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Schumacher, J.M.

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Johannes M. Schumacher

Amsterdam School of Economics, University of Amsterdam, Amsterdam, The Netherlands

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ABSTRACT

The bankruptcy problem is to divide a homogeneous divisible good (the "estate") between claimants, when the sum of the claims exceeds the value of the estate. When the problem is looked at from an ex-ante point of view (i.e. before the size of the estate is revealed), it is possible to formulate a notion of Pareto efficiency that is stronger than when the more common ex-post perspective is taken. Under the assumption of common beliefs, the strong notion of efficiency leads, in combination with the requirement that all claims should be fulfilled when the value of the estate is equal to the sum of the claims, to a uniquely defined division rule when utility functions for all agents are given. The resulting rule can be represented in the form of a parametric function. For the case in which all agents are equipped with the same utility function, the class of parametric functions that can be obtained in this way is characterized. In particular, it is shown that two well-known division rules for the bankruptcy problem, namely Constrained Equal Losses and Proportional Division, can be rationalized under strong Pareto efficiency by constant absolute risk aversion and constant relative risk aversion respectively.

1. Introduction

The present paper differs from the mainstream literature on estate division in terms of the formulation of the problem as well as in terms of the solution concept that is applied. These deviations are motivated in the following two subsections.

1.1. Why ex-ante?

The bankruptcy problem is concerned with the division of a homogeneous good among a number of agents who hold certain claims with respect to the amount to be divided (the "estate"), in a situation in which the sum of the claim values exceeds the value of the estate. In the ex-post formulation, which is the usual version, both the claim values and the size of the estate are considered as given. Taking the size of the estate as known is natural indeed, since typically this information *is* available at the point in time at which arbitration takes place. Nevertheless, the situation admits application of a principle of fairness used in contract law, which calls for an *ex-ante* point of view. In the terminology of contract law, the discovery that the size of the estate falls short of the sum of the claims can be viewed as an "unforeseen circumstance", which prevents implementation of the contract as originally agreed between the agents. Such unforeseen circumstances arise quite frequently in commercial business; indeed, it would not be practical for contract parties to negotiate in advance about what should be done in every scenario that might possibly occur, but surprising events do sometimes happen. In such cases, a court of arbitration may be called upon. In taking its decision, the court should be guided by a preformulated rule. How such a rule should be obtained is described by Goetz and Scott (1983, p. 971) as follows: "Ideally, the preformulated rules supplied by the state should mimic the agreements contracting parties would reach were they costlessly to bargain out each detail of the transaction." In other words, the court should *imagine* a situation in which the parties would have negotiated a more comprehensive contract covering a wide range of scenarios, including in particular the scenario that led to the court case. The decision taken by the court for a particular situation should be the one that the agents themselves would have agreed upon as the one that should be followed in this situation, *if* they would have negotiated the more comprehensive contract beforehand.[1] As a consequence, it is meaningful to take an ex-ante point of view, even in a situation that is in fact ex-post.

The rule applied by the court of arbitration is based on *hypothetical*, rather than actual, negotiations by the agents. Agents' preferences are *imputed* by the court, rather than reported by the agents themselves. The court does not need to model the negotiation process as such; instead, it may rely on an axiomatic characterization of the *ex-ante* negotiation outcome. This is in contrast to a large part of the literature on the bankruptcy problem, which focuses on axioms for the *ex-post* outcome (Thomson,

E-mail address: j.m.schumacher@uva.nl.

[1] The idea that a notion of justice can be formed by *suppressing* information is present in the philosophical literature as well. The idea is seen, for instance, in the "veil of ignorance" that appears in the work of Rawls (1971).

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2019, Ch. 3–12). An important difference is that the ex-ante approach is likely to bring in assumptions regarding agents' preferences, whereas such assumptions can be avoided in the axiomatic ex-post setting. A specific proposal for axiomatization of the ex-ante outcome will be made below; this makes use of the notion of expected utility, which is well-established in economics. Game-theoretic approaches to the bankruptcy problem (Thomson, 2019, Ch. 14) also make use of utility; however, this is frequently restricted to linear utility, and applied within an ex-post framework. An example of a class of bankruptcy rules that is defined on the basis of a general notion of utility is the class of equal-sacrifice rules that has been proposed by Young (1988). The motivation for utility functions in the present paper is different however; see Section 4.4 for a comparison with Young's approach.

From an operational perspective, the distinction between the ex-ante and ex-post viewpoints may appear rather subtle. Indeed, the estate value is usually taken as a parameter in the ex-post approach, so that the solution is presented as a multivariate function that specifies the amount that will be received by each of the agents (the awards vector) for any value that the estate may take. The solution in the ex-ante case is presented in exactly the same way. The difference lies in the way that solutions are constructed. In the ex-ante formulation, a stronger notion of efficiency can be applied, since trade-offs are possible between allocations at different estate values. This is further discussed in the next subsection.

1.2. Strong Pareto efficiency

Generally speaking, one can define weaker or stronger notions of Pareto efficiency by considering a smaller or a larger set of alternatives to a proposed solution. A solution that is Pareto efficient within a certain set may no longer be so if the range of competitors is extended by relaxing requirements that solutions should satisfy. In the specific context of the bankruptcy problem, a requirement that may be placed on solutions is that the allotments to agents should be equal to the claims when the estate value is equal to the sum of the claims. This condition will be called *compliance* in this paper. While it is a very natural restriction to impose, under certain circumstances agents may have reasons to deviate. To illustrate this, consider the following example of an ex-ante estate division problem. Suppose that there are only two possible estate values, namely 50 and 100, and that each of these occurs with probability 0.5. There are two agents A and B whose claims are 25 and 75, respectively. They are both subject to log utility so that their expected utilities under a given ex-ante division function $x = (x_A, x_B)$ are given by

$$U_i = 0.5 \log x_i(50) + 0.5 \log x_i(100), \quad i \in \{A, B\}.$$

A division function that satisfies compliance is given for instance by

$$x_A(50) = 5, \quad x_B(50) = 45, \quad x_A(100) = 25, \quad x_B(100) = 75. \tag{1.1}$$

The numerical values of the expected utilities are $U_A = 2.4142$, $U_B = 4.0621$. Under the constraint of compliance, no Pareto improvement of the rule (1.1) is possible. An alternative division function, not constrained by compliance, is the following:

$$x_A(50) = 8.5, \quad x_B(50) = 41.5, \quad x_A(100) = 17, \quad x_B(100) = 83. \tag{1.2}$$

The corresponding expected utilities are $U_A = 2.4866$, $U_B = 4.0723$. The unconstrained rule (1.2) is better for *both* agents than the constrained rule (1.1). The requirement that the sum of awards cannot exceed the value of the estate is still satisfied by the rule (1.2).

