla g(x) = \int_C \nabla f(x,y) dy \). Therefore,

\[
\int (g(x + z) - g(x) - \nabla g(x) \nabla \chi(z) )K(x, dz) = \\
\int \int_C (f(x + z, y) - f(x, y) - \nabla f(x, y) \nabla \chi(z) )dy K(x, dz).
\]

It remains to show that the order of integration can be interchanged by Fubini’s theorem. Fix \( x_0 \in E \). Since \( \int (|z|^2 \wedge 1)K(x_0, dz) < \infty \) and \( \text{supp} K(x_0, dz) \subset E - x_0 \), it suffices to show that

\[
|f(x_0 + z, y) - f(x_0, y) - \nabla f(x_0, y) \nabla \chi(z)| \leq M(|z|^2 \wedge 1), \tag{3.15}
\]

for some \( M > 0 \), for all \( z \in E - x_0, y \in C \). Recall that \( \chi(z) = z \) in a neighborhood of 0, so there exists \( \varepsilon > 0 \) such that we can write

\[
f(x_0 + z, y) - f(x_0, y) - \nabla f(x_0, y) \nabla \chi(z) = \sum_{i,j} \int_0^1 \int_0^1 f_{ij}(x_0 + stz, y)tz_i z_j ds dt,
\]

for all \( |z| \leq \varepsilon \), where \( f_{ij} \) is short-hand notation for \( \partial_x \partial_y f \). By assumption \( (x, y) \mapsto f_{ij}(x, y) \) is continuous. In particular it is bounded on \( B(x_0, \varepsilon) \times C \). It follows that

\[
|f(x_0 + z, y) - f(x_0, y) - \nabla f(x_0, y) \nabla \chi(z)| \leq M_1 |z|^2,
\]

for some \( M_1 > 0 \), for all \( |z| \leq \varepsilon, y \in C \). Since \( f \) and \( \chi \) are bounded and since \( y \mapsto \nabla f(x_0, y) \) is bounded on \( C \) by continuity, we have

\[
|f(x_0 + z, y) - f(x_0, y) - \nabla f(x_0, y) \nabla \chi(z)| \leq M_2,
\]

for some \( M_2 > 0 \), for all \( |z| > \varepsilon, y \in C \). This yields (3.15) and the lemma is proved.

Theorem 3.20. Let \((X, (\mathbb{P}_x)_{x \in E})\) be a strongly regular canonical Feller process, with infinitesimal generator \( \mathcal{A} \) and symbol \( \Psi \) given by (3.11), for measurable functions \( b : E \to \mathbb{R}^p, c : E \to S_p, \gamma : E \to \mathbb{R}_+ \), a transition kernel \( K \) from \( E \) to \( \mathbb{R}^p \setminus \{0\} \) with \( \text{supp} K(x, dz) \subset E - x \), and satisfying (2.1). In addition, let \( \mathcal{A}^2 : C_0^2(E) \to M(E) \) be the linear operator given by

\[
\mathcal{A}^2 f(x) = \nabla f(x)^\top b(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) - \gamma(x)f(x) \\
+ \int (f(x + z) - f(x) - \nabla f(x)^\top \chi(z))K(x, dz).
\]

Assume \( \mathcal{A}^2(C_0^2(E)) \subset C_0(E) \) as well as \( \mathcal{A}^2(H) \subset C_0(E) \), where \( H \) is given by (3.14). Then \( C^2_0(E) \subset D(\mathcal{A}) \) and \( \mathcal{A}|_{C^2_0(E)} = \mathcal{A}^2|_{C^2_0(E)} \). In particular, \( \mathbb{P}_x \) is a solution of the martingale problem for \( (\mathcal{A}^2, \delta_z) \) in \( \Omega \) given by (2.8), for \( x \in E \).
3. Markov processes

