On the dialectical foundations of mathematics

Damsma, D.

Publication date
2008

ON THE DIALECTICAL FOUNDATIONS OF MATHEMATICS[1]

Version 5.3

Dirk Damsma

August 2008

Abstract

This paper tracks the systematic dialectical determination of mathematical concepts in Hegel's *Encyclopädie der philosophischen Wissenschaften* (1830[3], 1817[1]) and investigates the insights that can be gained from such a perspective on the mathematical. To begin with, the determination of Numbers and arithmetical operations from Being shows that the One and the successor function have a qualitative base and need not be presupposed. It is also shown that even for infinite Intensive Magnitudes (cardinals) there exists an Extensive Magnitude through which they gain meaning. This makes the 'bad' in Hegel's 'bad infinity' a trifle problematic. Finally, if 'Dasein' is interpreted as the whole of perception in the present, Place can be viewed as the spatial Now, Motion as the passage from Place to Now and Matter as the actual (as opposed to observed) Presence of the natural realm.

[1] I am grateful to Christopher Arthur, Gerard Alberts, Marcel Boumans and Patrick Murray for their interest in the research that led to this paper and their valuable suggestions for reading. I also want to thank Wouter Krasser, Tijmen Daniels and Marcel Boumans for help with some of the mathematical points. Despite his illness Louk Fleischhacker read my paper thoroughly before he deceased. His comments have been very helpful and I will remember him with great gratitude. Furthermore, I am indebted to David Gray Carlson for his valuable comments on an earlier version of this paper, his enthusiastic support and his readiness to explain some of the puzzling passages in Hegel's *Wissenschaft der Logik* (1812, 1813, 1816) whenever I asked. Last, but not least I want to extend my gratitude to Geert Reuten for his continual support and his valuable comments on earlier versions of this paper. All remaining mistakes are my own.

Introduction

Mathematics, like any other science, cannot justify its own foundations. For example, it needs Numbers to build up the tools by which it can apprehend Numbers.[2] Due to this circularity, mathematicians are forced to work with notions whose genesis they cannot fathom. The mentioned circularity can be evaded by a reflection on the concepts mathematicians work with, rather than by working out their implications (for this is already the core business of mathematicians anyhow). Hegel's methodology of systematic dialectics is instrumental in this reflection, for it entails an ordering of concepts from abstract concepts to concrete instances. On the basis of this order, I will show that the concept of sets of elements stems from a failed attempt at making qualitative distinctions on the basis of quality alone. Further, the exhibition in this paper clarifies the proper use of finite as well as infinite cardinal and ordinal Numbers and shows that our awareness of Time and hence of Motion presupposes distinctions in Space.

Hegel discusses some important concepts of number theory and algebra (viz. Numbers and arithmetical operations) at length in the first part of his *Wissenschaft der Logik* (1812, 1813, 1816) and more succinctly in the first subdivision of the first part of the *Encyclopädie der philosophischen Wissenschaften* (1830^3, 1817^1).[3] Important concepts of geometry are discussed in the first subdivision of Hegel's philosophy of nature, which can be found in the second part of the *Encyclopädie,* but not in the *Wissenschaft*. So the *Encyclopädie* encompasses more mathematical concepts than the *Wissenschaft*, but discusses them more succinctly. Because of this completeness and because I have little to add to Carlson's comprehensive discussion of the *Wissenschaft* (2000, 2002, 2003a) I will confine myself to the *Encyclopädie* in this paper.

---

[2] In this paper, dialectically important concepts will always be written with a capital letter, enabling the reader to see whether a word is used dialectically or not. In German, all nouns are written with a capital letter. So, this practice (although common among native English speaking Hegelians) has no warrant in German (Inwood 1992: 6). However, since this linguistically questionable convention usually clarifies dialectical presentations significantly, I will adopt it here.

[3] Superscripts behind a publication year denote editions. The edition that was actually used is always cited first. Thus (1830^3, 1817^1) means that the current text relies on the third edition of the *Encyclopädie* and that the first edition of that work was published in 1817.

The central questions are how Hegel develops important mathematical concepts systematically out of other more abstract concepts, how this reflects on the meaning of these concepts and how this in turn reflects on the mathematics in which the concepts are utilized. In answering these questions it will be shown that mathematical concepts presuppose abstract concepts in common language. So the mathematical mindset and formal languages are predicated upon natural languages.

The first section of this paper deals with Hegel's methodology of systematic dialectics in general and that of the *Encyclopädie* specifically. Next, in section two, a representative part of the literature on Hegel and Mathematics is discussed. This helps position this paper and provides an idea of the potential uses of systematic dialectics with regard to the philosophy of mathematics.

Hegel's determination of the quantitative is discussed in the third section and his determination of mathematical mechanics in the fourth. The accounts given are neither quantitative nor mathematical. Rather, mathematical concepts, like Discrete and Continuous Magnitude, Number, Spatial Dimensions, the Point and the Line, are ordered along other concepts within Hegel's philosophical framework. In the concluding section the question will be answered what insights can be gained from this systematic dialectical perspective on the mathematical.

1. Systematic Dialectics and Hegel's Logic: Preliminary Notions

In his *Encyclopädie der Philosophischen Wissenschaften* (1830^3, 1817^1) Hegel takes on the description of all sciences and all knowledge in their interconnectedness. According to Hegel, this totality (and therefore the *Encyclopädie* itself) in turn consists of three parts, each of which is an object totality in its own right. An object totality is a part of the world that is encompassed by one universal principle without which the totality cannot exist (Reuten & Williams 1989: 16, 20). The first part is the logic, the philosophy of our most abstract ideas about reality and the concepts in which they are embodied.

At this level of abstraction, it is impossible to point at something in the world and say: '*that* is what this concept means'. Therefore, only the concepts themselves and their conceptual interrelations can be studied in the logic. Hence, Hegel calls this 'the science of the Idea in and for itself' (Hegel 1830[3], 1817[1]: §18, Geraets, Suchting and Harris' 1991 translation). This object totality is described at length in Hegel's *Wissenschaft der Logik* (1812, 1813, 1816). It is also the most abstract of the object totalities. Hence the *Encyclopädie* and the *Wissenschaft* start with the universal principle of the totality *of everything*, rather than of an *object* totality.[4] This universal principle is Being (Hegel 1830[3], 1817[1]: §86; Hegel 1812, 1813, 1816: 82-83, 1.1A).[5]

The second part of the *Encyclopädie* is the philosophy of nature. Our knowledge of nature cannot alter its basic laws, although it enables us to use them to our advantage. Nature therefore is separate from our ideas about it. This prompted Hegel to call this object totality 'the science of the Idea in its otherness' (Hegel 1830[3], 1817[1]: §18, Geraets, Suchting and Harris' 1991 translation). The concept that first describes this otherness is Space, for the distinctions in Space

---

[4] Note that 'theorizing from the totality of everything' is not the same as what Stephen Hawking calls 'a theory of everything'. The latter is an audacious attempt to formulate a theory that would fit all physical observations and of which all current partial theories (specifically general relativity and the partial theories of gravity and weak, strong and electromagnetic interactions between particles) can be shown to be special cases (Hawking 1998: 213). The method for arriving at this theory is still mainly inductive, whereas Hegel's Systematic Dialectic is neither deductive nor inductive. Rather, Hegel analyses concepts in language as labels for observations that at the same time produce observations. So for Hegel, the dichotomy between observations and theory does not exist.

[5] This paper is primarily concerned with the *Encyclopädie* (1830[3], 1817[1]), but sometimes I will also refer to the *Wissenschaft*. This latter work is divided first into parts, then into books, then into segments ('Abschnitte') and next into chapters. The chapters are subdivided into sections A, B and C which are usually, but not always, subdivided again into subsections a, b, and c. In order to enable comparisons to translations and to other editions, I will not only refer to the page number in the Suhrkamp (taschenbuch) edition of this book, but I will also specify the segment, the chapter, the section and the subsection, respectively. All references to the *Wissenschaft* are to the first book of the first part, so the part and the book in question need not be specified. Thus 1.1A means (part 1, book 1,) segment 1, chapter 1, section A (The first chapter of Hegel 1812, 1813, 1816 has no subsections). The *Encyclopädie* is divided into parts and subdivisions ('Abteilungen'). Just as with the *Wissenschaft*, the subdivisions are divided into sections A, B and C which are usually, but not always, subdivided again into subsections a, b, and c. But since it is partitioned into continuously numbered and sufficiently small §§, a reference to those (notation: §#) suffices to enable comparisons to translations and to other editions than the Lasson edition usually referred to in this paper. Finally, *Grundlinien der Philosophie des Rechts* (1821) (which will be referred to in examples only) is divided into parts and segments, but it too is partitioned into continuously numbered and sufficiently small §§, so a reference to those (§#) will suffice here as well.

are necessarily material in nature. Taking Space as the universal principle of this object totality therefore ensures this realm stays separate from our immaterial thoughts (Hegel 1830[3], 1817[1]: §254).

In the third part of the *Encyclopädie*, the philosophy of mind, and in his *Grundlinien der Philosophie des Rechts* (1821), Hegel sets out to describe that part of reality that is the result of human agency, viz. society (Hegel 1830[3], 1817[1]: §18). When we comprehend society, we can actively change it. Hegel therefore describes this object totality as the science ‘of the idea that returns into itself out of its otherness’ (Hegel 1830[3], 1817[1]: §18, Geraets, Suchting and Harris’ 1991 translation). In Hegel’s view the universal principle of human agency, hence the starting point for a comprehensive description of society is Freedom (Hegel 1830[3], 1817[1]: §382) and more specifically Free Will (1821: §4).

The idea is that an abstract principle such as Being, Space or Free Will already embodies all concrete instances of it, albeit implicitly, not immediately. There are a lot of concepts which are less abstract than their principle, but that are nevertheless far from concrete. For example, man is part of a family (meant in a rudimentary sense – man is not made in a factory but by two people), families are part of society, society is bound by law and morality and these, in Hegel’s view, are the result of human agency and thus a product of Free Will (1821: §4). Hegel wants to work backwards in the last sentence. That is, he first wants to show the most abstract instances of the universal principle (and as *instances* these are already more concrete than the principle), next the most abstract instances of these, and so on until something can be said about concrete and tangible things. This process is called concrete determination, for the concrete instances of a concept are determined from an abstract universal principle. If this concrete determination can be completed, it is claimed this proves that the abstractions utilized in the presentation are indeed suitable for understanding the object totality in question (Reuten & Williams 1989: 21-22).

