Affine Markov processes on a general state space
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Admissible parameter sets

In this chapter we work out the admissibility conditions as given in Definition 4.3 for affine processes with an arbitrary polyhedral and with an arbitrary quadratic state space, which complements the characterization of admissible parameter sets for the state space $\mathbb{R}_+^m \times \mathbb{R}^{p-m}$ and $S^p_+$, as derived in [17] and [10]. First we derive some preliminary results in Section 5.1, where we translate the positive maximum-principle as discussed in Section 2.4 in explicit boundary conditions for general processes. Here we essentially consider state spaces where the boundary can be transformed locally into the boundary of the state space $\mathbb{R}_+^m \times \mathbb{R}^{p-m}$. Our method to derive boundary conditions is an alternative to the analysis in [10], which is based on stochastic invariance results from [12]. In Section 5.2 we derive the admissibility conditions for a general polyhedral state space, see Theorem 5.12. For the proof we use results from convex analysis. Next we determine all possible quadratic state spaces in Section 5.3. We show that the parabolic state space and the Lorentz cone are the only possibilities for an affine process, for which we derive the required admissibility conditions in Sections 5.4 and 5.5, see Theorems 5.17 and 5.22.

We note that it suffices to characterize the admissible parameter sets up to an invertible affine transformation. The following proposition will be needed for this.

Proposition 5.1. Suppose $(X, (\mathbb{P}_x)_{x \in E})$ is an affine jump-diffusion with state space $E$ and parameters $(b(x), c(x), \gamma(x), K(x, dz))$. Let $\ell \in \mathbb{R}^p$ and $L \in \mathbb{R}^{p \times p}$ be non-singular. Write $Y = \ell + LX$, $\bar{E} = \ell + LE$, $Q_y = \mathbb{P}_{L^{-1}(y-\ell)}$ for $y \in \bar{E}$. Then $(Y, (Q_y)_{y \in \bar{E}})$ is an affine jump-diffusion with state space $\bar{E}$ and parameters...
\((\tilde{b}(y), \tilde{c}(y), \tilde{\gamma}(y), \tilde{K}(y, dz))\) given by

\[
\begin{align*}
\tilde{b}(y) &= Lb(L^{-1}(y - \ell)) \\
\tilde{c}(y) &= Lc(L^{-1}(y - \ell))L^\top \\
\tilde{\gamma}(y) &= \gamma(L^{-1}(y - \ell)) \\
\tilde{K}(y, dz) &= K(L^{-1}(y - \ell), L^{-1}dz),
\end{align*}
\]

with respect to the truncation function \(\tilde{\chi}(z) = L\chi(L^{-1}z)\).

**Proof.** Let \(\tilde{A}\) be given by (2.2) with \((\tilde{b}, \tilde{c}, \tilde{\gamma}, \tilde{K})\) instead of \((b, c, \gamma, K)\) and with the truncation function \(\tilde{\chi}\) instead of \(\chi\). We have to show that for all \(g \in C^2_c(\tilde{E})\) it holds that

\[
g(Y_t) - \int_0^t \tilde{A}g(Y_s)ds
\]

is a \(\mathbb{Q}_y\)-martingale for all \(y \in \tilde{E}\). Let \(g \in C^2_c(\tilde{E})\) be arbitrary. Define \(f \in C^2_c(\tilde{E})\) by \(f(x) = g(\ell + Lx)\) and let \(\mathcal{A}\) given by (2.2) be the generator of \(X\). Then it holds that

\[
f(X_t) - \int_0^t \mathcal{A}f(X_s)ds
\]

is a \(\mathbb{P}_x\)-martingale for all \(x \in E\). Note that \(\nabla f(x) = L\nabla g(y)\) and \(\nabla^2 f(x) = L^\top \nabla^2 g(y)L\) for \(y = Lx + \ell, x \in E\). Using this, one easily verifies that \(\mathcal{A}f(x) = \tilde{A}g(y)\) for \(y = Lx + \ell, x \in E\). This gives the result. \(\square\)

Throughout this chapter the following notation regarding matrices and vectors is used. Let \(p, q \in \mathbb{N}, P = \{1, \ldots, p\}, Q = \{1, \ldots, q\}, A \in \mathbb{R}^{p \times q}, I \subset P, J \subset Q\). Write \(I = \{i_1, \ldots, i_{\#I}\}, J = \{j_1, \ldots, j_{\#J}\}\), with \(i_1 \leq i_2 \leq \ldots \leq i_{\#I}\) and \(j_1 \leq j_2 \leq \ldots \leq j_{\#J}\). Then \(A_{IJ}\) denotes the \((\#I \times \#J)\)-matrix with elements \((A_{IJ})_{kl} = A_{i_k j_l}\). The same notation applies to matrix-valued functions \(\phi\), e.g. \(\phi_{IJ}(x)\) stands for \((\phi(x))_{IJ}\). If \(\#I = 1\), say \(I = \{i\}\), we write \(A_{iJ}\) instead. If \(J = Q\) then we write \(A_I\) instead. In particular, \(A_i\) denotes the \(i\)-th row of \(A\). The \(j\)-th column is denoted by \(A^j\) and the transpose of \(A\) is denoted by \(A^\top\). For \(a_1, \ldots, a_p \in \mathbb{R}\) we write \(\text{diag}(a_1, \ldots, a_p)\) for the \(p\)-dimensional diagonal matrix \(D\) with diagonal elements \(D_{ii} = a_i, i \in \mathbb{N}\). We also write \(\text{diag}(a)\) instead, where \(a\) denotes the vector with elements \(a_i\), sometimes explicitly denoted by \(a = \text{vec}(a_1, \ldots, a_p)\). The unique positive semi-definite square root of a positive semi-definite matrix \(A\) is denoted by \(A^{1/2}\).
5.1 Preliminaries

In Proposition 2.16 we derived necessary boundary conditions for the positive maximum principle for general processes on general state spaces. In this section we elaborate on this and derive necessary and sufficient conditions for polyhedral and quadratic state spaces, which we apply in the next sections to the affine setting. The key steps are Propositions 5.2 and 5.3 below. In the first one we consider $\mathbb{R}_+^m \times \mathbb{R}^{p-m}$, while in the second we consider smooth transformations of this state space.

**Proposition 5.2.** Suppose $E = \mathbb{R}_+^m \times \mathbb{R}^{p-m}$ and let $b \in \mathbb{R}^p$, $c \in S_+^p$ and $K$ be a measure on $\mathbb{R}^p \setminus \{0\}$. Let $O \subset \mathbb{R}^p$ be open and suppose $x_0 \in O \cap E$. Then for all $f \in C^2(O \cap E)$ such that $f(x_0) = \sup_{x \in O \cap E} f(x)$ it holds that

(i) $\nabla f(x_0)^\top c \nabla f(x_0) = 0$,

(ii) $\int |\nabla f(x_0)^\top \chi(z)| K(dz) < \infty$,

(iii) $\nabla f(x_0)^\top b - \int \nabla f(x_0)^\top \chi(z) K(dz) + \frac{1}{2} \text{tr} \left( \nabla^2 f(x_0) c \right) \leq 0$,

if and only if for all $i = 1, \ldots, m$ with $x_{0,i} = 0$ it holds that

1. $c_{ij} = c_{ji} = 0$, for all $j = 1, \ldots, p$,

2. $\int |\chi_i(z)| K(dz) < \infty$,

3. $b_i - \int \chi_i(z) K(dz) \geq 0$.

**Proof.** The “only if”-part follows from the proof of Proposition 2.16. We show the “if”-part. Note that if $x_0 \in E^0$, then all first-order derivatives of $f$ in $x_0$ are zero and $\nabla^2 f(x_0)$ is negative semi-definite, whence the conditions are trivially satisfied. Therefore, suppose $x_0 \in \partial E$. Define $M = \{i \leq m : x_{0,i} \in \partial E \}$. By permuting the indices, we may assume without loss of generality that $M = \{1, \ldots, q\}$, for some $1 \leq q \leq m$. Write $\partial_i f(x) = \partial_x f(x)$. It holds that $\partial_i f(x_0) \leq 0$ for all $i \in M$ and $\partial_j f(x_0) = 0$ for all $j \notin M$. The latter yields

$$c_i \nabla f(x_0) = \sum_{j=1}^p c_{ij} \partial_j f(x_0) = \sum_{j=1}^q c_{ij} \partial_j f(x_0) = 0,$$

for all $i = 1, \ldots, p$, by assumption [1]. This gives condition (i). Condition (ii) follows from assumption (2), as we have

$$\nabla f(x_0)^\top \chi(z) = \sum_{i=1}^q \partial_i f(x_0) \chi_i(z).$$
5. Admissible parameter sets

It remains to verify condition (iii). Suppose \( q < p \). Write \( N = \{ q + 1, \ldots, p \} \) and decompose \( x \in E \) as \( x = (x_M, x_N) \), so that \( x_0 = (0, x_{0,N}) \). There exists an open set \( B \subset \mathbb{R}_+^{m-q} \times \mathbb{R}^{p-m} \) containing \( x_{0,N} \) and such that \( (0, x_N) \in O \cap E \) for all \( x_N \in B \). Define the function

\[
g : B \to \mathbb{R} : x_N \mapsto f(0, x_N).
\]

Then \( g \) assumes a maximum at \( x_{0,N} \in B \). Therefore, the Hessian \( \nabla^2 g(x_0) \) is negative semi-definite. Note that by assumption (1) we have \( \text{tr} (\nabla^2 f(x_0)c) = \text{tr} (\nabla^2 g(x_{0,N})c_{NN}) \), which is non-positive, since \( c_{NN} \) is positive semi-definite. It follows that

\[
\nabla f(x_0)^\top b - \int \nabla f(x_0)^\top \chi(z)K(dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0)c) \\
= \sum_{i=1}^q \partial_i f(x_0) \left( b_i - \int \chi_i(z)K(dz) \right) + \frac{1}{2} \text{tr} (\nabla^2 g(x_0)c_{NN}) \\
\leq \sum_{i=1}^q \partial_i f(x_0) \left( b_i - \int \chi_i(z)K(dz) \right) \leq 0,
\]

in view of assumption (3) and the fact that \( \partial_i f(x_0) \leq 0 \) for \( i = 1, \ldots, q \). This yields condition (iii). If \( q = p \), then \( c = 0 \) and the above argument simplifies.

\[\square\]

**Proposition 5.3.** Let \( O \subset \mathbb{R}^p \) be open, \( E \subset \mathbb{R}^p \) arbitrary, \( x_0 \in O \cap E \) and suppose there exists a \( C^2 \)-bijection \( \phi : \mathbb{R}^p \to \mathbb{R}^p \) such that \( \phi(O \cap E) = B \cap (\mathbb{R}_+^m \times \mathbb{R}^{p-m}) \) for some open set \( B \subset \mathbb{R}^p \). Let \( b \in \mathbb{R}^p \), \( c \in S_p^+ \) and \( K \) be a measure on \( \mathbb{R}^p \setminus \{0\} \) satisfying

\[
\int (|z|^2 \wedge 1)K(dz) < \infty.
\]

Then for all \( f \in C^2(O \cap E) \) such that \( f(x_0) = \sup_{x \in O \cap E} f(x) \) the conditions (i)-(iii) of Proposition 5.2 are satisfied if and only if for all \( i = 1, \ldots, m \) with \( \phi_i(x_0) = 0 \) it holds that

1. \( \nabla \phi_i(x_0)^\top c \nabla \phi_j(x_0) = 0 \), for all \( j = 1, \ldots, p \),

