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Improving the Performance of CUSUM Charts

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The control chart is an important statistical technique that is used to monitor the quality of a process. Shewhart control charts are used to detect larger disturbances in the process parameters, whereas CUSUM and EWMA charts are meant for smaller and moderate changes. Runs rules schemes are generally used to enhance the performance of Shewhart control charts. In this study, we propose two runs rules schemes for the CUSUM charts. The performance of these two schemes is compared with the usual CUSUM, the weighted CUSUM, the fast initial response CUSUM and the usual EWMA schemes. The comparisons revealed that the proposed schemes perform better for small and moderate shifts, whereas they reasonably maintain their efficiency for large shifts as well. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: average run length (*ARL*); control charts; EWMA; runs rules schemes; in-control; out-of-control

1. Introduction

Production processes are subject to variations. These variations are mainly classified into two types, namely common cause variation and special cause variation (cf. Nolan and Provost[1]). Common cause variation always exists even if the process is designed very well and maintained very carefully. This variation is relatively small in magnitude, uncontrollable and due to many small unavoidable causes. A process is said to be in statistical control if only common cause variation is present. This variation is inherent to the process. If there exist other sources of variation that are not part of common causes, then the process is called out-of-control. This additional variation may come from one or more special (or assignable) causes associated with the machines, the operators, the materials, etc.

Statistical process control (SPC) is a collection of powerful tools that are useful in maintaining and improving process performance through the reduction of variability (cf. Does *et al.*[2]). This is done by collecting, organizing, analyzing and interpreting data so that the process can be maintained at its present level or improved to a higher level of quality. SPC is a strategy that can be applied to any process to reduce variation and it contains tools like histograms, check sheets, Pareto charts, cause and effect diagrams, defect concentration diagrams, scatter diagrams and control charts. This collection is formally known as the SPC tool-kit. The control chart is the most important device that helps to determine if a process is in-control.

The design of control charts is based on first identifying the distribution of the process characteristics followed by monitoring the stability of its parameters. In general, a control chart is a trend chart with three additional lines: the center line (*CL*), the upper control limit (*UCL*) and the lower control limit (*LCL*). These limits are chosen such that almost all of the data will lie between these limits as long as the process remains statistically in-control. In this paper, we study the control chart for the location parameter.

The performance of a control chart is generally measured in terms of the power and the average run length (*ARL*). The power of a control chart is defined as the probability of detecting an out-of-control signal, whereas the *ARL* represents the average number of samples required to signal an out-of-control situation in the process. There are two ways to obtain the *ARL* in a given situation. In case the distributional parameters are known, the run length distribution turns out to geometric and we may use $ARL = 1/q$ (where q represents the probability of exceeding the control limits). Alternatively, we may calculate the *ARL* using the true run length distribution in a given case. The in-control and out-of-control *ARL's* are denoted by ARL_0 and ARL_1, respectively.

There are three major categories of charts to monitor processes, namely Shewhart-type control charts, cumulative sum (CUSUM) charts and exponentially weighted moving average (EWMA) charts. The Shewhart-type control charts are quite good in detecting large shifts, whereas CUSUM and EWMA charts are effective for smaller shifts in the parameters of interest (generally location and spread parameters). The concern of this study is the performance of the CUSUM chart for the location parameter based on individual observations.

The CUSUM charts have the ability to address small shifts quite efficiently. In order to further enhance this ability, this study proposes two alternative schemes for CUSUM charts which help to minimize the ARL_1 for fixed ARL_0. This is done by exploiting

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runs rules schemes with the help of warning and action limits for CUSUM charts. An algorithm is developed to calculate the values of the *ARLs* for a given pair of warning and action limits. Using this algorithm, different pairs of warning and *ALs* are found that fix the ARL_0 at a desired level. Then out of these selected pairs, an optimal pair of warning and action limit is selected, which minimizes the value of the ARL_1.

2. CUSUM charts

This section provides a brief introduction of CUSUM charts and proposes two new schemes for the CUSUM charts.