Hegel makes this concrete determination by asking three questions about concepts encountered in the presentation. α) What does the concept mean in total conceptual isolation? How does it appear when viewed from the inside out? (E.g.

Free Will is a product of individual consciousness, which is universal and hence infinite when conceptually isolated from all other concepts (Hegel 1821: §5).) Next he asks himself: β) How does this concept appear when viewed from the outside in? How does it express itself in the world?[6] (E.g. a real individual's Will is not really universal, but constrained within a person, which is only one of many and hence finite (Hegel 1821: §6).)

As you can see, the answers to those questions usually involve oppositional concepts (e.g. Free Will exists only if it is also bounded). The tension between those concepts needs to be γ) resolved in order to make sure that the concept (e.g. Free Will) is not only a concept, but has empirical counterparts. Resolving an opposition of this sort involves either showing how the one half of the opposition becomes the other half, or showing how the two halves can coexist (e.g. individual Free Will, when constrained by others is only potentially free, it is a Possibility, which can only be actualized to the extent determined in the remainder of *Grundlinien der Philosophie des Rechts* (Hegel 1821: §7)). Sometimes the condition for coexistence cannot be found immediately. In that case there often are successive stages of coexistence. The first concepts in this succession only partially resolve the tension between α) and β), which is fully resolved at subsequent stages.

By asking the questions α) and β) again about the last of the concepts found under γ), a new opposition will generally be found, which can be resolved again, and so on. The universal principle and the concepts that arise from it dialectically are called moments. A moment is 'an element considered in itself, which can be

[6] Inside and outside correspond to the German terms Innere and Äußere. Hegel speaks of α) as inner reflection or reflection in itself ('innere Reflexion' or 'Reflexion in sich') and of β) as outer reflection or reflection in others ('äußere Reflexion' or 'Reflexion in Anderes') (Hegel 1830[3], 1817[1]: §113-120). It must be stressed that these terms indicate a reflection on one and the same concept. Thus, there are three sides to every story: α) the inside story, β) the outside story and γ) the truth. The truth of a concept is in the connection between both stories, in comprehending how the inside is responsible for the outside and the other way round, the way DNA and environmental factors make up an organism's phenotype. The 'inside' and 'outside' aspects of concepts are abstractly distinguishable, but not concretely separable.

conceptually isolated, and analysed as such, but which can have no isolated existence' (Reuten & Williams 1989: 22).[7]

Hegel does not usually make these three questions explicit in his *Encyclopädie*. He seems to regard some oppositions and their resolution as self-evident, for example the opposition between Being and Nothing and its resolution in Becoming and Presence. So, α), β) and γ) are not mentioned in section A, subsection a (Hegel 1830^3, 1817^1: §§86-88). From Presence until the start of section B, Quantity, he uses these questions explicitly as a tool to drive his presentation onwards (Hegel 1830^3, 1817^1: §§89-98). After that, the quantitative and its moments (including Measure) are again not explicitly discussed this way (Hegel 1830^3, 1817^1: §§99-107).

The first opposition arising from Space is once more determined explicitly through the questions α) and β) (Hegel 1830^3, 1817^1: §§255-256), but the first conditions of existence of this opposition are also discussed under β) (Hegel 1830^3, 1817^1: §256), whereas I contend they should be discussed under γ). In the remaining determination of the foundations of geometry the α), β) and γ) questions are left implicit (Hegel 1830^3, 1817^1: §§257-261). But, however implicit, these questions always linger in the background when Hegel determines the oppositions implicit in the presentation of the moments and their resolution.

In sections 3 and 4 of the current paper Hegel's ordering of concepts is largely preserved, but α), β) and γ) are consistently applied. This means among other things, that some moments are brought to bear under a different heading than Hegel did or would have done. It is hoped that the consistent application of the three questions mentioned, clarifies the presentation and the method used. It certainly makes Hegel's method more transparent.

[7] Instead, the existence of a moment is mediated ('vermittelt') by the moments that were posited before it and dialectically follow from it. For Hegel, mediation ('Vermittlung') contrasts with immediateness ('Unmittelbarkeit') (see page 4). In its immediateness Being is a distinctionless soup of everything and Free Will is utter and total chaos and anarchy, but when they are mediated by other concepts, the prospects for the logic and the philosophy of society respectively become less daunting.

2. Previous Literature on Hegel and Mathematics

The literature on Hegel and Mathematics falls roughly into two categories. First, some authors look for a philosophical understanding of mathematics in Hegel's works. Second, there are those that try to elucidate, comment upon and expand Hegel's views on mathematics and especially infinity.

The reason to look for a philosophy of mathematics in Hegel lies in the rigor and precision of mathematics and mathematical definitions. Once a concept or subject is rigorously defined, it is set apart from all possibilities that are not captured by the definition. When the mathematical implications of these definitions become clearer and clearer, so do their shortcomings. Thus, the rigorous definitions of mathematics call up their own negations (Paterson 1997a: 14; Tóth 1972: 36-38).

Tóth (1972) illustrates this point in relation to the development of non-Euclidian geometry. Interestingly, many authors, like Aristotle in the third century B.C. and Saccheri and Lambert in the 18th century A.D., already knew that a non-Euclidian geometry was possible in principle, but except for Aristotle they all dismissed this type of geometry as untrue (Tóth 1972: 20-23). Thus, the Euclidian system clearly calls up its own negation, even though this negation was only accepted as a true possibility in the 19th century A. D.. Within both axiomatic geometrical systems, the other system can be shown to be false, so the two are truly oppositional.[8] But this is only a problem if an ontological status is ascribed

[8] Mathematically, this is only true with regard to (one of) the *axioms* of both systems. Euclidian geometry accepts the fifth axiom, which states: 'given a line *l* and a point A not on the line, there is only one line through A which does not cross *l*'. If this axiom is rejected there are two possibilities:
1) In hyperbolic geometry there is an infinite number of lines through A that do not cross *l*;
2) in elliptical geometry all lines cross.
The first of these non-Euclidian possibilities implies an infinite (as opposed to one) number of parallels through A, while the second possibility implies that parallel lines are only parallel on a finite domain. So in terms of finite versus infinite the non-Euclidian axioms are truly opposed to Euclid's fifth axiom.

In mathematical practice, however, Euclidian geometry is a special case of elliptical geometry. Parallel lines on a globe, rather than a plane best represent the elliptical concept of parallelism. As the radius r of this globe approaches infinity, the elliptical system starts behaving like the Euclidian system. So the Euclidian system is the limit of the non-Euclidian elliptical system for $r \rightarrow \infty$. I am grateful to Louk Fleischhacker for help with these remarks.

to either of these systems of formal logic. If not, it is the positing of this opposition itself that might lead to a more comprehensive dialectical understanding of the nature of geometry (Tóth 1972: 36-40).

The fact that Hegelian philosophy can be used to make conceptual sense of the development of non-Euclidian geometry and the nature of geometry in general, is not to say that Hegel gave any account of non-Euclidian geometry in his writings. Rather, he 'fully accepted the essential validity of the Euclidian approach' (Paterson 2004/2005: 46), albeit that he criticized some of Euclid's proofs, especially when they involve superposition. Hegel's criticism follows from his contention that two distinct congruent triangles are conceptually the same. According to Hegel therefore, a pure mathematical triangle can only be congruent *with itself*. Hence, congruence must be proven from one triangle instead of from superposition of one triangle over another (Paterson 2004/2005: 37-39).

Paterson (1997a) discusses the problems that the formal systems have run into that were proposed in the 20th century as foundations for mathematics (Paterson 1997a: 3-10). Each of the proposed formal systems was inspired by different intuitions. In that sense each of them is deficient and can only be a foundational system for that part of mathematics that concurs with the foundational intuitions of that specific system (Paterson 1997a: 12).

As a solution to the problems that have arisen from this state of affairs, Paterson proposes to contrast the α) universality of mathematics itself (as a concept) with β) formal systems as particular instances of foundational systems, and to proceed dialectically towards the γ) singularity of natural numbers, sets and functions (Paterson 1997a: 12-14). In such an exhibition, 'the implicit conceptual content of the formal approach will be made explicit' (Paterson 1997a: 14) and 'the development will make conceptual sense of the insights which motivated the various foundational systems' (Paterson 1997a: 14-15).[9]

[9] On Paterson's website (http://home.olemiss.edu/~mmap/) the reader will find another five papers on the desirability and merits of a Hegelian philosophy of mathematics. Three of these (Paterson 1997b; 1999; 2000) are about the philosophy of Number, one is about the concept of the propositional calculus (Paterson 1994) and one is about the Hegelian philosophy of mathematics in general (Paterson 2002). In each of these, Hegelian philosophy is proposed as a solution to the

Hegel distinguishes between the bad or metaphysical and the true infinite. According to Hegel the latter concept is involved in the mathematical infinite (Lacroix 2000: 303). The bad infinite is a Magnitude beyond Quantity in that it is forever beyond the finite: whatever operations you perform using finite quantities; the result will always be finite again (Hegel 1812, 1813, 1816: 282, 2.2Bc; Lacroix 2000: 314). Hence the bad infinite is only a potential infinity that cannot be reached by finite mathematicians. It is beyond our grasp by its very definition.

All things in the world are finite, but this fact itself is infinite. 'Finite entities develop, change, pass away and give rise to other entities' (Inwood 1992: 295) ad infinitum. This passage itself is the basis for Hegel's conception of the true infinity (Hegel 1812, 1813, 1816: 163, 1.2Cc; Lacroix 2000: 315). Thus, the mathematician deals with 'finite objectivities that thought posits in its infinite self-development' (Lacroix 2000: 315). To Hegel the most important example of an application of the true infinite in the realm of Quantity is the differential calculus. At their limit the dy and dx in the ratio dy/dx are 'no Quanta anymore […] but have meaning only in their relation' (Hegel 1812, 1813, 1816: 295, 2.2Bc, my translation).[10] So dy and dx are not infinitesimally small but nonzero Quanta. Rather, it is the law that relates y and x that becomes apparent in the

---

problems 'which arise out of the existence in mathematics of self-referential, non-constructive concepts (such as *class*)' (Paterson 2002: 143).