2. \( \int |\nabla \phi_i(x_0)^\top \chi(z)|K(dz) < \infty \),

3. \( \nabla \phi_i(x_0)^\top b - \int \nabla \phi_i(x_0)^\top \chi(z)K(dz) + \frac{1}{2} \text{tr} (\nabla^2 \phi_i(x_0)c) \geq 0 \).
Proof. Define \( \tilde{b} \in \mathbb{R}^p \), \( \tilde{c} \in S_+^p \) and a measure \( \tilde{K} \) on \( \mathbb{R}^p \setminus \{0\} \) by
\[
\tilde{b}_i = \nabla \phi_i(x_0) \top \tilde{b} + \frac{1}{2} \text{tr} (\nabla^2 \phi_i(x_0) \tilde{c})
\]
\[
+ \int (\chi_i(\phi(x_0 + z) - \phi(x_0)) - \nabla \phi_i(x_0) \top \chi(z)) \, K(dz)
\]
\[
\tilde{c}_{ij} = \nabla \phi_i(x_0) \top c \nabla \phi_j(x_0)
\]
\[
\tilde{K}(G) = \int 1_G(\phi(x_0 + z) - \phi(x_0)) \, K(dz), \text{ for } G \in \mathcal{B}(\mathbb{R}^p \setminus \{0\}).
\]
Note that \( \tilde{b} \) is well-defined, since \( \int (|z|^2 + 1) \, K(dz) < \infty \) and \( \chi(z) = z \) in a neighborhood of 0. We claim that conditions (i)-(iii) of Proposition 5.2 hold for all \( f \in C^2(\phi(O \cap E)) \) with \( f(x_0) = \sup_{x \in O \cap E} f(x) \) if and only if
\begin{enumerate}[(i)]
\item \( \nabla g(y_0) \top \tilde{c} \nabla g(y_0) = 0 \),
\item \( \int |\sum_{i=1}^p \partial_i g(y_0) \top \nabla \phi_i(x_0) \chi(z)| \, K(dz) < \infty \),
\item \( \nabla g(y_0) \top \tilde{b} - \int \nabla g(y_0) \top \chi(z) \, K(dz) + \frac{1}{2} \text{tr} (\nabla^2 g(y_0) \tilde{c}) \leq 0 \),
\end{enumerate}
holds for all \( g \in C^2(\phi(O \cap E)) \) with \( g(y_0) = \sup_{y \in \phi(O \cap E)} g(y) \), where we put \( y_0 = \phi(x_0) \). This is tedious, but straightforward to verify, using the identities
\[
\frac{\partial}{\partial x_i} f(x) = \sum_j \frac{\partial}{\partial y_j} g(y) \frac{\partial}{\partial x_i} \phi_j(x)
\]
\[
\frac{\partial^2}{\partial x_i \partial x_j} f(x) = \sum_{k,l} \frac{\partial^2}{\partial y_k \partial y_l} g(y) \frac{\partial}{\partial x_i} \phi_k(x) \frac{\partial}{\partial x_j} \phi_l(x) + \sum_k \frac{\partial}{\partial y_k} g(y) \frac{\partial^2}{\partial x_i \partial x_j} \phi_k(x),
\]
where \( f \in C^2(O \cap E) \), \( g = f \circ \phi^{-1} \), \( x \in O \cap E \) and \( y = \phi(x) \). Note that condition (ii) is equivalent with
\[
\int |\nabla g(y_0) \top \chi(z)| \, \tilde{K}(dz) < \infty,
\]
as
\[
\int |\chi_i(\phi(x_0 + z) - \phi(x_0)) - \nabla \phi_i(x_0) \top \chi(z)| \, K(dz) < \infty, \text{ for all } i = 1, \ldots, p.
\]

Since \( \phi(O \cap E) = B \cap (\mathbb{R}_+^m \times \mathbb{R}^{p-m}) \) for some open set \( B \subset \mathbb{R}^p \), the result follows from Proposition 5.2. \( \square \)

In the following, we let \( \Omega \) be given by (2.8), \( A \) as in (2.2), we assume (2.17) and we let \( E \subset \mathbb{R}^p \) be of the form \( E = X \times \mathbb{R}^{p-m} \) as in (3.2). We write \( Q = \{2, \ldots, m\} \). We derive two existence results from Proposition 5.3. In the first we give necessary and sufficient conditions for a parabolic state space, in the second for the Lorentz cone. We apply these in Section 5.4 respectively Section 5.5 to obtain the admissibility conditions for the corresponding affine processes.
Corollary 5.4. Suppose
\[ X = \{ x \in \mathbb{R}^m : x_1 \geq \sum_{i=2}^{m} x_i^2 \}. \]

Define \( \Phi(x) = (1, -2x_Q, 0) \in \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R}^{p-m} \) for \( x \in E \). Then for all \( x \in E \) there exists a solution \( \mathbb{P}_x \) of the martingale problem for \((A, \delta_x)\) in \( \Omega \) if and only if for all \( x \in \partial E \) it holds that

1. \( c(x)\Phi(x) = 0 \),
2. \( \int \Phi(x)^\top \chi(z)|K(x, dz) < \infty \),
3. \( \Phi(x)^\top b(x) - \int \Phi(x)^\top \chi(z)K(x, dz) - \text{tr}(cQQ(x)) \geq 0 \).

Proof. This follows from Proposition 2.15 together with Proposition 5.3 with \( O = \mathbb{R}^p \), \( m = 1 \) and \( \phi \) given by \( \phi_1(x) = x_1 - \sum_{i=2}^{m} x_i^2 \), \( \phi_i(x) = x_i \) for \( i = 2, \ldots, p \).

Corollary 5.5. Suppose \( E \) is the Lorentz cone given as
\[ E = \{ x \in \mathbb{R}^p : x_1 \geq |x_Q| \}. \]

Define \( \Phi(x) = (2x_1, -2x_Q) \in \mathbb{R} \times \mathbb{R}^{p-1} \) for \( x \in E \). Then for all \( x \in E \) there exists a solution \( \mathbb{P}_x \) of the martingale problem for \((A, \delta_x)\) in \( \Omega \) if and only if for all \( x \in \partial E \setminus \{0\} \) it holds that

1. \( c(x)\Phi(x) = 0 \),
2. \( \int \Phi(x)^\top \chi(z)|K(x, dz) < \infty \),
3. \( \Phi(x)^\top b(x) - \int \Phi(x)^\top \chi(z)K(x, dz) + c_{11}(x) - \text{tr}(cQQ(x)) \geq 0 \).

and

1'. \( c(0) = 0 \),
2'. \( \int \chi(z)|K(0, dz) < \infty \),
3'. \( b(0) - \int \chi(z)K(0, dz) \in E \).

Proof. We apply Proposition 2.15 together with Proposition 5.3. Clearly the conditions of Proposition 2.15 only depend on the values of \( f \) in a neighborhood of \( x_0 \). Take an open set \( O \subset \{ x \in \mathbb{R}^p : x_1 > 0 \} \) with \( x_0 \in O \). Then there exists a \( C^2 \)-bijection \( \phi : \mathbb{R}^p \to \mathbb{R}^p \) such that for \( x \in O \) we have \( \phi_1(x) = x_1^2 - \sum_{i=2}^{p} x_i^2 \) and \( \phi_i(x) = x_i \), for \( i = 2, \ldots, p \). Note that \( \phi(O \cap E) = B \cap (\mathbb{R}_+ \times \mathbb{R}^{p-1}) \), for some open
set \( B \subset \mathbb{R}^p \). Hence we can apply Proposition 5.3 to deduce that the conditions of Proposition 2.15 are satisfied for \( x_0 \in \partial E \setminus \{0\} \) if and only if conditions (1)-(3) hold. The conditions of Proposition 2.15 are obviously satisfied for \( x_0 \in E^c \). It remains to show they are equivalent with (1')-(3') in case \( x_0 = 0 \).

Assume the conditions of Proposition 2.15 hold for all \( f \in C^2_c(E) \) such that \( f(0) = \sup_{x \in E} f(x) \geq 0 \). Let \( y \in E \) be arbitrary and define \( f(x) = -y^\top x \). Then \( f(x) \leq 0 \) for all \( x \in E \) by self-duality of the Lorentz cone \( E \), whence \( f(0) = \sup_{x \in E} f(x) \geq 0 \). It holds that \( \nabla f(0) = y \). Since \( y \in E \) is arbitrary, the conditions of Proposition 2.15 yield

1. \( c_i(0)y = 0 \) for all \( i = 1, \ldots, p \), \( y \in E \),
2. \( \int |y^\top \chi(z)|K(0, dz) < \infty \), for all \( y \in E \),
3. \( y^\top b(0) - \int y^\top \chi(z)K(0, dz) \in E \).

By (i) we have \( c_i(0)^\top \in E \) for all \( i \). Taking \( y = c_i(0)^\top \) gives \( c_i(0) = 0 \) for all \( i \), whence we have (1'). Likewise we see that (2') holds. By the self-duality of \( E \) we infer (3') from (iii).

Conversely, assume (1')-(3') holds. Let \( f \in C^2_c(E) \) be such that \( f(0) = \sup_{x \in E} f(x) \geq 0 \). Let \( y \in E \) be arbitrary. Define \( g : \mathbb{R}_+ \to \mathbb{R} : t \mapsto f(ty) \). Then \( 0 \geq \frac{\partial f}{\partial t} \bigg|_{t=0} g(t) = \nabla f(0)^\top y \). Hence \( \nabla f(0) \in -E \), since \( E \) is a self-dual cone. This yields the result.

### 5.2 Polyhedral state space

In this section we derive the explicit form of the admissible parameters for an affine process with a general polyhedral state space. Throughout we let \( E \subset \mathbb{R}^p \) be a (non-empty) polyhedron given by

\[
E = \bigcap_{i=1}^q \{ x \in \mathbb{R}^p : \zeta_i + \eta_i x \geq 0 \},
\]

with \( \zeta \in \mathbb{R}^q \), \( \eta \in \mathbb{R}^{q \times p} \), for some \( q \geq 1 \). We write \( u(x) = \zeta + \eta x \) and \( \partial E_i = E \cap \{ x : u_i(x) = 0 \} \). We start with a general result on affine functions that are positive on \( E \), using methods from convex analysis, see also [46].

**Proposition 5.6.** Suppose \( E \subset \{ x \in \mathbb{R}^p : d(x) \geq 0 \} \) for some affine function \( d(x) = a^\top x + b \) with \( a \in \mathbb{R}^p \), \( b \in \mathbb{R} \). Then there exist \( c \geq 0 \) and \( \lambda \in \mathbb{R}_+^q \), such that

\[
d(x) = \lambda^\top u(x) + c, \text{ for all } x \in \mathbb{R}^p.
\]
5. Admissible parameter sets

Proof. We give a proof by contradiction. Let

\[ K = \{(\eta^\top \lambda, \zeta^\top \lambda + c) \in \mathbb{R}^p \times \mathbb{R} : \lambda \in \mathbb{R}^q_+, c \geq 0\}. \]

Suppose \((a, b) \not\in K\). Since \(K\) is a closed convex set, \((a, b)\) is strictly separated from \(K\) by the Separating Hyperplane Theorem. Therefore, there exist \(y \in \mathbb{R}^p\) and \(y_0 \in \mathbb{R}\) such that \(\langle (y, y_0), (k, k_0) \rangle > \langle (y, y_0), (a, b) \rangle\) for all \((k, k_0) \in K\), i.e.

\[ k^\top y + k_0 y_0 > a^\top y + b y_0 \quad \text{for all} \quad (k, k_0) \in K. \]

In other words, for all \(\lambda_i \geq 0\) and \(c \geq 0\) we have

\[ \sum_i \lambda_i (\eta_i y + \zeta_i y_0) + cy_0 > a^\top y + b y_0. \]

It easily follows that

\[ a^\top y + b y_0 < 0 \quad (5.1) \]
\[ \eta_i y + \zeta_i y_0 \geq 0 \quad (5.2) \]
\[ y_0 \geq 0. \quad (5.3) \]

Using this we construct \(x \in E\) for which \(d(x) < 0\). Suppose \(y_0 > 0\). Then we take \(x = y/y_0\). Indeed, \(u_i(x) = (\eta_i y + \zeta_i y_0)/y_0 \geq 0\), so \(x \in E\). But \(d(x) = (a^\top y + b y_0)/y_0 < 0\), which is a contradiction. Suppose \(y_0 = 0\). Then we take an arbitrary \(x_0 \in X\) and let \(x_N = x_0 + N y\), with \(N \in \mathbb{N}\). Then \(u_i(x_N) = u_i(x_0) + N \eta_i y \geq 0\) for all \(N\), so \(x_N \in E\), but \(d(x_N) = d(x_0) + N a^\top y < 0\) for \(N\) big enough.

