



# Damping and dispersion of oscillating modes of a multicomponent ionic mixture in a magnetic field

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## Abstract

The collective-mode spectrum of a multicomponent magnetized ionic mixture for small wave number  $k$  is studied with the use of magnetohydrodynamics and formal kinetic theory. Apart from the usual thermal and diffusive modes, the spectrum contains a set of four oscillating modes. By evaluating the  $k^2$  contributions to the eigenfrequencies, the damping and the dispersion of these oscillating modes are determined. The long-range nature of the Coulomb interactions is shown to imply that Burnett terms with higher-order gradients in the linear phenomenological laws have to be taken into account in order to obtain a full description of all damping and dispersion effects.

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## 1. Introduction

The dynamical properties of classical Coulomb systems change considerably when external magnetic fields are applied. In particular, the spectrum of large-scale collective modes is strongly influenced by such fields. Whereas the modes associated to the longitudinal and the transverse degrees of freedom may be excited independently in an unmagnetized system, these degrees of freedom get coupled when a magnetic field is switched on, at least for modes propagating in an oblique direction with respect to the magnetic field. As a consequence, the well-known plasma oscillations and the viscous modes get mixed when the isotropy of the system is lost by applying a magnetic field.

The influence of magnetic fields on the mode spectrum of Coulomb systems has been studied in several model systems. For the one-component plasma model (OCP), which consists of charged particles moving through an inert neutralizing background of opposite charge, the changes in the mode spectrum have been studied by means of magnetohydrodynamics [1] and in the framework of formal kinetic theory [1–3]. Since the dynamics of the magnetized OCP model has an anomaly that is caused by a divergency in the heat conductivity [4,5], a full treatment of the effects of a magnetic field on the mode spectrum calls for a somewhat

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more general model. The multicomponent ionic mixture (MIM) in a magnetic field is quite useful in this respect [6].

In the magnetized MIM model several species of charged particles are present. If the charge-mass ratios of the particles are not all equal, the anomalous behaviour of the heat transport is no longer present, as has been shown in Ref. [6]. In that paper the modes of the magnetized MIM have been studied by formal kinetic theory. More recently [7], a magnetohydrodynamical approach has been used to determine the influence of the magnetic field on the modes. The results were found to corroborate those found in Ref. [6]. In the long-wavelength limit the spectrum of collective modes consists of four oscillating modes with a non-vanishing frequency, and of a set of coupled heat and diffusion modes with frequencies tending to 0 for vanishing wave number. By including terms of higher order in the wave number, the latter modes could be discerned. The second-order terms in the mode frequencies describe damping that is caused by dissipative transport phenomena.

In Refs. [6,7] the mode frequencies of the oscillating modes have been obtained in the limit of vanishing wave number only. Hence, dispersion effects for these modes could not be studied. Furthermore, damping effects that are of second order in the wave number were not treated either. In the present paper we wish to fill this gap by evaluating the second-order contributions to the oscillating-mode frequencies in a systematic way.

As in our previous paper [7] we shall use both a magnetohydrodynamical approach and formal kinetic theory. Interestingly enough, it turns out that standard magnetohydrodynamics is not enough to determine the frequencies of the oscillating modes in second order of the wave number. As we will show, so-called Burnett terms will be needed to obtain a consistent picture. For systems of neutral particles these Burnett terms [8–12], which describe non-local effects in transport phenomena, are of rather limited importance. They play a role only when one is interested in contributions to mode frequencies that are of higher than second order in the wave number. For charged-particle systems, however, the long-range nature of the Coulomb interaction leads to a reshuffling of terms in the magnetohydrodynamical equations, which enhances the relevance of the Burnett contributions.

The paper is organized as follows. In Section 2 the conservation laws and the balance equations of linearized magnetohydrodynamics for the magnetized MIM model are formulated. The linear phenomenological laws relating thermodynamic forces and flows are presented. In particular, the Burnett contributions will be written down explicitly. The Fourier-transformed versions of these magnetohydrodynamical equations are given in Section 3. The next two sections contain the analysis that leads to our main result, namely the eigenfrequencies of the oscillating modes up to second order in the wavelength. In Sections 6 and 7 a kinetic approach is used to rederive these frequencies and to obtain the kinetic equivalents of the transport coefficients, both in leading order and in the higher-order Burnett regime. A few auxiliary thermodynamical relations and the basic formalism of the kinetic eigenvalue problem are presented in two appendices.

## 2. Conservation laws, balance equations and phenomenological relations

An ionic mixture consists of several species of charged particles (with charges of the same sign), which move in an inert uniform neutralizing background. If a uniform external magnetostatic field is present as well, the motion of each of the particles is governed by the electrostatic forces from the other particles and from the background, and by the Lorentz force due to the external field. We assume that the motion of the particles is so slow that the magnetic fields generated by them can be disregarded.

In thermal equilibrium the particle density  $n_\sigma$  of species  $\sigma$  (with  $\sigma = 1, \dots, s$ ) is uniform and time independent. The total charge density  $q_v = \sum_\sigma e_\sigma n_\sigma$  (with  $e_\sigma$  the charge of a particle of species  $\sigma$ ) is matched by the background charge density  $-q_v$ . The total mass density  $m_v$  follows as  $\sum_\sigma m_\sigma n_\sigma$ , with  $m_\sigma$  the mass of a particle of species  $\sigma$ .

If equilibrium is slightly perturbed, the particle densities acquire deviations  $\delta n_\sigma(\mathbf{r}, t)$  that in general are non-uniform and time dependent. Moreover, each species in the perturbed system will get a non-uniform and time-dependent velocity  $\mathbf{v}_\sigma(\mathbf{r}, t)$ . In the following we assume that the perturbations vary slowly in space and time. The characteristic length  $l$  of the spatial variations should be large compared to the average interparticle distance  $n_\sigma^{-1/3}$ , for all  $\sigma$ . The characteristic time for temporal variations should be large compared to  $l/c$ , with  $c$  the velocity of light.

The continuity equation for species  $\sigma$  reads in first order of the perturbations [7]:

$$\frac{\partial}{\partial t} \delta n_\sigma(\mathbf{r}, t) = -n_\sigma \nabla \cdot \mathbf{v}(\mathbf{r}, t) - \frac{1}{m_\sigma} \nabla \cdot \mathbf{J}_\sigma(\mathbf{r}, t). \quad (2.1)$$

Here  $\mathbf{v}(\mathbf{r}, t) = \sum_\sigma m_\sigma n_\sigma \mathbf{v}_\sigma(\mathbf{r}, t) / m_v$  is the barycentric velocity and  $\mathbf{J}_\sigma = m_\sigma n_\sigma [\mathbf{v}_\sigma(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}, t)]$  the diffusion flow of species  $\sigma$  with respect to the barycentric motion. These flows satisfy the identity  $\sum_\sigma \mathbf{J}_\sigma(\mathbf{r}, t) = 0$ . Upon multiplying by  $e_\sigma$  and summing over  $\sigma$  we get the conservation law for the total charge:

$$\frac{\partial}{\partial t} \delta q_v(\mathbf{r}, t) = -q_v \nabla \cdot \mathbf{v}(\mathbf{r}, t) - \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \nabla \cdot \mathbf{J}_\sigma(\mathbf{r}, t). \quad (2.2)$$

As an independent set we will choose (2.2) in combination with (2.1) for  $\sigma \neq 1$ .

The time dependence of the barycentric motion is given by the equation of motion, which up to first order in the perturbations reads [7]

$$\begin{aligned} m_v \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = & -\nabla \delta p(\mathbf{r}, t) - \nabla \cdot \delta \mathbf{P}(\mathbf{r}, t) + q_v [\mathbf{E}(\mathbf{r}, t) + c^{-1} \mathbf{v}(\mathbf{r}, t) \wedge \mathbf{B}] \\ & + c^{-1} \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \mathbf{J}_\sigma(\mathbf{r}, t) \wedge \mathbf{B}. \end{aligned} \quad (2.3)$$

Here  $\delta p$  is the perturbation of the (local) equilibrium pressure, while  $\delta \mathbf{P}$  is the viscous pressure tensor. The electric field is determined by  $\delta q_v$  through Gauss's law  $\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \delta q_v(\mathbf{r}, t)$ . Furthermore,  $\mathbf{B}$  is the uniform external magnetostatic field.

The perturbation  $\delta \varepsilon$  of the internal energy density satisfies a balance equation, which gets the form of a conservation law when only terms up to first order in the perturbations are taken into account:

$$\frac{\partial}{\partial t} \delta \varepsilon(\mathbf{r}, t) = -h_v \nabla \cdot \mathbf{v}(\mathbf{r}, t) - \nabla \cdot \mathbf{J}_\varepsilon(\mathbf{r}, t). \quad (2.4)$$

Here  $h_v = \varepsilon + p$  is the enthalpy per volume, with  $\varepsilon$  the (uniform) equilibrium internal energy density and  $p$  the equilibrium pressure. Furthermore  $\mathbf{J}_\varepsilon$  is the heat flow.

The conservation laws and the equation of motion contain the heat flow  $\mathbf{J}_\varepsilon$ , the diffusion flows  $\mathbf{J}_\sigma$  and the viscous pressure tensor  $\delta \mathbf{P}$ . These quantities can be expressed in terms of thermodynamic forces by means of phenomenological relations [13]. The forces that drive the heat flow and the diffusion flows are [7]

$$\mathbf{X}_\varepsilon = \nabla \left( \frac{1}{T} \right), \quad (2.5)$$

$$\mathbf{X}_\sigma = -\frac{1}{T} \left[ \nabla \left( \frac{\mu_\sigma}{m_\sigma} - \frac{\mu_1}{m_1} \right) \right]_T + \frac{1}{T} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right). \quad (2.6)$$

The subscript  $T$  indicates that the gradients of the chemical potentials  $\mu_\sigma$  have to be taken at constant temperature  $T$ .

In the standard formalism of non-equilibrium thermodynamics the diffusion flows are written as linear combinations of  $\mathbf{X}_\varepsilon$  and  $\mathbf{X}_\sigma$ :

$$\mathbf{J}_\sigma = \mathbf{L}_{\sigma\varepsilon} \cdot \mathbf{X}_\varepsilon + \sum_{\sigma'(\neq 1)} \mathbf{L}_{\sigma\sigma'} \cdot \mathbf{X}_{\sigma'}. \quad (2.7)$$

The phenomenological coefficients  $\mathbf{L}_{\sigma\varepsilon}$  and  $\mathbf{L}_{\sigma\sigma'}$  are anisotropic tensors of second rank.

The modified heat flow  $\mathbf{J}'_\varepsilon = \mathbf{J}_\varepsilon - \sum_\sigma (h_\sigma / m_\sigma) \mathbf{J}_\sigma$ , with  $h_\sigma$  the partial specific enthalpy per particle, is a linear combination of the thermodynamic forces  $\mathbf{X}_\varepsilon$  and  $\mathbf{X}_\sigma$ :

$$\mathbf{J}'_\varepsilon = \mathbf{L}_{\varepsilon\varepsilon} \cdot \mathbf{X}_\varepsilon + \sum_{\sigma(\neq 1)} \mathbf{L}_{\varepsilon\sigma} \cdot \mathbf{X}_\sigma. \quad (2.8)$$

As a consequence, the heat flow  $\mathbf{J}_\varepsilon$  itself gets the form [7]

$$\mathbf{J}_\varepsilon = \bar{\mathbf{L}}_{\varepsilon\varepsilon} \cdot \mathbf{X}_\varepsilon + \sum_{\sigma(\neq 1)} \bar{\mathbf{L}}_{\varepsilon\sigma} \cdot \mathbf{X}_\sigma \quad (2.9)$$

with the tensorial coefficients

$$\bar{\mathbf{L}}_{\varepsilon\varepsilon} = \mathbf{L}_{\varepsilon\varepsilon} + \sum_{\sigma'(\neq 1)} \left( \frac{h_{\sigma'}}{m_{\sigma'}} - \frac{h_1}{m_1} \right) \mathbf{L}_{\sigma'\varepsilon}, \quad (2.10)$$

$$\bar{\mathbf{L}}_{\varepsilon\sigma} = \mathbf{L}_{\varepsilon\sigma} + \sum_{\sigma'(\neq 1)} \left( \frac{h_{\sigma'}}{m_{\sigma'}} - \frac{h_1}{m_1} \right) \mathbf{L}_{\sigma'\sigma}. \quad (2.11)$$

The tensors  $\mathbf{L}$  satisfy the Onsager relations:

$$L_{\varepsilon\varepsilon}^{ij}(\mathbf{B}) = L_{\varepsilon\varepsilon}^{ji}(-\mathbf{B}), \quad L_{\varepsilon\sigma}^{ij}(\mathbf{B}) = L_{\sigma\varepsilon}^{ji}(-\mathbf{B}), \quad L_{\sigma\sigma'}^{ij}(\mathbf{B}) = L_{\sigma'\sigma}^{ji}(-\mathbf{B}). \quad (2.12)$$

Similar relations hold for  $\bar{\mathbf{L}}$ .

Finally, in standard non-equilibrium thermodynamics the viscous pressure tensor is proportional to the thermodynamic force  $\mathbf{X}_v = -\nabla v$ :

$$\delta \mathbf{P} = \boldsymbol{\eta} : \mathbf{X}_v. \quad (2.13)$$

The symbol  $:$  denotes a double contraction, so that the  $ij$ -component of the right-hand side is  $\eta^{ijmn} X_v^{mn}$ . The components of the fourth-rank anisotropic viscosity tensor  $\boldsymbol{\eta}$  satisfy the Onsager relations:

$$\eta^{ijmn}(\mathbf{B}) = \eta^{nmij}(-\mathbf{B}). \quad (2.14)$$

Furthermore,  $\eta^{ijmn}$  is symmetric in  $ij$  and in  $mn$ .