The existence of these concepts leaves every consistent formal account of mathematics essentially incomplete. That is, for every formal system it can be proven that there exists a theorem that, although true, cannot be proven from the axioms of the system. This theorem, then, provably exists while being provably unprovable. The proof of this is known as the Gödel incompleteness theorems (Yourgrau 2005: 53-68; Hofstadter 1979: #).

For Gödel his result signalled that mathematical concepts exist objectively and not just analytically and thus that their meaning is not exhausted by their relations to other concepts (Yourgrau 2005: 170-171). Thus, '[s]ince [a concept] is not a child of our own imagination, we will never exhaust the information to be gained by different ways of approaching it' (Yourgrau 2005: 174). Although Gödel based his phenomenology on Husserl, this approach of concepts from different perspectives is presumably what he also admired in Hegel (c.f. Yourgrau 2005: 182).

At any rate, as Stachel remarks, 'the Gödel program in logic has interesting parallels with Hegel's dialectical method of subverting a philosophical system from within. Starting from the system's own premises, one demonstrates its inability to reach its own goals by exposing some contradiction between premises and goals. These contradictions are then "sublated" [i.e. resolved] by synthesis in some higher, more advanced system' (2007: 862).

[10] Quantum is Hegel's term for a specified Quantity. These Quanta have nothing to do with Planck's packages of specific amounts of light. Exactly what is at stake will be discussed at length in the next section. In that section and the section after that, all the capitalized concepts encountered thus far are elaborated upon.

expression dy/dx. Thus, while at this limit dy and dx disappear as specified Quantities, their relation reappears as a qualitatively different ratio. If y and x are positively related (e.g. through a successor function), it is this relation that is the true locus of the true (quantitative) infinite, because through it the finite Quantum x is ceaselessly led beyond itself into the bad potential infinite. Thus, if γ) true infinity is conceived of as the law that leads α) the finite Quantum x into the β) bad potential infinity it resolves the opposition between the two (Lacroix 2000: 311-315).

In 1994 Ellsworth de Slade wrote a study on the counterparts of Hegel's true infinity in his conception of infinitesimal mathematics. In 1986 Baer published an article on Hegel and Mathematics in general. In their texts both hail the result of the last paragraph as one of the most important insights Hegel has to offer in the field of mathematics (Ellsworth de Slade 1994: 213; Baer 1932: 112). Fleischhacker agrees with these authors that dy/dx is qualitatively different from other ratios, but disagrees with calling it a ratio. He argues that at its limit dy/dx is not a qualitatively different ratio, but a normal Quantum, whereas before the limit it was still a ratio. "Not dy and dx are 'the ghosts of deceased quanta', but dy/dx is the corpse of a deceased ratio" (Fleischhacker 1982: 148, my translation). However, as far as mathematical practice is concerned, all three authors are correct. That is, under some circumstances dy/dx is conceived of as a Quantum while under others it is best treated as a ratio.

Hegel's views on the infinite and infinitesimal mathematics are not intramathematical, but conceptual. However, as Wolff clearly shows in his 1986 text entitled *Hegel und Cauchy*, he was well versed in the research that mathematicians such as Lagrange and Cauchy have done on the subject. In this text Wolff traces how Cauchy influenced Hegel regarding the mathematical infinite and infinitesimal mathematics and discusses similarities and differences between the two (1986: 197-263).

Hegel's views on mathematics have also been an inspiration to Marx and Marxists. Kol'man and Yanovskaya (1931) discuss the nature and extent of the influence of the Hegelian philosophy of mathematics on Marxism-Leninism. [11] To them, as to Fleischhacker (1982), the most important merit of the Hegelian philosophy of Mathematics was his correct recognition of the subject matter of mathematics (1931: 5) (more about this in section 3.8).

But according to Kol'man and Yanovskaya, Hegel should not have stopped there. His dialectical perspective may have helped Hegel to correctly analyze the nature of mathematics and some of its problems and shortcomings, 'but as a bourgeois philosopher who only intends to explain the world and not to change it, he does not at all pose himself the task of transforming mathematics dialectically' (1931: 15). It goes without saying that Kol'man and Yanovskaya do not agree (1931: 14-18).

Finally, in the first three papers in a series on Hegel's *Wissenschaft der Logik* (2000, 2002, 2003a) Carlson gives a complete account of Hegel's determination of the quantitative. His treatment in these papers is very similar to mine. That is, all of Hegel's dialectical transformations and all of the important concepts in the *Wissenschaft* are discussed, explained and when appropriate, amended with modern-day insights. In the next two sections, I will do roughly the same for the *Encyclopädie*, although a different procedure for exhibition is adopted in those sections. Carlson exhibits Hegel's logic in the form of pictorial triads of overlapping concepts (2003a: 93-101), whereas I stick to the α) - β) - γ) -format explained in section 1.

Furthermore, the conceptual development in the *Encyclopädie* differs at a few crucial points from that in the *Wissenschaft* and this of course is reflected in the exhibition in section 3. Finally, the *Wissenschaft* does not encompass the philosophy of nature. As a result, Carlson does not discuss Hegel's determination of the concepts of mathematical mechanics, which I will do in section 4.

---

[11] Ernst Kol'man is also referred to as Kolman or Colman. Since he was a Russian mathematician, his name – which would otherwise be written in Cyrillic letters - is usually translated (or rather transcripted) along with the rest of his text, leading to the variations mentioned. By the same token, Sofya Yanovskaya is also referred to as Janovskaja or Ianovskaia.

3. Hegel's Determination of the Quantitative

In this section the method discussed in section 1 will be used to further exhibit the systematic dialectical determination of the quantitative. The mathematically important concepts here are Numbers and arithmetical operations. The main gist of this section is taken from subdivision 1 of part 1 (logic) of the *Encyclopädie* (Hegel 1830^3, 1817^1: §§84-111). Since the logic is the most abstract of the object totalities, this section begins with the universal principle of everything, Being. It will take ten (out of a total of 14) subsections to dialectically determine Number from Being.

A. Quality

3.1. Being

α) As mentioned in section 1, the universal principle of the totality of everything is Being, simply because everything we can perceive, think or imagine *is.* In total conceptual isolation this leaves us just about nowhere. Everything *is*, so viewed from the inside out, Being points to a total lack of further distinctions (Hegel 1812, 1813, 1816: 82, 1.1A; Hegel 1830^3, 1817^1: §86).[12] Hence, pure abstract Being is entirely imperceptible.

3.2. Nothing

β) To view Being from an outside perspective seems virtually impossible, for Being already encompasses everything. But by definition Being (or any other concept) does not encompass its opposite, Nothing (Hegel 1812, 1813, 1816: 83, 1.1B; Hegel 1830^3, 1817^1: §87). So Nothing stands outside of Being and to acknowledge Being we need an outside perspective. Hegel regards Nothing as the

---

[12] The logic in the *Wissenschaft der Logik* and the *Encyclopädie* is the same for Being, Nothing, Becoming, Presence ('Dasein') and perhaps Something and Others. After that, terms like finitude and infinity take a rather central stage in *Wissenschaft der Logik* but not in the *Encyclopädie.* Because I'm primarily concerned with the *Encyclopädie* here, I will only refer to the *Wissenschaft* up to the point where Something and Others are presented.

outside standpoint that enables this outside perspective and therefore concludes that it is empty observation and thought itself (Hegel 1812, 1813, 1816: 82-83, 1.1A-B), which is itself every inch a Being. The goal of his philosophy is to show how this is possible. That is, Hegel wants to develop an argument in which reality is not viewed from one of two poles, but in which the one pole can be shown to be part of the same totality as the other and in which the poles are mutually supportive so that the finished system is self-explanatory. Among other things, Hegel seeks to overcome the classic dichotomies between subjectivity and objectivity and between thought and things in this way (Inwood 1992: 16). Having the external standpoint Nothing doesn't help much, but at least one distinction can now be made: some things are and some are not.

Some have tried to formalize Hegel's dialectical logic using set theory (Baer 1932: 105; Kosok 1972; Priest 1989: 393-396). In set theoretical terminology one might say Hegel sought to resolve Russell's paradox long before it was formulated. This paradox runs: if V is the class of all sets (cf. Being), then its complement C\V (cf. Nothing) $\notin$ V because of the definition of C\V. But at the same time C\V $\in$ V because of the definition of V (Russell 1903: 527-528). Hegel resolves this paradox by initially placing Nothing outside of Being, while stating that Nothing is empty observation and thought itself. So initially C\V is conceptualized external to and independent of Being, but when we become conscious of the world, we come to realize that subjective Nothing is part of objective Being and conversely that the recognition of the objective (Being) requires the subjective (Nothing). So in truth the external independence posited does not hold. Surprisingly enough neither Kosok (1972) nor Baer (1932), nor Priest (1989) mentions this remarkable parallel.

3.3. Becoming

γ) Pure Being is as imperceptible as pure Nothing. If one tries to think of Being in all its entirety (so without any recourse to concrete examples), one might just as well think Nothing, because any real thought implies some distinction. Thus, the thought of pure Being immediately vanishes into pure Nothing and the other way round, in that neither of them can be a thought on its own (Hegel 1812, 1813, 1816: 83, 1.1C; Hegel 1830[3], 1817[1]: §87). What we *can* think is exactly this disappearance of the one in the other.

The process whereby the thought of Being vanishes into Nothing and the other way around, is Becoming. Thus, Becoming explicitly posits the non-thoughts of Being and Nothing as distinct and even oppositional concepts. At the current level of abstraction, Becoming is only change: Being Becomes Nothing and Nothing Becomes Being. Becoming is imperceptible because of this unceasing dynamism, but unlike Being and Nothing, Becoming can be thought (Hegel 1812, 1813, 1816: 83, 1.1C; Hegel 1830[3], 1817[1]: §88; Carlson 2000: 11-12; Carlson 2003b: 11-16).