As a consequence of the above result, we obtain Proposition \[5.7\] and Proposition \[5.9\] below, which we use later on to tackle the drift respectively the diffusion part of affine processes with a general polyhedral state space.

**Proposition 5.7.** Suppose \(\emptyset \neq \partial E_i \subset \{x : d(x) \geq 0\}\) for some \(i \leq q\), some affine function \(d(x) = a^\top x + b\) with \(a \in \mathbb{R}^p\), \(b \in \mathbb{R}\). Then there exist \(c \geq 0\) and \(\lambda \in \mathbb{R}^q\) with \(\lambda_j \geq 0\) for \(j \neq i\), such that

\[ d(x) = \lambda^\top u(x) + c. \]

Proof. Let \(u_0 := -u_i\). Then \(\partial E_i = \cap_{j=0}^q \{u_j \geq 0\}\) and \(d(x) \geq 0\) for \(x \in \partial E_i\). Hence we can apply Proposition \[5.6\] which gives the existence of \(\lambda_j \geq 0\) with \(j = 0, \ldots, q\) and \(c \geq 0\) such that

\[ d(x) = \sum_{j=0}^q \lambda_j u_j(x) + c = \sum_{j=1}^q \tilde{\lambda}_j u_j(x) + c, \]

with \(\tilde{\lambda}_j = \lambda_j \geq 0\) for \(j \neq i\) and \(\tilde{\lambda}_i = \lambda_i - \lambda_0\). \(\square\)
5.2. Polyhedral state space

Henceforth in this section, we assume that $q$ is minimal in the sense that $E$ is strictly contained in

$$\bigcap_{i \leq q, i \neq j} \{x \in \mathbb{R}^p : \zeta_i + \eta_i x \geq 0\}.$$  

for all $j \leq q$.

**Lemma 5.8.** It holds that $\partial E_i \neq \emptyset$ for all $i \leq q$.

**Proof.** Fix $i \leq q$. By minimality of $q$ we can choose $x \in \mathbb{R}^p$ such that $u_i(x) < 0$ and $u_j(x) \geq 0$ for all $j \neq i$. Let $y \in E$. Then $u_j(y) \geq 0$ for all $j$. For $t \in [0, 1]$ it holds that

$$u_j(tx + (1-t)y) = tu_j(x) + (1-t)u_j(y),$$

which is non-negative for $j \neq i$. For $t = u_i(y)/(u_i(y) - u_i(x))$ we have $u_i(tx + (1-t)y) = 0$, so $tx + (1-t)y \in \partial E_i$. \qed

**Proposition 5.9.** Suppose $\partial E_i \subset \{v \geq 0\}$ for some $i \leq q$, some affine function $v(x) = a^T x + b$ with $a \in \mathbb{R}^p$, $b \in \mathbb{R}$. Then there exists $\lambda_i \in \mathbb{R}$ such that $v(x) = \lambda_i u_i(x)$ for $x \in E$. If $E^c \neq \emptyset$, then $v(x) = \lambda_i u_i(x)$ for all $x \in \mathbb{R}^p$.

**Proof.** We have $\partial E_i \subset \{v \geq 0\}$ and $\partial E_i \subset \{-v \geq 0\}$. Applying Proposition 5.7 with $d = v$ respectively $d = -v$, we derive that

$$v(x) = \sum_{j=1}^q \lambda_j u_j(x) + c_1$$
$$-v(x) = \sum_{j=1}^q \mu_j u_j(x) + c_2,$$

for some $\lambda_j, \mu_j \in \mathbb{R}$ with $\lambda_j, \mu_j \geq 0$ for $j \neq i$ and $c_1, c_2 \geq 0$. Adding the equations in the above display gives

$$0 = \sum_{j=1}^q (\lambda_j + \mu_j) u_j(x) + c_1 + c_2.$$

By Lemma 5.8 we can choose $x \in \partial E_i$ and deduce that $c_1 = c_2 = 0$. So

$$-(\lambda_i + \mu_i) u_i(x) = \sum_{j \neq i} (\lambda_j + \mu_j) u_j(x).$$

By minimality of $q$ we can choose $x \in \mathbb{R}^p$ such that $u_i(x) < 0$ and $u_j(x) \geq 0$ for all $j \neq i$. This gives that $c := \lambda_i + \mu_i \geq 0$. If $c > 0$, then for $x \in E$ we have

$$0 \leq u_i(x) = -c^{-1} \sum_{j \neq i} (\lambda_j + \mu_j) u_j(x) \leq 0,$$
whence \( u_i(x) = 0 \) for \( x \in E \). This gives \( E = \partial E_i \subset \{ x : v(x) = 0 \} \), so that \( v(x) = u_i(x) = 0 \) for \( x \in E \). If \( c = 0 \), then \( \sum_{j \neq i}(\lambda_j + \mu_j)u_j(x) = 0 \) for all \( x \). This holds in particular for \( x \in E \), i.e. for \( x \) such that \( u_j(x) \geq 0 \) for all \( j \). Hence for \( x \in E \) we have \( \lambda_j u_j(x) = \mu_j u_j(x) = 0 \) for all \( j \neq i \), so

\[
v(x) = \sum_{j=1}^{q} \lambda_j u_j(x) + c_1 = \lambda_i u_i(x), \tag{5.4}
\]

for \( x \in E \). If \( E^o \neq \emptyset \), then choosing \( x \in E^o \) gives \( u_j(x) > 0 \) for all \( j \), which implies \( \lambda_j = 0 \) for all \( j \neq i \). Then (5.4) holds for all \( x \in \mathbb{R}^p \).

Having proved the above general results, we now turn to affine processes on a polyhedron. Using Proposition 5.9 together with a necessary boundary condition on the diffusion matrix \( c(x) \) of an affine process, we are able to characterize the form of \( c(x) \) and further specify the state space \( E \). The next proposition generalizes [23 Lemma 7.1] and considerably simplifies and improves upon the results given in the first part of the appendix in [18].

**Proposition 5.10.** Let \( c : \mathbb{R}^p \to S^p \) be affine and such that \( c(E) \subset S^p_+ \). Assume \( E^o \neq \emptyset \) and

\[
\forall i \leq q, \forall x \in \partial E_i : \eta_i c(x) = 0. \tag{5.5}
\]

Then there exists a non-singular \( L \in \mathbb{R}^{p \times p} \) and a vector \( \ell \in \mathbb{R}^p \) such that for all \( x \in \mathbb{R}^p \) we have

\[
LC(L^{-1}(x - \ell))L^\top = \begin{pmatrix}
\text{diag}(x_M, 0_N) & 0 \\
0 & \Psi(x_{M \cup N})
\end{pmatrix}, \tag{5.6}
\]

for some index sets \( M = \{1, \ldots, m\}, \ N = \{m + 1, \ldots, m + n\} \) and affine function \( \Psi \). In addition we have

\[
LE + \ell = \mathbb{R}^m_+ \times C \times \mathbb{R}^{p-m-n}, \tag{5.7}
\]

for some convex polyhedron \( C = \bigcap_{i=1}^{q-m} \{ y \in \mathbb{R}^n : \tilde{u}_i(y) \geq 0 \} \subset \mathbb{R}^n_+ \) with \( \tilde{u}_i(y) = y_i \) for \( i \leq n \).

**Proof.** We divide the proof into a couple of steps.

**Step 1.** There exists \( B \in \mathbb{R}^{q \times p} \) such that for \( i \leq q \) it holds that

\[
\eta_i c(x) = B_i u_i(x), \tag{5.8}
\]

\[
B_i \eta_i^{\top} > 0, \text{ if } B_i \neq 0, \tag{5.9}
\]

\[
B_i \eta_j^{\top} = 0, \text{ for } j \neq i. \tag{5.10}
\]

114  5. Admissible parameter sets
This is shown as follows. Fix $i \leq q$. By (5.5) and Proposition 5.9 there exists $B_i \in \mathbb{R}^{1 \times p}$ such that $\eta_i c^j(x) = B_{ij} u_i(x)$ for $j \in P$. By assumption there exists $x_0 \in E^\circ$. It holds that $u_i(x_0) > 0$, so we can write

$$B_i = u_i(x_0)^{-1} \eta_i c(x_0).$$

(5.11)

By positive semi-definiteness of $c(x_0)$, it holds that $B_i \eta_i^\top \geq 0$. We have $B_i \eta_i^\top = 0$ if and only if $\eta_i c(x_0) = 0$, i.e. $B_i = 0$. This yields (5.9). Moreover, if $j \neq i$, then by symmetry of $c(x)$ it holds that

$$B_i \eta_j^\top u_i(x) = \eta_i c(x) \eta_j^\top = B_j \eta_i^\top u_j(x),$$

for all $x \in \mathbb{R}^p$. This implies $B_i \eta_j^\top = 0$, since $q$ is minimal.

**Step 2.** Define $M = \{i : \eta_i B_i^\top \neq 0\}$ and permute indices such that $M = \{1, \ldots, m\}$, with $m = \#M$. Then $\eta_M$ has full-rank, in view of

$$\eta_M B_M^\top = \text{diag}(\eta_1 B_1^\top, \ldots, \eta_m B_m^\top).$$

Now take $N \subset \{m+1, \ldots, q\}$ such that $\eta_M \cup N$ has full-rank. By permuting the indices we may assume $N = \{m+1, \ldots, m+n\}$, with $n = \#N$.

**Step 3.** It holds that

$$c(x) = \sum_{i=1}^m (\eta_i B_i^\top)^{-1} B_i^\top B_i u_i(x) + \Phi(x),$$

for some affine function $\Phi : \mathbb{R}^p \to S^p$ satisfying $\eta \Phi(x) = 0$, for all $x$. This is an immediate consequence of Step 1. In addition, $\Phi$ can be written as a function of $u(x)$, since it is positive semi-definite on $E$.

**Step 4.** The result follows by taking $\ell \in \mathbb{R}^p$ with $\ell_M \cup N = \zeta_M \cup N$ and $L \in \mathbb{R}^{p \times p}$ non-singular with $L_M \cup N = \eta_M \cup N$ and the remaining rows orthogonal to the first $m+n$ rows. 

Proposition 5.7 together with a necessary boundary condition for the drift enables us to derive a kind of maximum principle, see the next proposition.

**Proposition 5.11.** Suppose $b : \mathbb{R}^p \to \mathbb{R}^p$ is affine and such that $\eta_i b(x) \geq 0$ for all $i \leq q, x \in \partial E_i$. Then for all $f \in C_0^2(E)$ and $x_0 \in E$ such that $f$ has a maximum at $x_0 \in E$, it holds that

$$\nabla f(x_0)^\top b(x_0) \leq 0.$$

**Proof.** Let $X_t$ be the unique solution to the ODE

$$dX_t = b(X_t)dt, \quad X_0 = x_0.$$
Proposition 5.10, we assume that the state space \( E \) is the "canonical" polyhedron \( R \). Admissibility conditions for general polyhedra, extending the conditions for the canonical polyhedron, do not exist an invertible affine transformation \( \phi \) in Proposition 5.10. In addition, without loss of generality we assume that there exists a convex polyhedron \( C \) with \( X = R \times C \). By Theorem 4.4 there exists a corresponding affine jump-diffusion with state space \( R \times C \). Necessarily this process equals \( u(X_t) \) in case the initial condition is \( u(x_0) \). This proves the claim.

We are now able to prove the main theorem of this section. We work out the admissibility conditions for general polyhedra, extending the conditions for the "canonical" polyhedron \( R \times R \) as given in [17, Theorem 2.4]. In view of Proposition 5.10 we assume that the state space \( E \) is of the form \( E = \mathcal{X} \times R^{q-m} \), with \( \mathcal{X} = R^n \times C \subset R^m \), with some \( n \leq m \) and \( C \subset R^{n-m} \) a convex polyhedron as in Proposition 5.10. In addition, without loss of generality we assume that there does not exist an invertible affine transformation \( \phi \) such that \( \phi(C) = R \times C' \) for some polyhedron \( C' \subset R^{m-n-1} \), i.e. \( n \) is maximal and \( C \) is minimal. We give the admissibility conditions up to an invariant transformation, i.e. an invertible affine transformation \( \phi \) such that \( \phi(E) \) is of the same form as \( E \).