2.1. The usual CUSUM scheme

CUSUM charts were first introduced by Page[3] to effectively address small parameter shifts. Shewhart-type control charts (e.g. the well-known $\bar{X}$-chart) efficiently detect large shifts, whereas for smaller shifts CUSUM and EWMA procedures are of more importance. The CUSUM procedures work by accumulating the deviations up and down from a target value (in general the mean of the in-control situation) for which we use the notations C^+ and C^-, respectively. Initially, C^+ and C^- are set to zero. The quantities C^+ and C^- are known as upper and lower CUSUM statistics, respectively, and these are defined as

$$\left.\begin{aligned} C_i^+ &= \max[0, X_i-(\mu_0+k)+C_{i-1}^+] \\ C_i^- &= \max[0, -X_i+(\mu_0-k)+C_{i-1}^-] \end{aligned}\right\} \tag{1}$$

where X_i denote the ith observation, μ_0 is the target value (mean) and k is known as the reference value which is chosen about half of the shift (in standard units) we want to detect; i.e. $k=\delta/2$, where δ equals the shift in standard units. Once the values of C^+ and C^- are calculated for each sample, then these values are plotted against the control limit h, where h is usually taken as 4 or 5 times the standard deviation of the process depending upon the pre-specified ARL_0. The details regarding the choice of h for different ARL_0s are available in the standard quality control literature, see e.g. Alwan[4].

An alternative way is to use the standardized CUSUM in which we replace X by $Z=(X-\mu_0)/\sigma_0$ (where μ_0 and σ_0 are the mean and standard deviation, respectively, of the in-control situation) in case of individual observations. The resulting standardized version of the CUSUM scheme is given as

$$\left.\begin{aligned} C_i^+ &= \max[0, Z_i-k+C_{i-1}^+] \\ C_i^- &= \max[0, -Z_i-k+C_{i-1}^-] \end{aligned}\right\} \tag{2}$$

and h is taken as simply 4 or 5 depending on the ARL_0. From now on, we shall use the standardized CUSUM version in our study. After deciding the value of h, the CUSUM chart may indicate an out-of-control signal when one or more points fall beyond the control limits.

The CUSUM chart structures discussed in this section so far and formally given in (1) and (2) are known as the usual CUSUM charts. Table I gives the *ARLs* for the two-sided CUSUM scheme with $k=0.5$ and $h=4$ or $h=5$ (cf. Alwan[4] and Lucas and Crosier[5]).

The results of Table I are based on the test that one point falling outside the limits indicates an out-of-control situation (the usual scheme of signaling). This test may be further extended to a set of rules named as sensitizing rules and runs rules schemes, which help to increase the sensitivity of the charts to detect out-of-control situations. The common set of sensitizing rules are (cf. Nelson[6]): 1 or more points outside the control limits; 2 out of 3 consecutive points outside the two sigma warning limits (*WLs*) but still inside the control limits; 4 out of 5 consecutive points beyond the one sigma limits but still inside the control limits; a run of 8 consecutive points on one side of the center line but still inside the control limits; 6 points in a row steadily increasing or decreasing but still inside the control limits; 14 points in a row alternating up and down but still inside the control limits. The basic principle underlying these runs rules is twofold. First, specific patterns of out-of-control conditions might be detected earlier, such as a small but persistent trend. Second, the decision rules are designed to have roughly the same (marginal) false alarm probability.

To enhance the performance of control charts, many researchers have used the idea of using different sensitizing rules and runs rules schemes with the Shewhart-type control charts, e.g. see Klein[7], Khoo[8], Koutras *et al.*[9] and Antzoulakos and Rakitzis[10]. The application of sensitizing rules causes an increase in false alarm rates, whereas the runs rules schemes take care of this issue. Klein[7], Khoo[8] and Antzoulakos and Rakitzis[10] suggested different runs rules schemes, namely r out of m or modified r out of m, to be used with the Shewhart-type control charts. They studied their performance and found that these r out of m runs rules schemes perform better as compared with the usual Shewhart-type control charts.

Table I. *ARL* values for the usual CUSUM scheme with $k=0.5$

δ	0	0.25	0.5	0.75	1	1.5	2	2.5	3
$h=4$	168	74.2	26.6	13.3	8.38	4.75	3.34	2.62	2.19
$h=5$	465	139	38.0	17.0	10.4	5.75	4.01	3.11	2.57

There is a variety of literature available on CUSUM charts, see e.g. Hawkins and Olwell[11] and the references therein. The existing approaches for the CUSUM charts use only the usual scheme of signaling an out-of-control situation. It is hard to find the application of the runs rules schemes with the CUSUM charts in the literature. However, Westgard *et al.*[12] studied some control rules using combined Shewhart–CUSUM structures. They proved superiority of this combined approach on the separate Shewhart's approach but ignored any comparison with the separate CUSUM application. Also their control rules considered only one point at a time for testing an out-of-control situation. The false alarm rates of their control rules kept fluctuating and no attempt was made to keep them fixed at a pre-specified level, which is very important for valid comparisons among different control rules/schemes.