For neutral-particle systems the above phenomenological relations are sufficiently general to determine the damping and dispersion of the collective modes up to second order in the wave number. However, for a system of charged particles this need not be true any longer, at least when various species of particles are present. Whereas the conventional phenomenological relations are still adequate to determine the modes of the OCP model [1–3], the standard formalism has to be generalized for the case of the MIM model. For that system the damping and dispersion of the diffusion and heat modes up to second order in the wave number are correctly found by inserting the above phenomenological relations in the balance equations, as shown in Ref. [7]. In contrast, one has to include Burnett terms with higher-order gradient operators in the phenomenological relations in order to determine the damping and dispersion of the oscillating modes up to second order in the wave number.

The relevance of higher-order Burnett terms becomes obvious upon inspecting the charge conservation law. When (2.7) with (2.6) is substituted in the last term of (2.2), the divergence of the diffusion flow  $\mathbf{J}_\sigma$  leads to a spatial derivative of the electric field. The long-range nature of the Coulomb field implies that such a derivative does not necessarily yield a contribution of first order in the wave number in Fourier language. In fact, from Gauss's law it follows that the spatial derivative of the electric field is proportional to  $\delta q_v$  itself, so that it is of the same order in the wave number as the left-hand side of (2.2). Hence, if one wishes to determine all contributions to the oscillating-mode frequencies up to second order in the wave number, one has to go further than the standard linear law (2.7). A similar effect is seen in the equation of motion (2.3), of which the last term is proportional to the diffusion flows. According to (2.6) and (2.7) the latter contain electrodynamic terms without gradient operators, so that in Fourier language terms appear that are of less than first order in the wave number.

From both these arguments it becomes clear that in order to determine the damping and dispersion of the collective modes up to second order in the wave number one is forced to generalize the phenomenological expression for the diffusion flow by including Burnett terms of higher order in the gradients. By invoking

parity invariance one finds the following general form of the diffusion flow up to second order in the gradients:

$$\mathbf{J}_\sigma = \mathbf{L}_{\sigma\varepsilon} \cdot \mathbf{X}_\varepsilon + \sum_{\sigma'(\neq 1)} \mathbf{L}_{\sigma\sigma'} \cdot \mathbf{X}_{\sigma'} + \sum_{\sigma'(\neq 1)} \mathbf{L}_{\sigma\sigma'}^{(c)} \dot{\mathbf{X}}_{\sigma'}^{(c)} + \mathbf{L}_{\sigma v}^{(c)} \dot{\mathbf{X}}_v^{(c)} \quad (2.15)$$

with the generalized thermodynamic forces:

$$\mathbf{X}_\sigma^{(c)} = \frac{1}{T} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \nabla \nabla \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right), \quad (2.16)$$

$$\mathbf{X}_v^{(c)} = -\frac{1}{T} \nabla \nabla \mathbf{v}. \quad (2.17)$$

Both  $\mathbf{L}_{\sigma\sigma'}^{(c)}$  and  $\mathbf{L}_{\sigma v}^{(c)}$  are fourth-rank tensors, with the following symmetry properties:  $L_{\sigma\sigma'}^{(c)ijmn}$  is symmetric in  $jm$ , while  $L_{\sigma v}^{(c)ijmn}$  is symmetric in its last three indices  $jmn$ . Furthermore,  $\mathbf{L}_{\sigma\sigma'}^{(c)}$  has the Onsager symmetry:

$$L_{\sigma\sigma'}^{(c)ijmn}(\mathbf{B}) = L_{\sigma'\sigma}^{(c)njmi}(-\mathbf{B}). \quad (2.18)$$

The symbol  $\dot{\mathbf{X}}_{\sigma'}^{(c)}$  in (2.15) indicates a triple contraction; the  $i$ -component of the last term, for instance, is  $L_{\sigma v}^{(c)ijmn} X_v^{(c)jmn}$ .

According to (2.15) with (2.17), the diffusion flow depends on the gradient of the thermodynamic force  $\mathbf{X}_v$ . Likewise, the viscous pressure gets coupled to the gradient of the force  $\mathbf{X}_\sigma$  when higher-order terms in the gradients are included:

$$\delta \mathbf{P} = \boldsymbol{\eta} : \mathbf{X}_v + \sum_{\sigma(\neq 1)} \mathbf{L}_{v\sigma}^{(c)} : \mathbf{X}_\sigma^{(c)} \quad (2.19)$$

with the generalized thermodynamic force

$$\mathbf{X}_\sigma^{(c)} = \frac{1}{T} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \nabla \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \wedge \mathbf{B} \right) \quad (2.20)$$

and a fourth-rank tensorial coefficient  $\mathbf{L}_{v\sigma}^{(c)}$ . It is related to  $\mathbf{L}_{\sigma v}^{(c)}$  by the Onsager relation

$$L_{v\sigma}^{(c)ijmn}(\mathbf{B}) = L_{\sigma v}^{(c)njmi}(-\mathbf{B}) \quad (2.21)$$

which is similar to (2.18). As a consequence,  $L_{v\sigma}^{(c)ijmn}$  is symmetric in  $ijm$ .

The generalized phenomenological relations (2.15) and (2.19) contain terms with higher-order gradients of the charge density  $\delta q_v$  and the barycentric velocity  $\mathbf{v}$ . These terms represent non-local effects in the phenomenological relations. Clearly, the diffusion flow and the viscous pressure at a certain position are partly determined by the charge density and the velocity in the neighbourhood of that position. The relevance of such non-local effects need not surprise for an ionic mixture in which long-range interparticle interactions are present.

### 3. Fourier transforms of the balance equations

To determine the properties of the collective modes we need the Fourier transforms of the balance equations. The Fourier transform of the particle conservation law (2.1) follows upon substitution of (2.15), with (2.5), (2.6), (2.16) and (2.17). Up to second order in the wave number we get

$$\begin{aligned} \frac{\partial}{\partial t} \delta n_\sigma = & -ik\omega_p \frac{n_\sigma}{q_v} \sqrt{m_v} \hat{\mathbf{k}} \cdot \mathbf{v} - k \frac{1}{m_\sigma T} \hat{\mathbf{k}} \cdot \mathbf{L}_\sigma \cdot \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \\ & + k^2 \frac{1}{m_\sigma} \hat{\mathbf{k}} \cdot \mathbf{L}_{\sigma\varepsilon} \cdot \hat{\mathbf{k}} \delta \left( \frac{1}{T} \right) - k^2 \frac{1}{m_\sigma T} \sum_{\sigma'(\neq 1)} \hat{\mathbf{k}} \cdot \mathbf{L}_{\sigma\sigma'} \cdot \hat{\mathbf{k}} \left( \frac{\delta \mu_{\sigma'}}{m_{\sigma'}} - \frac{\delta \mu_1}{m_1} \right)_T, \end{aligned} \quad (3.1)$$

Here  $\hat{\mathbf{k}} = \mathbf{k}/k$  is a unit vector in the direction of the wave vector. Furthermore, we introduced the following combination of phenomenological coefficients:

$$\mathbf{L}_\sigma = \sum_{\sigma'(\neq 1)} \left( \frac{e_{\sigma'}}{m_{\sigma'}} - \frac{e_1}{m_1} \right) \mathbf{L}_{\sigma\sigma'}. \quad (3.2)$$

The ratio of the plasma frequency  $\omega_p = q_v/\sqrt{m_v}$  and the cyclotron frequency  $\omega_c = q_v B/(m_v c)$  is denoted as  $\gamma = \omega_c/\omega_p$ . As a consequence of the long range of the Coulomb interactions, large-scale charge perturbations are suppressed effectively. In fact, one may prove [14] that in thermal equilibrium the Fourier-transformed fluctuations  $\delta q_v$  are proportional to  $k$ . For that reason, the ‘reduced’ charge-density perturbation  $\delta q_v/k$  will be treated as being of order  $k^0$ . It should be noted that upon adopting this convention all contributions from the generalized thermodynamic forces in (2.15) drop out from (3.1), as these become of higher than second order in the wave number.

The Fourier transform of the charge conservation law (2.2) is derived in a similar way. Dividing the conservation law by  $k$  so as to obtain the time derivative of  $\delta q_v/k$ , we see that the terms with the generalized thermodynamic forces in (2.15) contribute in second order of the wave number. We get

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\delta q_v}{k} \right) &= -i\omega_p \sqrt{m_v} \hat{\mathbf{k}} \cdot \mathbf{v} - \frac{1}{T} \hat{\mathbf{k}} \cdot \mathbf{L} \cdot \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \\ &\quad + k \hat{\mathbf{k}} \cdot \mathbf{L}_\varepsilon \cdot \hat{\mathbf{k}} \delta \left( \frac{1}{T} \right) - k \frac{1}{T} \sum_{\sigma(\neq 1)} \hat{\mathbf{k}} \cdot \mathbf{L}_\sigma \cdot \hat{\mathbf{k}} \left( \frac{\delta \mu_\sigma}{m_\sigma} - \frac{\delta \mu_1}{m_1} \right)_T \\ &\quad + k^2 \frac{1}{T} \hat{\mathbf{k}} \hat{\mathbf{k}} : \mathbf{L}^{(c)} : \hat{\mathbf{k}} \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \\ &\quad - ik^2 \frac{1}{\sqrt{m_v} T} \hat{\mathbf{k}} \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \sqrt{m_v} \mathbf{v}. \end{aligned} \quad (3.3)$$

In the first two lines new combinations of phenomenological coefficients show up:

$$\mathbf{L} = \sum_{\sigma(\neq 1)} \sum_{\sigma'(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \left( \frac{e_{\sigma'}}{m_{\sigma'}} - \frac{e_1}{m_1} \right) \mathbf{L}_{\sigma\sigma'}, \quad (3.4)$$

$$\mathbf{L}_\varepsilon = \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \mathbf{L}_{\sigma\varepsilon}. \quad (3.5)$$

Analogously, the terms in order  $k^2$  contain combinations of generalized phenomenological coefficients:

$$\mathbf{L}^{(c)} = \sum_{\sigma(\neq 1)} \sum_{\sigma'(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \left( \frac{e_{\sigma'}}{m_{\sigma'}} - \frac{e_1}{m_1} \right) \mathbf{L}_{\sigma\sigma'}^{(c)}, \quad (3.6)$$

$$\mathbf{L}_v^{(c)} = \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \mathbf{L}_{\sigma v}^{(c)}. \quad (3.7)$$

The Fourier transform of the equation of motion follows upon inserting the Fourier versions of (2.15) and (2.19) in (2.3). Using (2.5)–(2.6), (2.16)–(2.17) and (2.20) we get

$$\begin{aligned} \sqrt{m_v} \frac{\partial}{\partial t} \mathbf{v} &= -i\omega_p \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) + i\frac{\gamma}{T} \hat{\mathbf{B}} \wedge \mathbf{L} \cdot \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \\ &\quad - ik\omega_p \hat{\mathbf{k}} \frac{\delta p}{q_v} - ik\gamma \hat{\mathbf{B}} \wedge \mathbf{L}_\varepsilon \cdot \hat{\mathbf{k}} \delta \left( \frac{1}{T} \right) \end{aligned}$$

$$\begin{aligned}
& + ik \frac{\gamma}{T} \sum_{\sigma(\neq 1)} \hat{\mathbf{B}} \wedge \mathbf{L}_\sigma \cdot \hat{\mathbf{k}} \left( \frac{\delta\mu_\sigma}{m_\sigma} - \frac{\delta\mu_1}{m_1} \right)_T - k^2 \frac{1}{\sqrt{m_v}} \hat{\mathbf{k}} \cdot \boldsymbol{\eta} : \hat{\mathbf{k}} \mathbf{v} \\
& - ik^2 \frac{\gamma}{T} \hat{\mathbf{B}} \wedge \left[ \hat{\mathbf{k}} \cdot \mathbf{L}^{(c)} : \hat{\mathbf{k}} \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \right] \\
& - k^2 \frac{\gamma}{\sqrt{m_v} T} \hat{\mathbf{B}} \wedge [\hat{\mathbf{k}} \cdot \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} (\sqrt{m_v} \mathbf{v})] \\
& - ik^2 \frac{1}{\sqrt{m_v} T} \hat{\mathbf{k}} \cdot \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right)
\end{aligned} \tag{3.8}$$

with the abbreviation

$$\mathbf{L}_v^{(c)} = \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \mathbf{L}_{v\sigma}^{(c)}. \tag{3.9}$$

Finally, we have to determine the Fourier transform of the energy law (2.4). With the help of (2.9) we get

$$\begin{aligned}
\frac{\partial}{\partial t} \delta\varepsilon = & - ik\omega_p \frac{h_v}{q_v} \sqrt{m_v} \hat{\mathbf{k}} \cdot \mathbf{v} - k \frac{1}{T} \hat{\mathbf{k}} \cdot \bar{\mathbf{L}}_\varepsilon \cdot \left( \hat{\mathbf{k}} \frac{\delta q_v}{k} + i\gamma \sqrt{m_v} \mathbf{v} \wedge \hat{\mathbf{B}} \right) \\
& + k^2 \hat{\mathbf{k}} \cdot \bar{\mathbf{L}}_{\varepsilon\varepsilon} \cdot \hat{\mathbf{k}} \delta \left( \frac{1}{T} \right) - k^2 \frac{1}{T} \sum_{\sigma(\neq 1)} \hat{\mathbf{k}} \cdot \bar{\mathbf{L}}_{\varepsilon\sigma} \cdot \hat{\mathbf{k}} \left( \frac{\delta\mu_\sigma}{m_\sigma} - \frac{\delta\mu_1}{m_1} \right)_T
\end{aligned} \tag{3.10}$$

with  $h_v = \varepsilon + p$  the enthalpy per volume. The tensor  $\bar{\mathbf{L}}_\varepsilon$  is defined in terms of (2.11) in the same way as in (3.5):

$$\bar{\mathbf{L}}_\varepsilon = \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \bar{\mathbf{L}}_{\varepsilon\sigma}. \tag{3.11}$$

The conservation laws (3.1) (for  $\sigma \neq 1$ ), (3.3), (3.10) and the balance equation (3.8) form a complete set of coupled linear differential equations. They govern the time dependence of the perturbations in the partial densities, the charge density, the hydrodynamical velocity and the energy density. In the following, we shall determine the oscillating modes of the system for small wave numbers, by solving the associated eigenvalue problem in consecutive orders of the wave number.