The example above indicates a certain type of vulnerability of solutions that follow a particular constraint, but that are not Pareto efficient within an extended set of solutions that violate this constraint.[2] A solution may be said to be *strongly Pareto efficient* (with respect to a particular constraint) if it satisfies the constraint, and is Pareto efficient not only with respect to other solutions that satisfy that constraint but also with respect to a larger set of solutions. In this paper, the notion of strong Pareto efficiency with respect to the compliance constraint will be used. The larger solution set that is used to test efficiency is constrained only by the requirement that the sum of the awards should not exceed the estate value. Under regularity assumptions, it is shown below (Theorem 4.2) that, given preferences of agents in expected-utility form, the constraints of compliance and strong Pareto efficiency together determine a *unique* ex-ante division function.

The origins of the notion of strong Pareto efficiency can be traced back at least to work of Gale (1977) (see also Gale and Sobel, 1979). The combination of compliance and strong Pareto efficiency has been used in the actuarial literature as a solution concept in the work on risk sharing of Bühlmann and Jewell (1979), which builds on Gale (1977). In a different context (fixed-price equilibria), the same solution concept appears in Balasko (1979).

1.3. Related literature

The bankruptcy problem was originally formalized by O'Neill (1982) and is extensively surveyed in Thomson (2019). The distinction between the ex-ante and ex-post viewpoints is discussed for instance by Perles and Maschler (1981) in the context of Nash bargaining games, and by Myerson (1981) in the context of social choice functions. The ex-ante point of view in the bankruptcy problem arises naturally when the problem is considered with some form of uncertainty. Interval uncertainty concerning the estate value (i.e. the estate value is only known to lie in a certain interval, without a probabilistic model) is considered by Brânzei et al. (2003), Brânzei and Alparslan Gök (2008), who relate the problem to a cooperative interval game (Brânzei et al., 2010). Papers that address stochastic uncertainty with respect to the estate value include Habis and Herings (2013) and Koster and Boonen (2019). Studies have been made as well of situations in which also other features, such as claim sizes, are subject to uncertainty; see for instance Ertemel and Kumar (2018), Hougaard and Moulin (2018), and Xue (2018). The ex-ante viewpoint is also natural in studies of incentives created by arbitration rules, as for instance in Kıbrıs and Kıbrıs (2013), Karagözoğlu (2014), and Boonen (2019). In the present paper, incentive effects are left out of consideration.

Necessary and sufficient conditions for a division rule to be derivable from a measure of collective welfare or from the principle of equal sacrifice were obtained by Young (1987, Thm. 2) and Young (1988, Thm. 2) respectively. A similar investigation of rationalizability is undertaken in the present paper; see Section 6. The rationalization problem can also be stated in a broader sense, in which one seeks to characterize division rules that meet what might be called basic criteria of rationality, such as antisymmetry, transitivity, and independence of irrelevant alternatives; see for instance Peters and Wakker (1991), Kıbrıs (2012), and the references given therein.

[2] Of course, this vulnerability is only relevant with respect to constraints that *can* in principle be violated, unlike for instance the constraint in the bankruptcy problem that the sum of the awards does not exceed the estate value.

1.4. Contributions and outline of the paper

To the author's best knowledge, the division principle that is developed in this paper, based on strong efficiency and compliance, has not been proposed before in the literature on the bankruptcy problem. The main technical contributions of the paper are the following:

(i) to show that the proposed principle is valid, in the sense that it leads to a uniquely defined division rule;
(ii) to show that the divisions that follow from the proposed principle have a utilitarian interpretation, in the sense that they can also be obtained from optimization of a certain social welfare function, as well as an interpretation in terms of equal sacrifice;
(iii) under homogeneous utility, to describe the extent to which rationalizations are unique, and to give verifiable necessary and sufficient conditions for rationalizability of a given parametric division rule.

The organization of the paper is as follows. Terminology and notation are discussed in Section 2 and basic definitions are collected in Section 3. The design method is worked out in Section 4, where items (i) and (ii) above are covered. The treatment here allows agents to have different utility functions. This context is specialized in Section 5, using the concept of a "social norm utility function". Division rules in this framework are derived, and the uniqueness question (first part of item (iii) above) is answered. Finally, the second part of item (iii) is covered in Section 6, where various criteria for rationalizability under the principle proposed in this paper are given. Conclusions are stated in Section 7. Most of the proofs have been collected in Appendix.

2. Notation and terminology

The notation and terminology in this paper largely follow Thomson (2019), with a few idiosyncrasies that partly relate to the ex-ante point of view taken in this paper. The set of nonnegative real numbers and the set of positive real numbers are denoted by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively. The terms "increasing", "decreasing", and "monotonic" are used in the strict sense. For instance, when a function $f(x)$ is said to be increasing, this means that $f(y) > f(x)$ for all y with $y > x$. A function that satisfies $f(y) \geq f(x)$ for $y \geq x$ is said to be nondecreasing. On the other hand, the terms "convex" and "concave" are used in the non-strict sense. Strict convexity and concavity will be named as such.

2.1. Division rules

Throughout the paper, a finite set $N = \{1, 2, \ldots, n\}$ is supposed to be given, representing a group of agents (also more specifically called *claimants*) among whom an estate has to be divided. In the ex-ante point of view of the estate division problem, claimants are not concerned with the division of a *given* (fixed) amount of money, but rather with the choice of a division *function*.

Definition 2.1. A *division function defined on the interval* $[0, M]$ is a piecewise continuous[3] function s from $[0, M]$ to $\mathbb{R}_+^N$ such that

$$\sum_{i \in N} s_i(E) \leq E \quad \text{for all } E \in [0, M]. \tag{2.1}$$

For $i \in N$ and $0 \leq E \leq M$, the number $s_i(E)$ represents the amount received by agent i in case the estate value is equal to E. The piecewise continuity requirement in Definition 2.1 helps avoid the trivial source of nonuniqueness that arises in solutions of optimization problems when agents' preferences are specified by an integral across a continuum of possible future outcomes, and consequently the agents are indifferent with respect to modifications on a set of measure zero. The component functions $s_i(\cdot)$ of a division function s will be referred to as *allotment functions*. Under the efficiency requirements that will be used in this paper, the sum of the allotments must always be equal to the estate value. This is expressed as follows.

Definition 2.2. A division function defined on $[0, M]$ is said to achieve *balance* if the inequality in (2.1) is satisfied with equality for all $E \in [0, M]$.

In bankruptcy problems, there is a *claim* $c_i > 0$ associated to each agent. The vector c of claims is called the *claims vector*. The sum of the claims is denoted by c_N.