In this study, we analyze some of the r out of m runs rules schemes (such as 2/2 and 2/3 schemes) with CUSUM charts following Klein[7], Khoo[8], and Antzoulakos and Rakitzis[10] and compared their performance (in terms of the *ARL*) with some other schemes meant particularly for the smaller shifts.

2.2. The proposed schemes for the CUSUM charts

A process is called to be out-of-control when a point falls outside the control limits. Specific runs rules or extra sensitizing rules can be used in addition to enhance the power of detecting out-of-control situations. The CUSUM charts can also take benefit out of these runs rules schemes if properly applied with the CUSUM structures. Following Klein[7], Khoo[8], and Antzoulakos and Rakitzis[10], we propose here two runs rules schemes to be used with the CUSUM charts to monitor the location parameter. The proposed schemes are based on the following terms and definitions.

Action limit (*AL*): This is a threshold level for the value of CUSUM chart statistic. If some value of CUSUM statistic exceeds the *AL*, the process is called to be out-of-control. The value of the *AL* would be greater than the usual CUSUM critical limit h for a fixed ARL_0.

Warning limit (*WL*): This is a level for the value of the CUSUM chart statistic beyond which (but not crossing the *AL*) some pattern of consecutive points indicates an out-of-control situation. The value of the *WL* would be smaller than the usual CUSUM critical level h for a fixed ARL_0.

Using the above definitions, we propose the two runs rules schemes for the CUSUM chart as:

Scheme I: A process is said to be out-of-control, if one of the following four conditions is satisfied:

1. One point of C^+ falls outside the *AL*.
2. One point of C^- falls outside the *AL*.
3. Two out of two consecutive points of C^+ fall between the *WL* and the *AL*.
4. Two out of two consecutive points of C^- fall between the *WL* and the *AL*.

Scheme II: A process is said to be out-of-control if one of the following four conditions is satisfied:

1. One point of C^+ falls outside the *AL*.
2. One point of C^- falls outside the *AL*.
3. Two out of three consecutive points of C^+ fall between the *WL* and the *AL*.
4. Two out of three consecutive points of C^- fall between the *WL* and the *AL*.

Note that the values of the *WL* and the *AL* are proportional to the value of the *ARL* for a given shift; i.e. the *ARL* is higher if the values of the *WL* and the *AL* are higher and vice versa.

There are infinite pairs of *WL* and *AL* that fix the in-control ARL_0 at a desired level. The objective is to find those pairs of *AL* and *WL* that maintain the ARL_0 value at the desired level and at the same time minimize the ARL_1 value.

The *ARL* computations may be carried out using different approaches such as Integral Equations, Markov Chains, approximations and Monte Carlo simulations. Details regarding the first two may be seen in Lucas and Crosier[5] and Brook and Evans[13] and the references therein. An *ARL* approximation for the upper-sided CUSUM (ARL^+) and the lower-sided CUSUM (ARL^-) is given as (cf. Alwan[4]):

$$ARL^+ = [\exp\{-2(\delta-k)(h+1.166)\}+2(\delta-k)(h+1.166)-1]/(2(\delta-k))$$

and

$$ARL^- = [\exp\{-2(-\delta-k)(h+1.166)\}+2(-\delta-k)(h+1.166)-1]/(2(-\delta-k)^2)$$

The *ARL* for a two-sided CUSUM can be obtained by the following relation:

$$1/ARL = 1/ARL^+ + 1/ARL^-$$

Monte Carlo simulation is also a standard option to obtain approximations for the ARL, and we have adopted this approach in our study. For that purpose, we have developed a simulation algorithm using an add-in feature of Excel software that helps calculating the *ARLs*. The program may be obtained from the authors upon request.

2.3. Performance evaluation of the proposed schemes

The performance of the two proposed schemes for the CUSUM chart has been evaluated in terms of *ARL* under different in-control and out-of-control situations. To meet the desired objective, we have used our simulation algorithm to find the ARL_0 and ARL_1

Table II. *WL*, *AL* and ARL_1 values for the proposed scheme I at $ARL_0=168$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.42	4.8	71.8715	25.5644	13.5392	8.6598	5.0776	3.6786
3.44	4.6	72.258	25.6532	13.5	8.5682	5.0128	3.6072
3.48	4.4	71.936	25.5934	13.4956	8.516	4.936	3.5246
3.53	4.2	***71.399***	***25.3002***	***13.3322***	***8.4044***	***4.8282***	***3.423***