#### 4. Oscillating modes in zeroth and first order of the wave number

In zeroth order of the wave number, Eqs. (3.1) for the perturbations of the partial densities  $n_\sigma$  and (3.10) for the perturbation of the energy density  $\varepsilon$  get a simple form, as all terms at the right-hand side vanish for  $k$  tending to 0. Hence, we are left with the charge conservation law (3.3) and the equation of motion (3.8). The latter can be decomposed in three equations for independent components of the velocity by writing

$$\mathbf{v} = \frac{1}{k_\parallel^2} (\hat{\mathbf{k}}_\parallel \cdot \mathbf{v}) \hat{\mathbf{k}}_\parallel + \frac{1}{k_\perp^2} (\hat{\mathbf{k}}_\perp \cdot \mathbf{v}) \hat{\mathbf{k}}_\perp + \frac{1}{k_\perp^2} [(\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v}] \hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}. \tag{4.1}$$

The Fourier variable  $\mathbf{k}$  has been written as  $\mathbf{k} = \mathbf{k}_\parallel + \mathbf{k}_\perp$ , with  $\mathbf{B} \cdot \mathbf{k}_\perp = 0$  and  $\mathbf{B} \wedge \mathbf{k}_\parallel = 0$ . Furthermore, we introduced the notations  $\hat{\mathbf{k}}_\parallel = \mathbf{k}_\parallel/k$ ,  $\hat{\mathbf{k}}_\perp = \mathbf{k}_\perp/k$ ,  $k_\parallel = |\hat{\mathbf{k}}_\parallel|$  and  $k_\perp = |\hat{\mathbf{k}}_\perp|$ .

Using the above decomposition of (3.8) in zeroth order of  $k$  and combining the resulting three equations with the zeroth-order approximation to (3.3) one gets four coupled equations from which the oscillating modes  $a_{\lambda\rho}^{(0)}$  and the associated frequencies  $z_{\lambda\rho}^{(0)}$ , with  $\lambda = \pm 1$  and  $\rho = \pm 1$ , can be obtained [7]. The time dependence of the modes is determined by the eigenvalue equation

$$\frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} = -iz_{\lambda\rho}^{(0)} a_{\lambda\rho}^{(0)} \tag{4.2}$$

with frequencies  $z_{\lambda\rho}^{(0)}$  that are solutions of the quartic equation

$$[z_{\lambda\rho}^{(0)2} - cz_{\lambda\rho}^{(0)} - \omega_p^2 \hat{k}_{\parallel}^2][(z_{\lambda\rho}^{(0)} - i\gamma b')^2 - \gamma^2(\omega_p + b)^2] + z_{\lambda\rho}^{(0)} \hat{k}_{\perp}^2 \{ (z_{\lambda\rho}^{(0)} - i\gamma b') [b'^2 - (\omega_p + b)^2] - 2i\gamma(\omega_p + b)^2 b' \} = 0. \quad (4.3)$$

The coefficients  $b$ ,  $b'$  and  $c$  are defined as

$$b = -\frac{\gamma}{T} L^t, \quad b' = -\frac{\gamma}{T} L^{\perp}, \quad c = -\frac{i}{T} (\hat{k}_{\parallel}^2 L^{\parallel} + \hat{k}_{\perp}^2 L^{\perp}) \quad (4.4)$$

with  $L^{\parallel}$ ,  $L^{\perp}$  and  $L^t$  independent parameters that determine the anisotropic tensor  $\mathbf{L}$  through the decomposition

$$\mathbf{L} = L^{\parallel} \hat{\mathbf{B}}\hat{\mathbf{B}} + L^{\perp} (\mathbf{U} - \hat{\mathbf{B}}\hat{\mathbf{B}}) + L^t \boldsymbol{\varepsilon} \cdot \hat{\mathbf{B}} \quad (4.5)$$

with  $\boldsymbol{\varepsilon}$  the Levi–Civita tensor. Since  $b$  and  $b'$  are real, while  $c$  is imaginary, the four solutions of (4.3) come in pairs. Their labels are chosen such that one has  $z_{\lambda\rho}^* = -z_{\lambda,-\rho}$ .

The four mode amplitudes can be written as [7]

$$a_{\lambda\rho}^{(0)} = \frac{\delta q_v}{k} + \sqrt{m_v} v_{\lambda\rho} \cdot \mathbf{v}. \quad (4.6)$$

The vectors  $v_{\lambda\rho}$  have the form

$$\mathbf{v}_{\lambda\rho} = v_{\lambda\rho}^{\parallel} \hat{\mathbf{k}}_{\parallel} + v_{\lambda\rho}^{\perp} \hat{\mathbf{k}}_{\perp} + v_{\lambda\rho}^t \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}, \quad (4.7)$$

where the components are defined as

$$v_{\lambda\rho}^{\parallel} = \frac{\omega_p}{z_{\lambda\rho}^{(0)}}, \quad v_{\lambda\rho}^{\perp} = \frac{z_{\lambda\rho}^{(0)}(\omega_p + b)}{\Delta_{\lambda\rho}^{(0)}}, \quad v_{\lambda\rho}^t = \frac{z_{\lambda\rho}^{(0)} b' - i\gamma[b'^2 + (\omega_p + b)^2]}{\Delta_{\lambda\rho}^{(0)}} \quad (4.8)$$

with the abbreviation

$$\Delta_{\lambda\rho}^{(0)} = (z_{\lambda\rho}^{(0)} - i\gamma b')^2 - \gamma^2(\omega_p + b)^2. \quad (4.9)$$

To determine the first-order corrections to the above oscillating modes we write

$$a_{\lambda\rho}^{(1)} = a_{\lambda\rho}^{(0)} + k a_{\lambda\rho}^{[1]}. \quad (4.10)$$

Here and in the following, a superscript ( $n$ ) in round brackets denotes the sum of all contributions up to and including a given order  $n$  in  $k$ . A superscript [ $n$ ] in square brackets denotes the coefficient of the contribution of a specific order  $n$ . In particular, one has  $a_{\lambda\rho}^{(0)} = a_{\lambda\rho}^{[0]}$ . As an *Ansatz* we will try and solve a first-order eigenvalue equation of the form

$$\frac{\partial}{\partial t} a_{\lambda\rho}^{(1)} = -iz_{\lambda\rho}^{(0)} a_{\lambda\rho}^{(1)} \quad (4.11)$$

with the same zeroth-order frequency as in (4.2), and with a first-order term of the general form

$$a_{\lambda\rho}^{[1]} = A_{\lambda\rho,\varepsilon} \delta\varepsilon + \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \delta n_{\sigma}, \quad (4.12)$$

where the coefficients have to be determined as yet. Inserting (4.10) in (4.11) we find

$$\left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} \right]^{(0)} + k \left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} \right]^{[1]} + k \left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{[1]} \right]^{(0)} = -iz_{\lambda\rho}^{(0)} a_{\lambda\rho}^{(0)} - ikz_{\lambda\rho}^{(0)} a_{\lambda\rho}^{[1]} \quad (4.13)$$

up to first order in the wave number. Of course, the zeroth-order terms drop out in view of (4.2). Upon substitution of (4.12) and inspection of (3.1) and (3.10) one infers that the last term at the left-hand side



vanishes. Hence, we are left with the relation

$$\left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} \right]^{[1]} = -iz_{\lambda\rho}^{(0)} a_{\lambda\rho}^{[1]} \quad (4.14)$$

from which the coefficients  $A_{\lambda\rho,\varepsilon}$  and  $A_{\lambda\rho,\sigma}$  have to be determined.

To evaluate the left-hand side of (4.14) we need the conservation law for the charge density and the balance equation for the three components of the velocity up to first order in the wave number. These follow from (3.3) and (3.8), with (4.1) inserted. To obtain more suitable forms for these equations we rewrite the chemical potentials  $\mu_\sigma$  in (3.3) and (3.8) in terms of new chemical potentials  $\tilde{\mu}_\sigma = \mu_\sigma - (e_\sigma/e_1)\mu_1$  [15]. These are better suited to the description of an ionic mixture, which satisfies the constraint of overall charge neutrality. The resulting expressions for the perturbations of the chemical potentials are given in (A.2). Furthermore, we decompose the tensors  $\bar{\mathbf{L}}_\varepsilon$  and  $\mathbf{L}_\sigma$  in a similar way as in (4.5). In this way we arrive at the following forms of the balance equations:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\delta q_v}{k} \right) &= -ic \frac{\delta q_v}{k} - i\omega_p \sqrt{m_v} \hat{\mathbf{k}}_\parallel \cdot \mathbf{v} - i(\omega_p + b) \sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v} - ib' \sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v} \\ &+ ik \left[ Tc_\varepsilon \delta \left( \frac{1}{T} \right) - T \sum_{\sigma(\neq 1)} c_\sigma \delta \left( \frac{\tilde{\mu}_\sigma}{T} \right) - c \frac{\delta p}{q_v} \right], \end{aligned} \quad (4.15)$$

$$\sqrt{m_v} \frac{\partial}{\partial t} (\hat{\mathbf{k}}_\parallel \cdot \mathbf{v}) = -i\omega_p \hat{k}_\parallel^2 \frac{\delta q_v}{k} - ik\omega_p \hat{k}_\parallel^2 \frac{\delta p}{q_v}, \quad (4.16)$$

$$\begin{aligned} \sqrt{m_v} \frac{\partial}{\partial t} (\hat{\mathbf{k}}_\perp \cdot \mathbf{v}) &= -i(\omega_p + b) \hat{k}_\perp^2 \frac{\delta q_v}{k} + \gamma b' \sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v} - \gamma(\omega_p + b) \sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v} \\ &+ ik \hat{k}_\perp^2 \left[ T b_\varepsilon \delta \left( \frac{1}{T} \right) - T \sum_{\sigma(\neq 1)} b_\sigma \delta \left( \frac{\tilde{\mu}_\sigma}{T} \right) - (\omega_p + b) \frac{\delta p}{q_v} \right], \end{aligned} \quad (4.17)$$

$$\begin{aligned} \sqrt{m_v} \frac{\partial}{\partial t} [(\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v}] &= ib' \hat{k}_\perp^2 \frac{\delta q_v}{k} + \gamma(\omega_p + b) \sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v} + \gamma b' \sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v} \\ &- ik \hat{k}_\perp^2 \left[ T b'_\varepsilon \delta \left( \frac{1}{T} \right) - T \sum_{\sigma(\neq 1)} b'_\sigma \delta \left( \frac{\tilde{\mu}_\sigma}{T} \right) - b' \frac{\delta p}{q_v} \right]. \end{aligned} \quad (4.18)$$

The abbreviations  $b_\varepsilon$ ,  $b'_\varepsilon$ ,  $b_\sigma$ ,  $b'_\sigma$ ,  $c_\varepsilon$  and  $c_\sigma$  are combinations of the components of  $\mathbf{L}$ ,  $\bar{\mathbf{L}}_\varepsilon$  and  $\mathbf{L}_\sigma$  [7]:

$$b_\varepsilon = -\frac{\gamma}{T} \bar{L}'_\varepsilon - b \frac{h_v}{q_v}, \quad b'_\varepsilon = -\frac{\gamma}{T} \bar{L}^\perp_\varepsilon - b' \frac{h_v}{q_v}, \quad (4.19)$$

$$b_\sigma = -\frac{\gamma}{m_\sigma T} L'_\sigma - b \frac{n_\sigma}{q_v}, \quad b'_\sigma = -\frac{\gamma}{m_\sigma T} L^\perp_\sigma - b' \frac{n_\sigma}{q_v}, \quad (4.20)$$

$$c_\varepsilon = -\frac{i}{T} (\hat{k}_\parallel^2 \bar{L}'_\varepsilon + \hat{k}_\perp^2 \bar{L}^\perp_\varepsilon) - c \frac{h_v}{q_v}, \quad c_\sigma = -\frac{i}{m_\sigma T} (\hat{k}_\parallel^2 L'_\sigma + \hat{k}_\perp^2 L^\perp_\sigma) - c \frac{n_\sigma}{q_v} \quad (4.21)$$

with  $\sigma \neq 1$ .

The terms of order  $k$  in (4.15)–(4.18) contain the perturbations  $\delta(1/T)$ ,  $\delta(\tilde{\mu}_\sigma/T)$  and  $\delta p$ . These can be expressed in terms of the perturbations  $\delta n_\sigma$  and  $\delta\varepsilon$  by employing the thermodynamic relations (A.9), (A.10) and (A.13) of Appendix A of which only the terms of order  $k^0$  are needed for the moment. Substituting the results in (4.14) with (4.6), we arrive at an expression for  $a_{\lambda\rho}^{[1]}$ , which is indeed of the form of (4.12). Upon using (4.3) we obtain the following expression for the coefficients:

$$A_{\lambda\rho j} = \beta^{-1} \left( \frac{n}{2\beta q_v} M_{ej}^{-1} - s_j + \sum_{j'} M_{jj'}^{-1} F_{\lambda\rho j'} \right) \quad (4.22)$$

with  $j$  and  $j'$  equal to  $\varepsilon$  or  $\sigma (\neq 1)$ , with  $\beta = 1/(k_B T)$ , and with  $M_{jj'}^{-1}$  and  $s_j$  defined in (A.4) and (A.8). Furthermore, we introduced the abbreviation

$$F_{\lambda\rho j} = \frac{1}{z_{\lambda\rho}^{(0)}} [c_j + (v_{\lambda\rho}^\perp b_j - v_{\lambda\rho}^t b'_j) \hat{k}_\perp^2] \quad (4.23)$$

with components  $v_{\lambda\rho}^i$  defined in (4.8).