Definition 2.3. A *division rule* is a mapping that associates to any claims vector $c \in \mathbb{R}_{++}^N$ a division function defined on $[0, c_N]$.[4]

The division function that is associated by a division rule S to a claims vector c is denoted by $S(c; \cdot)$. The awards vector in case the estate value is equal to E is consequently written as $S(c; E)$.[5] A division rule can be viewed as a family of division functions parametrized by claims vectors.

In some cases, it may be desirable to take more information about agents explicitly into account, for instance seniority or contract date. Such information can be gathered in what might be called the "type" of an agent. The notion of *type space* has been introduced by Kaminski (2006). A function called "max" is used to indicate the maximum award that an agent of a given type can receive; this function can be thought of as extracting the claim size from the type information.

Definition 2.4. A *division method with type space T and maximum-award function* $\max : T \to \mathbb{R}_+$ is a mapping that associates to any type vector $t \in T^N$ a division function defined on $[0, \sum_{i \in N} \max(t_i)]$.

A division method can be viewed as a family of division functions parametrized by type vectors.

2.2. Parametric descriptions

Many well known division rules can be constructed by the "parametric" method that was introduced by Young (1987) and later generalized by Kaminski (2006). The development in the present paper leads naturally to representations in parametric form, as will be shown below. Formal definitions can be stated as follows; see also Thomson (2019, §2.2.2).

Definition 2.5. A *Young–Kaminski parametrization* is a triple $(T, \max, f)$ where T is a set called the *type space*, $\max\colon T \to \mathbb{R}_+$ is a function called the *maximum-award function*, and $f : T \times [0, 1] \to \mathbb{R}_+$ is a mapping such that, for each $t \in T$,

[3] In this paper, a real-valued function f defined on an interval $[a, b]$ is said to be piecewise continuous if there exists a partition $a = x_0 < x_1 < \cdots < x_k = b$ such that, for each $i = 0, \ldots, k-1$, the restriction of f to the interval (x_i, x_{i+1}) is continuous, and the limits $\lim_{x \downarrow x_i} f(x)$ and $\lim_{x \uparrow x_{i+1}} f(x)$ both exist.

[4] The definition can be stated more formally by introducing $\mathcal{S}$ as the set of all pairs (M, s) where $M > 0$ and s is a division function defined on $[0, M]$, and the mapping $m : (M, s) \mapsto M$ which projects an element of $\mathcal{S}$ to its first component. A division rule may then be defined as a mapping S from $\mathbb{R}_{++}^N$ to $\mathcal{S}$ such that $m(S(c)) = c_N$ for all $c \in \mathbb{R}_{++}^N$.

[5] The notational device of writing the awards vector as $S(c; E)$ instead of $S(c, E)$ as usual is meant to emphasize the ex-ante point of view in this paper, which takes the claims vector c as known and the estate value E as unknown.

(i) the mapping $f(t,\cdot) : [0,1] \to \mathbb{R}_+$ is continuous and nondecreasing;
(ii) $f(t,0) = 0$ and $f(t,1) = \max(t)$.

Definition 2.6. A *Young parametrization* is a Young–Kaminski parametrization $(T, \max, f)$ in which $T = \mathbb{R}_+$ and max is the identity function, i.e. $\max(c_0) = c_0$ for $c_0 \in \mathbb{R}_+$.

The mapping f in the definition above is formally a function of two variables, but it is more natural to look at it as a family of functions of c_0, parametrized by $\lambda \in [0,1]$. This point of view motivates referring to f as a "parametric function".[6] In the case of a Young parametrization, this function defines the parametrization completely. Standard examples of parametric functions can be described as follows, with omission of explicit parametrization.

(i) The family consisting of all linear functions with slope in [0, 1]. The corresponding division rule is the proportional rule (PRO).
(ii) The family consisting of all convex piecewise linear functions that take the value 0 at 0 and whose slope, at points of differentiability, is equal to either 0 or 1. The corresponding division rule is Constrained Equal Losses (CEL).
(iii) The family consisting of all concave piecewise linear functions that take the value 0 at 0 and whose slope, at points of differentiability, is equal to either 0 or 1. The corresponding division rule is Constrained Equal Awards (CEA).

The proposition below describes the way in which Young (-Kaminski) parametrizations can be used to obtain division methods.

Proposition 2.7. *Let $(T, \max, f)$ be a Young–Kaminski parametrization. For each $t \in T^N$, a mapping $S_f(t;\cdot) : [0,M] \to \mathbb{R}^N_+$ where $M = \sum_{i\in N} \max(t_i)$ can be unambiguously defined by*

$$(S_f(t;E))_i = f(t_i,\lambda) \quad (i \in N), \qquad \text{with } \lambda \text{ s.t. } \sum_{i\in N} f(t_i,\lambda) = E. \tag{2.2}$$

The mapping $S_f : t \mapsto S_f(t;\cdot)$ is a division method defined on $[0, M]$.

The proof of the proposition can be given in the same way as in the ex-post case; see Young (1987), Kaminski (2006), Thomson (2019, §2.2.2). The division method (2.2) will be referred to as the method *induced* by the Young–Kaminski parametrization $(T, \max, f)$. Analogously, in case of a Young parametrization $f(c_0,\lambda)$, the *rule induced by $f(c_0,\lambda)$* is the one given by (2.2) when specialized to Young parametrizations.

2.3. Further conventions

Suppose that g is a continuous decreasing function from $(0,\infty)$ to $(0,\infty)$.[7] For such a function, the limits $\lim_{x\downarrow 0} g(x)$ and $\lim_{x\to\infty} g(x)$ are well-defined (allowing the value ∞ for the limit at 0), since these numbers are equal to the supremum and the infimum respectively of the set $\{g(x) \mid x > 0\}$. It will be convenient to write

$$g(0) := \lim_{x\downarrow 0} g(x), \qquad g(\infty) := \lim_{x\to\infty} g(x)$$

and to extend in this way the function g to a function from $[0,\infty]$ to $[0,\infty]$ that will still be denoted by g (strictly speaking, this is an abuse of notation). As a mapping from $(0,\infty)$ to $(0,\infty)$, the function g has a standard functional inverse denoted by g^{-1}, which is defined as a function from $(g(\infty), g(0))$ to $(0,\infty)$ by the prescription $g(x) = y \Leftrightarrow x = g^{-1}(y)$. On the interval $(g(\infty), g(0))$, the inverse function is continuous and decreasing; the limit values are $\lim_{y\downarrow g(\infty)} g^{-1}(y) = \infty$ and $\lim_{y\uparrow g(0)} g^{-1}(y) = 0$. This function can be extended to a continuous and nonincreasing function from $[0,\infty]$ to $[0,\infty]$, which will still be denoted by g^{-1}, by defining $g^{-1}(y) = \infty$ for $0 \le y \le g(\infty)$, and $g^{-1}(y) = 0$ for $g(0) \le y \le \infty$. The function that is defined in this way will be called the *extended functional inverse* of g. Note that, for $x \in [0,\infty)$, we have

$$x = g^{-1}(y) \quad\Leftrightarrow\quad x > 0 \text{ and } g(x) = y,\ \text{ or } x = 0 \text{ and } g(x) \le y. \tag{2.3}$$

The logarithmic function can be extended to a continuous function from $[0,\infty]$ to $[-\infty,\infty]$ by defining $\log 0 = -\infty$ and $\log\infty = \infty$. Again, no distinction in notation is made between the usual and the extended version.