Table III. *WL*, *AL* and ARL_1 values for the proposed scheme II at $ARL_0=168$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.5	4.44	***71.489***	25.3786	13.3984	8.462	4.9412	3.5406
3.6	4.19	72.938	***25.3676***	13.3524	8.3828	4.83	3.424
3.7	4.08	73.1095	25.3692	13.3058	8.3442	4.7772	3.3762
3.8	4.03	73.589	25.4026	***13.2766***	***8.3156***	***4.75***	***3.3474***

Table IV. *WL*, *AL* and ARL_1 values for the proposed scheme I at $ARL_0=200$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.9	4.24	82.9524	28.6972	***13.8006***	8.9038	***4.9088***	***3.4918***
3.8	4.29	84.537	28.736	14.065	***8.6606***	5.002	3.5234
3.7	4.4	82.1152	***28.4956***	13.8216	8.8124	4.9928	3.561
3.6	4.77	84.1504	28.716	13.9612	8.9812	5.2276	3.7252
3.57	∞	***79.4742***	28.9396	14.2622	9.213	5.5104	4.076

Table V. *WL*, *AL* and ARL_1 values for the proposed scheme II at $ARL_0=200$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.9	4.23	82.975	***28.206***	***13.7702***	8.869	4.9782	***3.4346***
3.8	4.28	***81.4026***	28.3564	13.9032	***8.6994***	***4.966***	3.5028
3.7	4.6	82.7322	28.8236	14.0778	8.8744	5.1404	3.687
3.64	∞	81.5176	29.1382	14.2656	9.1142	5.4572	4.1198

values for the pairs of *WL* and *AL*. We have generated for different values of δ 10 000 samples of size *n* from $N(\mu+\delta\sigma, \sigma)$ and we have calculated the statistics C_i^+ and C_i^- for all the samples. Here μ and σ refer to the mean and the standard deviation of the process under study and δ the amount of shift in μ. The value of δ indicates the state of control for our process, i.e. $\delta=0$ implies that the process mean μ is in-control and $\delta\neq 0$ that the process mean μ is out-of-control. Without loss of generality, we have taken $\mu=0$ and $\sigma=1$ in our simulations. After getting 10 000 samples, we have applied all the four conditions of the two proposed runs rules schemes (i.e. Schemes I and II). In this way, the run lengths are found for the two proposed schemes. This procedure is repeated 5000 times and each time the run lengths are computed for both schemes. By taking the average of these run lengths, we obtain the *ARLs* for the two schemes.

To evaluate the performance of the two proposed schemes I and II, we will report the results for the values of the ARL_0 equal to 168, 200 and 500. Other values of the ARL_0 can be easily obtained. The choices made will show the performance of the two schemes and enable us to make comparisons with the results of other schemes and approaches from the literature. By fixing the ARL_0 at a desired level for the proposed schemes I and II, we are able to obtain pairs of *WL* and *AL* using our algorithm. Then for these pairs of *WL* and *AL*, we have obtained the ARL_1 at different values of δ for both the schemes. The results of the *WL* and the *AL* along with their corresponding ARL_1 values for the above-mentioned pre-specified ARL_0s are provided in Tables II–VII for both schemes. In Tables II–VII, the first two columns contain the *WL* and *AL* pairs that fix the ARL_0 value at a specified desired level and the remaining columns give the corresponding ARL_1 values. The values in the bold and italic text in these tables indicate the smallest ARL_1 value for a given shift.

Some researchers (e.g. Antzoulakos and Rakitzis[10]) suggest also to report the standard deviations of the run lengths along with the *ARL* values to describe more about the run length behavior. Moreover, Palm[14] and Shmueli and Cohen[15] highlighted the importance of percentile points of the run length distribution and suggested to report them for the interest of practitioners.