3.4. Presence

γ) Paradoxically, the requirement of dynamism inherent in Becoming means that we must give a further *static* determination of Being, for if there is change, here and now must be different from there and then. This static determination of Being is Presence – my translation of Hegel's 'Dasein'. [13] So Presence is Becoming taken statically. Because of this stasis, Presence can finally be perceived (albeit very abstractly), or more precisely, it is the whole of perception itself, but at the same time the concept helps us to keep in mind that everything we perceive is continually undergoing change (Hegel 1812, 1813, 1816: 113, 1.1C; Hegel 1830[3], 1817[1]: §89).[14]

[13] The correct translation of 'Dasein' is something of a debate among English-writing Hegelians. Geraets, Suchting and Harris have stayed very close to the German term with the artificially sounding 'Being-there' (1992: 145). Others either stress the connotation of a particular space and time (as I am doing here) or the connotation of Being. Those that use Existence instead of Presence stress this latter connotation. Other alternatives are: 'Being Determinate' (Wallace 1873: 133, §89), 'Determinate Being' (Carlson 2000) and 'Prevalence' (Reuten 2005: 79).

[14] Heisenberg's uncertainty principle states that the variance of a particle's position (its Presence) times the variance of its speed (its Becoming) times its mass can never be smaller than Planck's constant (Hawking 1998: 72). So, we cannot know both at the same time. Rather, we need to alternate between both magnitudes in order to say anything conclusive at all. Mathematically, at the level of Becoming, speed (or whatever 'change') is the dependent variable, whereas at the level of Presence it is the independent variable. This happens because at the level of Becoming, the most we can say about reality mathematically, is that the development of the totality of all things (let's call it speed or velocity – v -, although other types of 'change' are also possible) is a function f of time t, $v = f(t)$. If you know the speed of something, you can find the distance it travelled by

<u>3.5. Something and Other</u>

α) Because Presence is Being's static determination it must be limited in either space or time or both.[15] This very finiteness further determines it as Something (Hegel 1812, 1813, 1816: 125-126, 1.2Ba; Hegel 1830[3], 1817[1]: §91).[16] The difference between Presence and Something is that Something's determination is *fixed* in time and space, while Presence is Becoming taken statically at *every* point in time and/or space. In other words, although Presence provides a static view of Becoming, it moves along with it.

β) If here-and-now is taken as the further determination of Presence as Something, the determination of there-and-then is abandoned. Yet Becoming requires Presence to be further determined as both, for if Something is fixed as here-and-now *or* there-and-then, there cannot be change. Hence, the thought of determined Presence as Something requires the thought of its Other. This Other then, is what Something is not. It is beyond Something's limits (Hegel 1830[3], 1817[1]: §§91-92).

This negative definition of Other means that the two concepts are only determined vis-à-vis each other (Hegel 1812, 1813, 1816: 125-126, 1.2Ba). At this

---

integrating the function for its speed by time over the interval t_0 till now. This yields: $vt = x_t = F(t)$, in which x_t stands for the distance travelled. If the speed v was constant in the first function this would have ran: $v = f(t) = v_0$ and its integer would be: $x_t = F(t) = v_0t$. So the dependent variable v became the independent variable through our change of focus.

Of course at this level of abstraction Becoming is an indeterminate Becoming of the universe as a whole. Consequently, if Presence is seen as the integer of this process, it must be the integer from the big bang, t_0, till now. This knowledge however does not help us make any real calculations yet, because 'the theory of everything' (see footnote 4) doesn't exist yet and we do not know what the universe was like at the big bang and even if we did know, there is still a host of possible ways in which it might have developed due to Heisenberg's uncertainty principle. I am grateful to Maurice Bos for suggesting this point and his extended loan of his Hawking (1998) volume. Eric Halmans and Wouter Krasser guarded me against overgeneralizations.

[15] Space and Time are only determined in the philosophy of nature (see section 4 of this paper), i.e. after the completion of the logic. So, all we can say at this stage is that Presence is finite, period. For reasons of readability and accessibility I added the considerations of here and now (in space and time).

[16] Note that a linguistic link can now be seen to exist between Nothing and Something. Where Nothing denoted an emptiness of observation and thought, Something is the first category that provides the mind with the possibility of *Some* (instead of *No*) concrete content, quality. This link cannot be made in German. Thus dialectics provides different insights, depending on the language used.

level of abstraction they are not yet determinations in their own right. As a first instance of Presence, Something is finitely confined by its very determinateness (or quality). But since Presence is necessarily Becoming and thus never stable, Something will sooner or later pass over into its Other. [17] Something then is firstly finite, and secondly alterable (Hegel 1830^3, 1817^1: §§92-93).

3.6. One and Many Ones

γ) Something denotes no more than a one-sided static determination of Presence. But due to Becoming, static determinations cannot last. So each Something Becomes its Other. This Other however, is itself also a one-sided determination of Presence, so it too may be taken as Something. Hence, in the process of Becoming, Something and Other are conceptually reunited as One (Hegel 1830^3, 1817^1: §96).

γ) Because every Other may be taken as Something, it again determines an Other vis-à-vis itself and this process may be reiterated indefinitely. Thus, by reuniting Something with its Other the concept One automatically leads us to acknowledge Many Ones. So by being ceaseless but yet requiring a further static determination, Becoming constitutes a *series* of static entities: Many Ones. Although each One in this series is a self-contained unit that excludes Other Ones from itself, it is also true that they can only be acknowledged as Ones because of the principle that generated the series (i.e. Becoming) (Hegel 1830^3, 1817^1: §97).

Bringing the concepts One and Many Ones to bear under γ) is a major digression from the treatment those terms get in the *Encyclopädie*. In that book, Hegel first introduces Being-for-self ('Fürsichsein') under γ) as the union of Something and Others (1830^3, 1817^1: §95). He generally uses this term to denote how reality appears to us before we comprehend it as being mediated by and itself

[17] Physicists have formulated this same point as the law of conservation of energy and mass. Put crudely, this law states that for everything that appears (as Something's Other) an equivalent amount of energy or mass disappears (from Something). Thus, ex nihilo creation of energy or mass is ruled out. Hence, each Something must have an Other. Still, the concepts of appearance and disappearance are opposites. But the law that links both concepts is not contradictory in a formal sense, for those concepts, *as concepts* do not work on the same concrete object. So the opposition between concepts used in a systematic dialectical presentation has nothing to do with formal contradictions (Wolff 1979: 342).

mediating the other moments. Since this is still the beginning of the Logic, we have little else but indeterminate, accidental appearance by which to distinguish Something from its Other and since both Something and its Other may be determined as Something, they can be distinguished from each other by their Being-for-self (their appearance) only.

Next, under α) Hegel contends that any Being-for-self is One and under β) he clarifies that there must be Many Ones, which as self-contained units are Repulsive of one another (Hegel 1830^3, 1817^1: §§96-97). Then, under γ) he continues that Ones are not only self-contained units, but also Ones. As such, each One is conceptually the same as each Other One and they have a relation not of Repulsion, but of Attraction (see section 3.7) (Hegel 1830^3, 1817^1: §98).

The major problem with this treatment is that the concepts of Repulsion and Attraction are opposites. So instead of resolving a dialectical opposition under γ), on this occasion Hegel introduces one there. He rarely, if at all, does this in the rest of the *Encyclopädie*, so in order to keep this passage consistent with the rest of the presentation in this section, I juggled the α), β) and γ) around a little bit. It must be stressed that this operation (although I contend it is an improvement) doesn't leave the meaning of the terms in question entirely unchanged.

3.7. Attraction and Repulsion

β) As self-contained units that exclude other Ones from themselves, the Ones are Repulsive towards each other. Through this Repulsion or reciprocal exclusion they can be conceived of as Many. Repulsion then, is the condition of existence of the Many (Hegel 1830^3, 1817^1: §97).

α) While ceaseless Becoming led us to acknowledge Many Ones, it also implies a certain indeterminacy concerning the limit to the One. That is, when everything always changes, it is not clear where Something ends and its Other begins. The limit to the One then, is equally indeterminate. Still the finite determinateness of Presence requires that the One be finite. So the placement of the required limit is entirely a matter of arbitrary external reflection. In that this is an arbitrary operation, one might conceive of many more Ones within this arbitrarily set limit. In this sense, the external reflection makes One out of the Many and posits the One as divisible. As such, it posits the Many Ones as mutually Attracting units, rather than self-contained mutually Repulsive Ones (cf. Hegel 1830^3, 1817^1: §98). In a sense then, the immanent indeterminacy of Becoming gets further articulated in Attraction, while the determinacy of Something is further articulated in Repulsion.

As was alluded to in section 3.6, Hegel regards the conceptual sameness of the Ones as the locus of Attraction instead of the indeterminacy of their Becoming. The difference between his and my treatment stems from my juggling around of α), β) and γ). Also, the conceptual sameness of the Ones in Hegel's treatment seems to imply a regress towards only the one Presence, whereas I think the point should be that positing a One requires an arbitrary external reflection. This reading seems to be confirmed by the conceptual development towards subsequent moments such as Discrete and Continuous Magnitude.

B. Quantity

3.8. Quantity

γ) In the realm of Quality we established that the Many Ones are self-contained units through Repulsion, whose oneness can nevertheless only be determined through an arbitrary external reflection (i.e. Attraction). With this we have entered the realm of Quantity. Thus, Quantity is an external reflection on a multitude of elements that are distinguishable as Many through Repulsion, but arbitrarily divisible through Attraction (Hegel 1830^3, 1817^1: §99).

Alberts, who writes on the nature of mathematization (1998: 18-30), Fleischhacker in his search for the object of mathematics (1982) and Dijkgraaf, who defines mathematics as 'the science of patterns and relations' (2001: 7, my translation) would all agree with this result. That is, most branches of mathematics (with the notable exception of topology) presuppose related elements. These relations may be studied for their own sake or they may give rise to patterns from which other relations can be discerned by external reflection. Thus, like Hegel, these authors consider 'an external reflection on many distinguishable but divisible elements' an apt description of the object and nature of mathematics and mathematical abstractions (Alberts 1998: 20, 27-28; Baer 1932: 104; Dijkgraaf 2001: 7; Fleischhacker 1982: 16-17). Hegel's position is remarkable for, in his time, Quantity was conceived of as a property of things rather than an external relation between indeterminate, abstract elements.[18]

3.9. Continuous and Discrete Magnitude

α) The indeterminacy of the limit to the One, means that the quantitative One is not only divisible - i.e. confined to the set of rational numbers Q -, but entirely Continuous - encompassing all real numbers R. Hence, Quantity (in its moment of Attraction) is essentially given as a Continuous Magnitude (Hegel 1830^3, 1817^1: § 100).