Subscript notation will be used for partial derivatives. This means for instance that, given a sufficiently smooth function $f = f(x,y)$, its partial derivative $\frac{\partial^3 f}{\partial^2 x\,\partial y}(x,y)$ will be written as $f_{xxy}(x,y)$. Sometimes the arguments are dropped as well, so that in that case the notation is simply f_{xxy}.

To a function $f(x,y)$ of two variables that is monotonic in both variables and that has continuous second partial derivatives, one can associate a function $f^\times(x,y)$ as follows:

$$f^\times(x,y) = \frac{f_{xy}(x,y)}{f_x(x,y)f_y(x,y)}\,. \tag{2.4}$$

The derived function $f^\times$ occurs naturally in the problem discussed in Section 6. It will be referred to below as the *normalized cross derivative* of the function f, in view of the fact that $f_x(x,y)$ and $f_y(x,y)$ represent the scalings that should be applied to the x and y variable respectively, locally around (x,y), to ensure that a small increment in x (resp. y) leads to an increment of the same size in $f(x,y)$. It is readily verified that the following property holds with respect to monotonic transformations of variables: if g is defined by $g(x,y) = f(h_1(x), h_2(y))$ where h_1 and h_2 are smooth monotonic functions, then $g^\times(x,y) = f^\times(h_1(x), h_2(y))$.

3. Definitions

In the ex-ante framework, division problems are stated without specification of the estate value, and the solution is a division *function* (a vector-valued function defined on an interval of the form $[0, M]$), rather than a *division* (a vector). Preferences of agents must be specified across allotment functions.[8] This can be done in several ways; in this paper, for concreteness, the classical expected utility (EU) model (Savage, 1954) will be used. Conditions on utility functions and subjective probability measures that are required to make the model meaningful will be kept tacit for the moment; specific regularity assumptions are stated in Definition 3.3. It will be convenient to allow utility functions to take the value $-\infty$.

[6] A more intrinsic definition of families of functions that can be parametrized subject to conditions (i) and (ii) in Definition 2.5 is possible by making use of the notion of "linear continuum" from point set topology (see for instance Munkres, 2000, §3.24).

[7] The notational conventions introduced in this paragraph will in particular be applied to marginal utilities, i.e. derivatives of utility functions.

[8] Recall that an allotment function is a component of a division function. If one wants to allow other-regarding preferences, perhaps in analogy with the "no-envy" condition that plays an important role in the theory of fair division (Brams and Taylor, 1996), then preferences should be defined across division functions.

Definition 3.1. An *EU division problem* is a collection $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ consisting of a nonempty finite set N, a positive number M, and, for each $i \in N$, a utility function u_i defined on $[0, \infty)$ with values in $[-\infty, \infty)$, as well as a probability measure μ_i on $[0, M]$.

Given an EU division problem $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ and a division function s defined on $[0, M]$, one can define for each agent $i \in N$ a quantity $U_i(s)$ by

$$U_i(s) = \int_0^M u_i(s_i(E))\, d\mu_i(E). \tag{3.1}$$

This number will be called the *felicity* of agent i. The use of this terminology implies in no way a claim that a scale for "happiness" can be given that would allow interpersonal comparisons; felicities, although expressed as numbers, will only be used for ordinal purposes below. In particular, the notion of Pareto efficiency requires only ordinal properties.

Definition 3.2. Let an EU division problem $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ be given. A division function s defined on $[0, M]$ is said to be *Pareto efficient* if there does not exist another division function $\tilde{s} : [0, M] \to \mathbb{R}^N_+$ such that $U_i(\tilde{s}) \geq U_i(s)$ for all $i \in N$, and $U_i(\tilde{s}) > U_i(s)$ for at least one agent $i \in N$.

For convenience, rather strong regularity assumptions as stated in the definition below will be used throughout the paper.

Definition 3.3. An EU division problem $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ is said to be *regular* if, for all $i \in N$,

(i) the restriction of $u_i(\cdot)$ to $(0, \infty)$ is finite-valued, increasing, strictly concave, and twice continuously differentiable, and $u_i(0) = \lim_{x\downarrow 0} u_i(x)$;
(ii) the probability measure μ_i is absolutely continuous with respect to Lebesgue measure on $[0, M]$, with positive density.

The following definitions state homogeneity properties.

Definition 3.4. An EU division problem $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ is said to satisfy *homogeneous utilities* if there is a utility function $u : [0, \infty) \to [-\infty, \infty)$ such that $u_i = u$ for all $i \in N$.

The common utility function u that appears in the definition above will be referred to as a *social-norm utility*.

Definition 3.5. An EU division problem $\big(N, M, (u_i, \mu_i)_{i\in N}\big)$ is said to satisfy *homogeneous beliefs* if there is a probability measure μ defined on $[0, M]$ such that $\mu_i = \mu$ for all $i \in N$.

In the usual terminology, an "estate division problem" is characterized by the presence of additional information, namely the claim sizes of agents.

Definition 3.6. An *EU estate division problem* is a collection $\big(N, c, (u_i, \mu_i)_{i\in N}\big)$ with $c \in \mathbb{R}^N_{++}$, such that $\big(N, c_N, (u_i, \mu_i)_{i\in N}\big)$ is an EU division problem.

Terminology that has been introduced above for division problems (regularity, homogeneity) will likewise be used for estate division problems. The key properties of division functions for estate division problems that will be used in this paper are formally stated in the definition below. The term "strongly efficient" is used for reasons that were discussed in the introduction.

Definition 3.7. Let an EU estate division problem $\big(N, c, (u_i, \mu_i)_{i\in N}\big)$ be given. A division function s defined on $[0, c_N]$ is said to satisfy

(i) *Strong Efficiency* if s is Pareto efficient for the associated EU division problem;
(ii) *Compliance* if $s(c_N) = c$.

These terms are extended to division rules and division methods in the natural way: a division rule or method is said to be strongly efficient (compliant) if the division functions generated by it are strongly efficient (compliant).