Table VI. *WL*, *AL* and ARL_1 values for the proposed scheme I at $ARL_0 = 500$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
4.8	5.12	***141.1114***	38.5986	***17.3916***	***10.5176***	5.9052	***4.0574***
4.7	5.2	150.3718	38.5942	17.5288	10.5994	***5.8984***	4.14
4.6	5.39	145.1886	***38.1954***	17.468	10.5584	6.0066	4.2374
4.49	∞	146.564	38.4918	17.7254	10.8566	6.3326	4.6894

Table VII. *WL*, *AL* and ARL_1 values for the proposed scheme II at $ARL_0 = 500$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
4.8	5.11	***139.7048***	38.8562	17.4586	***10.5056***	***5.8222***	***4.0776***
4.7	5.19	142.1588	***37.9752***	***17.2674***	10.5826	5.8716	4.1036
4.6	5.5	145.7868	38.3342	17.3938	10.734	6.0526	4.2726
4.54	∞	149.0352	39.9042	17.5682	10.9658	6.4506	4.873

Table VIII. *SDRL* values for the proposed scheme I at $ARL_0 = 168$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.42	4.8	66.87329	21.7684	8.957528	4.559059	2.007781	1.132236
3.44	4.6	67.29742	21.83755	8.960896	4.551032	2.022932	1.139635
3.48	4.4	66.42193	21.79842	9.011627	4.559507	2.028084	1.160028
3.53	4.2	66.27938	21.7872	8.95986	4.544424	2.017002	1.158766

Table IX. *SDRL* values for the proposed scheme II at $ARL_0 = 168$

Limits		δ					
WL	*AL*	0.25	0.5	0.75	1	1.5	2
3.5	4.44	64.32835	21.53114	8.897433	4.605649	2.00533	1.154564
3.6	4.19	67.08282	21.7383	8.971818	4.634151	2.023442	1.16073
3.7	4.08	67.98399	21.78837	8.981226	4.671583	2.035286	1.167624
3.8	4.03	67.62782	21.92499	8.94629	4.675251	2.036057	1.166956

Therefore, the standard deviations (denoted by *SDRL*) and the percentile points (denoted by P_i) of the run length distribution are also computed for proposed schemes I and II. The results of *SDRL* and P_i (for $i = 10, 25, 50, 75, 90$) are provided in Tables VIII–XI for the two proposed schemes at $ARL_0 = 168$. For the other values of ARL_0, similar tables can be easily obtained.

The standard errors of the results reported in Tables II–XI are expected to remain around 1% (in relative terms) as we have checked by repeating our simulation results. We have also replicated the results Table I for $h = 4$ of the usual CUSUM scheme using our simulation routine and obtained almost the same results, which ensures the validity of the algorithm developed in Excel and the simulation results obtained from it.

We have observed for the two proposed schemes I and II that:

(i) many pairs of *WL* and *AL* may be found that fix the ARL_0 at a desired level, but the optimum choice helps minimizing the ARL_1 value (cf. Tables II–VII);
(ii) the two proposed schemes perform very good at detecting small and moderate shifts while maintaining their ability to address the large shifts as well (cf. Tables II–VII);
(iii) the *SDRL* decreases if the value of δ increases for both schemes I and II (cf. Tables VIII and IX);
(iv) the run length distribution of both the schemes are positively skewed (cf. Tables X and XI);
(v) the two proposed schemes I and II are almost equally efficient for the shifts of small, moderate and large magnitude and hence may be used as a replacement of each other at least for normally distributed processes;
(vi) with an increase in the value of δ the ARL_1 decreases rapidly for both the schemes, at a fixed value of ARL_0;
(vii) with a decrease in the value of ARL_0, the ARL_1 decreases quickly for both the schemes for a given value of δ (cf. Tables II–VII);
(viii) the proposed schemes I and II may be extended to more generalized schemes (as given in Klein[7], Khoo[8] and Antzoulakos and Rakitzis[10]).

Table X. Percentile points for the proposed scheme I at $ARL_0 = 168$

Limits			δ					
WL	*AL*	Percentiles	0.25	0.5	0.75	1	1.5	2
3.42	4.8	P_{10}	10	6	5	4	3	2
		P_{25}	24	10	7	5	4	3
		P_{50}	53	19	11	8	5	3
		P_{75}	98	34	17	11	6	4
		P_{90}	157	53	26	15	8	5
3.44	4.6	P_{10}	10	6	5	4	3	2
		P_{25}	23.25	10	7	5	4	3
		P_{50}	53	19	11	7	5	3
		P_{75}	98	34	17	11	6	4
		P_{90}	158.9	53	26	15	8	5
3.48	4.4	P_{10}	10	6	5	4	3	2
		P_{25}	23	10	7	5	3.25	3
		P_{50}	53	19	11	7	5	3
		P_{75}	98	34	17	11	6	4
		P_{90}	156.9	54	26	15	8	5
3.53	4.2	P_{10}	10	6	5	4	3	2
		P_{25}	23	10	7	5	3	3
		P_{50}	52	19	11	7	4	3
		P_{75}	97	33	17	11	6	4
		P_{90}	156	53	25	15	7	5