**Theorem 5.12.** Suppose \( E = \mathcal{X} \times R^{q-m} \) is a polyhedron with \( \mathcal{X} = R^n \times C \) as given as above. The parameter set \( (a^i, A^i, \gamma^i, K^i) \) is admissible if and only if the following properties hold, possibly after an invariant transformation.

(A) The killing part \( \gamma(x) = \gamma^0 + \sum_{i=1}^p \gamma^i x_i \) can be written as

\[ \gamma(x) = \lambda_0 + \lambda^\top u(x), \]

for some \( \lambda_0 \geq 0, \lambda \in R_+^q \).

(B) The jump part \(|z|^2 \land 1)K(x, dz) = K^0(dz) + \sum_{i=1}^p K^i(dz)x_i \) can be written as

\[ (|z|^2 \land 1)K(x, dz) = \mu^0(dz) + \sum_{i=1}^q \mu^i(dz)u_i(x), \]

for some finite positive measures \( \mu^i \) with \( \text{supp} \mu^i \subset K \) for some convex cone \( K \subset E \) such that \( E + K \subset E \). In addition, \( \int |\eta_i\chi(z)|(|z|^2 \land 1)^{-1}\mu^j(dz) < \infty \) for all \( i = 1, \ldots, q, j = 0, \ldots, q \) with \( j \neq i \).
Proof. We first prove the “only if”-part using Proposition 2.16. Since non-negative numbers with the above property. Therefore, $\mu \in G$ additivity of $\mu$ for some $K$ and $\delta$ holds that $i$ Proposition 2.16 (iii) together with Proposition 5.7 yields (D), since for all $i$ Proposition 2.16 (i) and Proposition 5.10 together with Proposition 5.1. Finally, for all $i$ in view of Lemma 4.7. Proposition 2.16 (ii) gives

\[ \int (|z|^2 \wedge 1) K(x, dz) = \mu^0(G) + \sum_{i=1}^{q} \mu^i(G) u_i(x), \]

for some $\mu^i(G) \geq 0$, for all $G \in B(\mathbb{R}^p \setminus \{0\})$. The coefficients $\mu^i(G)$ are the unique non-negative numbers with the above property. Therefore, $\mu^i(\emptyset) = 0$ and $\sigma$-additivity of $G \mapsto K(x, G)$ carries over to $G \mapsto \mu^i(G)$, so that $\mu^i$ are finite positive measures. Their support is contained in some convex cone $\mathcal{K} \subseteq E$ with $E + \mathcal{K} \subseteq E$, in view of Lemma 4.7. Proposition 2.16 (ii) gives

\[ \int |\eta_i \chi(z)|( |z|^2 \wedge 1)^{-1} \mu^0(dz) + \sum_{j=1}^{q} \int |\eta_i \chi(z)|( |z|^2 \wedge 1)^{-1} \mu^j(dz) u_j(x) < \infty, \]

for all $i = 1, \ldots, q$, $x \in \partial E_i$, which proves (B). Necessity of (C) follows from Proposition 2.16 (i) and Proposition 5.10 together with Proposition 5.1. Finally, Proposition 2.16 (iii) together with Proposition 5.7 yields (D), since for all $i$ it holds that

\[ d(x) := \eta_i b(x) + \int \eta_i \chi(z)(|z|^2 \wedge 1)^{-1} \mu^0(dz) + \sum_{j \neq i} \int \eta_i \chi(z)(|z|^2 \wedge 1)^{-1} \mu^j(dz) u_j(x) \]

is a (well-defined) affine function satisfying $d(x) \geq 0$ for all $x \in \partial E_i$.

We now show the “if”-part. Let $f \in C^2_0(E)$ assume a non-negative maximum at $x_0 \in E$. Property (a) of Definition 4.3 is obvious. Note that $\nabla f(x_0)^{\top}$ is in the linear span of $L := \{ \eta_i : x_0 \in \partial E_i \}$, since

\[ 0 = \partial_v f(x_0 + tv)|_{t=0} = \nabla f(x_0)^{\top} v, \]
for all $v$ orthogonal to $L$. Hence (b) follows from (B). It remains to show (c). By the same argument as in the proof of Proposition 5.2 we have $\text{tr} \left( \nabla^2 f(x_0) c(x_0) \right) \leq 0$. To show $\nabla f(x_0)^\top b(x_0) - \int \nabla f(x_0)^\top \chi(z) K(x_0, dz) \leq 0$ we argue as follows. Since $\text{supp} K(x, dz) \subset K \subset E$ for some cone $K$, we have $\eta_i z \geq 0$ for all $i$ and $z \in \text{supp} K(x, dz)$ (as $u_i(z) = \zeta_i + \eta z \geq 0$ for all $z \in E$). Let $\varepsilon > 0$ and write

$$b_\varepsilon(x) = b(x) - \int_{\{|z| > \varepsilon\}} \chi(z) K(x, dz).$$

Then for all $i = 1, \ldots, q$ it holds that

$$\eta_i b_\varepsilon(x) = \eta_i b(x) - \int_{\{|z| > \varepsilon\}} \eta_i \chi(z) K(x, dz) \geq \eta_i b(x) - \int \eta_i \chi(z) K(x, dz),$$

for $\varepsilon > 0$ small enough, since $\chi(z) = z \in E$ in a neighborhood of 0. Hence by (D) we have $\eta_i b_\varepsilon(x) \geq 0$ for all $i$ and $x \in \partial E_i$. Proposition 5.11 yields

$$\nabla f(x_0)^\top b_\varepsilon(x_0) \leq 0.$$

The left-hand side equals

$$\nabla f(x_0)^\top b(x_0) - \int_{\{|z| > \varepsilon\}} \nabla f(x_0)^\top \chi(z) K(x_0, dz),$$

so dominated convergence gives the result when we let $\varepsilon$ tend to zero (recall that we already showed that $\int |\nabla f(x_0)^\top \chi(z)| K(x_0, dz) < \infty$).

**Remark 5.13.** An important difference between the admissibility conditions for the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^{p-m}$ and a general polyhedron is the form of the diffusion matrix. In case $\mathcal{X} = \mathbb{R}_+^m$, it is easy to see that $\Psi(\mathcal{X}) \subset S^p_+$ if and only if $\Psi(x_I) = B^0 + \sum_{i=1}^m B^i x_i$ with $B^i \in S^{p-m}_+$. In general such a characterization is not possible. More precise, for $\Psi(\mathcal{X}) \subset S^p_+$ it is not necessary that $\Psi(x_I)$ can be written as $B^0 + \sum_{i=1}^m B^i u_i(x)$ for some $B^i \in S^{p-m}_+$. For example, let $\mathcal{X} = \bigcap_{i=1}^3 \{ x \in \mathbb{R}^2 : u_i(x) \geq 0 \} \subset \mathbb{R}^2$ with $u_1(x) = x_1$, $u_2(x) = x_2$, $u_3(x) = x_1 + x_2 - \frac{3}{2}$ and take

$$\Psi(x) = \begin{pmatrix} x_1 + \frac{1}{2} & 1 \\ 1 & x_2 + \frac{1}{2} \end{pmatrix}.$$ 

Then $\Psi(\mathcal{X}) \subset S^2_+$, but one verifies that $\Psi(x)$ is not of the form $B^0 + \sum_{i=1}^3 B^i u_i(x)$ for $B^i \in S^2_+$. 

5.3 Characterizing all quadratic state spaces

In this section we consider quadratic spaces as defined below and determine all possible quadratic state spaces for an affine process. As we will see, the only possibilities are the parabolic state space and the Lorentz cone. Our analysis extends the classification of [27] for the two-dimensional case to higher dimensions.

Let \( E \subset \mathbb{R}^p \) be closed and convex with non-empty interior. We say \( E \) is a quadratic state space if there exists a quadratic function \( \Phi : \mathbb{R}^p \rightarrow \mathbb{R} \) given by

\[
\Phi(x) = x^\top M x + v^\top x + w,
\]

for some symmetric non-zero \( M \in \mathbb{R}^{p \times p} \), a vector \( v \in \mathbb{R}^p \) and a constant \( w \in \mathbb{R} \), such that \( \partial E \subset \{ x : \Phi(x) = 0 \} \) and \( E^\circ \subset \{ x : \Phi(x) \neq 0 \} \), or equivalently, \( E^\circ \) is a connected component of \( \{ x : \Phi(x) \neq 0 \} \) (i.e. a maximal connected subset of \( \{ x : \Phi(x) \neq 0 \} \)). By the convexity of \( E \), there are only three possibilities for \( \Phi \), as we show in the following proposition.

**Proposition 5.14.** Let \( E \subset \mathbb{R}^p \) be convex and assume \( E^\circ \) is a non-empty connected component of \( \{ \Phi \neq 0 \} \), with \( \Phi \) given by (5.12). Then, up to an invertible affine transformation, \( \Phi(x) \) is of one of the following forms:

(i) \( \Phi(x) = x_1 - \sum_{i=2}^q x_i^2 \),

(ii) \( \Phi(x) = \sum_{i=1}^q x_i^2 + d \),

(iii) \( \Phi(x) = x_1^2 - \sum_{i=2}^q x_i^2 + d \),

for some \( d \in \mathbb{R} \), \( q \leq p \).

**Proof.** Since \( M \) is symmetric, it is diagonalizable by an orthogonal matrix. By further scaling one can take the diagonal elements equal to \( -1, 0 \) or \( 1 \). Using the equality

\[
x^\top x + v^\top x = (x^\top + \frac{1}{2} v^\top)(x + \frac{1}{2} v) - \frac{1}{4} v^\top v,
\]

we deduce that, up to an invertible affine transformation, the quadratic function \( \Phi \) is of the form

\[
\Phi(x) = x_1 - \sum_{i \in Q} x_i^2 + \sum_{i \in Q'} x_i^2,
\]

(5.13)
or
\[
\Phi(x) = x_1^2 - \sum_{i \in Q} x_i^2 + \sum_{i \in Q'} x_i^2 + d, \tag{5.14}
\]
for some disjoint sets \(Q, Q' \subset \{2, \ldots, p\}\) and \(d \in \mathbb{R}\). If \(\Phi\) is of the form \(5.13\), then \(E\) is of the form
\[
E = \{x \in \mathbb{R}^p : x_1 \geq \sum_{i \in Q} x_i^2 - \sum_{i \in Q'} x_i^2\},
\]
possibly after replacing \(x_1\) by \(-x_1\) and interchanging \(Q\) and \(Q'\). Convexity of \(E\) yields that the Hessian of \(\sum_{i \in Q} x_i^2 - \sum_{i \in Q'} x_i^2\) is positive semi-definite, which implies that \(Q' = \emptyset\). Permuting coordinates gives \(Q = \{2, \ldots, q\}\) with \(q = \#Q + 1\).