4. Division under strong efficiency and compliance

The main purpose of this section is to show that, subject to homogeneous beliefs and given utility functions of the agents, the combination of strong efficiency and compliance leads to a uniquely defined division function. The rule that is obtained in this way is called the *SEC rule*. The section begins with a standard result on risk sharing known as Borch's theorem. Using this, the main uniqueness result is derived in Theorem 4.2. After a few comments on properties of the SEC rule, Proposition 4.7 shows a formulation of the rule in terms of a Young–Kaminski parametrization. Subsequently, it is shown how the SEC rule can be interpreted as a welfare-maximizing rule and as an equal-sacrifice rule. A numerical illustration of the rule is provided at the end of Section 4.3.

4.1. Borch's theorem

For a class of division problems that is closely related to the class of problems considered here, Borch (1962) has shown how to parametrize the ex-ante efficient solutions under EU preferences. His result is fundamental for risk sharing and is well known in actuarial science. In the standard Borch theorem, division is described in terms of *random variables*, which represent the allotments to the agents, and whose sum must be equal to a given random variable, namely the total risk. For the purposes of the present paper, an adaptation is needed which describes the allotments to agents more explicitly as being obtained by applying a *division function* to the estate. A second adaptation is needed in order to accommodate nonnegativity constraints on divisions. For details of the proof in the random-variable setting, one may for instance refer to DuMouchel (1968), Gerber (1978), Gerber and Pafumi (1998) and Barrieu and Scandolo (2008). In particular, Gerber (1978) deals with nonnegativity constraints. Since the formulation of Borch's theorem as given here differs from the usual one, an independent proof is provided in Appendix A.1. Recall the notational conventions that were introduced in the first paragraph of Section 2.3.

Theorem 4.1. *Let $\big(N, M, (u_i)_{i\in N}, \mu\big)$ be a regular EU division problem with homogeneous beliefs. A division function $s\colon [0, M] \to \mathbb{R}^n_+$ is Pareto efficient if and only if there exist a nonempty index set $N' \subset N$, positive constants $(\alpha_i)_{i\in N'}$, and a continuous function $\lambda : [0, M] \to [0, \infty]$ such that, for $i \in N'$,*

$$\alpha_i u_i'(s_i(E)) \leq \lambda(E) \quad \text{for all } E \in [0, M] \tag{4.1a}$$

$$\alpha_i u_i'(s_i(E)) = \lambda(E) \quad \text{for all } E \in [0, M] \text{ such that } s_i(E) > 0 \tag{4.1b}$$

and, for $i \notin N'$, $s_i(E) = 0$ for all $E \in [0, M]$.

Making use of the concept of the extended functional inverse as defined in Section 2.3, one can replace the two lines in (4.1) by a single one:

$$s_i(E) = (u_i')^{-1}\big(\lambda(E)/\alpha_i\big) \qquad \text{for all } E \in [0, M]. \tag{4.2}$$

Note that the function $\lambda(\cdot)$ and the constants α_i appearing in the above conditions are not uniquely determined: if $\lambda(\cdot)$ and α_i ($i \in N'$) are such that (4.1) is satisfied, then the same holds for $\eta\lambda(\cdot)$ and $\eta\alpha_i$, where η is any positive constant.

4.2. The SEC rule

The following theorem is essential for the present paper.

Theorem 4.2. *For every regular EU estate division problem with homogeneous beliefs, there exists a uniquely determined division function that is both strongly efficient and compliant.*

Proof. Let $(N, c, (u_i)_{i\in N}, \mu)$ be a regular EU problem with homogeneous beliefs. A division function that is both strongly efficient and compliant can be constructed as follows. On the interval $[1, \infty]$, define a function φ by

$$\varphi(\lambda) = \sum_{i\in N}(u_i')^{-1}\big(\lambda u_i'(c_i)\big). \tag{4.3}$$

We have $\varphi(1) = c_N$, and $\varphi(\lambda) = 0$ for all $\lambda \geq \Lambda$, where Λ (possibly equal to ∞) is defined by $\Lambda = \max_{i\in N} u_i'(0)/u_i'(c_i)$. If $\lambda < \Lambda$, then there is at least one $i \in N$ such that $\lambda u_i'(c_i) < u_i'(0)$, so that $\varphi(\lambda) > 0$ for $\lambda \in [1, \Lambda)$. Moreover, the function $\varphi(\lambda)$ is continuous and decreasing on the interval $[1, \Lambda]$. This shows that a continuous decreasing function $\lambda : [0, c_N] \to [1, \Lambda]$ can be defined implicitly by

$$\varphi(\lambda(E)) = E. \tag{4.4}$$

Now define a function $s\colon [0, c_N] \to \mathbb{R}^N_+$ by

$$s_i(E) = (u_i')^{-1}(\lambda(E)u_i'(c_i)) \qquad (0 \leq E \leq c_N,\ i \in N). \tag{4.5}$$

Note that $\sum_{i\in N} s_i(E) = \varphi(\lambda(E)) = E$ for all $0 \leq E \leq c_N$, so that the function s is a division function. Moreover, s is of the form described in Theorem 4.1, with $N' = N$ and $\alpha_i = 1/u_i'(c_i)$, and therefore it is strongly efficient. Furthermore, since $\lambda(c_N) = 1$, compliance is satisfied as well.

To show the uniqueness, suppose that another compliant and strongly efficient division function is given by $\tilde{s}(\cdot)$. Efficiency implies that the conditions of Theorem 4.1 are satisfied for $\tilde{s}$, say with a function $\tilde{\lambda}(\cdot)$ and constants $\tilde{\alpha}_i$. Due to the compliance requirement, the index set N' that is mentioned in the theorem must be equal to the set N of all agents. Applying the condition (4.1b) at $E = c_N$, one finds (using also the compliance condition) that $\tilde{\alpha}_i u_i'(c_i) = \tilde{\lambda}(c_N)$. The condition $\sum_{i\in N}(u_i')^{-1}\big(\tilde{\lambda}(E)/\tilde{\alpha}_i\big) = E$, which follows from the balance requirement, can therefore also be written as $\sum_{i\in N}(u_i')^{-1}\big(\tilde{\lambda}(E)u_i'(c_i)/\tilde{\lambda}(c_N)\big) = E$, or in other words as $\varphi(\tilde{\lambda}(E)/\tilde{\lambda}(c_N)) = E$ for all $0 \leq E \leq c_N$. It follows that $\tilde{\lambda}(E)/\tilde{\lambda}(c_N) = \lambda(E)$ for $0 < E \leq c_N$. Inserting the relations $\tilde{\alpha}_i = \tilde{\lambda}(c_N)/u_i'(c_i)$ and $\tilde{\lambda}(E) = \lambda(E)/\tilde{\lambda}(c_N)$ in Borch's equation (4.2), one finds that $\tilde{s}(E) = s(E)$ for all $E > 0$. Of course, for $E = 0$, the divisions must be equal as well. □