Proof. Throughout we make use of the equality
\[
\int A^z f_{y,iz}(x)g(z)dz = A^x \left( \int f_{y,iz}(x)g(z)dz \right),
\]
for \( y \in U^o, g \in C_c^\infty(\mathbb{R}^{p-m}), \)
\[
(3.16)
\]
which is a consequence of Lemmas \[3.18\] and \[3.19\]. We first show that \( \mathcal{H} \subset D(A) \) and that
\[
Ah = A^zh, \text{ for all } h \in \mathcal{H}.
\]
\[
(3.17)
\]
Let \( y \in U^o, g \in C_c(\mathbb{R}^{p-m}) \) and take \( h(x) = \int f_{y,iz}(x)g(z)dz \). Then \( h \in C_0(E) \) by Lemma \[3.16\]. Note that for \( u \in U \times i\mathbb{R}^{p-m} \) we can write
\[
A^z f_u(x) = f_u(x) \left( u^\top b(x) + \frac{1}{2} u^\top c(x)u - \gamma(x) \right) + \int (e^{u^\top z} - 1 - u^\top \chi(z))K(x, dz).
\]
\[
(3.18)
\]
Fubini and strong regularity of \( X \) yields
\[
\frac{1}{t}(P_th(x) - h(x)) = \frac{1}{t}(P_th_{y,iz}(x) - f_{y,iz}(x))g(z)dz \to \int \Psi(y,iz,x)f_{y,iz}(x)g(z)dz,
\]
as \( t \downarrow 0 \), for all \( x \in E \). From \[3.11\] we infer that \( \Psi(u,x)f_u(x) = A^z f_u(x) \) for \( u \in U^o \times i\mathbb{R}^{p-m}, x \in E \). Together with \[3.16\] this gives
\[
\frac{1}{t}(P_th(x) - h(x)) \to A^zh(x),
\]
as \( t \downarrow 0 \), for all \( x \in E \). Since \((P_t)_{t\geq 0}\) is a Feller semi-group with generator \( A \) and \( A^zh \in C_0(E) \) by assumption, the pointwise convergence in the above display suffices for uniform convergence, see [48, Lemma 31.7]. Hence \( h \in D(A) \) and \[3.17\] holds.

Let \( T_n \) be the stopping time given by \[2.7\], \( n \in \mathbb{N} \). By Dynkin’s formula [34, Lemma 19.21], it holds for all \( h \in \mathcal{H} \) that
\[
h(X_t^{T_n}) - h(X_0) - \int_0^{t\wedge T_n} Ah(X_s)ds
\]
is a \( \mathbb{P}_x \)-martingale for all \( x \in E \). Let \((y,iz_0) \in U^o \times i\mathbb{R}^{p-m}\) be arbitrary. Take a sequence of non-negative functions \((g_k)\) in \( C_c^\infty(\mathbb{R}^{p-m}) \) with support contained in the unit ball \( B(0,1) \) in \( \mathbb{R}^{p-m} \), with \( \int g_k(z)dz = 1 \) for all \( k \in \mathbb{N} \) and such that \( g_k(z)dz \) converges weakly to \( \delta_{z_0}(dz) \). We define a sequence \((h_k)\) in \( \mathcal{H} \) by
\[
h_k(x) = \int f_{y,iz}(x)g_k(z)dz.
\]
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We show that

\[ h_k(X_t^{T_n}) - \int_0^{t \wedge T_n} \mathcal{A}^2 h_k(X_s) ds \xrightarrow{L^1} f_{y,iz_0}(X_t^{T_n}) - \int_0^{t \wedge T_n} \mathcal{A}^2 f_{y,iz_0}(X_s) ds, \tag{3.19} \]

as \( k \to \infty \), so that the right-hand side, being an \( L^1 \)-limit of a sequence of \( \mathbb{P}_x \)-martingales, is a \( \mathbb{P}_x \)-martingale itself for all \( x \in E \). Since \( h_k(x) \to f_{y,iz_0}(x) \) and \( \|h_k\|_\infty = \|f_y\|_\infty < \infty \), we have

\[ h_k(X_t^{T_n}) \xrightarrow{L^1} f_{y,iz_0}(X_t^{T_n}) \]

as \( k \to \infty \). By (3.16) it holds that

\[ \int_0^{t \wedge T_n} \mathcal{A}^2 f_{y,iz}(X_s) g_k(z) dzds = \int_0^{t \wedge T_n} \mathcal{A}^2 f_{y,iz_0}(X_s) g_k(z) dzds. \]

From (3.18) together with (2.1) it follows that \( z \mapsto \mathcal{A}^2 f_{y,iz}(x) \) is continuous, whence bounded on \( B(0,1) \). Hence

\[ \int \mathcal{A}^2 f_{y,iz}(x) g_k(z) dz \to \mathcal{A}^2 f_{y,iz_0}(x), \]

as \( g_k(z) dz \xrightarrow{w} \delta_{z_0}(dz) \). It also follows from (3.18) together with (2.1) that

\[ \sup_{|x| \leq n, |z| \leq 1} |\mathcal{A}^2 f_{y,iz}(x)| := C < \infty. \]

This gives

\[ \left| \int \mathcal{A}^2 f_{y,iz}(X_s) g_k(z) dz \right| \leq C \int g_k(z) dz = C, \]

for \( s < T_n \). From dominated convergence we deduce that

\[ \int_0^{t \wedge T_n} \mathcal{A}^2 f_{y,iz}(X_s) g_k(z) dzds \xrightarrow{L^1} \int_0^{t \wedge T_n} \mathcal{A}^2 f_{y,iz_0}(X_s) ds, \]

whence we have that (3.19) holds.