Now assume \(\Phi\) is of the form \(5.14\). We have to show that either \(Q' = \emptyset\) or \(\#Q \leq 1\). Assume \(Q' \cup Q \neq \emptyset\). Define the function \(f\) by \(f(x_{Q \cup Q'}) = \sum_{i \in Q} x_i^2 - \sum_{i \in Q'} x_i^2 - d\). One verifies that there are two possible forms \(E\) can assume (possibly after replacing \(x_1\) by \(-x_1\)), namely
\[
E = \{x \in \mathbb{R}^p : x_1 \geq f(x_{Q \cup Q'})^{1/2}, x_{Q \cup Q'} \in K\}
\]
or
\[
E = \{x \in \mathbb{R}^p : |x_1| \leq f(x_{Q \cup Q'})^{1/2}, x_{Q \cup Q'} \in K\},
\]
where \(K\) is convex and with \(K^o\) a non-empty connected component of \(\{f > 0\}\). By convexity of \(E\), we have in the first case that the Hessian of \(f(x_{Q \cup Q'})^{1/2}\) is positive semi-definite, while in the second case the Hessian is negative semi-definite. Now suppose \(Q' \neq \emptyset\). We show that in that case \(\#Q \leq 1\). Let \(x_{Q \cup Q'} \in K^o\). For \(i \in Q'\) we have
\[
\frac{\partial^2}{\partial x_i^2} f(x_{Q \cup Q'})^{1/2} = -f(x_{Q \cup Q'})^{-1/2} - x_i^2 f(x_{Q \cup Q'})^{-3/2} < 0,
\]
wherefore necessarily the Hessian is negative semi-definite. For \(i \in Q\) we have
\[
\frac{\partial^2}{\partial x_i^2} f(x_{Q \cup Q'})^{1/2} = f(x_{Q \cup Q'})^{-1/2} - x_i^2 f(x_{Q \cup Q'})^{-3/2}, \tag{5.15}
\]
which therefore also has to be negative. Now if \(\#Q > 1\), then there exists \(i, j \in Q\) with \(i \neq j\). For such \(i, j\) it holds that
\[
\frac{\partial^2}{\partial x_i \partial x_j} f(x_{Q \cup Q'})^{1/2} = -f(x_{Q \cup Q'})^{-3/2} x_i x_j,
\]
whence
\[
\det \left( \begin{array}{ccc} \frac{\partial^2}{\partial x_i^2} & \frac{\partial^2}{\partial x_i \partial x_j} \\ \frac{\partial^2}{\partial x_j \partial x_i} & \frac{\partial^2}{\partial x_j^2} \end{array} \right) f(x_{Q \cup Q'})^{-1/2} = f(x_{Q \cup Q'})^{-1/2} \left( f(x_{Q \cup Q'})^{-1/2} - (x_i^2 + x_j^2) f(x_{Q \cup Q'})^{-3/2} \right),
\]
which is negative as (5.15) is negative for \( i \in Q \). This contradicts the negative semi-definiteness of the Hessian of \( f(x_{Q \cup Q'})^{1/2} \). Thus it holds that \( \#Q \leq 1 \), as we needed to show.

Next we eliminate possibility (ii) for all \( d \) and (iii) for \( d \leq 0 \), by exploiting the necessary boundary condition for the diffusion matrix \( c(x) \). For this we need the following lemma.

Lemma 5.15. Let \( \Psi : \mathbb{R}^p \to S^p : x \mapsto A^0 + \sum_{k=1}^{p} A^k x_k \) with \( A^k \in S^p \) and assume \( x^\top \Psi(x) = 0 \) for all \( x \in \mathbb{R}^p \). Then \( \Psi(x) = 0 \) for all \( x \in \mathbb{R}^p \).

Proof. We show that \( A^k = 0 \) for \( k \geq 0 \). For all \( j = 1, \ldots, p \) and \( x \in \mathbb{R}^p \) it holds that

\[
0 = x^\top \Psi_j(x) = (x^\top (A^0 + \sum_{k=1}^{p} A^k x_k))_j = \sum_{i=1}^{p} x_i A^0_{ij} + \sum_{k=1}^{p} \sum_{i=1}^{p} x_i x_k A^k_{ij}
\]

\[
= \sum_{i=1}^{p} x_i A^0_{ij} + \sum_{1 \leq i < k \leq p} x_i x_k (A^k_{ij} + A^k_{kj})
\]

\[
+ \sum_{i=1}^{p} x_i^2 A^i_{ij}.
\]

Hence for all \( i, j = 1, \ldots, p \) we have \( A^0_{ij} = A^i_{ij} = 0 \) and \( A^k_{ij} = -A^k_{kj} \) for \( k \neq i \). Since \( A^k \) is symmetric, the latter gives \( A^k_{ij} = -A^k_{ij} \) for \( k \neq i \). So if we permute the indices \( i, j, k \) by the cycle \( (i \mapsto j, j \mapsto k, k \mapsto i) \), then \( A^k_{ij} \) gets a minus sign. Permuting the indices repeatedly we obtain

\[
A^i_{ij} = -A^i_{jk} = A^j_{ki} = -A^k_{ij},
\]

which implies \( A^k_{ij} = 0 \) for all \( i, j \) and \( k \neq i \). Hence \( A^k = 0 \) for all \( k \), as we have already shown that \( A^i_{ij} = 0 \) for all \( i, j \). \( \square \)

Proposition 5.16. Suppose \( (X, (\mathbb{P}_x)_{x \in E}) \) is an affine jump-diffusion with parameters \( (b(x), c(x), \gamma(x), K(x, dz)) \) given by (4.3), with \( E \) of the form (3.2) and \( X^0 \) a connected component of \( \{ \Phi \neq 0 \} \), for some quadratic function \( \Phi \) given by (5.12). In addition, suppose \( c_{II}(x) \neq 0 \) for some \( x \in E \). Then, up to an invertible affine transformation, \( \Phi \) is of the form

\[
\Phi(x) = x_1 - \sum_{i=2}^{m} x_i^2 \quad \text{or} \quad \Phi(x) = x_1^2 - \sum_{i=2}^{m} x_i^2.
\]
5. Admissible parameter sets

Proof. By Corollary 4.22 we may assume without loss of generality that \( \mathcal{X} = E \), i.e. \( m = p \). There exists an invertible affine transformation such that \( \Phi \) is of one of the three forms as stated in Proposition 5.14. We show that \( \Phi \) cannot be of the second form for all \( d \in \mathbb{R} \) and not of the third form unless \( d = 0 \). Let \( x \in \partial E \) be arbitrary. Recall that \( U \) given by (3.3) is assumed to have non-empty interior. Therefore, all the components \( x_i \) for \( i \leq p \) are present in the expression for \( \Phi(x) \), so that \( q = p \). In addition, \( \Phi(x) = \sum_{i=1}^{p} x_i^2 + d \) with \( d \geq 0 \) is impossible, as this would imply \( X = \mathbb{R}^p \).

Now suppose \( \Phi(x) = \sum_{i=1}^{p} x_i^2 + d \) with \( d < 0 \). Then \( E = \{ \Phi \leq 0 \} \), whence \( \Phi(x) = \sup_{y \in E} \Phi(y) = 0 \) for all \( x \in \partial E = \{ \Phi = 0 \} \). Define \( \Psi(x) : \mathbb{R}^p \to S^p : x \mapsto A_0 + \sum_{k=1}^{p} A_k x_k \), so that \( \Psi|_E = c \). By the admissibility conditions as given in Definition 4.3 it follows that \( x^\top \Psi_i(x) = 0 \), for all \( i \) and all \( x \in \mathbb{R}^p \) such that \( \Phi(x) = 0 \).

Regarding \( \Phi(x) \) as a univariate polynomial in \( x_1 \), we see that it has distinct roots \( \pm \left(-d - \sum_{j=2}^{p} x_j^2 \right)^{1/2} \), for \( x \) close enough to zero. Therefore, these are also the roots of \( x^\top \Psi_i(x) \). Since the latter has maximal degree 2, it follows that \( x^\top \Psi_i(x) = P_i \Phi(x) \), for all \( i \) and \( x \in \mathbb{R}^p \), for some constant \( P_i \). Note that the right-hand side of the above display has a constant term \( P_i d \). Since the left-hand side only contains multiples of \( x \), this yields \( P_i = 0 \) for all \( i \), so that \( x^\top \Psi(x) = 0 \) for all \( x \in \mathbb{R}^p \). Lemma 5.15 yields that \( \Psi(x) = 0 \) for all \( x \), which contradicts the assumption that \( c(x) \neq 0 \) for some \( x \in E \).

Likewise we show that \( \Phi(x) = x_1^2 - \sum_{i=2}^{p} x_i^2 + d \) with \( d \neq 0 \) is impossible. Indeed, suppose \( \Phi \) is of this form. Write \( Q = \{ 2, \ldots, p \} \) and define the function \( f \) by \( f(x_Q) = \sum_{i \in Q} x_i^2 - d \). Then convexity of \( E \) yields

\[
E = \{ x \in \mathbb{R}^p : x_1 \geq f(x_Q), x_Q \in K \},
\]

where \( K \) is convex and with \( K^o \) a non-empty connected component of \( \{ f > 0 \} \). It holds that \( E \subset \{ \Phi \geq 0 \} \), so \( -\Phi \) assumes a non-negative maximum at the boundary. The admissibility conditions yield

\[
( -x_1, x_Q )^\top \Psi_i(x) = 0,
\]

for all \( i \) and all \( x \in E \) such that \( \Phi(x) = 0 \), where \( \Psi \) is defined as before. If \( x \in E \) and \( \Phi(x) = 0 \), then \( x_1 = f(x_Q)^{1/2} \). Hence \( f(x_Q)^{1/2} \) is a root of \( x^\top \Psi_i(x) \), regarded as a univariate polynomial in \( x_1 \).
Since the degree does not exceed 2, necessarily also \(-f(x_Q)^{1/2}\) is a root, so that 
\[x^\top \Psi_i(x) = P_i \Phi(x)\] for some constant \(P_i\), for all \(x \in \mathbb{R}^p\). The rest of the argument

is verbatim the previous paragraph.

From Proposition 5.16 it follows that in order to characterize all affine jump-diffusions with a quadratic boundary \(\partial E \subset \{x : \Phi(x) = 0\}\), there are two cases to consider, namely \(\Phi(x) = x_1 - \sum_{i=2}^m x_i^2\) and \(\Phi(x) = x_1^2 - \sum_{i=2}^m x_i^2\), for some \(m \leq p\). In the next two sections we work out the corresponding admissibility conditions.

**5.4 Parabolic state space**

In this section we state the admissibility conditions for the parabolic state space \(E = \mathcal{X} \times \mathbb{R}^{p-m}\) with \(\mathcal{X}\) given by

\[\mathcal{X} = \{x \in \mathbb{R}^m : x_1 \geq \sum_{i=2}^m x_i^2\}\]

For characterizing the diffusion matrix, we introduce the following notation. For \(x \in \mathbb{R}^p\) we write \(y = (x_2, \ldots, x_m)\) and we define affine matrix-valued functions \(\zeta\) and \(\eta\) by

\[
\zeta(x) = \begin{pmatrix} 4x_1 & 2y^\top \\ 2y & I \end{pmatrix}, \quad \eta(x) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ T_{12}(y) & T_{13}(y) & \cdots & T_{m-2,m-1}(y) \end{pmatrix},
\]

with \(T_{ij} : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}\) for \(1 \leq i < j < m\) given by \(T_{ij}(y)_i = y_j, T_{ij}(y)_j = -y_i, T_{ij}(y)_k = 0\) for \(k \neq i, j\). For example, for \(m = 4\) we have

\[
\eta(x) = \begin{pmatrix} 0 & 0 & 0 \\ y_2 & y_3 & 0 \\ -y_1 & 0 & y_3 \\ 0 & -y_1 & -y_2 \end{pmatrix}.
\]

As in the previous section, we give the admissible parameters up to an invariant transformation, i.e. an invertible affine transformation that leaves the state space \(E\) unaltered. In addition, we assume that the diffusion part for the first \(m\) components does not vanish on the whole of \(E\). The following theorem extends [27, Proposition 2] from the 2-dimensional continuous case to higher dimensions including jumps and killing. As in Chapter 4 we write \(I = \{1, \ldots, m\}, J = \{m+1, \ldots, p\}\) and we write \(Q = I \setminus \{1\}\).
Theorem 5.17. Suppose $E = \mathcal{X} \times \mathbb{R}^{p-m}$ with $\mathcal{X}$ parabolic. The parameter set $(a^i, A^i, \gamma^i, K^i)$ is admissible with $A^i_1 + \sum_{i=1}^p A^i_i x_i \neq 0$ for some $x \in E$, if and only if the following properties hold, possibly after an invariant transformation.