Given a group of agents with utility functions u_i and homogeneous beliefs, a division rule S as in Definition 2.3 can now be obtained by associating to a given claims vector c the division function defined by (4.3), (4.4), and (4.5). This division rule is called the SEC rule (strongly efficient and compliant). The rule can be stated as follows in a form that avoids the use of the extended functional inverse: given a regular EU problem with homogeneous beliefs $(N, c, (u_i)_{i\in N}, \mu)$ and an estate value $E \in [0, c_N]$, the division of the estate according to the SEC rule is the unique vector $(x_1, \ldots, x_n) \in \mathbb{R}^N_+$ that satisfies

$$\sum_{i\in N} x_i = E, \text{ and } \exists\lambda \in [1, \infty]\colon \forall i \in N\colon\ \ x_i > 0 \text{ and}$$
$$\frac{u_i'(x_i)}{u_i'(c_i)} = \lambda, \quad \text{or } x_i = 0 \text{ and } \frac{u_i'(x_i)}{u_i'(c_i)} \leq \lambda. \tag{4.6}$$

Remark 4.3. A division rule is said to satisfy Claims Boundedness if agents do not receive more than their claim values, assuming that the estate value does not exceed the sum of the claims. The function $\lambda(E)$ defined by (4.4) satisfies $\lambda(E) \geq 1$ for all $0 \leq E \leq c_N$, so that the allotment $s_i(E)$ defined in (4.5) satisfies $s_i(E) \leq c_i$ for all $0 \leq E \leq c_N$. In other words, SEC rules satisfy Claims Boundedness, even though the Compliance axiom only requires this property to hold at $E = c_N$. As is seen from (4.5), the SEC division rule $S(c; E)$ is jointly continuous in E and c. In other words, SEC rules satisfy the Continuity property (Thomson, 2019, p. 63). Moreover, for values of E and c for which $S_i(c; E) > 0$, the awards $S_i(c; E)$ are increasing functions of E. Consequently, SEC rules satisfy Null-Compensation-Conditional Strict Endowment Monotonicity (Thomson, 2019, p. 96). Furthermore, for a given estate value and a given claims vector, the expression (4.5) shows that $s_i(E) > s_j(E)$ for $i, j \in N$ such that $c_i > c_j$ and $s_i(E) > 0$. It follows that SEC rules satisfy Order Preservation in Awards (Thomson, 2019, p. 89), and in fact strict order preservation holds except in the case of claimants who receive zero awards.

Remark 4.4. The observation that strong efficiency in combination with a single equality constraint for each agent can lead to unique solutions appears in Gale (1977); see also Gale and Sobel (1979, 1982) and Bühlmann and Jewell (1978, 1979). The equality constraint, which represents a fairness condition, is expressed in terms of expectation under a measure which in the work of Gale and co-authors is assumed to be the same as the measure that is used in the specification of agents' preferences. This assumption is used in their proof technique, which is based on a clever transformation from the original multi-objective problem to an associated single-objective problem. Bühlmann and Jewell use essentially the same method of proof, while weakening the assumption that the two measures should be the same to only the requirement that they should be equivalent.[9] More recently, Pazdera et al. (2017) developed a different approach, which does not require the equivalence assumption. The analogous equality constraint in the contract completion approach to the bankruptcy problem, as developed in the present paper, is expressed in terms of evaluation at the point c_N, which may be viewed as expectation with respect to the measure that consists of a point mass concentrated at c_N. This measure is *not* equivalent to measures that are typically used to define preferences via (3.1). The proof of Theorem 4.2 above therefore follows Pazdera et al. (2017), with substantial simplifications that are possible due to the special structure of the bankruptcy problem. In particular, the weights α_i can be determined easily, in contrast to the situation in the cited papers where the determination of the weights α_i is nontrivial and calls for an iterative solution process.

Remark 4.5. Suppose that the utility functions $u_i(x)$ are replaced by $\tilde{u}_i(x) = a_i u_i(x) + b_i$ for positive $a_1, \ldots, a_n$ and arbitrary $b_1, \ldots, b_n$. These replacements generate a monotonic transformation of agents' felicities. Given the uniqueness of solutions and the fact that the properties of strong efficiency and compliance are not affected by such monotonic transformations, one should expect that the solution will not be changed by substituting $\tilde{u}_i(x)$ for $u_i(x)$. This property can indeed readily be verified directly from (4.6).

Remark 4.6. In the definition of a SEC rule as stated above, it is assumed that agents do not adapt their utility functions

[9] Recall that two measures P and Q defined on the same measurable space are said to be equivalent if, for any measurable set A, $P(A) = 0$ implies $Q(A) = 0$ and vice versa.

to the claim sizes. This can be viewed as a form of rationality. In the context of taxation problems as discussed by Young (1988, 1990),[10] the assumption implies that agents derive utility from post-tax income, without regard to pre-tax income. Casual observation may suggest, however, that the difference between pre-tax income and post-tax income does have an impact on the subjective welfare of at least some people. Such effects may be modeled by allowing claim-dependent utility functions. Under preference specifications of this form, the construction based on (4.6) can still be used to define a division rule. Depending on what forms of dependence on claim values are admitted, the class of division rules that can be obtained in this way can be much wider than the class of division rules that are obtainable from claim-independent utility functions.

There are several ways to describe the division function (4.5) in terms of a Young–Kaminski parametrization; for instance it may be done as follows. Recall the definition of the generalized functional inverse in Section 2.3. Define a triple $(T, \max, f)$ in the following way:

(i) the type space T consists of the continuous decreasing functions t from $(0, \infty)$ to $(0, \infty)$ such that $t(0) > 1 > t(\infty)$;
(ii) the maximum-award function $\max : T \to \mathbb{R}_+$ is given by $\max(t) = t^{-1}(1)$;
(iii) the parametric function $f : T \times [0, 1] \to \mathbb{R}_+$ is given by

$$f(t, \lambda) = t^{-1}(h(\lambda)) \qquad (\lambda \in [0, 1], \quad t \in T) \tag{4.7}$$

where h is any continuous decreasing function that maps $[0, 1]$ onto $[1, \infty]$ (for instance $h(\lambda) = 1/\lambda$ will do).

The terms used already anticipate that the triple that is defined in this way is a Young–Kaminski parametrization, which is part of the claim of the following proposition.

Proposition 4.7. *The triple $(T, \max, f)$ given by (i)–(iii) above is a Young–Kaminski parametrization. For any regular EU estate division problem with homogeneous beliefs $\big(N, c, (u_i)_{i\in N}, \mu\big)$, the functions t_i defined by*

$$t_i(x) = \frac{u_i'(x)}{u_i'(c_i)} \qquad (0 < x < \infty) \tag{4.8}$$

are elements of the type space T defined in (i). When the types of agents are defined in this way, the division function determined by the parametrization (i)–(iii) via Proposition 2.7 coincides with the one given by the SEC rule (4.6).