## 5. Oscillating-mode frequencies in second order of the wave number

Having determined the oscillating modes up to first order in the wave number, we now turn to the contributions of second order in  $k$ . In particular, we want to evaluate the mode frequencies in second order, as these determine the dispersion and damping effects. The mode amplitudes in second order are less interesting. Fortunately, their explicit form is not needed in the following, as we shall see.

The second-order contributions  $z_{\lambda\rho}^{[2]}$  to the eigenfrequencies of the oscillating modes follow from the eigenvalue equation

$$\frac{\partial}{\partial t} a_{\lambda\rho}^{(2)} = -i z_{\lambda\rho}^{(2)} a_{\lambda\rho}^{(2)} \quad (5.1)$$

with the mode frequencies

$$z_{\lambda\rho}^{(2)} = z_{\lambda\rho}^{(0)} + k^2 z_{\lambda\rho}^{[2]} \quad (5.2)$$

and the mode amplitudes

$$a_{\lambda\rho}^{(2)} = a_{\lambda\rho}^{(0)} + k a_{\lambda\rho}^{[1]} + k^2 a_{\lambda\rho}^{[2]}. \quad (5.3)$$

The zeroth- and first-order terms in the amplitudes have been given in (4.6) and (4.12) with (4.22). The second-order terms have the general form

$$a_{\lambda\rho}^{[2]} = A_{\lambda\rho,q} \frac{\delta q_v}{k} + A_{\lambda\rho}^\parallel \sqrt{m_v} \hat{\mathbf{k}}_\parallel \cdot \mathbf{v} + A_{\lambda\rho}^\perp \sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v} + A_{\lambda\rho}^t \sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v} \quad (5.4)$$

with coefficients  $A_{\lambda\rho}$  that are as yet unknown.

Substitution of (5.2) and (5.3) into (5.1) yields the second-order equation

$$\left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} \right]^{[2]} + \left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{[1]} \right]^{[1]} + \left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{[2]} \right]^{(0)} = -i z_{\lambda\rho}^{(0)} a_{\lambda\rho}^{[2]} - i z_{\lambda\rho}^{[2]} a_{\lambda\rho}^{(0)}. \quad (5.5)$$

The right-hand side is a linear combination of the four independent perturbations  $\delta q_v/k$ ,  $\sqrt{m_v} \hat{\mathbf{k}}_\parallel \cdot \mathbf{v}$ ,  $\sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v}$  and  $\sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v}$ , as follows by inserting (4.6)–(4.7) and (5.4). Likewise, the third term at the left-hand side of (5.5) can be written in terms of these independent perturbations by inserting (5.4) and substituting the zeroth-order contributions of (4.15)–(4.18). In the second term at the left-hand side of (5.5) we substitute (4.12) and use (3.1), (3.10). The first-order contributions in the latter equations can be rewritten as linear combinations of the independent perturbations in the way shown in (6.1)–(6.2) of Ref. [7]. Finally, we have to consider the first term at the left-hand side of (5.5). Formally, it can be written as a linear combination of the independent perturbations as well:

$$\left[ \frac{\partial}{\partial t} a_{\lambda\rho}^{(0)} \right]^{[2]} = D_{\lambda\rho,q} \frac{\delta q_v}{k} + D_{\lambda\rho}^\parallel \sqrt{m_v} \hat{\mathbf{k}}_\parallel \cdot \mathbf{v} + D_{\lambda\rho}^\perp \sqrt{m_v} \hat{\mathbf{k}}_\perp \cdot \mathbf{v} + D_{\lambda\rho}^t \sqrt{m_v} (\hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}}) \cdot \mathbf{v}. \quad (5.6)$$

The coefficients  $D_{\lambda\rho}$  will be determined later.

Having taken the above steps we are in a position to compare the contributions of the four independent perturbations at both sides of (5.5). Upon equating the coefficients we arrive at the following

set of equalities:

$$\begin{aligned} & (c - z_{\lambda\rho}^{(0)})A_{\lambda\rho,q} + \omega_p \hat{k}_{\parallel}^2 A_{\lambda\rho}^{\parallel} + (\omega_p + b) \hat{k}_{\perp}^2 A_{\lambda\rho}^{\perp} - b' \hat{k}_{\perp}^2 A_{\lambda\rho}^t \\ & = z_{\lambda\rho}^{[2]} - iD_{\lambda\rho,q} - A_{\lambda\rho,\varepsilon} \left( c_{\varepsilon} + c \frac{h_v}{q_v} \right) - \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \left( c_{\sigma} + c \frac{n_{\sigma}}{q_v} \right), \end{aligned} \quad (5.7)$$

$$\omega_p A_{\lambda\rho,q} - z_{\lambda\rho}^{(0)} A_{\lambda\rho}^{\parallel} = z_{\lambda\rho}^{[2]} v_{\lambda\rho}^{\parallel} - iD_{\lambda\rho}^{\parallel} - A_{\lambda\rho,\varepsilon} \omega_p \frac{h_v}{q_v} - \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \omega_p \frac{n_{\sigma}}{q_v}, \quad (5.8)$$

$$\begin{aligned} & (\omega_p + b)A_{\lambda\rho,q} + (i\gamma b' - z_{\lambda\rho}^{(0)})A_{\lambda\rho}^{\perp} + i\gamma(\omega_p + b)A_{\lambda\rho}^t \\ & = z_{\lambda\rho}^{[2]} v_{\lambda\rho}^{\perp} - iD_{\lambda\rho}^{\perp} - A_{\lambda\rho,\varepsilon} \left[ b_{\varepsilon} + (\omega_p + b) \frac{h_v}{q_v} \right] - \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \left[ b_{\sigma} + (\omega_p + b) \frac{n_{\sigma}}{q_v} \right], \end{aligned} \quad (5.9)$$

$$\begin{aligned} & b' A_{\lambda\rho,q} - i\gamma(\omega_p + b)A_{\lambda\rho}^{\perp} + (i\gamma b' - z_{\lambda\rho}^{(0)})A_{\lambda\rho}^t \\ & = z_{\lambda\rho}^{[2]} v_{\lambda\rho}^t - iD_{\lambda\rho}^t - A_{\lambda\rho,\varepsilon} \left( b'_{\varepsilon} + b' \frac{h_v}{q_v} \right) - \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \left( b'_{\sigma} + b' \frac{n_{\sigma}}{q_v} \right). \end{aligned} \quad (5.10)$$

The above system of equations for the four unknowns  $A_{\lambda\rho,q}$ ,  $A_{\lambda\rho}^{\parallel}$ ,  $A_{\lambda\rho}^{\perp}$  and  $A_{\lambda\rho}^t$  turns out to be a dependent set. In fact, the determinant of the matrix of coefficients appearing at the left-hand sides is exactly equal to the left-hand side of the quartic equation (4.3), so that it vanishes. The mutual dependence of the equations is a mathematical reflection of the fact that the amplitudes of the modes are fixed up to a multiplicative constant only. Because the system is a dependent set, the matrix has got a left-eigenvector with eigenvalue 0. This four-dimensional vector is found as

$$(1, v_{\lambda\rho}^{\parallel} \hat{k}_{\parallel}^2, v_{\lambda\rho}^{\perp} \hat{k}_{\perp}^2, -v_{\lambda\rho}^t \hat{k}_{\perp}^2). \quad (5.11)$$

The second-order contributions  $z_{\lambda\rho}^{[2]}$  to the eigenfrequencies of the oscillating modes now follow by taking the inner product of (5.11) with the set of Eqs. (5.7)–(5.10). The left-hand side of the resulting equation vanishes by construction, so that we are left with an equality from which  $z_{\lambda\rho}^{[2]}$  can be determined. Using (4.3) to eliminate  $c$  we find:

$$\begin{aligned} z_{\lambda\rho}^{[2]} = N_{\lambda\rho} \left\{ z_{\lambda\rho}^{(0)} \left[ A_{\lambda\rho,\varepsilon} \left( F_{\lambda\rho,\varepsilon} + \frac{h_v}{q_v} \right) + \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma} \left( F_{\lambda\rho,\sigma} + \frac{n_{\sigma}}{q_v} \right) \right] \right. \\ \left. + iD_{\lambda\rho,q} + iv_{\lambda\rho}^{\parallel} \hat{k}_{\parallel}^2 D_{\lambda\rho}^{\parallel} + iv_{\lambda\rho}^{\perp} \hat{k}_{\perp}^2 D_{\lambda\rho}^{\perp} - iv_{\lambda\rho}^t \hat{k}_{\perp}^2 D_{\lambda\rho}^t \right\}, \end{aligned} \quad (5.12)$$

where the constant  $N_{\lambda\rho}$  is defined as

$$N_{\lambda\rho} = [1 + (v_{\lambda\rho}^{\parallel})^2 \hat{k}_{\parallel}^2 + (v_{\lambda\rho}^{\perp})^2 \hat{k}_{\perp}^2 - (v_{\lambda\rho}^t)^2 \hat{k}_{\perp}^2]^{-1}. \quad (5.13)$$

The expression (5.12) still contains the coefficients  $D_{\lambda\rho}$  that have been introduced in (5.6). These will now be determined by substituting (4.6) in the left-hand side of (5.6). Employing the balance equations (4.15)–(4.18) and using the thermodynamical relations of Appendix A, we encounter contributions that follow from the terms proportional to  $\delta q_v$  in (A.9), (A.10) and (A.13). The ensuing term in  $D_{\lambda\rho,q}$  is

$$D'_{\lambda\rho,q} = \frac{i}{\beta} z_{\lambda\rho}^{(0)} \left( \sum_j F_{\lambda\rho,j} W_j + \frac{n}{2\beta q_v} W_{\varepsilon} - \frac{1}{2q_v e_1} \right). \quad (5.14)$$

A further contribution  $D''_{\lambda\rho,q}$  to  $D_{\lambda\rho,q}$  follows by considering the terms of order  $k^2$  in (3.3) and (3.8). These yield

$$D''_{\lambda\rho,q} = -\frac{i}{\sqrt{m_v T}} v_{\lambda\rho} \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}} + \frac{1}{T} (\hat{\mathbf{k}} - i\gamma v_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}}. \quad (5.15)$$

Likewise, we find for the other coefficients:

$$D_{\lambda\rho}^{\parallel} = -\frac{1}{m_v \hat{k}_{\parallel}^2} \mathbf{v}_{\lambda\rho} \hat{\mathbf{k}} : \boldsymbol{\eta} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\parallel} - \frac{i}{\sqrt{m_v} T \hat{k}_{\parallel}^2} (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\parallel}, \tag{5.16}$$

$$D_{\lambda\rho}^{\perp} = -\frac{1}{m_v \hat{k}_{\perp}^2} \mathbf{v}_{\lambda\rho} \hat{\mathbf{k}} : \boldsymbol{\eta} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\perp} + \frac{i\gamma}{T \hat{k}_{\perp}^2} (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}^{(c)} : \hat{\mathbf{k}} (\hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}) - \frac{i}{\sqrt{m_v} T \hat{k}_{\perp}^2} (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\perp} + \frac{\gamma}{\sqrt{m_v} T \hat{k}_{\perp}^2} \mathbf{v}_{\lambda\rho} \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} (\hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}), \tag{5.17}$$

$$D_{\lambda\rho}^t = -\frac{1}{m_v \hat{k}_{\perp}^2} \mathbf{v}_{\lambda\rho} \hat{\mathbf{k}} : \boldsymbol{\eta} : \hat{\mathbf{k}} (\hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}) - \frac{i\gamma}{T \hat{k}_{\perp}^2} (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\perp} - \frac{i}{\sqrt{m_v} T \hat{k}_{\perp}^2} (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho} \wedge \hat{\mathbf{B}}) \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} (\hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}) - \frac{\gamma}{\sqrt{m_v} T \hat{k}_{\perp}^2} \mathbf{v}_{\lambda\rho} \hat{\mathbf{k}} : \mathbf{L}_v^{(c)} : \hat{\mathbf{k}} \hat{\mathbf{k}}_{\perp}. \tag{5.18}$$

The fourth-rank tensors in these expressions may be decomposed in terms of their invariant parts in a way analogous to (4.5). The expressions for the coefficients  $D_{\lambda\rho}$  that result by substituting these decompositions are rather unwieldy and will not be presented here.

Apart from the coefficients  $D_{\lambda\rho}$ , expression (5.12) also depends on the coefficients  $A_{\lambda\rho}$  for which the explicit form (4.22) is available. Upon inserting the latter and combining the resulting contributions with those from (5.14) we arrive at the final form for the second-order terms in the eigenfrequencies of the oscillating modes:

$$z_{\lambda\rho}^{[2]} = N_{\lambda\rho} \left\{ z_{\lambda\rho}^{(0)} \left[ \frac{c_s^2}{\omega_p^2} + \frac{2}{\beta} \sum_j \left( \frac{n}{2\beta q_v} M_{ej}^{-1} - s_j \right) F_{\lambda\rho,j} + \frac{1}{\beta} \sum_{j,j'} F_{\lambda\rho,j} M_{jj'}^{-1} F_{\lambda\rho,j'} \right] + iD''_{\lambda\rho,q} + i\mathbf{v}_{\lambda\rho}^{\parallel} \hat{k}_{\parallel}^2 D_{\lambda\rho}^{\parallel} + i\mathbf{v}_{\lambda\rho}^{\perp} \hat{k}_{\perp}^2 D_{\lambda\rho}^{\perp} - i\mathbf{v}_{\lambda\rho}^t \hat{k}_{\perp}^2 D_{\lambda\rho}^t \right\}, \tag{5.19}$$

where we have used (A.18) as well. Our result shows that the damping and the dispersion of the oscillating modes in second order of the wave number is caused by several rather different mechanisms. The first term, with the sound velocity  $c_s$ , depends on equilibrium properties only. Of course, dispersion of collective modes that is governed by the sound velocity is a well-known effect, both in plasmas and in systems of neutral particles. The remaining terms in the first line are linear or quadratic in  $F_{\lambda\rho}$ . According to (4.23) these coefficients are linear combinations of  $b_j$ ,  $b'_j$  and  $c_j$ , which have been defined in (4.19)–(4.21) with (4.4). They are governed by non-equilibrium processes through the components of the phenomenological tensors  $\mathbf{L}$ ,  $\mathbf{L}_e$  and  $\mathbf{L}_\sigma$ , which are linear combinations of  $\mathbf{L}_{\sigma\sigma'}$  and  $\mathbf{L}_{e\sigma}$ . The latter occur in the linear laws (2.7) and (2.9) for the diffusion flows and the heat flow. Finally, the second line of (5.19) depends on non-equilibrium processes as well via the coefficients  $D_{\lambda\rho}$ . From (5.15)–(5.18) we infer that viscous effects play a role here. However, these are not the only dissipative phenomena that are relevant for the damping and the dispersion. Higher-order couplings between thermodynamic forces and flows, as described by Burnett terms, are important as well.