Table XI. Percentile points for the proposed scheme II at $ARL_0 = 168$

Limits			δ					
WL	*AL*	Percentiles	0.25	0.5	0.75	1	1.5	2
3.5	4.44	P_{10}	11	6	5	4	3	2
		P_{25}	24	10	7	5	4	3
		P_{50}	53	19	11	7	5	3
		P_{75}	98	34	17	10	6	4
		P_{90}	155.9	53	26	14	8	5
3.6	4.19	P_{10}	11	6	5	4	3	2
		P_{25}	24	10	7	5	3	3
		P_{50}	54	19	11	7	4	3
		P_{75}	99	34	17	10	6	4
		P_{90}	158.9	53	25	14	7	5
3.7	4.08	P_{10}	11	6	5	4	3	2
		P_{25}	23.25	10	7	5	3	3
		P_{50}	54	19	11	7	4	3
		P_{75}	99	34	17	10	6	4
		P_{90}	160	53	25	14	7	5
3.8	4.03	P_{10}	11	6	5	4	3	2
		P_{25}	24	10	7	5	3	3
		P_{50}	55	19	11	7	4	3
		P_{75}	101	34	17	10	6	4
		P_{90}	161.9	53	25	14	7	5

3. Comparisons

In this section, we compare the performance of the proposed schemes I and II with some existing schemes for detecting small, moderate and large shifts. The *ARL* is used as a performance measure for all the schemes under discussion. The existing schemes we have considered for comparison purpose include the usual CUSUM scheme of Page[3], the weighted CUSUM scheme of

Table XII. *ARL* values for the symmetric two-sided weighted CUSUM scheme at $ARL_0=500$

$k=0.5$		δ			
γ	h	0.5	1	1.5	2
0.7	3.16	86.30	15.90	6.08	3.52
0.8	3.46	70.20	13.30	5.66	***3.50***
0.9	3.97	54.40	***11.40***	***5.50***	3.60
1.0	5.09	***39.00***	10.50	5.81	4.02

Table XIII. *ARL* values for two-sided EWMA chart with $L=3$ and $\lambda=0.25$

δ	0	0.25	0.5	0.75	1	1.5	2
ARLs	502.9	171.09	48.45	20.16	11.15	5.47	3.62

Table XIV. *ARLs* for FIR CUSUM scheme with $C_0=h/4$ and $k=0.5$

δ	0	0.25	0.5	1	1.5	2
$h=4$, $C_0=1$	163	71.1	24.4	7.04	3.85	2.7

Yashchin[16], the EWMA scheme given in Crowder[17] and the fast initial response (FIR) CUSUM scheme of Lucas and Crosier[5]. The ARL_1 results for the above-mentioned schemes are provided in Tables I and XII–XIV at some selective values of ARL_0, which will be used for the comparisons.

Now we present a comparative analysis of the proposed schemes with the existing schemes one by one.

Proposed vs the Usual CUSUM: The usual CUSUM scheme of Page[3] accumulates the up and down deviations from the target and is quite efficient at detecting small shifts. Table I provides the *ARL* performance of the usual CUSUM scheme. Tables II–VII provide the *ARL* performances of the two proposed schemes. The results of these tables advocate that the proposed schemes are better compared with the usual CUSUM scheme for small shifts, whereas for moderate and large shifts their performances almost coincide.

Proposed vs the Weighted CUSUM: Yashchin[16] presented a class of weighted control schemes that generalize the basic CUSUM technique by assigning different weights to the past information used in the usual CUSUM statistic. The *ARL* performance of the weighted CUSUM scheme is given in Table XII, where γ represents the weight and the other terms as defined earlier in this paper. Tables VI and VII provide the *ARL* performance of the proposed schemes at $ARL_0=500$; hence, these tables can be used to compare the proposed schemes with the weighted CUSUM scheme. By comparing the results of Tables VI, VII and XII, we can see that the proposed schemes perform better than the weighted CUSUM for small and moderate shifts. Particularly, when γ is small the performance of our proposed schemes is significantly better than that of the weighted CUSUM scheme. However for $\gamma=1$, the weighted CUSUM scheme is the same as the usual CUSUM; hence, the comments of the proposed versus the usual CUSUM scheme hold here as well.

