Cauchy’s functional equation

We adapt well-known results for Cauchy’s functional equation (see e.g. [1]), which we apply in Chapter 4 to extend exponential affine expressions beyond their initial domains.

Lemma A.1. Suppose \( f : [0, 1] \rightarrow \mathbb{R} \) is a bounded function that satisfies Cauchy’s functional equation \( f(t + s) = f(t) + f(s) \) for \( t, s \in [0, 1] \) with \( t + s \in [0, 1] \). Then \( f \) is linear, i.e. \( f(t) = f(1)t \) for all \( t \in [0, 1] \).

Proof. Define \( g(t) = f(t) - f(1)t \). Then \( g(t) \) also satisfies Cauchy’s functional equation and \( g(1) = 0 \). We have to show that \( g(t) = 0 \) for all \( t \in [0, 1] \). For \( n, m \in \mathbb{N} \) with \( m \leq n \) we deduce from Cauchy’s functional equation that

\[
ng(1/n) = g(n \cdot 1/n) = g(1) = 0
\]

and

\[
g(m/n) = g(m \cdot 1/n) = mg(1/n) = 0.
\]

Hence \( g(t) = 0 \) for all \( t \in \mathbb{Q} \cap [0, 1] \). Since \( f \) is bounded, \( g \) is also bounded. Let \( M = \sup_{t \in [0, 1]} |g(t)| \) and suppose \( M > 0 \). Then there exists \( t_0 \in [0, 1] \) such that \( |g(t_0)| > M/2 \). Let \( q \in \mathbb{Q} \cap [0, 1/2] \) be such that \( t_0 - q \in [0, 1/2] \). Since \( g(q) = 0 \), it follows that

\[
g(2(t_0 - q)) = 2g(t_0 - q) = 2(g(t_0 - q) + g(q)) = 2g(t_0).
\]

Hence \( |g(2(t_0 - q))| > M \), which contradicts the definition of \( M \).

\[ \square \]
Lemma A.2. Let $E$ be closed convex with $0 \in E^\circ$ and let $g : E \to \mathbb{C}$ be either a continuous function or a real valued, non-negative function that is bounded in a neighborhood of 0. If $g$ satisfies Cauchy’s functional equation

$$g(x + y) = g(x)g(y), \quad \text{for all } x, y \in E \text{ with } x + y \in E,$$

(A.1)

then either $g = 0$ or $g(x) = \exp(u^\top x)$ for some $u \in \mathbb{C}^p$.

Proof. Suppose $g(x_0) = 0$ for some $x_0 \in E$. We first show that $g(0) = 0$. Since $0 \in E^\circ$ and $E$ is convex, it holds that $tx_0 \in E$ for all $t \in [0,1]$. We have $0 = g(x_0) = g(x_0/2^k)$ for all $k \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that $-x_0/2^k \in E$. This gives $g(0) = g(-x_0/2^k)g(x_0/2^k) = 0$. Let $\varepsilon > 0$ be such that $B(0,\varepsilon) \subset E$. We show that $g(x) = 0$ for all $x \in B(0,\varepsilon/2)$. Let $x_0 \in B(0,\varepsilon/2)$. Then we have $g(x_0)g(-x_0) = g(0) = 0$, so either $g(x_0) = 0$ or $g(-x_0) = 0$. Suppose $g(-x_0) = 0$. Since $2x_0 \in E$, it follows that

$$g(x_0) = g(-x_0 + 2x_0) = g(-x_0)g(2x_0) = 0.$$

Hence $g(x) = 0$ for all $x \in B(0,\varepsilon/2)$. For arbitrary $x \in E$ we have $g(x) = g(tx)g((1-t)x)$ for all $t \in [0,1]$. Since $tx \in B(0,\varepsilon/2)$ for small $t$, it follows that $g(x) = 0$.

Now suppose $g(x) \neq 0$ for all $x \in E$. By assumption, there exists $\varepsilon > 0$ such that for $t \in [-\varepsilon,\varepsilon]$ we have $te_i \in E$ and $t \mapsto g(te_i)$ is bounded, for all $i \leq p$. Fix $i \leq p$ and define $f : [0,1] \to \mathbb{C}$ by

$$f(t) = g(\varepsilon te_i).$$

(A.2)

Since $g(x) \neq 0$ for all $x \in E$, we can write

$$f(t) = \phi(t) e^{\psi(t)i},$$

(A.3)

for some strictly positive, bounded function $\phi$ and real valued function $\psi$ that satisfy

$$\phi(t + s) = \phi(t)\phi(s)$$

$$\psi(t + s) = \psi(t) + \psi(s) + 2k\pi,$$

for $t, s \in [0,1]$ such that $t + s \in [0,1]$, with $k$ some integer, possibly depending on $t$ and $s$. The assumptions allow us to choose $k$ constant. Indeed, if $g$ is real-valued and non-negative, then we can chose $\psi$ and $k$ equal to 0. If $g$ is continuous, then we can chose $\psi$ continuous by taking the distinguished logarithm (see [7].
Theorem 7.6.2), so that necessarily \( k \) is constant. Since \( \phi \) is bounded and strictly positive, it is also bounded away from 0, as

\[
\phi(t) = \phi(1)/\phi(1-t).
\]

Applying Lemma A.1 to \( \log \phi \) and to \( \psi - 2k\pi \) yields \( \phi(t) = e^{at} \) and \( \psi(t) = bt + 2k\pi \), for some \( a, b \in \mathbb{R} \). Substituting these in (A.3) and (A.2) we derive the existence of \( c_i \in \mathbb{C} \) for \( i \leq p \) such that

\[
g(te_i) = e^{c_i t},
\]

for \( t \in [0, \varepsilon] \). For \( t \in [-\varepsilon, 0] \) we also obtain

\[
g(te_i) = g(0)/g(-te_i) = e^{c_i t}.
\]

Now it follows that for all \( x \in E \) we have \( g(x) = e^{c^\top x} \). Indeed, take \( x \in E \) and let \( n \in \mathbb{N} \) be such that \( |x_i/n| \leq \varepsilon \). Then it holds that

\[
g(x) = g(nx/n) = g(x/n)^n = g\left(\sum_{i=1}^{p} x_i e_i/n\right)^n = \prod_{i=1}^{p} g(x_i/n)^n = \prod_{i=1}^{p} (e^{c_i x_i/n})^n = e^{c^\top x}.
\]

\( \square \)