For the one-component plasma the expression (5.19) gets a simpler form. For that system  $F_{\lambda\rho,j}$  vanishes, so that in the first line only the term with the sound velocity survives. Furthermore, the Burnett contributions disappear from the coefficients  $D_{\lambda\rho}$ , so that only the viscous terms in (5.16)–(5.18) remain. The resulting expression agrees with that found in Refs. [1,2].

### 6. Relation to kinetic theory

The collective modes of an ionic mixture can also be derived with the help of formal kinetic theory [6,7]. In that approach the starting point is the set of microscopic balance equations. From these one obtains the collective modes by employing a projection-operator technique.

The microscopic partial particle density is  $n_\sigma^m(\mathbf{r}) = \sum_\alpha \delta(\mathbf{r} - \mathbf{r}_{\sigma\alpha})$ , with  $\mathbf{r}_{\sigma\alpha}$  the position of particle  $\alpha$  of component  $\sigma$ . The Fourier transform  $\delta n_\sigma^m(\mathbf{k})$  of the perturbation  $n_\sigma^m(\mathbf{r}) - \langle n_\sigma^m(\mathbf{r}) \rangle$  of the microscopic partial particle density, with  $\langle n_\sigma^m(\mathbf{r}) \rangle$  its uniform thermal equilibrium average, reads

$$\delta n_\sigma^m(\mathbf{k}) = \sum_\alpha e^{-i\mathbf{k} \cdot \mathbf{r}_{\sigma\alpha}}. \quad (6.1)$$

The Fourier transform of the perturbation of the microscopic charge density is  $\delta q_v^m(\mathbf{k}) = \sum_\sigma e_\sigma \delta n_\sigma^m(\mathbf{k})$ . Furthermore, the microscopic partial momentum density of species  $\sigma$  in Fourier space is given by

$$\mathbf{g}_\sigma^m(\mathbf{k}) = \sum_\alpha \mathbf{p}_{\sigma\alpha} e^{-i\mathbf{k} \cdot \mathbf{r}_{\sigma\alpha}} \quad (6.2)$$

with  $\mathbf{p}_{\sigma\alpha}$  the momentum of particle  $\alpha$  of component  $\sigma$ . Finally, the perturbation of the microscopic energy density in Fourier space will be denoted by  $\delta \varepsilon^m(\mathbf{k})$ .

The conservation laws for  $\delta n_\sigma^m(\mathbf{k})$  and  $\delta \varepsilon^m(\mathbf{k})$ , and the balance equation for the microscopic total momentum density  $\mathbf{g}^m(\mathbf{k}) = \sum_\sigma \mathbf{g}_\sigma^m(\mathbf{k})$  read [6]

$$\mathcal{L} \delta n_\sigma^m(\mathbf{k}) = -\frac{\mathbf{k}}{m_\sigma} \cdot \mathbf{g}_\sigma^m(\mathbf{k}), \quad (6.3)$$

$$\mathcal{L} \mathbf{g}^m(\mathbf{k}) = -\mathbf{k} \cdot \delta \boldsymbol{\tau}^m(\mathbf{k}) - q_v \frac{\mathbf{k}}{k^2} \delta q_v^m(\mathbf{k}) - i \sum_\sigma \frac{e_\sigma}{m_\sigma c} \mathbf{g}_\sigma^m(\mathbf{k}) \wedge \mathbf{B}, \quad (6.4)$$

$$\mathcal{L} \delta \varepsilon^m(\mathbf{k}) = -\mathbf{k} \cdot \mathbf{j}_\varepsilon^m(\mathbf{k}) \quad (6.5)$$

with  $\delta \boldsymbol{\tau}^m(\mathbf{k})$  the Fourier-transformed perturbation of the microscopic pressure tensor and  $\mathbf{j}_\varepsilon^m(\mathbf{k})$  the microscopic energy-current density in Fourier space. The Liouville operator in phase space  $\mathcal{L}$  follows by writing the time derivative  $\dot{F}$  of an arbitrary function  $F$  as  $\dot{F} = i\mathcal{L}F$ .

The collective modes are specific independent linear combinations of the microscopic basic quantities

$$a_i^b(\mathbf{k}) \in \left\{ \frac{1}{k} \delta q_v^m(\mathbf{k}), \mathbf{g}^m(\mathbf{k}), \delta \varepsilon^m(\mathbf{k}), \delta n_2^m(\mathbf{k}), \dots, \delta n_s^m(\mathbf{k}) \right\} \quad (6.6)$$

with  $i = 0, 1, 2, 3, \varepsilon, \sigma (\neq 1)$ . The microscopic charge density is divided by the wave number, since the fluctuations in the charge density vanish in the long-wavelength limit, as has been remarked already in Section 3. Indeed, in leading order of the wave number one has

$$\frac{1}{V} \langle [\delta q_v^m(\mathbf{k})]^* \delta q_v^m(\mathbf{k}) \rangle = \frac{k^2}{\beta} \quad (6.7)$$

with  $V$  the volume of the system. The fluctuation formulas for the ionic mixture have been given in Refs. [6,14].

To derive the collective modes one may use a projection-operator technique. The one-sided Fourier transform of the time-dependent collective mode  $a_i^m(\mathbf{k}, t)$  is given by

$$a_i^m(\mathbf{k}, z) \equiv -i \int_0^\infty dt e^{izt} a_i^m(\mathbf{k}, t) = \frac{1}{z + \mathcal{L}} a_i^m(\mathbf{k}) \quad (6.8)$$

which is regular for  $z$  in the upper halfplane. By introducing a projection operator  $P$  that projects an arbitrary phase function  $f(\mathbf{k})$  on the space spanned by the basis (6.6) and its complement  $Q = 1 - P$ , one arrives at an eigenvalue problem for the modes  $a_i^m(\mathbf{k})$  and their frequencies  $z_i^m(\mathbf{k})$ :

$$-\frac{1}{V} \langle [a_j^b(\mathbf{k})]^* \mathcal{L} a_i^m(\mathbf{k}) \rangle + \frac{1}{V} \left\langle [a_j^b(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^m(\mathbf{k}) + Q \mathcal{L} Q} Q \mathcal{L} a_i^m(\mathbf{k}) \right\rangle = z_i^m(\mathbf{k}) \frac{1}{V} \langle [a_j^b(\mathbf{k})]^* a_i^m(\mathbf{k}) \rangle \quad (6.9)$$

for all basis functions  $a_j^b(\mathbf{k})$ . Near  $z = z_i^m(\mathbf{k})$  the projection of (6.8) has a pole structure of the form

$$P \frac{1}{z + \mathcal{L}} a_i^m(\mathbf{k}) = \frac{1}{z - z_i^m(\mathbf{k})} a_i^m(\mathbf{k}). \quad (6.10)$$

The eigenvalue problem (6.9) leads to a set of  $s + 4$  modes. Apart from a heat mode and  $s - 1$  diffusion modes one finds four modes that are the kinetic counterparts of the oscillating modes. In zeroth order of the wave number their frequencies  $z_{\lambda\rho}^{m(0)}$  follow from a relation of the form (4.3). The parameters  $b$ ,  $b'$  and  $c$  are replaced by  $z$ -dependent functions which follow from the identities [7]

$$\frac{\beta}{V} \left\langle \left[ \frac{\delta q_v^m(\mathbf{k})}{k} \right]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \frac{\delta q_v^m(\mathbf{k})}{k} \right\rangle^{(0)} = c(z), \quad (6.11)$$

$$\frac{\beta}{\sqrt{m_v} V} \left\langle [\mathbf{g}^m(\mathbf{k})]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \frac{\delta q_v^m(\mathbf{k})}{k} \right\rangle^{(0)} = b(z) \hat{\mathbf{k}}_{\perp} + b'(z) \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}, \quad (6.12)$$

$$\frac{\beta}{m_v V} \left\langle [\mathbf{g}^m(\mathbf{k})]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \mathbf{g}^m(\mathbf{k}) \right\rangle^{(0)} = -i\gamma b(z) \boldsymbol{\varepsilon} \cdot \hat{\mathbf{B}} + i\gamma b'(z) (\mathbf{U} - \hat{\mathbf{B}} \hat{\mathbf{B}}). \quad (6.13)$$

The consistency of the second and third of these definitions has been demonstrated in Ref. [7].

The amplitudes  $a_{\lambda\rho}^m(\mathbf{k})$  of the associated modes in zeroth order of the wave number are given by

$$a_{\lambda\rho}^{m(0)}(\mathbf{k}) = \frac{\delta q_v^m(\mathbf{k})}{k} + \frac{1}{\sqrt{m_v}} \mathbf{v}_{\lambda\rho}^m(\mathbf{k}) \cdot \mathbf{g}^m(\mathbf{k}). \quad (6.14)$$

The vector  $\mathbf{v}_{\lambda\rho}^m(\mathbf{k})$  has the same form as (4.7)–(4.8), with the parameters  $b$ ,  $b'$  and  $c$  replaced by the corresponding functions  $b(z)$ ,  $b'(z)$  and  $c(z)$  for  $z = z_{\lambda\rho}^{m(0)}$ .

The first-order contributions to the mode amplitudes have the general form (cf. (4.12))

$$a_{\lambda\rho}^{m(1)}(\mathbf{k}) = A_{\lambda\rho,\varepsilon}^m \delta \varepsilon^m(\mathbf{k}) + \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma}^m \delta n_{\sigma}^m(\mathbf{k}) \quad (6.15)$$

with coefficients  $A_{\lambda\rho,j}^m$  that have to be determined by substituting (6.14) and (6.15) into (6.9) for  $i = \lambda\rho$  and  $j = \varepsilon, \sigma$ . Employing the fluctuation formulas [6,14] and the thermodynamic formulas of Appendix A, we find a set of inhomogeneous linear equations for the unknown  $A_{\lambda\rho,j}^m$ , with a coefficient matrix given by (A.4). Solving these equations we arrive at expressions for  $A_{\lambda\rho,j}^m$  of the same form as (4.22), with  $F_{\lambda\rho,j}$  replaced by  $F_{\lambda\rho,j}^m$ , with a similar structure as in (4.23). The kinetic counterparts of (4.19)–(4.21) are [7]:

$$\frac{\beta}{V} \left\langle \left[ \frac{\delta q_v^m(\mathbf{k})}{k} \right]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \left[ \delta \varepsilon^m(\mathbf{k}) - \frac{h_v}{q_v} \delta q_v^m(\mathbf{k}) \right] \right\rangle^{[1]} = c_{\varepsilon}(z), \quad (6.16)$$

$$\frac{\beta}{V} \left\langle \left[ \frac{\delta q_v^m(\mathbf{k})}{k} \right]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \left[ \delta n_{\sigma}^m(\mathbf{k}) - \frac{n_{\sigma}}{q_v} \delta q_v^m(\mathbf{k}) \right] \right\rangle^{[1]} = c_{\sigma}(z), \quad (6.17)$$

$$\frac{\beta}{\sqrt{m_v} V} \left\langle [\mathbf{g}^m(\mathbf{k})]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \left[ \delta \varepsilon^m(\mathbf{k}) - \frac{h_v}{q_v} \delta q_v^m(\mathbf{k}) \right] \right\rangle^{[1]} = b_{\varepsilon}(z) \hat{\mathbf{k}}_{\perp} + b'_{\varepsilon}(z) \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}, \quad (6.18)$$

$$\frac{\beta}{\sqrt{m_v} V} \left\langle [\mathbf{g}^m(\mathbf{k})]^* \mathcal{L} \mathcal{Q} \frac{1}{z + Q \mathcal{L} \mathcal{Q}} \mathcal{Q} \mathcal{L} \left[ \delta n_{\sigma}^m(\mathbf{k}) - \frac{n_{\sigma}}{q_v} \delta q_v^m(\mathbf{k}) \right] \right\rangle^{[1]} = b_{\sigma}(z) \hat{\mathbf{k}}_{\perp} + b'_{\sigma}(z) \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}. \quad (6.19)$$

The oscillating mode amplitudes  $a_{\lambda\rho}^{m(1)}$  up to first order in the wave number have been obtained as the sum of (6.14) and (6.15). In the following we shall also need the thermal and diffusive mode amplitudes up to first order in  $k$ . These have been derived in [7] as

$$a_{\varepsilon}^{m(1)}(\mathbf{k}) = \delta \varepsilon^m(\mathbf{k}) - \frac{h_v}{q_v} \delta q_v^m(\mathbf{k}) + k \frac{1}{\sqrt{m_v}} A_{\varepsilon}^m \cdot \mathbf{g}^m(\mathbf{k}), \quad (6.20)$$

$$a_{\sigma}^{m(1)}(\mathbf{k}) = \delta n_{\sigma}^m(\mathbf{k}) - \frac{n_{\sigma}}{q_v} \delta q_v^m(\mathbf{k}) + k \frac{1}{\sqrt{m_v}} A_{\sigma}^m \cdot \mathbf{g}^m(\mathbf{k}). \quad (6.21)$$

The coefficients  $A_j^m$  have the decomposition

$$A_j^m = A_j^{m\parallel} \hat{\mathbf{k}}_{\parallel} + A_j^{m\perp} \hat{\mathbf{k}}_{\perp} + A_j^{m\prime} \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}} \tag{6.22}$$

with

$$A_j^{m\parallel} = \frac{1}{\omega_p \hat{k}_{\parallel}^2} \left( \frac{i}{\gamma} b_j' \hat{k}_{\perp}^2 - c_j \right), \quad A_j^{m\perp} = \frac{i b_j b' - b_j'(\omega_p + b)}{\gamma (\omega_p + b)^2 + b'^2},$$

$$A_j^{m\prime} = \frac{i b_j(\omega_p + b) + b_j' b'}{\gamma (\omega_p + b)^2 + b'^2} \tag{6.23}$$

for  $j = \varepsilon, \sigma (\neq 1)$ . The functions  $b, b', b_j, b_j'$  and  $c_j$  are given by (6.12), (6.16)–(6.19). Here  $z$  is equal to the infinitesimal value  $i0$ , since the frequencies  $z_j^{m(0)}$  of the heat and diffusion modes vanish in lowest order of  $k$ .