β) However, since it is imperative that the One be limited one way or another, it must be possible to stipulate Discrete elements (hence a Discrete Magnitude) within the Continuous Magnitude. In that a Discrete Magnitude excludes other Magnitudes from itself, it is the Quantitative determination of the Repulsion of the Many Ones vis-à-vis each other. Quantity then, is essentially Discrete and Continuous at the same time (Hegel 1830^3, 1817^1: § 100).

3.10. Quantum [19]

γ) Quantum is a specified Quantity. In the first instance, it is a Discrete Magnitude in that whatever specification is given excludes the other specifications from itself. However, the *range* of possible limitations to the One is Continuous. Depending on how the Continuous Magnitude is limited to arbitrarily fence of a One, the same Magnitude may have every imaginable Discrete size. Thus, the Discrete size of the Quantum crucially depends upon the way the Continuous

---

[18] I am grateful to Marcel Boumans for making this remark.

[19] The main gist of this section was taken from an addition ('Zusatz'), which cannot be found in the German edition of the *Encyclopädie* (1830^3, 1817^1) that I've used so far, but which can be found in Geraets, Suchting and Harris' (1991) translation.

Magnitude is arbitrarily chopped up into Many Ones. So Quantum unites both moments (Hegel 1830^{3}, 1817^{1}: §101).[20]

3.11. Unit and Amount

α) Unit is Quantity taken in its moment of Attraction. Hence, the Unit determines the limit to the One. But since this limit is indeterminate, the One is Continuous in itself as well as into Other Ones and thus may be divided into as many smaller Ones as we please. So, a dozen, a pair, a hundred, a million etcetera may all serve as Unit (Hegel 1830^{3}, 1817^{1}: §102).

β) From the standpoint of Repulsion, however, Ones are self-contained Units that exclude the Other Ones from themselves. As such they are distinguishable as Many, so their Amount may be determined (e.g. by counting) (Hegel 1830^{3}, 1817^{1}: §102).

3.12. Number

γ) When you specify a certain Amount of a certain Unit, the result is a Number. With Number we shed the last vestiges of quality. It derives its meaning from its relations to other Numbers instead of from its relations to other concepts, as will be elaborated upon in the next section (Hegel 1830^{3}, 1817^{1}: §102).

---

[20] Nowadays, physicists tend to think of a quantum as the smallest possible package of light or other waves that can be radiated (cf. footnote 10). Low frequency quanta have little energy and high frequency quanta have high energy. Low energy quanta have little influence on a particle's speed, but hardly illuminate its position. For high-energy quanta it is the other way round. This so-called quantum hypothesis led Heisenberg to formulate his uncertainty principle (see footnote 14). An implication of this principle is that the laws of Newtonian physics break down for particles that are smaller than a certain limit. Below this limit nature can no longer be described by deterministic laws, but by chance only (Hawking 1998: 69-73). It must be stressed that this modern interpretation of the quantum is *not* what is at stake here.

At this point, Hegel introduces what I call a 'side dialectic'.[21] That is, he sidesteps his main argument for a while in order to develop a further understanding of Numbers and arithmetical operations. In a sense he views this realm as an object totality in its own right. The principle of this object totality 'can only be found in the determinations that are contained within the concept of [N]umber itself.' (Hegel 1830[3], 1817[1]: §102, Geraets, Suchting and Harris' 1991 translation). These are: α) Unit and β) Amount. The γ) resolution of these two oppositional concepts in the 'side dialectic' cannot be given in just two moments. Instead, Hegel distinguishes four moments, three of which are arithmetical operations that he orders according to the degree to which the opposition between Unit and Amount is preserved in that operation.

γ) First of all, as Many Ones as we please, have to be taken together and Numerated. Numeration prepares a colligation of Many Ones for a quantitative treatment. However, further arithmetical operations can only be performed upon them, if what is taken together is already numerical (Hegel 1830[3], 1817[1]: §102). For example, if you count the elements in a set, 1, 2, 3, …, n, you have made that set quantitative, but no further calculations can be performed upon it without a further Magnitude: a set of m elements to be added to it, or b colligations (i.e. sets) of n elements, etc. Without this further Magnitude all that is established is that all the numerated elements can be taken together as a Unit, while they are still numerable as Amount (which implies that the elements themselves are also Units). So, it is still mainly their Unit that is established so far.

---

[21] Within his §§, Hegel always first explains his main argument (in this case how Unit and Amount spring from Quantum and Number). In his *Encyclopädie* he usually, but not always, widens the left margin next. In the wider margined section he explains himself further either to all readers that need further explanation or to a specific group of readers such as mathematicians. The mentioned 'side dialectic' was taken from such a section.

There are similar sections in the *Wissenschaft der Logik*, which Hegel calls remarks ('Anmerkungen'). In the Petry (1970) translation of the second part of the *Encyclopädie*, the wider margined sections are also separated off from the main text under the heading 'Remark'. In the posthumously published fourth edition of the *Encyclopädie* (1832) the first part of which was translated by Geraets, Suchting and Harris (1992), you will also find additions ('Zusätze'), which are based on lecture notes of Hegel's students. In the Geraets, Suchting and Harris (1992) translation, like in the original, these are distinguished from the main text by the use of a smaller font. Petry (1970) also had the additions printed in a smaller font.

γ) Next, sets generally consist of unequal amounts of elements. So a different Number is applicable to each lot. Counting these lots together is the first arithmetical operation: Addition (Hegel 1830[3], 1817[1]: §102).

γ) Secondly, Numbers may be considered as equal rather than different (i.e. b sets of n elements), so that the moment of Unit is itself an Amount (that is, each of b sets, or units, itself consists of an Amount of n elements). Saying something quantitative about these equal Numbers is Multiplication. In Multiplications, Unit may be taken for Amount and the other way around, because $b \cdot n = n \cdot b$. That is, in multiplication the two moments are still distinguished, but it does no longer matter what you regard as Unit and what as Amount (Hegel 1830[3], 1817[1]: §102).

γ) Thirdly, in Raising to even powers the distinction between Unit and Amount can no longer be made, because in Raising to even powers every Number only bears on itself. This means that Unit and Amount are completely equal under this operation. Hegel therefore concludes that the opposition between those two moments is entirely resolved in Raising to even powers, so there can be no other modes of calculation. However, Unit and Amount can be taken together as well as taken apart. So next to the three positive arithmetical operations discussed, there are also three negative ones (viz. Subtraction, Division and Taking the n-root) (Hegel 1830[3], 1817[1]: §102).

When a Number is Raised to an odd power however, Unit and Amount are again unequal. If a number is Raised to the power of three, for example, this can be written as $b \cdot b^2$. In b^2 Amount equals Unit, thus forming a new Amount. With b^2 being the Amount, b must be the Unit in this expression. Thus, b^2 and b can no longer be the same ontologically when Raising to an odd power, so the difference between Unit and Amount resurfaces (Carlson 2002: 36).

3.13. Intensive and Extensive Magnitude

α) Expressing a certain Amount of a certain Unit is to specify a Discrete Number. In the first instance this Number is set apart from other Numbers. As such the Number is an Intensive Magnitude or Degree (Hegel 1830[3], 1817[1]: §103; Hegel 1812, 1813, 1816: 250, 2.2Ba).

β) In itself however, this Intensive Magnitude is entirely meaningless. It derives its meaning from what it is not, from what lies beyond it. That is, a Number is a Degree or Intensive Magnitude on an Extensive scale. Without this scale the Intensive Magnitude would not make sense. A hundred for example is what it is because it is one more than 99 and one less than 101. So a Number is only defined through its relations to other Numbers (Hegel 1812, 1813, 1816: 256, 2.2Bb). In this sense the quality of the Quantum is external to itself. This externality is explicitly expressed in Extensive Magnitude.

The Extensive Magnitude is always beyond the Intensive Magnitude posited and the fact that every 'beyond' (e.g. 101 in the example above) can again be taken as an Intensive Magnitude, implies an infinite progression towards a beyond beyond every beyond. Thus, the Extensive Magnitude progresses towards a bad potential infinity (Hegel 1830^3, 1817^1: §104).

This progression, however, is the result of transfinite iterations of one and the same operation. E.g. assuming an element 1 as an Intensive Magnitude and a function that adds 1 to it leaves you with the – denumerable infinite – set of natural numbers N.[22] So to get a series one needs to assume an Intensive Magnitude of the Quantum and a function that specifies how to arrive at the other elements in the series. As was indicated in section 2, this function is the locus of the true quantitative (mathematical) infinity (Hegel 1812, 1813, 1816: 260-264, 276-278, 2.2Ca-2.2Cc; Fleischhacker 1982: 143-147; Lacroix 2000: 311-315). In the *Encyclopädie* however, Hegel only mentions the infinite quantitative progression that takes place in Extensive Magnitude. He does not go into its resolution the way he does in the *Wissenschaft*.

C. Measure

[22] Mathematically, the set N requires three axioms. First, there is an element $1 \in N$. Second, each element $\in N$ has a successor that is exactly 1 element larger than the previous element. Third, all elements thus obtained $\in N$. The set Z is obtained by expanding N with zero and the negative numbers. Next, Q is obtained by dividing Z by N. I am grateful to Wouter Krasser for help with this example.

3.14. Measure

γ) Our careful examination of the realm of Quality left us with a multitude of elements distinguishable through Repulsion, but arbitrarily divisible through Attraction. Thus, the basis for the quantitative was a result of the failed attempt at making Qualitative distinctions through the examination of Quality alone. In the realm of Quantity however, we established that an Intensive Magnitude must go beyond itself into its Extensive Magnitude to end up beyond Quantity. So in the end, all distinctions that were posited in the realm of Quantity dissolve in the bad potential infinity. So neither the qualitative nor the quantitative realm can stand on its own. Each ends up as the other and has its meaning in that other. This inextricable relation between the two realms is expressed in Measure. As such, Measure is a qualitative Quantum (Hegel 1830^3, 1817^1: §106-107).