The proof of the proposition is in Appendix A.2.

4.3. Utilitarian interpretation

For division rules given by a Young parametrization, it is shown in Young (1987) that an interpretation is possible in terms of a social welfare function. An analogous statement holds for SEC rules. The proof of the following theorem is in Appendix A.3.

Theorem 4.8. *Let $\big(N, c, (u_i)_{i\in N}, \mu\big)$ be a regular EU estate division problem with homogeneous beliefs. For any estate value $0 \le E \le c_N$, there exists a unique solution $(x_1, \dots, x_n)$ to the optimization problem*

$$\sum_{i\in N} \frac{u_i(x_i)}{u_i'(c_i)} \to \max \quad \textit{subject to} \quad \sum_{i\in N} x_i = E, \; x_i \ge 0 \;\; (i \in N). \tag{4.9}$$

The division given by this solution coincides with the one given by the SEC rule.

The theorem shows that the SEC division can be viewed as a weighted-utilitarian solution, with weights given by the inverses of marginal utility at the claim values. In other words, a social planner desiring to follow the principles of strong efficiency and compliance can do so by maximizing a Benthamite welfare function in which the agents' individual utilities are normalized in such a way that marginal utility at the claim value is equal for all agents. The motivation of the SEC principle as given earlier has, in philosophical terms, a contractarian feel to it, since it is based on a form of Pareto optimality combined with the simple requirement that all claims should be fulfilled when this is possible. Theorem 4.8 shows that the SEC rule can also be obtained from a utilitarian perspective. However, it may not be easy to motivate the weights in (4.9) (or equivalently, the normalizations applied to utility functions) from a purely utilitarian perspective.

Looking at the weights in (4.9), one may be led to the suspicion that the SEC scheme will favor the rich. Indeed, wealthier agents are likely to have lower marginal utilities, and therefore receive larger weights. A more detailed analysis produces a different picture, however. Suppose that the estate value is sufficiently close to the sum of the claim values, so that the nonnegativity constraints are not active. The Karush–Kuhn–Tucker conditions for the optimization problem can then be written in the form of $n + 1$ equations in $n + 1$ unknowns, with the estate value E as a parameter:

$$u_i'(x_i) = \lambda u_i'(c_i), \qquad \sum_{i\in N} x_i = E. \tag{4.10}$$

This set of equations defines the awards x_i and the Lagrange multiplier λ as functions of E. Writing $x_i = s_i(E)$ (to adapt to earlier notation) and $\lambda = \lambda(E)$, one finds by differentiation with respect to E[11]:

$$s_i'(E) = \lambda'(E)\,\frac{u_i'(c_i)}{u_i''(s_i(E))}\,, \qquad \sum_{i\in N} s_i'(E) = 1. \tag{4.11}$$

At $E = c_N$, we have $s_i(E) = c_i$ for all $i \in N$, so that from the above one finds

$$s_i'(c_N) = \frac{u_i'(c_i)/u_i''(c_i)}{\sum_{i\in N} u_i'(c_i)/u_i''(c_i)} = \frac{\tau_i(c_i)}{\sum_{i\in N} \tau_i(c_i)} \tag{4.12}$$

where $\tau_i(x) := -u_i'(x)/u_i''(x)$ is the risk tolerance function of agent i, which is the reciprocal of agent i's Arrow–Pratt measure of absolute risk aversion. This implies that the estates close to the level c_N are divided by the SEC rule in such a way that agents' losses are proportional to their risk tolerances at the claim values.

A numerical illustration is given in Fig. 1.[12] The figure refers to a situation in which there are two agents whose preferences across awards are both described by power utility (CRRA), i.e. $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$ where γ_i is the coefficient of relative risk aversion of agent i, and both agents have equal claims. It can be verified that the SEC division depends only on the ratio of the coefficients of relative risk aversion. The division functions are shown for the case in which the ratio of the coefficients is equal to 2, and also for the case in which the ratio is 10. It

[10] The analogy between taxation and estate division can be constructed as follows: pre-tax income corresponds to claim value, post-tax income to allotment, and total pre-tax income minus required tax revenue corresponds to the estate value.

[11] A similar differential equation was obtained by Bühlmann (1984, Section 5) in the context of risk sharing.

[12] For the purpose of numerical calculation of division functions according to the SEC rule, it is not necessary to carry out the computation of $\lambda(E)$ as might be suggested by (4.4). Instead, one can parametrize the awards functions as well as the estate value in terms of λ. That is, one can compute the function values $z_{ij} = (u_i')^{-1}(\lambda_j u_i'(c_i))$ on a grid of values λ_j in the interval $[1, \max_{i\in N} u_i'(0)/u_i'(c_i)]$, and then plot the vectors $z_{i\cdot}$ against their sum. This is especially convenient when the inverse marginal utilities are available in analytic form.

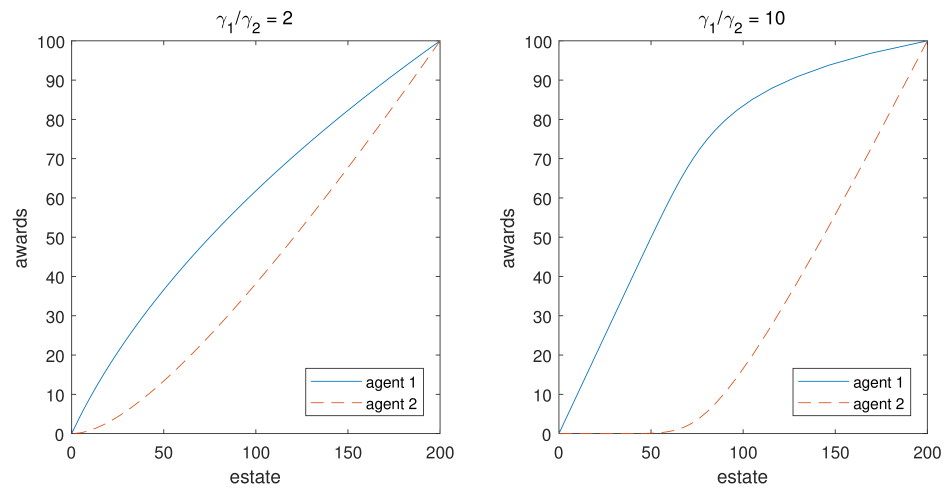

Fig. 1. SEC divisions on the basis of power utility. The coefficients of relative risk aversion of the two agents are given by γ_1 and γ_2. The claim value is 100 for both agents.

is seen that the SEC rule favors the more risk averse agent.[13] The result may be contrasted with the well known fact (see for instance Osborne and Rubinstein, 1994, § 15.2.3, Gintis, 2000, § 15.3) that the more risk averse agent is at a disadvantage in the ex-post Nash bargaining solution.