### 7. Kinetic derivation of the frequencies in second order of the wave number

The frequencies  $z_i^{m[2]}$  in second order of the wave number can be obtained without having to solve the full eigenvalue problem (6.9) in second order. In fact, it can be shown (see Appendix B) that knowledge of the mode amplitudes up to first order is enough to evaluate the second-order frequencies, since the latter follow as

$$z_i^{m[2]} = -z_i^{m(0)} \frac{1}{V} \langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* a_i^{m(1)}(\mathbf{k}) \rangle^{[2]} - \frac{1}{V} \langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* \mathcal{L} a_i^{m(1)}(\mathbf{k}) \rangle^{[2]} + \frac{1}{V} \left\langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^{m(0)} + Q \mathcal{L} Q} Q \mathcal{L} a_i^{m(1)}(\mathbf{k}) \right\rangle^{[2]}. \tag{7.1}$$

In this expression we have introduced the adjoint modes  $\bar{a}_i^m(\mathbf{k})$ . These adjoints are specific independent linear combinations of the basis functions  $a_i^b(\mathbf{k})$  (6.6). They are defined by the orthonormality relations

$$\frac{1}{V} \langle [\bar{a}_i^m(\mathbf{k})]^* a_j^m(\mathbf{k}) \rangle = \delta_{ij}. \tag{7.2}$$

From Eq. (7.1) it follows that the adjoints of the modes (6.14)–(6.15) up to first order in the wave number have to be determined. In deriving these adjoints from (7.2) it will be assumed that the transport properties  $b(z), b'(z), b_j(z), b_j'(z)$  and  $c_j(z)$  are slowly varying functions of  $z$ , which can be treated as constants. In zeroth order the adjoints  $\bar{a}_{\lambda\rho}^{m(0)}(\mathbf{k})$  are given by the general expression [6]

$$\bar{a}_{\lambda\rho}^{m(0)}(\mathbf{k}) = \beta (N_{\lambda\rho}^m)^* \left[ \frac{\delta q_v^m(\mathbf{k})}{k} + \frac{1}{\sqrt{m_v}} \bar{v}_{\lambda\rho}^m(\mathbf{k}) \cdot \mathbf{g}^m(\mathbf{k}) \right], \tag{7.3}$$

where the vector  $\bar{v}_{\lambda\rho}^m(\mathbf{k})$  can be written as

$$\bar{v}_{\lambda\rho}^m = \bar{v}_{\lambda\rho}^{m\parallel} \hat{\mathbf{k}}_{\parallel} + \bar{v}_{\lambda\rho}^{m\perp} \hat{\mathbf{k}}_{\perp} + \bar{v}_{\lambda\rho}^{m\prime} \hat{\mathbf{k}}_{\perp} \wedge \hat{\mathbf{B}}. \tag{7.4}$$

The normalization constant  $N_{\lambda\rho}^m$  and the coefficients  $\bar{v}_{\lambda\rho}^{m\parallel}, \bar{v}_{\lambda\rho}^{m\perp}$  and  $\bar{v}_{\lambda\rho}^{m\prime}$  follow from the orthonormality relations in zeroth order:

$$\frac{1}{V} \langle [\bar{a}_{\lambda\rho}^{m(0)}(\mathbf{k})]^* a_{\mu\sigma}^{m(0)}(\mathbf{k}) \rangle^{(0)} = \delta_{\lambda\mu} \delta_{\rho\sigma} \tag{7.5}$$

with  $\lambda = \pm 1, \rho = \pm 1, \mu = \pm 1$  and  $\sigma = \pm 1$ . With the help of the fluctuation formulas [6] these orthonormality relations can be brought into the form

$$N_{\lambda\rho}^m \{ 1 + [\bar{v}_{\lambda\rho}^m(\mathbf{k})]^* \cdot \mathbf{v}_{\mu\sigma}^m(\mathbf{k}) \} = \delta_{\lambda\mu} \delta_{\rho\sigma}. \tag{7.6}$$

The coefficients  $\bar{v}_{\lambda\rho}^{m\parallel}, \bar{v}_{\lambda\rho}^{m\perp}$  and  $\bar{v}_{\lambda\rho}^{m\prime}$  may be obtained from this equality by choosing combinations of indices  $(\lambda\rho)$  that differ from  $(\mu\sigma)$ . In that case the orthonormality relations read

$$(\bar{v}_{\lambda\rho}^{m\parallel})^* v_{\mu\sigma}^{m\parallel} \hat{k}_{\parallel}^2 + (\bar{v}_{\lambda\rho}^{m\perp})^* v_{\mu\sigma}^{m\perp} \hat{k}_{\perp}^2 + (\bar{v}_{\lambda\rho}^{m\prime})^* v_{\mu\sigma}^{m\prime} \hat{k}_{\perp}^2 = -1, \quad (\lambda\rho) \neq (\mu\sigma). \tag{7.7}$$

Let us now consider two different solutions  $z_{\lambda\rho}^{m(0)}$  and  $z_{\mu\sigma}^{m(0)}$  of the quartic equation (4.3). An alternative form of the latter is

$$c - z_{\lambda\rho}^{m(0)} = -\omega_p v_{\lambda\rho}^{m\parallel} \hat{k}_{\parallel}^2 - (\omega_p + b) v_{\lambda\rho}^{m\perp} \hat{k}_{\perp}^2 + b' v_{\lambda\rho}^{m\perp} \hat{k}_{\perp}^2. \quad (7.8)$$

Subtraction of this equality from its analogue with  $(\lambda\rho) \rightarrow (\mu\sigma)$  and division by  $z_{\lambda\rho}^{m(0)} - z_{\mu\sigma}^{m(0)}$  yields the identity

$$v_{\lambda\rho}^{m\parallel} v_{\mu\sigma}^{m\parallel} \hat{k}_{\parallel}^2 + v_{\lambda\rho}^{m\perp} v_{\mu\sigma}^{m\perp} \hat{k}_{\perp}^2 - v_{\lambda\rho}^{m\perp} v_{\mu\sigma}^{m\perp} \hat{k}_{\perp}^2 = -1, \quad (\lambda\rho) \neq (\mu\sigma). \quad (7.9)$$

By comparison with (7.7) one infers that the coefficients  $\bar{v}_{\lambda\rho}^{m\parallel}$ ,  $\bar{v}_{\lambda\rho}^{m\perp}$  and  $\bar{v}_{\lambda\rho}^{m\perp}$  are given by

$$\bar{v}_{\lambda\rho}^{m\parallel} = (v_{\lambda\rho}^{m\parallel})^*, \quad \bar{v}_{\lambda\rho}^{m\perp} = (v_{\lambda\rho}^{m\perp})^*, \quad \bar{v}_{\lambda\rho}^{m\perp} = -(v_{\lambda\rho}^{m\perp})^*. \quad (7.10)$$

The normalization constant  $N_{\lambda\rho}^m$  follows from (7.6) for  $(\lambda\rho) = (\mu\sigma)$ . It has the same structure as (5.13).

The contribution of first order in the adjoint  $\bar{a}_{\lambda\rho}^{m[1]}(\mathbf{k})$  takes the general form

$$\bar{a}_{\lambda\rho}^{m[1]}(\mathbf{k}) = \beta (N_{\lambda\rho}^m)^* \left[ \bar{A}_{\lambda\rho,\varepsilon}^m \delta\varepsilon^m(\mathbf{k}) + \sum_{\sigma(\neq 1)} \bar{A}_{\lambda\rho,\sigma}^m \delta n_{\sigma}^m(\mathbf{k}) \right]. \quad (7.11)$$

The coefficients  $\bar{A}_{\lambda\rho,\varepsilon}^m$  and  $\bar{A}_{\lambda\rho,\sigma}^m$  can be obtained from the orthogonality relations

$$\frac{1}{V} \langle [\bar{a}_{\lambda\rho}^{m(1)}(\mathbf{k})]^* a_j^{m(1)}(\mathbf{k}) \rangle^{(1)} = 0 \quad (7.12)$$

with  $\lambda = \pm 1$ ,  $\rho = \pm 1$  and  $j = \varepsilon, \sigma(\neq 1)$ . Substituting (7.3), (7.11) and (6.20), (6.21) into (7.12) and employing the fluctuation formulas [6,14] we get a set of inhomogeneous linear equations for the unknown coefficients  $\bar{A}_{\lambda\rho,j}^m$ . Using the expressions (6.23) for the coefficients  $A_j^{m\parallel}$ ,  $A_j^{m\perp}$  and  $A_j^{m\perp}$  we obtain the solutions:

$$\bar{A}_{\lambda\rho,j}^m = (A_{\lambda\rho,j}^m)^* \quad (7.13)$$

with  $j = \varepsilon, \sigma(\neq 1)$ .

Having succeeded in deriving the adjoints up to first order in  $k$  we are ready to evaluate the terms of second order  $z_{\lambda\rho}^{m[2]}$  in the mode frequencies. In the first term of (7.1) we employ the fluctuation formulas and the thermodynamic relations of Appendix A up to second order in  $k$ . In particular, we use the Stillinger–Lovett relation for the ionic mixture, as given in formula (6.6) of Ref. [14], and the closely connected fluctuation formula involving the pressure and the charge density (see (7.9) of [14]). As a result we get from the first term of (7.1):

$$N_{\lambda\rho}^m z_{\lambda\rho}^{m(0)} \left( -\beta \sum_{jj'} A_{\lambda\rho,j}^m M_{jj'} A_{\lambda\rho,j'}^m - 2 \sum_{jj'} A_{\lambda\rho,j}^m M_{jj'} s_{j'} - 2 A_{\lambda\rho,\varepsilon}^m \frac{4\varepsilon}{3q_v} \right. \\ \left. - 2 \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma}^m \frac{n_{\sigma}}{q_v} - \frac{1}{\beta} \sum_{jj'} s_j M_{jj'} s_{j'} + \frac{4\varepsilon}{9q_v^2} + \frac{n}{2\beta q_v^2} \right). \quad (7.14)$$

A more useful form, which partly resembles (5.12), is found by employing (A.11) and the kinetic analogue of (4.22):

$$-N_{\lambda\rho}^m z_{\lambda\rho}^{m(0)} \left[ A_{\lambda\rho,\varepsilon}^m \left( F_{\lambda\rho,\varepsilon}^m + \frac{h_v}{q_v} \right) + \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma}^m \left( F_{\lambda\rho,\sigma}^m + \frac{n_{\sigma}}{q_v} \right) + \frac{1}{\beta} \left( \sum_j F_{\lambda\rho,j}^m W_j + \frac{n}{2\beta q_v} W_{\varepsilon} - \frac{1}{2q_v e_1} \right) \right]. \quad (7.15)$$

In the second line we recognize the kinetic counterpart  $D_{\lambda\rho,q}^{m\perp}$  of (5.14).