(A) The killing parameters satisfy $\gamma^0 \geq 0, \gamma^1 \geq 0, \gamma^i = 0$ for $i > m$ and

$$4\gamma^0 \gamma^1 \geq \sum_{i=2}^m (\gamma^i)^2,$$

(B) For the jump parameters it holds that $K^0(G) \geq 0, K^1(G) \geq 0, K^i(G) = 0$ for $i > m$ and

$$4K^0(G)K^1(G) \geq \sum_{i=2}^m (K^i(G))^2,$$

for all $G \in B(\mathbb{R}^p \setminus \{0\})$. In addition, $\text{supp} |K^i| \subset \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p-m}$ and $\int_{\{|z_1| \leq 1\}} |z_1||z|^2 |K^i|(dz) < \infty$, for all $i$.

(C) The diffusion part $c(x) = A^0 + \sum_{i=1}^p A^i x_i$ is of the form

$$c(x) = \left( \begin{array}{c} \zeta(x) \\ M^\top \eta(x)^\top \\ \eta(x)M \\ B(x) \end{array} \right),$$

for some $M \in \mathbb{R}^{p-m} \times \mathbb{R}^{p-m}$, with $B(x)$ affine and

$$B(x) \geq M^\top \eta(x)^\top \eta(x)M, \text{ for all } x \in E.$$

(D) Let $\alpha^0 \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}^{p \times p}$ be given by $\alpha^0_i = a^0_i - \int \chi_1(z)(|z|^2 \wedge 1)^{-1} K^i(dz)$ and $\alpha^i_j = a^i_j$ for $i = 0, \ldots, p, j \neq 1$. Then these drift parameters satisfy

$$\alpha_{1J} = 0$$
$$\alpha_{Q1} = 0$$
$$\alpha_{QQ} = \frac{1}{2} \alpha_{11}I - \frac{1}{2} \text{diag}(d)$$
$$\alpha_{1i} = 2\alpha^0_i, \text{ for } q + 2 \leq i \leq m$$
$$\alpha^0_i \geq m - 1 + \sum_{i=1}^q \frac{1}{4} d_i^{-1} (\alpha_{1,i+1} - 2\alpha^0_{i+1})^2,$$

for some vector $d \in \mathbb{R}^{m-1}$ with $d_i > 0$ for $i \leq q$ and $d_i = 0$ for $i > q$, for some $q \leq m - 1$.

Proof. Define $\Phi(x) = (1, -2y, 0)$ and let $b(x), c(x), \gamma(x), K(x, dz)$ be given by (4.3). In view of Corollary 5.3 and positivity of $c, \gamma$ and $K$, we have to verify that the above parameter restrictions are equivalent with the following properties:
5.4. Parabolic state space

(i) $c(x) \geq 0$ for all $x \in E$, $c_{11}(x_0) \neq 0$ for some $x_0 \in E$, and $c(x)\Phi(x) = 0$ for all $x \in \mathbb{R}^p$ such that $x_1 = y^\top y$.

(ii) $K(x, dz) \geq 0$, supp $K(x, dz) \subset E - x$ for $x \in E$ and $\int |\Phi(x)^\top \chi(z)|K(x, dz) < \infty$ for all $x \in \mathbb{R}^p$ such that $x_1 = y^\top y$.

(iii) $\gamma(x) \geq 0$ for all $x \in E$.

(iv) $\Phi(x)^\top b(x) - \int \Phi(x)^\top \chi(z)K(x, dz) - \text{tr}(c_{\mathcal{QQ}}(x)) \geq 0$ for all $x \in \mathbb{R}^p$ such that $x_1 = y^\top y$.

We treat each item separately in the next subsections.

5.4.1 Admissibility for the diffusion parameters

In this section we show the equivalence between (C) and (i) (up to an invariant transformation). We first prove that (C) implies (i). Suppose $c(x)$ is of the form (5.16) and (5.17) holds. To show that $c(x) \geq 0$ for all $x \in E$, it suffices to find a square root $\sigma(x)$ for $c(x)$, i.e. a matrix $\sigma(x)$ such that $\sigma(x)\sigma(x)^\top = c(x)$ for all $x \in E$. Write

$$\xi(x) = \begin{pmatrix} 2\sqrt{x_1 - y^\top y} & 2y^\top \\ 0 & 1 \end{pmatrix},$$

for $x \in E$. Note that $\xi(x)\eta(x) = \eta(x)$. Since $B(x) - M^\top \eta(x)\eta(x)^\top M^\top$ is positive semi-definite, it admits a square root, so that we can define

$$\sigma(x) = \begin{pmatrix} \xi(x) & 0 \\ M^\top \eta(x)^\top & (B(x) - M^\top \eta(x)\eta(x)^\top M^\top)^{1/2} \end{pmatrix}.$$

One easily verifies that $\sigma(x)\sigma(x)^\top = c(x)$ for all $x \in E$, whence $c(x)$ is positive semi-definite. It is also clear that $c(x)(1, -2y, 0) = 0$ whenever $x_1 = y^\top y$. Thus we have shown that (C) implies (i).

Next we show the other direction, for which we need the following two lemmas.

Lemma 5.18. Consider the linear space

$$\mathcal{L} = \left\{ a : \mathbb{R}^p \to \mathbb{R}^m \text{ affine} \mid \begin{pmatrix} 1 & -2y^\top \end{pmatrix} a(x) = 0 \text{ for all } x \text{ with } x_1 = y^\top y \right\}.$$  

(5.18)

Then a basis for $\mathcal{L}$ is formed by the columns of $\zeta$ and $\eta$.

Proof. Clearly these columns are linearly independent elements of $\mathcal{L}$. To prove that they span $\mathcal{L}$ we use a dimension argument. Let $\text{Aff}(\mathbb{R}^p, \mathbb{R}^m)$ denote the space
of affine functions from $\mathbb{R}^p$ to $\mathbb{R}^m$ and let $\text{Quadr}(\mathbb{R}^p, \mathbb{R})/(x_1 - y^\top y)$ be the space of quadratic functions from $\mathbb{R}^p$ to $\mathbb{R}$, modulo $x_1 - y^\top y$ (that is, $\phi$ and $\psi$ are equivalent if $\phi(x) - \psi(x) = k(x_1 - y^\top y)$ for some constant $k$). Consider the linear operator

$$
L : \text{Aff}(\mathbb{R}^p, \mathbb{R}^m) \to \text{Quadr}(\mathbb{R}^p, \mathbb{R})/(x_1 - y^\top y) : a(x) \mapsto \begin{pmatrix} 1 & -2y^\top \end{pmatrix} a(x),
$$

and note that $\mathcal{L} = \ker L$. By the dimension theorem for linear operators, we have

$$
dim \text{Aff}(\mathbb{R}^p, \mathbb{R}^m) = dim \ker L + dim \text{im } L.
$$

It holds that $dim \text{Aff}(\mathbb{R}^p, \mathbb{R}^m) = pm + m$. Since $x_1 \equiv y^\top y$, a basis for $\text{im } L$ is given by

$$\{1, x_2, \ldots, x_p\} \cup \{x_i x_j : 2 \leq i \leq m, 1 \leq j \leq p\},$$

whence

$$dim \text{im } L = p + (m - 1)p - \binom{m - 1}{2} = pm - \binom{m - 1}{2}.$$

It follows that $dim \ker L = m + \frac{1}{2}(m - 1)(m - 2)$, which is the number of columns in $\zeta$ and $\eta$. Thus the columns span the kernel of $L$. $\Box$

**Lemma 5.19.** Let $\mathcal{L}$ be defined by (5.18) and suppose $M : \mathbb{R}^p \to S^m$ is affine. If the columns of $M$ are in $\mathcal{L}$, then $M = c\zeta$ for some $c \in \mathbb{R}$.

**Proof.** By Lemma 5.18 there exist matrices $A$ and $B$ such that

$$M(x) = \zeta(x) A + \eta(x) B.$$

Write $T(y) = (T_{ij}(y))_{1 \leq i < j < m}$ and $B = \begin{pmatrix} B^1 & \Bar{B} \end{pmatrix}$. Then the above display reads

$$M(x) = \begin{pmatrix} 4x_1 A_{11} + 2y^\top A_{Q1} & 4x_1 A_{1Q} + 2y^\top A_{QQ} \\
2y A_{11} + A_{Q1} + T(y) B^1 & 2y A_{1Q} + A_{QQ} + T(y) \Bar{B} \end{pmatrix}.$$

Since $M(x)$ is symmetric it immediately follows that $A_{1Q} = 0$ and $A_{Q1} = 0$. We have to show that $N(x) := M(x) - A_{11} \zeta(x)$ is zero. Note that $N$ is symmetric and that

$$N(x) = \begin{pmatrix} 0 & 2y^\top C \\
T(y) B^1 & C + T(y) \Bar{B} \end{pmatrix},$$

with $C = A_{QQ} - A_{11} I$. This yields $C = C^\top$ and $T(y) B^1 = 2Cy$. Since $y^\top T(y) = 0$, the latter implies $y^\top Cy = 0$ for all $y \in \mathbb{R}^{p-1}$, whence $C = 0$, as $C$ is symmetric. Thus $A_{QQ} = A_{11} I$ and it remains to show that $\Bar{B} = 0$.

It holds that $T(y) \Bar{B}$ is symmetric and $y^\top T(y) \Bar{B} = 0$. Lemma 5.15 yields $T(y) \Bar{B} = 0$, whence $\Bar{B} = 0$ by linear independence of the columns of $T(y)$, as we needed to prove. $\Box$
Proposition 5.20. Suppose $c : \mathbb{R}^p \to \mathbb{S}_+^p$ is affine and such that $c(x)(1, -2y, 0) = 0$ whenever $x_1 = y^\top y$. Then necessarily $c$ is of the form

$$c(x) = \begin{pmatrix} k\zeta(x) & N(x) \\ N(x)^\top & B(x) \end{pmatrix},$$

(5.19)

for some $k \geq 0$, with $N(x) = \zeta(x)M_1 + \eta(x)M_2$ for some matrices $M_1, M_2$ and $B : \mathbb{R}^p \to \mathbb{S}^{(p-m)}$ affine. Moreover, if $k = 1$ and $M_1 = 0$, then it holds that

$$B(x) - M_2^\top \eta(x)^\top \eta(x)M_2 \geq 0,$$

for all $x \in E$.

Proof. The first part follows from Lemma 5.18 and Lemma 5.19. It remains to show (5.17). Suppose $k = 1$ and $M_1 = 0$. By positive semi-definiteness of $c$, we have

$$0 \leq \begin{pmatrix} v^\top & w^\top \end{pmatrix} c(x) \begin{pmatrix} v \\ w \end{pmatrix} = v^\top \zeta(x)v + 2v^\top \eta(x)M_2w + w^\top B(x)w,$$

for all $v \in \mathbb{R}^m$, $w \in \mathbb{R}^{p-m}$, $x \in E$. Fix $w \in \mathbb{R}^{p-m}$, let $x \in E$ be arbitrary and take $v = -\eta(x)M_2w$. Noting that $\zeta(x)\eta(x) = \eta(x)$ for all $x \in \mathbb{R}^p$, the above display then reads

$$w^\top B(x)w - w^\top M_2^\top \eta(x)^\top \eta(x)M_2w \geq 0,$$

which gives the result.

In view of the above proposition, (i) implies that $c(x)$ is of the form (5.19). It remains to find an invariant transformation such that we can take $k = 1$ and $M_1 = 0$, in other words, we have to find a non-singular matrix $L \in \mathbb{R}^{p \times p}$, a vector $\ell \in \mathbb{R}^p$ such that

$$L c(L^{-1}x - \ell)L^\top = \begin{pmatrix} \zeta(x) & \eta(x)M_2 \\ M_2^\top \eta(x)^\top & B(x) \end{pmatrix}, \quad \text{and} \quad LE + \ell = E,$$

(5.20)

see Proposition 5.1. Since by assumption $c_{11}(x_0) \neq 0$ for some $x_0 \in E$, it holds that $k > 0$. Define matrices

$$L_1 = \begin{pmatrix} k^{-1} & 0 & 0 \\ 0 & k^{-1/2}I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad L_2 = \begin{pmatrix} I & 0 \\ -M_1^\top & I \end{pmatrix}.$$
The invariant transformation \( x \mapsto L_1 x \) gives (5.19) with \( k = 1 \), while \( x \mapsto L_2 x \) gives (5.19) with \( M_1 = 0 \), so that the composition \( x \mapsto L_2 L_1 x \) is the transformation we are looking for. The reader verifies that (5.20) indeed holds for \( L = L_2 L_1 \), \( \ell = 0 \). Thus we have shown the equivalence between (C) and (i), up to an invariant transformation.