4. Hegel's Determination of Mathematical Mechanics

In this section the concepts of mathematical mechanics, which encompasses geometry, are dialectically determined. Mathematically, the most important ones here are mathematical Space, the Point, the Line, the Plane and the 'spatial figure' (i.e. Distinct Space). The argument largely depends upon part 2 (the philosophy of nature), subdivision 1, section A of the *Encyclopädie* (Hegel 1830^3, 1817^1: §§253-261).

A. Space and Time

4.1. Space

Space is the universal principle of all material things, because all material things are spatial.[23] So the natural sciences, which study material observables, presuppose Space. As such their object resides not in the realm of quality, which

[23] As a universal principle that is not, like Being, itself part of an opposition, Space is in neither of the categories α), β), or γ).

is internal to Something, but in the realm of external relations, i.e. Quantity, only (cf. section 3.8). In conceptual isolation Space is distinctionless, continuous (Hegel 1830[3], 1817[1]: §254) and infinite.

4.2. Spatial Dimensions

α) The first distinction that can be made within Space is between the three Spatial Dimensions, height, length and breadth. Each of these is determined vis-à-vis the other two, but other than that it does not matter at all what direction you called height, length or breadth in the first place (Hegel 1830[3], 1817[1]: §255).

4.3. The Point

β) The Point resides in Space but it is not spatial for it has neither area, nor body. One might also say that the point is 'given entirely as limit' (Paterson 2004/2005: 18).[24] As such it can function as a reference point for qualitative distinctions in Space, but it cannot itself have material Presence ('Dasein') in Space.

Mathematically, the reference Point can only have the coordinates (0,0,0), since the dimensions height, length and breadth are still indeterminate. Hence the orientation of the mathematical axes cannot be determined yet, but there can be a mathematical origin (the Point 0,0,0) around which the axes may pivot freely.

4.4. The Line

γ) The opposition between Spatial Dimensions and the Point is only partially resolved in the Line (Hegel 1830[3], 1817[1]: §256). A Line is defined by one direction and at least one Point it passes through.[25] It is Dimensional in that it has

---

[24] Paterson's 2004/2005 paper is entitled: *Hegel's Early Geometry*. In it, Paterson mainly discusses Hegel's *Geometrische Studien* (abbreviated by Paterson to GS) and his *Dissertatio Philosophica de Orbitus Planetarum*. However, 'the discussion of GS […] is fundamental for all of [Hegel's] later thought on the subject' (Paterson 2004/2005: 2) and the dialectic of Point, Plane and Distinct Space (or 'solid') is basically the same in his early and mature writings. Hence, Paterson's comments are also helpful to elucidate the *Encyclopädie's* account of geometry.

[25] Hegel would not have accepted that a (straight) line can be defined by two points it passes through, because those points must be points in Space and understanding Space involves the Spatial Dimensions *and* the Point. Without the Spatial Dimensions therefore the two points are not necessarily points in space, so it is not clear what they define, but it cannot be a line in space (cf. Hegel 1830[3], 1817[1]: §256, where he criticizes Kant's definition of a straight line).

direction and it is positional in that it passes through a Point (and from a perspective at a right angle to the line it *is* a Point). But, just like the Point, the Line has neither area, nor body and it too is not spatial in that sense, so it still does not allow us to make any true qualitative distinctions in Space. The Line then, is the extension (unlimiting) of the Point in Space. But because it too is not spatial, it still is itself a limit, which requires further unlimiting in order to gain positive Being (Paterson 2004/2005: 29).

Mathematically, it is natural to call the Line either height, length or width. In distinctionless Space this is still a matter of choice or convention.

4.5. The Plane

γ) The opposition between Spatial Dimensions and the Point is further resolved in the Plane (Hegel 1830[3], 1817[1]: §256). Two directions and at least one Point can define a Plane. The Point then defines the position of the Plane and the two directions define its orientation. The unbounded Plane can divide the unbounded Space into two, so the Plane – other than the Point and the Line – provides the first distinction of Space. However, the Plane has no spatial existence as well, for it has an (unbounded) area, but still no body. Thus, although the Plane is the positive Being of the Line it is still a limit or negative Being at the same time (Paterson 2004/2005: 29).

Since positing the Plane involves two different directions, it must involve two dimensions as well. By definition, the third dimension is at a right angle with this Plane. Therefore, all dimensions are determined in positing the Plane. This means that the whole coordinate system is now determined.

4.6. Distinct Space

γ) At least four (flat) Planes can form an enclosing surface that separates of a Distinct (part of) Space (‘einzeln Raum’) (Hegel 1830[3], 1817[1]: §256). This Distinct Space finally has spatial existence, for it has area and content. Since Hegel recognizes only three Spatial Dimensions (section 4.2.), unlimiting Distinct Space leads to its expansion or contraction without further conceptual change

(Paterson 2004/2005: 33-34). So, in positing Distinct Space the opposition between the indeterminate Spatial Dimensions and the non-spatial Point is fully resolved.

The dialectic of Line-Plane-Distinct Space can also be viewed as a process of integration. That is, integrating the formula for a Line, $y = a\,x + b$, gives us the area of the Plane beneath the graph of the line, $y = \frac{1}{2}\,a\,x^2 + b\,x$. Integrating this again yields the content of Distinct Space, $y = {}^{1}/_{6}\,a\,x^3 + \frac{1}{2}\,b\,x^2$ (Paterson 2004/2005: 54). To do this, however, is to presuppose the existence of a coordinate system, whereas this can only be determined *after* the Plane has been posited.

4.7. Time

γ) In section 3.4 it was said that a Presence is here and now, but the concepts here and now themselves were left implicit. Distinct Space allows us to separate of an abstract part of Space that may be designated 'here'. When we look at a Distinct part of Space, we may perceive Something. But Something will eventually and inescapably Become its Other through Becoming. So when we confine ourselves to some designated Distinct part of Space, we may observe Becoming in action. It is this change in Space that constitutes our awareness of Time.[26] Time then, is observed Becoming ('das angeschaute Werden') (Hegel 1830^3, 1817^1: §258; Inwood 1992: 295).[27]

4.8. Temporal Dimensions

[26] Inwood writes on this subject: '[T]he measurement of time, and our perception of its passage, require movement in *space*, esp. of the heavenly bodies.' (1992: 295). True as this may be, to think of Time like this distorts the presentation, for Motion and Matter are presented *after* Time.

[27] The view that our consciousness of time first and foremost requires a notion of process (i.e. Becoming) in conventional (or conceptual – as opposed to absolute) Distinct Space, remarkably squares with the philosophical moral that Stachel draws *contra* Yourgrau (2005) from the post-Einstein time-relativistic philosophical debate on the objective reality of time in four-dimensional space-time. As he puts it: 'The philosophical moral I draw from this discussion is that process is primary and absolute, while its division into spatial states evolving over time is secondary and always relative to the choice of some frame of reference, local or global. Translated into the language of relativity theory, space-time takes precedence over space and time' (2007: 867).

α) Our awareness of Time involves three Temporal Dimensions: the present, the future and the past (Hegel 1830[3], 1817[1]: §259). The present is clearly linked to Presence. Thus the present is long enough to Become aware of whatever is present (i.e. Presence), but short enough to minimize the changes in the Presence under scrutiny.

Not surprisingly this means that the present is the 'shortest' of the three Temporal Dimensions. In the past there were a series of changes that led Presence to Become the present Something and there are a series of changes that will shape the future of the Other Ones that will Become of it. Hence there can be a lot of changes to Ones in the past and in the future, but close to none in the present. In other words, the past and future are defined by change, while the present is an almost changeless period.

I am aware that treating Time as if it consists of three dimensions instead of only one must seem quaint. What is more, it is incongruent with the basically Euclidian determination of the Point, Line, Plane and Distinct Space in the previous sections. However, you might think of a Dimension not only as a direction, but more generally as an unbounded medium in which something can develop. As such, a Dimension is Extensive Magnitude taken spatially or temporally.

We have seen that Presence, and hence the present, moves along with Becoming. In Hegel's defense therefore, one might say that Presence is indeed just as resilient to change as the spatial dimensions, so it can be thought of as an unbounded medium and hence as a dimension. Even if everything that is, has some definite beginning in the Big Bang, the past is just as unbounded, for the present continually leaves more of it behind while it is Becoming. Finally, the Becoming of the present can only be ceaseless if the future is unbounded. So one may indeed think of the past, present and future as unbounded mediums.

<u>*4.9. Now*</u>

β) Absolute changelessness is impossible in any period, no matter how short. So if we want to interpret the present as a changeless period, we need to shorten it infinitesimally. An infinitesimally short and hence truly changeless period is Now (Hegel 1830^{3}, 1817^{1}: §259). Like the Point is *in* Space, but not spatial, Now is *in* Time, but not temporal.

4.10. Place

γ) Space in its continuous emptiness and infinity can be subdivided into Distinct Spaces (sections 4.1-4.6). Those Distinct Spaces, however, are still empty and indifferent with respect to their content. But the abstract requirement that all we can perceive, think or imagine, is Presence in a state of Becoming (section 3.3-3.4) not only constitutes Time in Distinct Space, it also means that any Distinct Space in Time contains some kind of Presence. It is the awareness of this Presence that enables us to speak of a Distinct Space as a Place. So while Time is observed Becoming, Place is observed Presence. As such, it is the union of Distinct Space, Time, the Temporal Dimensions and Now, or the union of determinate Distinct Space, here, and Now (Hegel 1830^{3}, 1817^{1}: §260).[28]

4.11. Motion

γ) Positing Place as the union of here and Now, still does not position that Place spatially, because the location of the Point and Distinct Space in empty, infinite and continuous Space cannot be determined for lack of a reference location which is not arbitrarily determined. Therefore every Place is just the same as every other Place, the way Something and its Other are the same as (Many) Ones.

Because Distinct Spaces are arbitrarily fenced of parts of empty, infinite and continuous Space, there is nothing inherent in them that prohibits any One to roam freely from one such Space to another. Hence the Ones can change Place. When this happens this constitutes our awareness of Time and when it does not it constitutes our awareness of Place. As we have seen in section 4.9, true changelessness only happens in the Now. Hence Place is the spatial Now. Since

[28] This latter remark is taken from an addition ('Zusatz') to the main text. So it is not found in Hegel's own writing, but was added posthumously on the basis of transcripts of his lectures (see footnote 19).