In the second term of (7.1) we insert the laws (6.3) and (6.4). Using the fluctuation formulas once again, we find

$$2N_{\lambda\rho}^m \omega_p \left( \frac{h_v}{q_v} A_{\lambda\rho,\varepsilon}^m + \sum_{\sigma(\neq 1)} \frac{n_{\sigma}}{q_v} A_{\lambda\rho,\sigma}^m \right) \hat{\mathbf{k}} \cdot \mathbf{v}_{\lambda\rho}^m. \quad (7.16)$$



Finally, we have to evaluate the last term of (7.1). Inserting (6.14), (6.15), (7.3) and (7.11), we obtain a sum of several contributions. The first of these is obtained upon substituting the zeroth-order mode amplitudes (6.14) and (7.3) in (7.1). Using (6.3)–(6.4) and the identities  $Q\delta q_v^m(\mathbf{k}) = 0$  and  $Qg^m(\mathbf{k}) = 0$ , we get

$$\begin{aligned} & \frac{i}{T} N_{\lambda\rho}^m (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho}^m \wedge \hat{\mathbf{B}}) \cdot (\hat{\mathbf{k}} \cdot \mathbf{L}^{m(c)} \cdot \hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}} + i\gamma \bar{\mathbf{v}}_{\lambda\rho}^{m*} \wedge \hat{\mathbf{B}}) \\ & + \frac{1}{T\sqrt{m_v}} N_{\lambda\rho}^m (\hat{\mathbf{k}} - i\gamma \mathbf{v}_{\lambda\rho}^m \wedge \hat{\mathbf{B}}) \cdot (\hat{\mathbf{k}} \cdot \mathbf{L}_v^{m(c)} \cdot \hat{\mathbf{k}}) \cdot \bar{\mathbf{v}}_{\lambda\rho}^{m*} \\ & + \frac{1}{T\sqrt{m_v}} N_{\lambda\rho}^m \mathbf{v}_{\lambda\rho}^m \cdot (\hat{\mathbf{k}} \cdot \mathbf{L}_v^{m(c)} \cdot \hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}} + i\gamma \bar{\mathbf{v}}_{\lambda\rho}^{m*} \wedge \hat{\mathbf{B}}) \\ & - \frac{i}{m_v} N_{\lambda\rho}^m \mathbf{v}_{\lambda\rho}^m \cdot (\hat{\mathbf{k}} \cdot \boldsymbol{\eta}^m \cdot \hat{\mathbf{k}}) \cdot \bar{\mathbf{v}}_{\lambda\rho}^{m*}. \end{aligned} \quad (7.17)$$

Here we introduced the microscopic equivalents of the transport quantities  $\mathbf{L}_{\sigma\sigma'}^{(c)}$ ,  $\mathbf{L}_{\sigma v}^{(c)}$ ,  $\mathbf{L}_{v\sigma}^{(c)}$  ( $\sigma, \sigma' \neq 1$ ) and  $\boldsymbol{\eta}$ . They are defined as

$$\frac{1}{V} \left\langle \left[ (g_\sigma^m(\mathbf{k}))^j \right]^* Q \frac{1}{z + Q\mathcal{L}Q} Q (g_\sigma^m(\mathbf{k}))^i \right\rangle^{[2]} = ik_B (\hat{\mathbf{k}} \cdot \mathbf{L}_{\sigma\sigma'}^{m(c)}(z) \cdot \hat{\mathbf{k}})^{ij}, \quad (7.18)$$

$$\frac{1}{V} \left\langle \left[ (\hat{\mathbf{k}} \cdot \delta\boldsymbol{\tau}^m(\mathbf{k}))^j \right]^* Q \frac{1}{z + Q\mathcal{L}Q} Q (g_\sigma^m(\mathbf{k}))^i \right\rangle^{[1]} = k_B (\hat{\mathbf{k}} \cdot \mathbf{L}_{\sigma v}^{m(c)}(z) \cdot \hat{\mathbf{k}})^{ij}, \quad (7.19)$$

$$\frac{1}{V} \left\langle \left[ (g_\sigma^m(\mathbf{k}))^j \right]^* Q \frac{1}{z + Q\mathcal{L}Q} Q (\hat{\mathbf{k}} \cdot \delta\boldsymbol{\tau}^m(\mathbf{k}))^i \right\rangle^{[1]} = k_B (\hat{\mathbf{k}} \cdot \mathbf{L}_{v\sigma}^{m(c)}(z) \cdot \hat{\mathbf{k}})^{ij}, \quad (7.20)$$

$$\frac{\beta}{V} \left\langle \left[ (\hat{\mathbf{k}} \cdot \delta\boldsymbol{\tau}^m(\mathbf{k}))^j \right]^* Q \frac{1}{z + Q\mathcal{L}Q} Q (\hat{\mathbf{k}} \cdot \delta\boldsymbol{\tau}^m(\mathbf{k}))^i \right\rangle^{(0)} = -i (\hat{\mathbf{k}} \cdot \delta\boldsymbol{\eta}^m(z) \cdot \hat{\mathbf{k}})^{ij}, \quad (7.21)$$

with  $i, j = 1, 2, 3$ . In (7.17) we employed reduced transport quantities that are defined in a way analogous to (3.6), (3.7) and (3.9). All these quantities are to be taken at  $z = z_{\lambda\rho}^{m(0)}$ . The contractions of the vectors  $\hat{\mathbf{k}}$  in (7.17)–(7.21) are to be taken with respect to the two inner indices of the fourth-rank tensors; the right-hand side of (7.18), for instance, equals  $ik_B k^p (\mathbf{L}_{\sigma\sigma'}^{m(c)}(z))^{ipqj} k^q$ . The expression (7.17) can be rewritten by inserting (7.4) with (7.10). In this way we get

$$iN_{\lambda\rho}^m (D_{\lambda\rho,q}^{m''} + v_{\lambda\rho}^{m\parallel} \hat{k}_\parallel^2 D_{\lambda\rho}^{m\parallel} + v_{\lambda\rho}^{m\perp} \hat{k}_\perp^2 D_{\lambda\rho}^{m\perp} - v_{\lambda\rho}^{m\parallel} \hat{k}_\perp^2 D_{\lambda\rho}^{m\parallel}), \quad (7.22)$$

where the coefficients  $D_{\lambda\rho,q}^{m''}$ ,  $D_{\lambda\rho}^{m\parallel}$ ,  $D_{\lambda\rho}^{m\perp}$  and  $D_{\lambda\rho}^{m\parallel}$  have the same form as their magnetohydrodynamical counterparts (5.15)–(5.18).

Further contributions from the last term of (7.1) arise by inserting either (6.14) and (7.11) or (6.15) and (7.3). Employing (6.11)–(6.13) and (6.16)–(6.19) we get

$$\begin{aligned} & 2N_{\lambda\rho}^m A_{\lambda\rho,\varepsilon}^m \left\{ c_\varepsilon + \frac{h_v}{q_v} c + \mathbf{v}_{\lambda\rho}^m \cdot \left[ \left( b_\varepsilon + \frac{h_v}{q_v} b \right) \hat{\mathbf{k}}_\perp - \left( b'_\varepsilon + \frac{h_v}{q_v} b' \right) \hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}} \right] \right\} \\ & + 2N_{\lambda\rho}^m \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma}^m \left\{ c_\sigma + \frac{n_\sigma}{q_v} c + \mathbf{v}_{\lambda\rho}^m \cdot \left[ \left( b_\sigma + \frac{n_\sigma}{q_v} b \right) \hat{\mathbf{k}}_\perp - \left( b'_\sigma + \frac{n_\sigma}{q_v} b' \right) \hat{\mathbf{k}}_\perp \wedge \hat{\mathbf{B}} \right] \right\}, \end{aligned} \quad (7.23)$$

where (7.10) and (7.13) have been used. As before, all quantities  $b$ ,  $b'$ ,  $c$ ,  $b_j$ ,  $b'_j$  and  $c_j$  are taken for  $z = z_{\lambda\rho}^{m(0)}$ . By means of the kinetic analogue of (4.23) we may write  $c_j$  in terms of  $F_{\lambda\rho,j}^m$ . Furthermore, the quartic

equation (4.3) can be used to eliminate  $c$ . As a result, (7.23) gets the form

$$2N_{\lambda\rho}^m A_{\lambda\rho,\varepsilon}^m \left[ z_{\lambda\rho}^{m(0)} \left( F_{\lambda\rho,\varepsilon}^m + \frac{h_v}{q_v} \right) - \omega_p \frac{h_v}{q_v} \hat{\mathbf{k}} \cdot \mathbf{v}_{\lambda\rho}^m \right] + 2N_{\lambda\rho}^m \sum_{\sigma(\neq 1)} A_{\lambda\rho,\varepsilon}^m \left[ z_{\lambda\rho}^{m(0)} \left( F_{\lambda\rho,\sigma}^m + \frac{n_\sigma}{q_v} \right) - \omega_p \frac{n_\sigma}{q_v} \hat{\mathbf{k}} \cdot \mathbf{v}_{\lambda\rho}^m \right]. \quad (7.24)$$

By adding (7.15), (7.16), (7.22) and (7.24) we find an expression for the second-order frequencies that is very similar to (5.12)

$$z_{\lambda\rho}^{m[2]} = N_{\lambda\rho}^m \left\{ z_{\lambda\rho}^{m(0)} \left[ A_{\lambda\rho,\varepsilon}^m \left( F_{\lambda\rho,\varepsilon}^m + \frac{h_v}{q_v} \right) + \sum_{\sigma(\neq 1)} A_{\lambda\rho,\sigma}^m \left( F_{\lambda\rho,\sigma}^m + \frac{n_\sigma}{q_v} \right) \right] + iD_{\lambda\rho,q}^m + i v_{\lambda\rho}^{m\parallel} \hat{k}_\parallel^2 D_{\lambda\rho}^{m\parallel} + i v_{\lambda\rho}^{m\perp} \hat{k}_\perp^2 D_{\lambda\rho}^{m\perp} - i v_{\lambda\rho}^{m\parallel} \hat{k}_\perp^2 D_{\lambda\rho}^{m\parallel} \right\}. \quad (7.25)$$

Note that in this kinetic derivation the coefficients  $A_{\lambda\rho,j}^m$  and  $F_{\lambda\rho,j}^m$  take the role of their magnetohydrodynamical counterparts  $A_{\lambda\rho,j}$  and  $F_{\lambda\rho,j}$  (with  $j = \varepsilon, \sigma(\neq 1)$ ). Likewise, the coefficients  $D_{\lambda\rho,q}^m$ ,  $D_{\lambda\rho}^{m\parallel}$ ,  $D_{\lambda\rho}^{m\perp}$  and  $D_{\lambda\rho}^{m\parallel}$  replace their magnetohydrodynamical equivalents (5.14)–(5.18). Upon substituting the kinetic analogues of (4.22) in (7.25) we finally arrive at the kinetic expression for the second-order contributions to the eigenfrequencies of the oscillating modes:

$$z_{\lambda\rho}^{m[2]} = N_{\lambda\rho}^m \left\{ z_{\lambda\rho}^{m(0)} \left[ \frac{c_s^2}{\omega_p^2} + \frac{2}{\beta} \sum_j \left( \frac{n}{2\beta q_v} M_{ej}^{-1} - s_j \right) F_{\lambda\rho,j}^m + \frac{1}{\beta} \sum_{j,j'} F_{\lambda\rho,j}^m M_{jj'}^{-1} F_{\lambda\rho,j'}^m \right] + iD_{\lambda\rho,q}^{m''} + i v_{\lambda\rho}^{m\parallel} \hat{k}_\parallel^2 D_{\lambda\rho}^{m\parallel} + i v_{\lambda\rho}^{m\perp} \hat{k}_\perp^2 D_{\lambda\rho}^{m\perp} - i v_{\lambda\rho}^{m\parallel} \hat{k}_\perp^2 D_{\lambda\rho}^{m\parallel} \right\}. \quad (7.26)$$

This expression has the same form as (5.19). Hence, the kinetic approach corroborates the magnetohydrodynamical results for the damping and the dispersion of the oscillating modes, at least up to second order in the wavelength.

## 8. Conclusion

In this paper we have shown how the spectrum of collective modes of an ionic mixture with an arbitrary number of species can be analyzed by means of magnetohydrodynamical and kinetic methods. We have derived explicit expressions for the eigenfrequencies of the oscillating modes up to second order. Since the second-order contributions to the frequencies determine the damping and the dispersion of the collective modes, our results give useful information on the dynamical behaviour of ionic mixtures. As we have seen, the damping and dispersion phenomena are governed by dissipative mechanisms that go beyond those of standard non-equilibrium thermodynamics. In fact, the well-known phenomenological laws for the diffusive flows and the viscous pressure are not sufficient to account for the full effects of dispersion and damping of the oscillating modes. Whereas the dissipation implied by these standard laws yields the damping and dispersion of the heat mode and the diffusion modes, as shown in our previous paper [7], we had to generalize the standard laws by including higher-order Burnett terms in order to get a complete picture of the damping and the dispersion of the oscillating modes up to second order in the wave number.

Burnett terms have been discussed previously in the context of systems of neutral particles [8–12]. For such systems the Burnett terms generally lead to small effects in the dynamical behaviour, so that they can usually be neglected. In the present case of an ionic mixture this is no longer true: in a systematic treatment of the collective modes the Burnett terms cannot be disregarded. The technical reason for the necessity to incorporate Burnett terms when investigating the dynamics of an ionic mixture turns out to be the long-range nature of the Coulomb interactions. In an ionic mixture the long-range electric fields generated by charge fluctuations are driving agents in the thermodynamical forces. These electric fields lead to terms of zeroth

order in the wave number after Fourier transformation. To be consistent up to second order in the wave number one has to include terms with gradients of the electric field in the thermodynamic forces. Hence, the phenomenological laws, which connect these thermodynamical forces to the thermodynamic flows, must be used in a form which accounts for such field gradient terms. We have to conclude that Burnett terms cannot be neglected in systems with long-range forces.

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### Appendix A. Thermodynamics for an ionic mixture

An ionic mixture satisfies the global constraint of charge neutrality. Therefore, it is convenient to introduce instead of  $\mu_\sigma$  a set of new chemical potentials:

$$\tilde{\mu}_\sigma = \mu_\sigma - \frac{e_\sigma}{e_1} \mu_1 \quad (\text{A.1})$$

for  $\sigma \neq 1$ . As shown in Ref. [15], these potentials arise naturally as Lagrange multipliers in a grand-canonical ensemble with a charge-neutrality constraint. The combination of perturbations of the chemical potentials  $\mu_\sigma$  occurring in the conservation laws and the balance equations of Section 3 can be rewritten in terms of these new chemical potentials as [7]

$$\begin{aligned} \left( \frac{\delta \mu_\sigma}{m_\sigma} - \frac{\delta \mu_1}{m_1} \right)_T &= -T \left[ \frac{h_\sigma}{m_\sigma} - \frac{h_1}{m_1} - \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \frac{h_v}{q_v} \right] \delta \left( \frac{1}{T} \right) \\ &+ T \sum_{\sigma' (\neq 1)} \left[ \frac{1}{m_{\sigma'}} \delta_{\sigma\sigma'} - \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \frac{n_{\sigma'}}{q_v} \right] \delta \left( \frac{\tilde{\mu}_{\sigma'}}{T} \right) + \left( \frac{e_\sigma}{m_\sigma} - \frac{e_1}{m_1} \right) \frac{\delta p}{q_v} \end{aligned} \quad (\text{A.2})$$

for  $\sigma \neq 1$ .