5.4.2 Admissibility for the killing parameters

Here we show the equivalence between (A) and (iii). Suppose (A) holds. If \( \gamma_1 = 0 \), then (A) yields \( \gamma_i = 0 \) for all \( i = 1, \ldots, p \), so that \( \gamma(x) = \gamma^0 \geq 0 \) for all \( x \). If \( \gamma_1 > 0 \), then for \( x \in E \) we can write

\[
\gamma(x) = \gamma^0 + \sum_{i=1}^{m} \gamma_i x_i \geq \gamma^0 + \sum_{i=2}^{m} (\gamma_1 x_i^2 + \gamma_i x_i) \geq \gamma^0 - \sum_{i=2}^{m} \frac{(\gamma_i)^2}{4\gamma_1},
\]

which is non-negative by (A). Conversely, suppose (iii) holds. Since \( 0 \in E \) and since we can choose \( x_1 \) arbitrarily large, we necessarily have \( \gamma^0 \geq 0 \) and \( \gamma_i \geq 0 \). If \( \gamma_1 = 0 \), then necessarily \( \gamma_i = 0 \) for all \( i = 1, \ldots, p \), as we can choose \( x_i \) arbitrarily large. Hence we have (A). If \( \gamma_1 > 0 \), then we define \( x \in E \) by

\[
x_1 = \sum_{i=2}^{m} x_i^2 \text{ and } x_i = -\frac{\gamma_i}{2\gamma_1}, \text{ for } i > 1.
\]

It holds that

\[
0 \leq \gamma(x) = \gamma^0 - \sum_{i=2}^{m} \frac{(\gamma_i)^2}{4\gamma_1},
\]

which yields (A).

5.4.3 Admissibility for the jump parameters

The first part of the equivalence between (B) and (ii) follows by similar arguments as in the previous paragraph. It remains to show the equivalence

\[
\text{supp } |K^i| \subset \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p-m}, \forall i, \iff \text{supp } K(x, dz) \subset E - x, \forall x \in E
\]

and the equivalence

\[
\int_{|z_1| \leq 1} |z||z|^{-2}|K^i|(dz) < \infty, \forall i, \iff \int |\Phi(x)^\top \chi(z)||K(x, dz) < \infty, \forall x \in \partial E.
\]

By Lemma 4.7 it holds that supp \( K(x, dz) \subset E - x \) for all \( x \in E \) if and only if supp \( K(x, dz) \subset C \) for all \( x \in E \), for some closed convex cone \( C \subset E \) with \( E + C \subset E \). The maximal cone that is contained in \( E \) is \( \mathbb{R}_+ \times \{0\} \times \mathbb{R}^{p-m} \). This proves the first equivalence. The second equivalence is straightforward.
5.4.4 Admissibility for the drift parameters

Assume that (C) holds. We conclude the proof of Theorem 5.17 by showing the equivalence between (D) and (iv), up to an invariant transformation that leaves the diffusion matrix unaltered. We have \( \text{tr}(cQQ(x)) = m - 1 \) for all \( x \), so the boundary condition (iv) for the drift reads

\[
y^\top (\alpha_{11}I - 2\alpha_Q) y + (\alpha_{1Q} - 2\alpha_0^\top) y + \alpha_0^0 - m + 1 \geq 0, \quad \text{for all } y \in \mathbb{R}^{m-1}.
\]

For this it is necessary that \( M := \alpha_{11}I - 2\alpha_Q \) is positive semi-definite. Moreover, if \( y \) is in the kernel of \( M \), then \( y \) should also be in the kernel of \( \alpha_{1Q} - 2\alpha_0^\top \). We can diagonalize \( M \) by an orthogonal matrix \( O \), so \( D = OMO^\top \) is diagonal with positive diagonal elements \( d_i \) for \( i \leq q \) and \( d_i = 0 \) for \( i > q \), for some \( q \leq m - 1 \). Applying the orthogonal transformation \( y \mapsto Oy \), the above condition becomes

\[
q \sum_{i=1}^{q} d_i y_i^2 + q \sum_{i=1}^{q} (\alpha_{1,i+1} - 2\alpha_0^0) y_i + \alpha_0^0 - m + 1 \geq 0, \quad \text{for all } y,
\]

in view of Lemma 5.21 below. We can write the left-hand side as

\[
q \sum_{i=1}^{q} d_i (y_i + \frac{1}{2} d_i^{-1} (\alpha_{1,i+1} - 2\alpha_0^0))^2 - \frac{1}{4} q \sum_{i=1}^{q} d_i^{-1} (\alpha_{1,i+1} - 2\alpha_0^0) + \alpha_0^0 - m + 1,
\]

which is non-negative for all \( y \) if and only if

\[
-\frac{1}{4} q \sum_{i=1}^{q} d_i^{-1} (\alpha_{1,i+1} - 2\alpha_0^0)^2 + \alpha_0^0 - m + 1 \geq 0.
\]

This yields the result.

**Lemma 5.21.** Let \( \alpha^0 \) and \( \alpha \) be given as in Theorem 5.17 and \( c(x) \) be given by (5.16). Let \( O \in \mathbb{R}^{(m-1) \times (m-1)} \) be orthogonal. Then the invariant transformation given by \( y \mapsto Oy \) and \( x_i \mapsto x_i \) for \( i \notin Q \), leaves \( c(x) \) unaltered and transforms \( \alpha^0 \) and \( \alpha \) into \( U\alpha^0 \) and \( U\alpha U^\top \), with \( U \in \mathbb{R}^{p \times p} \) given by

\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & O & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

**Proof.** This is a consequence of Proposition 5.1 and the fact that \( \chi_1(z) \) only depends on \( z_1 \). \( \square \)
5. Admissible parameter sets

5.5 The Lorentz cone

In this section we work out the admissibility conditions for the Lorentz cone $E$ given by

$$E = \{ x \in \mathbb{R}^p : x_1 \geq |(x_2, \ldots, x_p)| \}.$$

As in the previous section, we introduce some notation for characterizing the diffusion matrix. We write $y = (x_2, \ldots, x_p)$ and let $\zeta(x)$ be given by

$$\zeta(x) = \begin{pmatrix} x_1 & y^\top \\ y & x_1 I \end{pmatrix},$$

and $\eta$ and $T$ be given as in the previous section. In addition, we define affine functions $\rho(i) : \mathbb{R}^p \to S^p$ for $i = 1, \ldots, p - 1$ by

$$\rho(i)_{i+1} : x \mapsto \begin{pmatrix} x_1 & y^\top \end{pmatrix},$$

$$\rho(i)_{11} : x \mapsto y_i,$$

$$\rho(i)_{jj} : x \mapsto -y_i, \text{ for } j \neq 1, i + 1,$$

$$\rho(i)_{jk} : x \mapsto 0, \text{ if } j, k \neq i + 1 \text{ and } j \neq k.$$

For example, for $p = 4$ we have

$$\rho(1)(x) = \begin{pmatrix} y_1 & x_1 & 0 & 0 \\ x_1 & y_1 & y_2 & y_3 \\ 0 & y_2 & -y_1 & 0 \\ 0 & y_3 & 0 & -y_1 \end{pmatrix}, \quad \rho(2)(x) = \begin{pmatrix} y_2 & 0 & x_1 & 0 \\ 0 & -y_2 & y_1 & 0 \\ x_1 & y_1 & y_2 & y_3 \\ 0 & 0 & y_3 & -y_2 \end{pmatrix},$$

$$\rho(3)(x) = \begin{pmatrix} y_3 & 0 & 0 & x_1 \\ 0 & -y_3 & 0 & y_1 \\ 0 & 0 & -y_3 & y_2 \\ x_1 & y_1 & y_2 & y_3 \end{pmatrix}.$$
(A) The killing parameters satisfy $\gamma^0 \geq 0$ and $(\gamma^1, \ldots, \gamma^p) \in E$.

(B) For the jump parameters it holds for all $G \in \mathcal{B}(\mathbb{R}^p \setminus \{0\})$ that $K^0(G) \geq 0$ and $(K^1(G), \ldots, K^p(G)) \in E$.

In addition, $\text{supp } |K^i| \subset E$ and we have $\int_{|z| \leq 1} |z|^{-1} K^0(dz) < \infty$.

(C) The diffusion part $c(x) = A^0 + \sum_{i=1}^p A^i x_i$ is of the form

$$c(x) = v_1 \zeta(x) + \sum_{i=1}^{p-1} v_{i+1} \rho(i)(x),$$

for some $v \in E$.

(D) Let $\alpha^0 \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}^{p \times p}$ be given by $\alpha^j_i = a^j_i - \int \chi_i(z) (|z| \wedge 1)^{-2} K^j(dz)$ for all $i = 1, \ldots, p$, $j = 0, \ldots, p$. Then these drift parameters satisfy

$$\alpha_{11} I - \alpha_{QQ} = \text{diag}(d) \text{ for some } d \in \mathbb{R}^{p-1}_{+} \quad (5.21)$$

$$\sum_{i=1}^{p-1} (d_i y_i^2 + (\alpha_{1,i+1} - \alpha_{i+1,1}) y_i) \geq 0 \text{ for all } y \in \mathbb{R}^{p-1} \text{ with } |y| = 1 \quad (5.22)$$

$$\alpha^0 - (p-2)v \in E, \quad (5.23)$$

where $v$ is given as in [C].

Proof. Define $\Phi(x) = (2x_1, -2y)$ and let $b(x), c(x), \gamma(x), K(x, dz)$ be given by [4.3]. In view of Corollary 5.5, we have to verify that the above parameter restrictions are equivalent with the following properties:

(i) $c(x) \geq 0$ for all $x \in \mathbb{R}^p$, $c(0) = 0$, $c(x_0) \neq 0$ for some $x_0 \in E$, and $c(x)\Phi(x) = 0$ for all $x \in \mathbb{R}^p$ such that $x_1 = |y|$. 

(ii) $K(x, dz) \geq 0$, supp $K(x, dz) \subset E - x$ for all $x \in E$, $\int |\Phi(x)^\top \chi(z)|K(x, dz) < \infty$ for all $x \in \mathbb{R}^p$ such that $x_1 = |y|$, and $\int \chi(z)K(0, dz) < \infty$.

(iii) $\gamma(x) \geq 0$ for all $x \in E$.

(iv) $\Phi(x)^\top b(x) - \int \Phi(x)^\top \chi(z)K(x, dz) + c_{11}(x) - \text{tr } (c_{QQ}(x)) \geq 0$ for all $x \in \mathbb{R}^p$ such that $x_1 = y^\top y$, and $b(0) - \int \chi(z)K(0, dz) \in E$.

We treat each item separately in the next subsections. \qed
Remark 5.23. Condition \((5.22)\) can be worked out using a Lagrange multiplier, as follows. Let \(w \in \mathbb{R}^{p-1}\) be given by \(w_i = a_{1,i+1} - a_{i+1,1}\) for \(i = 1, \ldots, p - 1\) and write
\[
f(y) = \sum_{i=1}^{p-1} (d_i y_i^2 + w_i y_i).
\]
Then \(f(y) \geq 0\) for all \(|y| = 1\) if and only if the critical values of
\[
g(y, \lambda) := f(y) + \lambda(y^\top y - 1)
\]
are non-negative. The critical points \((y, \lambda)\) of \(g\) satisfy
\[
2(d_i + \lambda) y_i + w_i = 0, \quad \text{for all } i, \quad (5.24)
\]
\[
y^\top y = 1, \quad (5.25)
\]
so that the critical values of \(g\) are equal to
\[
g(y, \lambda) = \sum_{i=1}^{p-1} ((d_i + \lambda) y_i + \frac{1}{2} w_i) y_i + \frac{1}{2} w^\top y - \lambda = \frac{1}{2} w^\top y - \lambda.
\]
Therefore, condition \((5.22)\) holds if and only if for all \((y, \lambda) \in \mathbb{R}^p\) satisfying \((5.24)\) and \((5.25)\) we have
\[
\frac{1}{2} w^\top y - \lambda \geq 0. \quad (5.26)
\]
As an example we work this out for the case \(d_i = d_1\) for all \(i\). Without loss of generality we may assume \(w_i \neq 0\) for all \(i\). Then \((5.24)\) gives \(y = -\frac{1}{2}(d_1 + \lambda)^{-1} w\). Substituting this expression in \((5.25)\) yields \(\lambda = -d_1 \pm \frac{1}{2}|w|\) and \(y = \mp w/|w|\). From this one infers that \((5.26)\) holds if and only if \(d_1 \geq |w|\).