Now is infinitesimally short, it immediately passes over into Time and in Time the Ones change their Place, thus constituting Motion (Hegel 1830^3, 1817^1: §261).[29]

<u>*4.12 Matter*</u>

γ) The passage from the spatial Now, Place, to the temporal Space, Now, i.e. Motion, still leaves every One in a notorious state of flux. Motion then is not observed Becoming (i.e. Time), but the Becoming of the natural realm as a whole. Therefore it is not the qualitative Becoming of Presence and Something and Others, but a quantitative Becoming. Quantitative Becoming is the Becoming of everything spatial, i.e. Matter. Matter therefore is the actual Presence (and not only the observed Presence) of the natural realm (Hegel 1830^3, 1817^1: §261).

Hegel does not explicitly phrase these points like this. He does call Time 'observed Becoming' ('das angeschaute Werden') (Hegel 1830^3, 1817^1: §258) and Motion 'this Becoming' (Hegel 1830^3, 1817^1: §260), but the comparisons to Presence are the result of interpretations enabled by the immanent link between Presence and Present, which does not exist between the German terms 'Dasein' and 'Gegenwart' respectively.

Summary and Conclusions: How This Dialectic Reflects on Mathematics

In this paper I have dealt with the question what insights can be gained from a systematic dialectical perspective on the conceptual foundations of set theory and mathematical mechanics. In systematic dialectic methodology words are ordered according to their internal meanings and conceptual interrelations. This is done by positing and resolving opposites.

In section 3 we established that the non-thought of Being immediately Becomes the non-thought of Nothing. Thus, Becoming is the first real thought in the presentation. The dynamism inherent in Becoming requires a further *static*

[29] It may also be the case that Place changes its Ones. That is, there is no way to tell if things only move relative to one another in a motionless coordinate system or whether that coordinate system is itself moving, so that some things that appear to be moving are actually standing still.

determination of Being as Presence – my translation of Hegel's 'Dasein'. Presence moves along with Becoming. If it did not, that is if its determination were to be fixed at some point, it is Something.

Because Something's determination is fixed, it can be left behind in the process of Becoming. When Something is left there-and-then, its Other is here-and-now. But by Being here and now, this Other is itself Something. Hence Something and it's Other are the same in the concept One. In that the process by which Something Becomes its Other, which, taken as Something, again has an Other, is a *ceaseless* Becoming, there must be Many Ones.

As self-contained units, these Ones are distinguishable As Many through a relation of Repulsion, but because everything is in a notorious state of flux through Becoming, it is not clear where Something ends and its Other begins. Hence the One is only limited through an arbitrary external reflection. In that this limit is arbitrary, one might think of many more Ones within the unit One. So, from the standpoint of Becoming the Many Ones have a relation of Attraction instead of Repulsion.

The presentation of the realm of Quality so far only succeeded in positing that there must be Many Ones, but that the limit to the One is entirely arbitrary. Thus, all that we are left with is an external reflection on a multitude of elements that are distinguishable as Many through Repulsion, but arbitrarily divisible through Attraction. With it, we have entered the realm of Quantity. Hegel was ahead of his time in arriving at this conclusion. Quantity is a Discrete Magnitude of elements, but because the limit to these elements can only be arbitrarily determined through external reflection, it must be a Continuous Magnitude at the same time.

A Quantum is a specified Quantity. In the first instance, it is a Discrete Magnitude that excludes other Magnitudes from itself. However, the *range* of possible Magnitudes is Continuous. That is, depending on how the One is limited, the same Magnitude may have every imaginable Discrete size. As we have seen, if the One is taken in its moment of Attraction, one may think of Many more Ones within any Unit One. Thus quantified, the One is the Unit of a Magnitude. Next, the Amount of a certain Unit can be determined (e.g. counted) and expressed in a Number.

In the first instance a Number is expressed as an Intensive Magnitude. This Intensive Magnitude derives its meaning from what it is not, that is, from the Extensive Magnitude it excludes. So, to specify a Quantum is to specify a Degree or Intensive Magnitude on an Extensive scale. In that an Intensive Magnitude must continually go beyond itself into its Extensive Magnitude to gain meaning, the Extensive Magnitude progresses towards a bad potential infinity, which is no longer quantitative.

So just as our reflection on Quality led us into the realm of Quantity, our reflection on Quantity led us back into the Qualitative unreachable and unnamable bad potential infinity. The relation of both realms is made explicit in Measure. As such, Measure is a qualitative Quantum.

The conclusions that will be drawn from the presentation of section 3 are largely based on Baer (1932). However, Baer does not present them systematically. Rather, he presupposes that the reader is quite well versed in Hegelian philosophy and proceeds to explain where Hegel stood in the mathematical debates of his time. Baer concludes that Hegel was ahead of his time on more than one occasion and that when his opinion differs from more modern views he is usually loyal to the ideas of the mathematicians of his time (1932: 109, 112-113). By contrast, the current paper is philosophical rather than historical in nature and aims at a wider audience.

The first mathematical problem that is solved by the systematic dialectical exhibition of section 3 is that of number theory. To apprehend numbers with the tools of mathematics, we need to have those tools first. But to build up these tools, especially induction, we first need to have all natural numbers (the complete set N) (Baer 1932: 113). In mathematics this problem is ‘solved’ by assuming a ‘one’ and a function that increases the assumed one by one (i.e. a successor function). Although this is a way to build up the full set of natural numbers, it does not explain the origin of the mathematical mindset.

The exhibition above offers a solution to this problem by showing that a reflection on the realm of quality automatically leads the presentation into the realm of Quantity. This happens because the attempt at truly qualitative distinctions fails, which leaves us with an external reflection on sets of arbitrarily chosen elements One (Q.E.D.). Hegel was ahead of his time in recognizing this as the subject matter of mathematics. In that the elements One must eventually Become an Other One, they form the Many Discrete Ones upon which the natural numbers are based. In that the elements are arbitrarily chosen and divisible, this set is further determined as the Continuum R.[30]

Thus, for Hegel the quantitative is only the necessary external reflection on the realm of Quality, without which neither realm can exist. This notion must appear alien to most modern mathematicians, for as long as statements are correctly deduced from given axioms, they do not generally care about their ontological import. In short, mathematicians nowadays do not inquire after the things themselves but only after (hypothetical) relations between them (Baer 1932: 108). In doing so they treat the quantitative as if it were a finished actuality (Fleischhacker 1982: 125).

Now, if we accept Hegel's view that the Quantitative exists because of the failed attempt at making qualitative distinctions on the basis of quality alone, this means that quantitative mathematics (including set theory) can never fully be this entirely free-floating subject that some modern mathematicians have made it into. Its basis (viz. the rationale behind sets of elements) namely is still qualitative.

On the other hand, the quantitative was determined as the realm of external reflection on a multitude of distinguishable yet arbitrarily divisible elements. This means that there is considerable scope to escape this qualitative basis by studying quantitative relations on their own account as most modern mathematicians do.

[30] One might comment this only means Numbers are now justified linguistically rather than mathematically. So instead of solving the problem, dialectics has only shifted its locus. However, a central thesis of dialectics is that thought presupposes concepts. So it is due to the concepts one has, that one can make the transition to mathematical thinking at all. It is the task of dialectics to make people conscious of this and to make them aware of the unconscious processes that must have happened in their brain before they started specializing on the basis of certain fixed definitions and methodologies. I am grateful to Wouter Krasser for insightful discussions on this topic.

But without the qualitative there would be no quantitative, hence there would be no Numbers and no mathematics. According to Kol'man and Yanovskaya this insight is one of the greatest merits of Hegelian philosophy in the field of mathematics (1931: 2, 5).

The second thing that is clarified by the presentation in section 3 is the proper use of ordinal and cardinal Numbers. In the 'side dialectic' of section 3.11 it was said that Numeration prepares a qualitative colligation of Many Ones for a quantitative treatment. In the first instance Numeration involves ordinal Numbers. If one counts the Amount of elements in a finite set you have to begin somewhere, so while counting, you implicitly call One element the first, One the second etc. But if the Numeration is complete, we have arrived at an Intensive Magnitude (Baer 1932: 115).

Within that Intensive Magnitude it no longer matters which element we counted first and which second, so the arbitrary distinctions between the elements that were created by the form of the series (e.g. 1,2,3,…,n) of the concomitant Extensive Magnitude have now disappeared. Every element in a set of size n may be the n^{th} element. Then the size of the set (n) is a cardinal Number. This cardinal Number is the Intensive Magnitude of the finite set. As such the size of the set is itself a Unit: it expresses the number of elements it contains while denying them autonomy.

If you count the Amount of elements in the infinite set of natural numbers N you count not to n but to ω. By analogy to finite sets, ω is the infinite ordinal number associated with the last step in a – physically impossible - complete numeration of all elements in N.[31] The smallest infinite cardinal Number $\aleph_0$, which is associated with ω, measures the size (cardinality) or Intensive Magnitude of N. If a set is expanded with all subsets contained within it, this is equivalent to raising 2 to the power of the number of elements in that set. Thus, the Intensive Magnitude of the power set of N, P(N) (i.e. the continuum R) is $2^{\aleph_0}$ and that of

[31] If you would count first odds and then evens, you would reach the ω element twice, because both sets correspond bijectively to the set N. So both odds and evens are of the same size as N. This is why infinite ordinal and cardinal numbers need to be more strictly distinguished from each other than finite ones (Horsten 2004: 25).

P(R) is 2 to the power of $2^{\aleph_0}$, etcetera (Horsten 2004: 26). It can be proven that each set thus obtained is of a higher order of infinity than the previous set. This implies that infinite cardinal Numbers may themselves be ranked in a well ordering.[32] So just as transfinite iterations of a successor function lead a finite Intensive Magnitude into the bad potential infinity associated with its Extensive Magnitude, transfinite iterations of the power operation lead an infinite Intensive Magnitude into the bad - or infinitely worse - potential infinity associated with the size of the class of all sets V. In short: even for an infinite Intensive Magnitude there exists an Extensive Magnitude through which it gains meaning.