The perturbations  $\delta(1/T)$  and  $\delta(\tilde{\mu}_{\sigma'}/T)$  (with  $\sigma' \neq 1$ ) in (A.2) have to be represented as linear combinations of the basic perturbations  $\delta\varepsilon$ ,  $\delta n_\sigma$  and  $\delta q_v$ . This can be achieved by starting from the identity [7]

$$\begin{pmatrix} \delta\varepsilon \\ \delta n_\sigma \end{pmatrix} = \mathbf{M} \begin{pmatrix} -\delta\beta \\ \delta(\beta\tilde{\mu}_\sigma) \end{pmatrix} + \mathbf{V} \delta q_v \quad (\text{A.3})$$

with the matrix:

$$\mathbf{M} = \begin{pmatrix} -\partial\varepsilon/\partial\beta & -\partial n_{\sigma'}/\partial\beta \\ -\partial n_\sigma/\partial\beta & \partial n_\sigma/\partial(\beta\tilde{\mu}_{\sigma'}) \end{pmatrix} \quad (\text{A.4})$$

and the vector

$$\mathbf{V} = \begin{pmatrix} \partial\varepsilon/\partial q_v \\ \partial n_\sigma/\partial q_v \end{pmatrix}. \quad (\text{A.5})$$

All partial derivatives are defined in terms of the independent variables  $\beta = 1/(k_B T)$ ,  $\beta\tilde{\mu}_\sigma$  (with  $\sigma \neq 1$ ) and  $q_v$ . We used the Maxwell relation  $\partial\varepsilon/\partial(\beta\tilde{\mu}_\sigma) = -\partial n_\sigma/\partial\beta$ .

The elements of  $\mathbf{V}$  can be evaluated by substituting the equation of state  $p = \frac{1}{3}\varepsilon + \frac{1}{2}nk_B T$  into the formulas (3.22) and (3.18) of Ref. [15], and replacing  $n$  by  $-\sum_{\sigma(\neq 1)}(e_\sigma/e_1 - 1)n_\sigma + q_v/e_1$ . Treating  $\beta$ ,  $\beta\tilde{\mu}_\sigma$  ( $\sigma \neq 1$ ) and  $q_v$  as independent variables, we get

$$\frac{\partial\varepsilon}{\partial q_v} = M_{\varepsilon\varepsilon} s_\varepsilon + \sum_{\sigma(\neq 1)} M_{\varepsilon\sigma} s_\sigma + \frac{4\varepsilon}{3q_v}, \quad (\text{A.6})$$

$$\frac{\partial n_\sigma}{\partial q_v} = M_{\sigma\varepsilon} s_\varepsilon + \sum_{\sigma'(\neq 1)} M_{\sigma\sigma'} s_{\sigma'} + \frac{n_\sigma}{q_v} \quad (\text{A.7})$$

with  $\sigma \neq 1$ , and with the abbreviations:

$$s_\varepsilon = -\frac{\beta}{3q_v}, \quad s_\sigma = \frac{1}{2q_v} \left( \frac{e_\sigma}{e_1} - 1 \right). \quad (\text{A.8})$$

Inverting (A.3) and substituting (A.6) and (A.7) we finally get

$$-\delta\beta = M_{\varepsilon\varepsilon}^{-1} \delta\varepsilon + \sum_{\sigma(\neq 1)} M_{\varepsilon\sigma}^{-1} \delta n_\sigma - k W_\varepsilon \frac{\delta q_v}{k}, \quad (\text{A.9})$$

$$\delta(\beta \tilde{\mu}_\sigma) = M_{\sigma\varepsilon}^{-1} \delta\varepsilon + \sum_{\sigma'(\neq 1)} M_{\sigma\sigma'}^{-1} \delta n_{\sigma'} - k W_\sigma \frac{\delta q_v}{k} \quad (\text{A.10})$$

with  $W_j$  defined as

$$W_j = \sum_j M_{jj}^{-1} V_j = s_j + M_{j\varepsilon}^{-1} \frac{4\varepsilon}{3q_v} + \sum_{\sigma'(\neq 1)} M_{j\sigma'}^{-1} \frac{n_{\sigma'}}{q_v} \quad (\text{A.11})$$

for  $j$  and  $j'$  equal to  $\varepsilon$  or  $\sigma(\neq 1)$ .

The variation in the pressure  $\delta p$  can likewise be expressed in terms of variations  $\delta\varepsilon$ ,  $\delta n_\sigma$  and  $\delta q_v/k$ . In fact, the perturbation of the pressure follows from the equation of state as

$$\delta p = \frac{1}{3} \delta\varepsilon - \frac{1}{2\beta} \sum_{\sigma(\neq 1)} \left( \frac{e_\sigma}{e_1} - 1 \right) \delta n_\sigma - \frac{n}{2\beta^2} \delta\beta + \frac{1}{2e_1\beta} \delta q_v. \quad (\text{A.12})$$

Upon inserting expression (A.9) for  $\delta\beta$  we obtain

$$\beta \frac{\delta p}{q_v} = \left( \frac{n}{2\beta q_v} M_{\varepsilon\varepsilon}^{-1} - s_\varepsilon \right) \delta\varepsilon + \sum_{\sigma(\neq 1)} \left( \frac{n}{2\beta q_v} M_{\varepsilon\sigma}^{-1} - s_\sigma \right) \delta n_\sigma + k \left( \frac{1}{2q_v e_1} - \frac{n}{2\beta q_v} W_\varepsilon \right) \frac{\delta q_v}{k}. \quad (\text{A.13})$$

The inverse matrix element  $M_{\varepsilon\varepsilon}^{-1}$  is directly related to the specific heat  $c_v$  per particle:

$$c_v = T \left( \frac{\partial s}{\partial T} \right)_{q_v, \{n_\sigma/q_v\}} = \frac{\beta^2}{n M_{\varepsilon\varepsilon}^{-1}} k_B \quad (\text{A.14})$$

with  $s$  the entropy per particle. Likewise, one may derive that the specific heat  $c_p$  per particle at constant pressure is given as

$$c_p = T \left( \frac{\partial s}{\partial T} \right)_{p, \{n_\sigma/q_v\}} = \frac{9n^2 M_{\varepsilon\varepsilon}^{-1} + 2\beta^2 (8\beta\varepsilon + 15n)}{2n(8\beta\varepsilon + 9n) M_{\varepsilon\varepsilon}^{-1} - 4n\beta^2} k_B. \quad (\text{A.15})$$

Furthermore, the isothermal compressibility  $\kappa_T$  can be introduced by any of the following equivalent definitions:

$$\kappa_T = \frac{1}{q_v} \left( \frac{\partial q_v}{\partial p} \right)_{T, \{n_\sigma/q_v\}} = \frac{1}{n} \left( \frac{\partial n}{\partial p} \right)_{T, \{n_\sigma/q_v\}} = \frac{1}{m_v} \left( \frac{\partial m_v}{\partial p} \right)_{T, \{n_\sigma/q_v\}}, \quad (\text{A.16})$$

where use has been made of the identity  $dn_\sigma = (n_\sigma/q_v) dq_v$  at fixed concentrations  $n_\sigma/n$  (or fixed  $n_\sigma/q_v$ ). In terms of the inverse matrix element  $M_{\varepsilon\varepsilon}^{-1}$  the compressibility reads

$$\kappa_T = \frac{18\beta M_{\varepsilon\varepsilon}^{-1}}{(8\beta\varepsilon + 9n) M_{\varepsilon\varepsilon}^{-1} - 2\beta^2}. \quad (\text{A.17})$$

Combining these results one finds the sound velocity as

$$c_s^2 = \frac{c_p}{m_v c_v \kappa_T} = \frac{n^2}{4\beta^3 m_v} M_{\varepsilon\varepsilon}^{-1} + \frac{4\varepsilon}{9m_v} + \frac{5n}{6\beta m_v}. \quad (\text{A.18})$$

Alternatively, the right-hand side may be written in terms of the enthalpy per volume  $h_v = \frac{4}{3}\varepsilon + \frac{1}{2}n/\beta$ .

### Appendix B. The eigenvalue problem in kinetic theory

In the main text it has been stated that the frequencies  $z_i^m(\mathbf{k})$  and their corresponding modes  $a_i^m(\mathbf{k})$  follow from the eigenvalue problem (6.9) in arbitrary order of  $k$ . However, for our purposes it is more convenient to introduce adjoint modes  $\bar{a}_i^m(\mathbf{k})$ . These adjoints are particular independent linear combinations of the basis functions  $a_i^b(\mathbf{k})$ . The adjoints follow from the orthonormality condition

$$\frac{1}{V} \langle [\bar{a}_i^m(\mathbf{k})]^* a_j^m(\mathbf{k}) \rangle = \delta_{ij}. \quad (\text{B.1})$$

With the use of these adjoints, the eigenvalue problem (6.9) up to second order in the wave number may be written as

$$z_i^{m(2)} \delta_{ij} = -\frac{1}{V} \langle [\bar{a}_i^{m(2)}(\mathbf{k})]^* \mathcal{L} a_j^{m(2)}(\mathbf{k}) \rangle^{(2)} + \frac{1}{V} \left\langle [\bar{a}_i^{m(2)}(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^{m(2)} + Q \mathcal{L} Q} Q \mathcal{L} a_j^{m(2)}(\mathbf{k}) \right\rangle^{(2)}. \quad (\text{B.2})$$

In this expression we have made explicit, through the notation between round brackets in the superscripts, that the thermal averages  $\langle \dots \rangle$  have to be evaluated up to second order in the wave number. Contrary to what one might expect from the above expression, the modes and their adjoints up to *first* order in  $k$  are sufficient to obtain the frequencies up to *second* order. This will be demonstrated below.

Let us expand the second-order modes  $a_i^{m(2)}(\mathbf{k})$  in terms of their first-order counterparts  $a_i^{m(1)}(\mathbf{k})$ . Formally this expansion can be written as

$$a_i^{m(2)}(\mathbf{k}) = \sum_j C_{ij} a_j^{m(1)}(\mathbf{k}), \quad (\text{B.3})$$

where the coefficients  $C_{ij}$  have the general form

$$C_{ij} = \delta_{ij} + k^2 C_{ij}^{[2]}. \quad (\text{B.4})$$

In the same way we can expand the adjoints:

$$\bar{a}_i^{m(2)}(\mathbf{k}) = \sum_j D_{ij} \bar{a}_j^{m(1)}(\mathbf{k}) \quad (\text{B.5})$$

with

$$D_{ij} = \delta_{ij} + k^2 D_{ij}^{[2]}. \quad (\text{B.6})$$

The modes  $a_i^{m(1)}(\mathbf{k})$  and their adjoints are orthogonal up to first order in the wave number, so that one has

$$\frac{1}{V} \langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* a_j^{m(1)}(\mathbf{k}) \rangle^{(2)} = \delta_{ij} + k^2 F_{ij}^{[2]}. \quad (\text{B.7})$$

From the orthogonality of the modes  $a_i^{m(2)}(\mathbf{k})$  and their adjoints in second order it then follows that the coefficients  $F_{ij}^{[2]}$  obey the relation

$$F_{ij}^{[2]} = -(D_{ij}^{[2]})^* - C_{ji}^{[2]}. \quad (\text{B.8})$$

Substitution of the expansions (B.3) and (B.5) into the eigenvalue equation (B.2) yields up to second order in  $k$ :

$$z_i^{m(2)} \delta_{ij} = \sum_{k,l} D_{ik}^* C_{jl} \left\{ -\frac{1}{V} \langle [\bar{a}_k^{m(1)}(\mathbf{k})]^* \mathcal{L} a_l^{m(1)}(\mathbf{k}) \rangle^{(2)} + \frac{1}{V} \left\langle [\bar{a}_k^{m(1)}(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^{m(2)} + Q \mathcal{L} Q} Q \mathcal{L} a_l^{m(1)}(\mathbf{k}) \right\rangle^{(2)} \right\}. \quad (\text{B.9})$$

Upon approximating the frequencies  $z_i^{m(2)}$  by  $z_i^{m(0)}$  in the denominator of the second term at the right-hand side, we may write the expression between curly brackets as

$$z_i^{m(1)} \delta_{kl} + k^2 G_{kl}^{[2]} \equiv -\frac{1}{V} \langle [\bar{a}_k^{m(1)}(\mathbf{k})]^* \mathcal{L} a_l^{m(1)}(\mathbf{k}) \rangle^{(2)} + \frac{1}{V} \left\langle [\bar{a}_k^{m(1)}(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^{m(0)} + Q \mathcal{L} Q} Q \mathcal{L} a_l^{m(1)}(\mathbf{k}) \right\rangle^{(2)} \quad (\text{B.10})$$

which defines  $G_{kl}^{[2]}$ . Inserting (B.10) in (B.9) and using relations (B.4), (B.6) and (B.8), we obtain for  $i = j$ :

$$z_i^{m[2]} = -z_i^{m(0)} F_{ii}^{[2]} + G_{ii}^{[2]} \quad (\text{B.11})$$

or, with (B.7) and again (B.10)

$$z_i^{m[2]} = -z_i^{m(0)} \frac{1}{V} \langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* a_i^{m(1)}(\mathbf{k}) \rangle^{[2]} - \frac{1}{V} \langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* \mathcal{L} a_i^{m(1)}(\mathbf{k}) \rangle^{[2]} + \frac{1}{V} \left\langle [\bar{a}_i^{m(1)}(\mathbf{k})]^* \mathcal{L} Q \frac{1}{z_i^{m(0)} + Q \mathcal{L} Q} Q \mathcal{L} a_i^{m(1)}(\mathbf{k}) \right\rangle^{[2]}. \quad (\text{B.12})$$

From this expression for the eigenvalue problem it is obvious that the modes and their adjoints up to first order in the wave number are sufficient to determine the second-order contributions to the frequencies.

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