Proposed vs the Usual EWMA: Crowder[17] gave a simple method for studying the run length distribution of the usual EWMA chart. Table XIII presents some selective *ARLs* of the EWMA chart for $\lambda=0.25$ and $L=3$, where λ is weighting constant and L is the control limits coefficient. As the proposed schemes have $ARL_0=500$ in Tables VI and VII, we use these tables for a comparison with the EWMA chart. From Tables VI, VII and XIII we see that for $\delta=0.25$, the usual EWMA chart has an ARL_1 value of 171.09 whereas the proposed schemes are minimizing the same ARL_1 value around 140. This shows that the proposed schemes perform better than the usual EWMA scheme for $\delta=0.25$. The same superiority also holds for all $0.25\leq\delta\leq 1$. However for $\delta>1$, the proposed schemes and the usual EWMA scheme have almost the same behavior as can be easily seen from the corresponding tables.

Proposed vs the FIR CUSUM: Lucas and Crosier[5] presented the FIR CUSUM which gives a head start value, say C_0, to the usual CUSUM statistic. A standard CUSUM has $C_0^+=C_0^-=0$, whereas an FIR CUSUM sets C_0^+ and C_0^- to some nonzero value. Table XIV presents the *ARLs* for the FIR CUSUM at $h=4$ and $C_0=1$ for discussion and comparison purposes. The FIR CUSUM scheme decreases the ARL_1 values as compared with those of the usual CUSUM scheme at the cost of reduction in ARL_0 value from 168 to 163 (see Table I vs Table XIV), which is generally undesirable in sensitive processes (e.g. those directly related to intensive care units which are highly time sensitive, cf. Bonetti *et al.*[18]). The ARL_1 results given in Tables II and III of the proposed schemes are also obtained for $ARL_0=168$ and hence can be used for comparison purposes here. Looking at the ARL_1 results of Tables II and III, we can see that almost the same amount of reduction in ARL_1 may be achieved, as obtained by FIR CUSUM, using the proposed schemes without paying any cost in terms of a decrease in ARL_0 value and the need of a head start value.

In brief the proposed schemes have shown better performance for the smaller values of δ (i.e. smaller shifts), which is the main concern of CUSUM charts, whereas for larger values of δ the proposed schemes can perform equally well as the other schemes. The better performance can be further enhanced with the help of other runs rules schemes of Khoo[8] and Antzoulakos and Rakitzis[10].

4. Illustrative examples

To illustrate the application of the proposed CUSUM schemes, we use the same method as in Khoo[8]. Two data sets are simulated consisting of some in-control and some out-of-control sample points. For the first data set we have generated 50 observations in total, of which the first 20 observations are from $N(0, 1)$ (showing the in-control situation) and the remaining 30 observations are generated from $N(0.25, 1)$ (showing a small shift in the mean level) whereas for the second data set we have generated 30 observations in total, of which the first 20 observations are from $N(0, 1)$ (showing the in-control situation) and the remaining 10 observations are generated from $N(1, 1)$ (showing a moderate shift in the mean level).

The two proposed CUSUM schemes of this study (i.e. schemes I and II) are applied to the above-mentioned two data sets. Additionally, the usual CUSUM scheme is also applied to these two data sets for illustration and comparison purposes. The CUSUM statistics are computed for the two data sets and are plotted against the respective control limits used with the three CUSUM schemes by fixing the ARL_0 at 168. For the usual CUSUM scheme, $h=4$ is used as the control limit to have $ARL_0=168$. For the proposed scheme I, $WL=3.53$ and $AL=4.2$ (for both the data sets) are used whereas for the proposed scheme II $WL=3.5$ and $AL=4.44$ for data set 1 and $WL=3.8$ and $AL=4.03$ for dataset 2 are used to have the ARL_0 value equal to 168 for both the schemes. The graphical displays of the three CUSUM schemes for the two data sets are given in the following two figures.

Figure 1 exhibits the behavior of data set 1 where a small mean shift was introduced, whereas Figure 2 exhibits the behavior of data set 2 where a moderate mean shift was introduced. In Figure 1, we see that an out-of-control signal is received at the sample point # 43 by the usual CUSUM scheme (i.e. one out-of-control signal), at the sample points # 38, 39, 40, 41, 42 and 43 by the proposed scheme I (i.e. six out-of-control signals) and at the sample point # 38, 39, 40, 41, 42 and 43 by the proposed scheme II (i.e. six out-of-control signals). Similarly, in Figure 2 the out-of-control signals are received at the sample points # 28 and 30 by the usual CUSUM scheme (i.e. two out-of-control signals), at the sample points # 28 and 30 by the proposed scheme I (i.e. two out-of-control signals) and at the sample point # 30 by the proposed scheme II (i.e. one out-of-control signal).