'[S]ince the power set of N for example, contains an enormous Amount of infinite sets as elements that therefore must be seen as complete, 'finished', limited objects', this way of thinking implies 'the existence of an enormous amount and enormously big *actually infinite sets*' (Horsten 2004: 27, my translation).[33] This fact, together with the fact that even infinite Numbers can be ordered to fit Hegel's conceptual apparatus, at least partially dispenses with the 'badness' of Hegel's bad potential infinity. Because the founding father of set theory, Georg Cantor, was born after Hegel's death, Hegel cannot possibly have been aware of these points.

Finally, section four showed it would be misguided to think of the Line as the result of the movement of a Point or of the Plane as resulting from a moving Line. This is because Time and Motion presuppose Distinct Space. But the real novelty with regard to the determination of mathematical mechanics in this paper stems from the chosen translation of 'Dasein' as Presence which enabled me to describe Place as observed Presence, (the spatial Now), Motion as the passage from the spatial Now (Place) to the temporal Space (Now) and Matter as the actual Presence (as opposed to the observed Presence) of the natural realm.

---

[32] It is remarkable that within the traditional Zermelo-Frankel axioms + Choice (ZFC), only the power operation can be proven to change the order of infinity of an already infinite set. Yet the continuum hypothesis, which states that the size of R corresponds to the second smallest infinite cardinal number $\aleph_1$, is independent of ZFC and can be proven true nor false. So, although a well-ordered list of infinite cardinal Numbers may be obtained by reiterating the power operation, it can only be labeled $\aleph_0$, $\aleph_1$, $\aleph_2$, etcetera, if the continuum hypothesis is adopted (Horsten 2004: 93-94).

[33] I am indebted to Marcel Boumans and Tijmen Daniels for help with these points.

References

Superscripts behind a publication year denote editions. The edition that was actually used is always cited first. Thus Hegel, Georg W. F. (1830^3, 1817^1) means that I relied on the third edition of the *Encyclopädie* throughout the article and that the first edition of that work was published in 1817. If a work was published in several parts, the publication years of the different parts are cited without superscripts.

Alberts, Gerard (1998). *Jaren van berekening; Toepassingsgerichte initiatieven in de Nederlandse wiskunde-beoefening 1945-1960*. Amsterdam: Amsterdam University Press.

Baer, Reinhold (1932). Hegel und die Mathematik. In Veröffentlichungen des Internationalen Hegelbundes, *Verhandlungen des Zweiten Hegelkongresses vom 18. bis 21. Oktober 1931*. Berlin & Tübingen: s.n., 104-120.

Carlson, David G. (2000). Hegel's Theory of Quality. *Public Law Research Paper, 17*. New York: Cardozo Law School. Consulted version: http://ssrn.com/abstract=241950.

Carlson, David G. (2002). Hegel's Theory of Quantity. *Cardozo Law Review 24, 6*. Consulted version: http://ssrn.com/abstract=326822.

Carlson, David G. (2003a). Hegel's Theory of Measure. *Public Law Research Paper, 66*. New York: Cardozo Law School. Consulted version: http://ssrn.com/abstract=413602.

Carlson, David G. (2003b). The Antepenultimacy of the Beginning in Hegel's Science of Logic. *Public Law Research Paper, 74*. New York: Cardozo Law School. Consulted version: http://ssrn.com/abstract=425122.

Dijkgraaf, Robbert (2001). *Symmetrie; Een inleiding in de wiskunde*. Amsterdam: Faculteit Natuurwetenschappen, Wiskunde en Informatica, Universiteit van Amsterdam.

Ellsworth de Slade, H. (1994). *Das wahrhafte Unendliche und die Unendlichkeit der Mathematik; Eine Studie zu den Entsprechungen zwischen Hegels Bestimmung des wahrhaften Unendlichen in der "Wissenschaft der Logik" und seiner Auffassung der Infinitisimal-Mathematik.* Heidelberg: Philosophisch-Historischen Fakultät der Ruprecht-Karls-Universität Heidelberg.

Fleischhacker, Louk E. (1982). *Over de grenzen van de kwantiteit*. Amsterdam: Mathematisch Centrum.

Hawking, Stephen (1998). *Het Heelal: Verleden en Toekomst van Ruimte en Tijd.* Amsterdam: Uitgeverij Bert Bakker.

Hegel, Georg W. F. (1812, 1813, 1816). *Wissenschaft der Logik I; Erster Teil, Die objective Logik, Erstes Buch*. Moldenhauer, Eva. & Michel, Karl M. (Ed.). Frankfurt am Main: Suhrkamp Taschenbuch Verlag. Engl. transl., Miller, Arnold V. (1969). *Hegel's Science of Logic*. St. Leonards: George Allen & Unwin.

Hegel, Georg W. F. (1821). *Naturrecht und Staatswissenschaft im Grundrisse* or *Grundlinien der Philosophie des Rechts.* Berlijn: Nicolai'schen Buchhandlung. Lasson, Georg (Ed. 1930). Leipzig: Verlag von Felix Meiner.

Hegel, Georg W. F. (1830[3], 1817[1]). *Encyklopädie der Philosophischen Wissenschaften im Grundrisse*. Heidelberg: Verwaltung des Oßwald'schen Verlags (C.F. Winter.). Lasson, Georg (Ed. 1930). Leipzig: Verlag von Felix Meiner. Engl. transl. 1: Geraets, Theodore F.; Suchting, Wallis A. & Harris, Henry S. (1991). *The Encyclopaedia Logic (with the Zusätze); Part I of the* Encyclopaedia of Philosophical Sciences *with the Zusätze*. Indianapolis: Hackett Publishing Company, Inc. Engl. transl. 2, Wallace, William (1873). *Hegel's Logic*. Oxford: Oxford University Press.

Hofstadter, Douglas R. (1979). *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Basic Books.

Horsten, Leon (2004). *Eindig, Oneindig, meer dan Oneindig: grondslagen van de wiskundige wetenschappen.* Utrecht: Epsilon Uitgaven

Inwood, Michael (1992). *A Hegel Dictionary*. Oxford: Blackwell.

Kol'man, Ernst & Yanovskaya, Sofya (1931). Hegel & Mathematics. In Yanovskaya, S. (Ed. 1983), *Marx's Mathematical Manuscripts.* London: New Park Publications. Consulted version: www.marxists.org/reference/subject/philosophy/works/ru/kolman.htm.

Kosok, Michael (1972). The Formalization of Hegel's Dialectical Logic; Its Formal Structure, Logical Interpretation and Intuitive Foundation. In MacIntyre, Alasdair C. (Ed.), *Hegel; a Collection of Critical Essays.* London: University of Notre Dame Press, 237-287.

Lacroix, Alain (2000). The Mathematical Infinite in Hegel. *The Philosophical Forum, 3-4, XXXI,* 298-327. Consulted version: www.blackwell-synergy.com/doi/pdf/10.1111/0031-806X.00043

Paterson, Alan L. T. (1994). *The Concept of the Propositional Calculus*. Preprint. Consulted version: http://home.olemiss.edu/~mmap/HPROPC.pdf.

Paterson, Alan L. T. (1997a). Towards a Hegelian philosophy of mathematics. *Idealistic Studies, 27,* 1-10. Consulted version: http://home.olemiss.edu/~mmap/TOHEGFF.pdf.

Paterson, Alan L. T. (1997b). Self-reference and the natural numbers as the logic of Dasein. *Hegel Studien, 32,* 93-121. Consulted version: http://home.olemiss.edu/~mmap/HEGSTUFR.pdf.

Paterson, Alan L. T. (1999) *Frege and Hegel on concepts and number*. Preprint. Consulted version: http://home.olemiss.edu/~mmap/nnfregh.doc.

Paterson, Alan L. T. (2000). The Successor Function and Induction Principle in a Hegelian Philosophy of Number. *Idealistic Studies 30 (1)*, 25-61. Consulted version: http://home.olemiss.edu/~mmap/inductio.doc.

Paterson, Alan L. T. (2002). Does Hegel have anything to say to modern mathematical philosophy? *Idealistic Studies, 32*, 143-158. Consulted version: http://home.olemiss.edu/~mmap/sjp6.doc.

Paterson, Alan L. T. (2004/2005). Hegel's Early Geometry. *Hegel Studien 39/40,* 61-124. Consulted version: http://home.olemiss.edu/~mmap/Heggeom7.doc.

Priest, Graham (1989). Dialectic and dialetheic. *Science & Society, 4, 53,* 388-415.

Reuten, Geert, & Williams, Michael (1989). *Value-Form and the State; The tendencies of accumulation and the determination of economic policy in capitalist society*. New York: Routledge.

Reuten, Geert (2005). Money as constituent of value; the ideal introversive substance and the ideal extroversive form of value in 'Capital'. In Moseley, Fred (Ed.), *Marx's Theory of Money: Modern Appraisals*. London/New York: Palgrave MacMillan, 78-92.

Russell, Bertrand (1903). *The Principles of Mathematics: vol. 1*. Cambridge: Cambridge University Press.

Stachel, John (2007). Book Review: A World Without Time: The Forgotten Legacy of Gödel and Einstein. *Notices of the American Mathematical Society, 54*, 861-868.

Tóth, Imre. (1972). *Die nicht-euklidische Geometrie in der Phänomenologie des Geistes; Wissenschaftstheoretische Betrachtungen zur Entwicklungsgeschichte der Mathematik.* Frankfurt am Main: Horst Heiderhoff Verlag.

Wolff, Michael (1979). Über das Verhältnis zwischen logischem und dialektischem Widerspruch. In: Hegel Gesellschaft, *Hegel Jahrbuch.* München: Hegel Gesellschaft, 340-348.

Wolff, Michael (1986). Hegel und Cauchy; Eine Untersuchung zur Philosophie und Geschichte der Mathematik. In Horstmann, Rolf-Peter & Petry, Michael J. (Eds.), *Hegels Philosophie der Natur: Beziehungen zwischen empirischer und spekulativer Naturerkenntnis.* Stuttgart: Klett-Cotta, 197-263.

Yourgrau, Palle (2005). A *World without Time: The Forgotten Legacy of Gödel and Einstein*. New York: Basis Books.