It remains to show that \( f_u(X_t^{T_n}) - \int_0^{t \wedge T_n} \mathcal{A}^2 f_u(X_s) ds \) is a \( \mathbb{P}_x \)-martingale for all \( x \in E, u \in i\mathbb{R}^p, n \in \mathbb{N} \). Indeed, in that case, Proposition 2.5 yields that \( \mathbb{P}_x \) is a solution of the martingale problem for \( (\mathcal{A}^2, \delta_x) \) for all \( x \in E \), while Lemma 2.11 gives

\[ \mathcal{A}^2 f(x) = \partial^+_t |_{t=0} \mathbb{E}_x f(X_t) = \partial^+_t |_{t=0} P_t f(x), \quad \text{for} \ f \in C^2_c(E). \]

Since \( (P_t)_{t \geq 0} \) is a Feller semi-group with generator \( \mathcal{A} \) and \( \mathcal{A}^2(C_0(E)) \subset C_0(E) \) by assumption, again the pointwise convergence in the above display suffices for
uniform convergence, see [48, Lemma 31.7], whence \( f \in \mathcal{D}(\mathcal{A}) \) and \( \mathcal{A}^2 f = \mathcal{A} f \) for \( f \in C^2_c(E) \).

Let \( u_0 \in \mathbb{i}\mathbb{R}^p \) be arbitrary and take a sequence \((u_k)\) in \( U^o \times \mathbb{i}\mathbb{R}^{p-m} \) that converges to \( u_0 \) and with the property that

\[
\sup_{k \in \mathbb{N}} \|u_k\|_{\infty} < \infty, \tag{3.20}
\]

which is possible in view of the proof of Lemma 3.10. Since \( \chi(z) = z \) in a neighborhood of 0, there exists \( \varepsilon > 0 \) such that for \( |z| \leq \varepsilon \) it holds that

\[
|e^{u_k^\top z} - 1 - u_k^\top \chi(z)| \leq |u_k|^2 |z|^2 \int_0^1 \int_0^1 \exp(u_k^\top zst) t \, ds \, dt \leq C|z|^2,
\]

for some constant \( C > 0 \), uniformly in \( k \). Also, since \( \chi \) is bounded and since we have (3.20), there exists \( C > 0 \) such that for all \( x \in E \), \( |z| > \varepsilon \) with \( z \in E - x \), \( k \in \mathbb{N} \), we have that

\[
|f_{u_k}(x)(e^{u_k^\top z} - 1 - u_k^\top \chi(z))| = |f_{u_k}(x + z) - f_{u_k}(x)(1 + u_k^\top \chi(z))| \leq C.
\]

By assumption \( \text{supp} \, K(x, dz) \subset E - x \), whence by the previous we have

\[
|f_{u_k}(x) \int (e^{u_k^\top z} - 1 - u_k^\top \chi(z))K(x, dz)| \leq C \int (|z|^2 \wedge 1)K(x, dz),
\]

for some \( C > 0 \), for all \( x \in E \). Hence (3.18) together with (2.1) gives

\[
\sup_{|x| \leq n, k \in \mathbb{N}} \mathcal{A}^2 f_{u_k}(x) < \infty,
\]

while dominated convergence gives

\[
\int (e^{u_k^\top z} - 1 - u_k^\top \chi(z))K(x, dz) \rightarrow \int (e^{u_0^\top z} - 1 - u_0^\top \chi(z))K(x, dz),
\]

as \( k \to \infty \). Hence \( \mathcal{A}^2 f_{u_k}(x) \to \mathcal{A}^2 f_{u_0}(x) \) for all \( x \in E \), as \( k \to \infty \), in view of (3.18). Using dominated convergence again, we obtain

\[
f_{u_k}(X^T_n) - \int_0^{t^{\wedge}T_n} \mathcal{A}^2 f_{u_k}(X_s) \, ds \xrightarrow{L^1} f_u(X^T_n) - \int_0^{t^{\wedge}T_n} \mathcal{A}^2 f_u(X_s) \, ds,
\]

whence the right-hand side, being an \( L^1 \)-limit of a sequence of \( \mathbb{P}_x \)-martingales, is a \( \mathbb{P}_x \)-martingale itself for all \( x \in E \), as we needed to show.