It is evident from the above figures that the proposed schemes have more out-of-control signals than the usual CUSUM scheme for the data set 1 where small mean shift was present, whereas the proposed schemes have signaled almost the same number of times as the usual CUSUM for the data set 2 where moderate mean shift was present. It is to be noted that these signaling performances of the proposed schemes versus the usual CUSUM scheme are in accordance with the findings of Section 3 where

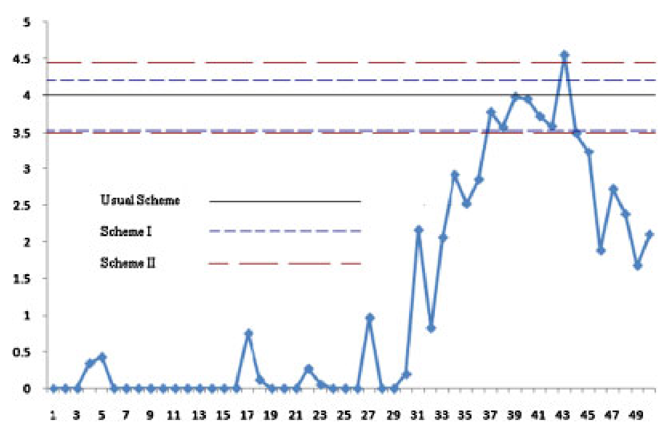

Figure 1. CUSUM chart of the usual scheme and the proposed schemes I and II for the data set 1

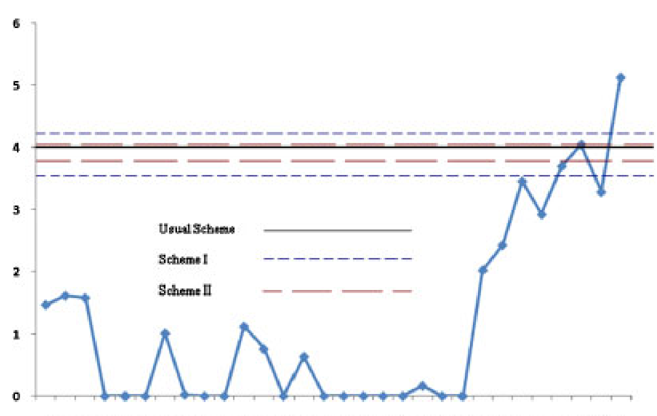

Figure 2. CUSUM chart of the usual scheme and the proposed schemes I and II for the data set 2

we found that the proposed schemes are more efficient than the usual CUSUM scheme for small shifts while almost equally good for other shifts.

5. Summary, conclusions and recommendations

SPC is a merger of many useful statistical techniques that help to distinguish the variations in a process. Out of these techniques, the control chart is the most important and commonly used tool. To control the process location parameter, the $\bar{X}$-chart is frequently used and is considered effective for detecting the larger shifts (around 3 sigma) in the location parameter. In order to make the Shewhart $\bar{X}$-chart more sensitive for moderate and smaller shifts, extra sensitizing rules have been proposed with its usual design structure at the cost of an increase in false alarm rate. The issue of increased false alarm rate can be handled by using the runs rules schemes.

For small shifts, CUSUM charts and EWMA charts are considered most effective. The efficiency of these charts can also be increased by using different sensitizing rules and runs rules schemes with their usual design structure. We have proposed two runs rules schemes namely, scheme I and scheme II, in this study for CUSUM charts for the location parameter. The proposed scheme I signals an out-of-control situation if one CUSUM statistic crosses the *AL* or two consecutive points fall between the *WL* and the *AL*. Similarly, the proposed scheme II signals an out-of-control situation if one CUSUM statistic crosses the *AL* or two out of three consecutive points fall between the *WL* and the *AL*. By investigating the performance of the two proposed schemes and by comparing them with some existing schemes we found that the proposed schemes I and II have the ability to perform better for smaller and moderate shifts while reasonably maintaining their efficiency for the large shifts as well.

To make the CUSUM charts even more efficient, some other sensitizing rules/runs rules schemes can be used with the CUSUM structure on the similar lines as followed in this study. We can also boost up the performance of EWMA charts by applying these rules/schemes with the conventional structure of EWMA-type control charts. The proposals and the recommendations of this study can also be extended for the attribute control charts based on CUSUM and EWMA patterns. Some of these investigations are in progress focusing their issues under different application environments.

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