Remark 5.24. There is a close connection between affine processes on the Lorentz cone and matrix-valued affine processes living on the cone of positive semi-definite matrices as treated in \([10]\). Indeed, let \(X\) be a 3-dimensional affine process on the Lorentz cone. Then we can construct a matrix-valued affine process on \(S^3_+\) by
\[
\begin{pmatrix}
X_1 - X_2 & X_3 \\
X_3 & X_1 + X_2
\end{pmatrix},
\]
see also \([26, \text{Example 1}]\).

5.5.1 Admissibility for the diffusion parameters

The equivalence between \((\text{C})\) and \((i)\) (up to an invariant transformation) is a consequence of Proposition 5.29 below, together with its proof. We first prove the following two lemmas, which are similar to Lemma 5.18 and Lemma 5.19.
Lemma 5.25. Consider the linear space

\[ \mathcal{L} = \left\{ a : \mathbb{R}^p \to \mathbb{R}^p \text{ affine} \mid \begin{pmatrix} x_1 & -y^\top \end{pmatrix} a(x) = 0 \text{ for all } x \text{ with } x_1^2 = y^\top y \right\}. \]

(5.27)

Then a basis for \( \mathcal{L} \) is formed by the columns of \( \zeta \) and \( \eta \).

Proof. Similar to the proof of Lemma 5.18. \( \square \)

Lemma 5.26. Consider the linear space

\[ \mathcal{M} = \left\{ M : \mathbb{R}^p \to S^p \text{ affine} \mid M^i \in \mathcal{L} \text{ for all } x, i \right\} \]

(5.28)

with \( \mathcal{L} \) defined by (5.27). Then a basis for \( \mathcal{M} \) is given by

\[ \mathcal{B} = \{ \zeta, \rho(1), \ldots, \rho(p-1) \}. \]

Proof. Clearly the elements of \( \mathcal{B} \) are linearly independent elements of \( \mathcal{M} \). It remains to show that they span \( \mathcal{M} \). Let \( M \in \mathcal{M} \) be arbitrary. By Lemma 5.25 there exist matrices \( A \) and \( B \) such that

\[ M(x) = \zeta(x)A + \eta(x)B. \]

Write \( T(y) = (T_{ij}(y))_{1 \leq i < j < p} \) as in Section 5.4 and let \( B = \begin{pmatrix} B^1 & \tilde{B} \end{pmatrix} \). Then the above display reads

\[ M(x) = \begin{pmatrix} x_1A_{11} + y^\top A_{Q1} & x_1A_{1Q} + y^\top A_{QQ} \\ yA_{11} + x_1A_{Q1} + T(y)B^1 & yA_{1Q} + x_1A_{QQ} + T(y)\tilde{B} \end{pmatrix}. \]

Symmetry of \( M(x) \) yields

\[ A_{1Q} = A_{Q1}^\top, \]
\[ A_{QQ} = A_{QQ}^\top, \]
\[ yA_{11} + T(y)B^1 = A_{QQ}^\top y, \]
\[ yA_{1Q} + T(y)\tilde{B} = (yA_{1Q} + T(y)\tilde{B})^\top. \]

Since \( y^\top T(y) = 0 \), the second equation together with the third gives

\[ 0 = y^\top T(y)B^1 = y^\top (A_{QQ} - A_{11}I)y, \]

which implies \( A_{QQ} - A_{11}I = 0 \), as \( A_{QQ} - A_{11}I \) is symmetric and thus diagonalizable by an orthogonal matrix. Define

\[ N = M - A_{11}\zeta - \sum_{i \in Q} A_i\rho(i). \]
Then $N \in \mathcal{M}$ and $N$ is of the form

$$N(x) = \begin{pmatrix} 0 & 0 \\ 0 & \sum_{k \in Q} C^k y_k \end{pmatrix},$$

for some matrices $C^k \in S^p_+$. By Lemma 5.15 it follows that $N = 0$. 

For the proof of Proposition 5.29 we use the following lemmas.

**Lemma 5.27.** For all $x \in E$ it holds that $\zeta(x) \geq 0$.

*Proof.* Let $v = (v_1, v_Q) \in \mathbb{R}^p$. Then we can write

$$v^\top \zeta(x)v = x_1(v_1^2 + |v_Q|^2) + 2v_1y^\top v_Q.$$

If $x \in E$, then $x_1 \geq |y|$, whence Cauchy-Schwarz gives

$$x_1v^\top v + 2v_1y^\top v_Q \geq |y|(v_1^2 + |v_Q|^2) + 2v_1y^\top v_Q \geq |y|(v_1^2 + |v_Q|^2) - 2|v_1||y||v_Q| = |y||(|v_1| - |v_Q|)^2 \geq 0.$$

This proves the result. \qed

**Lemma 5.28.** For all $x \in E$ it holds that $\zeta(x) + \rho(1)(x) \geq 0$.

*Proof.* Let $v = (v_1, v_2, w) \in \mathbb{R}^p$, write $x = (x_1, y)$ and $y = (y_1, y_R)$. Then we can write

$$v^\top \rho(1)(x)v = y_1(v_1^2 + v_2^2) + 2x_1v_1v_2 + 2v_2y^\top R w - y_1|w|^2.$$

Consequently,

$$v^\top (\zeta(x) + \rho(1)(x))v = x_1(v_1^2 + v_2^2 + |w|^2) + 2y_1v_1v_2 + 2v_1y^\top R w$$

$$+ y_1(v_1^2 + v_2^2) + 2x_1v_1v_2 + 2v_2y^\top R w - y_1|w|^2$$

$$= (x_1 + y_1)(v_1 + v_2)^2 + (x_1 - y_1)|w|^2 + 2(v_1 + v_2)y^\top R w.$$

If $x \in E$, then $x_1 \geq |y|$, whence Cauchy-Schwarz gives

$$v^\top (\zeta(x) + \rho(1)(x))v \geq (|y| + y_1)(v_1 + v_2)^2 + (|y| - y_1)|w|^2 - 2|v_1 + v_2||y_R||w|$$

$$= \left(|y| + y_1^{1/2}v_1 + v_2 - (|y| - y_1)^{1/2}|w|\right)^2 \geq 0.$$

This proves the result. \qed

**Proposition 5.29.** Let $z \in \mathbb{R}^p$. We have $k(x, z) := z_1\zeta(x) + \sum_{i=1}^{p-1} z_i \rho(i)(x) \geq 0$ for all $x \in E$ if and only if $z \in E$. 

5.5.3 Admissibility for the drift parameters

Assume that (C) holds. We conclude the proof of Theorem 5.22 by showing the equivalence between (A) and (iii) and part of the equivalence between (B) and (ii). The remaining part follows from Lemma 4.7.

Proof. The “only if”-part follows from the observation that $k(x, z)_{11} = x^\top z$, together with the self-duality of $E$. For the “if”-part it suffices to show that $k(x, z) \geq 0$ for all $x \in E$ and $z \in \mathbb{R}^p$ with $z_1 = 1$, $|z_Q| = 1$, since $\zeta(x) \geq 0$ for all $x \in E$ by Lemma 5.27. Write $c(x) = \zeta(x) + \rho(1)(x)$. Then $c(x) \geq 0$ for all $x \in E$ by Lemma 5.28. Let $|z_Q| = 1$ be arbitrary and let $L \in \mathbb{R}^{p \times p}$ be of the form

$$L = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix},$$

with $O$ orthogonal and $O^1 = z_Q$. Then $Lx \in E$ for all $x \in E$, so that $L^\top c(Lx)L \geq 0$ for all $x \in E$. We conclude the proof by showing that $L^\top c(Lx)L = \zeta(x) + \rho(x)$. It holds that

$$(L^\top c(Lx)L)_{11} = \zeta_{11}(x) + \sum_{i=1}^{p-1} z_{i+1} \rho(i)_{11}(Lx) = x_1 + z_Q^\top O y = x_1 + y_1.$$ 

In addition, we have $L^\top c(Lx)L \in \mathcal{M}$, where $\mathcal{M}$ is given by (5.28). Therefore, by Lemma 5.26 we can write $L^\top c(Lx)L$ as a linear combination of $\zeta(x)$ and $\rho(1)(x), \ldots, \rho(p - m)(x)$. By the above display we necessarily have $L^\top c(Lx)L = \zeta(x) + \rho(x)$ for all $x$, as we needed to show. \hfill \Box

5.5.2 Admissibility for the jump and killing parameters

The facts that $0 \in E$ and that $E$ is a self-dual cone immediately yield the equivalence of (A) and (iii) and part of the equivalence between (B) and (ii). The remaining part follows from Lemma 4.7.

5.5.3 Admissibility for the drift parameters

Assume that (C) holds. We conclude the proof of Theorem 5.22 by showing the equivalence between (D) and (iv). We have

$$c_{11}(x) - \text{tr}(c_{QQ}(x)) = v_1 x_1 - v_Q^\top y - \left( (p - 1)v_1 x_1 - \sum_{i=2}^{p} (3 - p)v_i x_i \right)$$

$$= (2 - p)(v_1 x_1 - v_Q^\top y),$$

so the boundary condition (iv) for the drift reads

$$x_1(b_1(x) - \int \chi_1(z) K(x, dz)) - y(b_Q(x) - \int \chi_Q(z) K(x, dz)) + c_{11}(x) - \text{tr} c_{QQ}(x)$$

$$= x_1(\alpha_{11} x_1 + \alpha_{1Q} y + \alpha_1^0) - y(\alpha_{Q1} x_1 + \alpha_{QQ} y + \alpha_Q^0) + (2 - p)(v_1 x_1 - v_Q^\top y)$$

$$= y^\top (\alpha_{11} I - \alpha_{QQ}) y + x_1(\alpha_{1Q} - \alpha_{Q1}) y + (\alpha_0^0 - (p - 2)v)^\top (x_1, -y) \geq 0,$$
for all $x_1 = |y|$, and in addition $\alpha^0 \in E$. For the above display it is sufficient and necessary that both

$$y^T (\alpha_{11} I - \alpha_{QQ}) y + (\alpha_{1Q} - \alpha_{Q1}^T) y \geq 0, \text{ for all } y \in \mathbb{R}^{p-m} \text{ with } |y| = 1, \quad (5.29)$$

and

$$(\alpha^0 - (p-2)v)^T (|y|, y) \geq 0, \text{ for all } y \in \mathbb{R}^{p-m}. \quad (5.30)$$

Assertion (5.30) holds if and only if $\alpha^0 - (p-2)v \in E$. Indeed, sufficiency follows from self-duality of $E$, while necessity follows by taking $y = -(\alpha^0_Q - (p-2)v_Q)$. Note that since $v \in E$, we also have $\alpha^0 \in E$ if $\alpha^0 - (p-2)v \in E$.

For assertion (5.29) it is necessary that $M := \alpha_{11} I - \alpha_{QQ}$ is positive semi-definite. In that case $M$ is diagonalizable by an orthogonal matrix $O$, i.e. $OMO^T = \text{diag}(d)$ with $d \in \mathbb{R}^{p-1}_+$. Applying the invariant transformation $x_1 \mapsto x_1, y \mapsto Oy$ we obtain the condition

$$\sum_{i=1}^{p-1} d_i y_i^2 + (\alpha_{1Q} - \alpha_{Q1}^T) y \geq 0, \text{ for all } y \in \mathbb{R}^{p-1} \text{ with } |y| = 1,$$

by a similar argument as in Lemma 5.21. This yields the result.