4.4. Equal sacrifice interpretation

The SEC rule can be compared to the notion of "equal sacrifice" that has been discussed by Young (1987, 1988, 1990). Equal absolute sacrifice means that there exists $\lambda \in \mathbb{R}$ such that

$$u_i(c_i) - u_i(x_i) = \lambda \quad \text{for all } i. \tag{4.13}$$

Equal proportional sacrifice requires that there exists $\lambda \in \mathbb{R}$ satisfying

$$\frac{u_i(c_i) - u_i(x_i)}{u_i(c_i)} = \lambda \quad \text{for all } i. \tag{4.14}$$

As noted in Young (1988), the difference between the two formulations is inessential in the sense that, if the utility functions u_i provide a rationalization for a given division function on the basis of the equal proportional sacrifice principle, then the utility functions $\log u_i$ rationalize the same division function on the basis of the equal absolute sacrifice principle.[14] This justifies speaking about the principle of equal sacrifice without further qualification.

The SEC rule (4.6) calls for an allocation such that there exists $\lambda \in \mathbb{R}$ satisfying

$$\begin{aligned} u_i'(x_i)/u_i'(c_i) \; &= \lambda \quad \text{for all } i \text{ such that } x_i > 0 \\ &\leq \lambda \quad \text{for all } i \text{ such that } x_i = 0. \end{aligned} \tag{4.15}$$

The rule shares with the principle of equal proportional sacrifice the desirable property that it is insensitive to multiplication of utility functions by a positive constant that may be different for different agents. An alternative representation, in terms of the Arrow–Pratt measure of risk aversion, is shown in the following proposition. The proof of the proposition is in Appendix A.4.

Proposition 4.9. *Let a regular EU estate division problem* $\big(N, c, (u_i)_{i\in N}, \mu\big)$ *with homogeneous beliefs be given. For* $i \in N$, *let* $r_i(x) = -u_i''(x)/u_i'(x)$ *denote the Arrow–Pratt measure of risk aversion associated to the utility function* $u_i(x)$. *For any given estate value* $0 \leq E \leq c_N$, *there exist a uniquely determined vector* $(x_1, \ldots, x_n) \in \mathbb{R}_+^N$ *and a constant* $\lambda \in \mathbb{R}$ *such that* $\sum_{i\in N} x_i = E$ *and, for all* $i \in N$,

$$\begin{aligned} &\textit{either} \quad x_i > 0 \textit{ and } \int_{x_i}^{c_i} r_i(x)\,dx = \lambda, \\ &\quad\textit{or} \quad x_i = 0 \textit{ and } \int_{x_i}^{c_i} r_i(x)\,dx \leq \lambda. \end{aligned} \tag{4.16}$$

The division $(x_1, \ldots, x_n)$ *coincides with the one given by the SEC rule.*

The integral appearing in (4.16) extends from the award x_i to the claim c_i and has the Arrow–Pratt function as its integrand; hence it may be viewed as a risk-aversion-weighted measure of the distance between agent i's claim and the amount that the agent actually receives. The proposition shows that an interpretation of the SEC rule as an equal sacrifice rule is possible when sacrifice is measured in terms of this integral, with the proviso that agents who receive nothing are always taken to have sacrificed already enough. The proposition shows moreover that division according to the SEC rule is unchanged when the risk aversions of all agents are multiplied by the same constant.

5. Homogeneous utility

Considerations of efficiency, consistency, and impartiality support the argument that settlement courts should work on the basis of criteria that are as "objective" as possible. The set of relevant attributes of agents should preferably be small, in order to reduce the costly and sensitive task of gathering and interpreting information concerning the competing claimants. As argued by Young (1990, p. 254), instead of thinking of the utility functions u_i as faithful representations of the preferences of distinct individuals, one can also let the utility functions u_i be all the same, and interpret the common utility function as a "social norm". The use of social-norm utility does mean that the model

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[13] Associating a higher value of the coefficient γ to lower wealth is not straightforward, since γ represents risk aversion *relative* to wealth. However, the utilities used in the SEC rule are defined across outcomes of the estate division, not across total wealth. When the claims that agents hold are relatively small compared to their total wealths, a less wealthy agent will therefore appear as a more risk averse claimant, even in terms of relative risk aversion. The strong aversion to low payoffs that is expressed by applying power utility to estate division can be interpreted as representing a psychological benchmark effect with respect to the zero payoff level.

[14] However, it may happen that $\log u(x)$ is concave while $u(x)$ is not. Therefore there may still be a difference in rationalizability on the basis of the two principles if the requirement is imposed, as is often done, that utility functions should be concave.

should not be applied without modification to situations in which there are categorical differences between claimants.

When a social-norm utility function is fixed, Young's principle of equal sacrifice leads to a division rule in the sense of Definition 2.3, depending on claim sizes only. In other words, the principle of equal sacrifice defines a mapping from social-norm utility functions to division rules. Similarly, the SEC principle defines a mapping from social-norm utility functions to division rules. The nature of this mapping will be studied in this section and in Section 6.

5.1. Representation by a parametric function

When beliefs are homogeneous and a social-norm utility function is taken as given, then all information that is relevant for the SEC rule is contained in the claims vector, so that the Young–Kaminski representation (4.7) can be replaced by a Young parametrization, as shown in the following proposition.

Proposition 5.1. *Let (N, c, u, μ) be a regular EU estate division problem with homogeneous beliefs and homogeneous utilities. Let h be any continuous decreasing function that maps the interval $[0, 1]$ onto the interval $[1, u'(0)/u'(\infty)]$, where $u'(0)/u'(\infty)$ is taken to be equal to ∞ when $u'(0) = \infty$ and/or $u'(\infty) = 0$. Then the parametric function $f : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ defined by*

$$f(c_0, \lambda) = (u')^{-1}\big(h(\lambda)u'(c_0)\big) \tag{5.1}$$

represents the SEC division rule defined by (4.6).

In view of Proposition 2.7, the division function that is associated by the SEC rule to a given claims vector c can be described, in the case of homogeneous utility, as follows. First define a function $\lambda(E)$, for $0 \le E \le c_N$, by the implicit specification

$$\sum_{i\in N}(u')^{-1}\big(\lambda(E)u'(c_i)\big) = E, \qquad 1 \le \lambda(E) \le \max_{i\in N} \frac{u'(0)}{u'(c_i)}. \tag{5.2}$$

Then define the division function by

$$s_i(E) = (u')^{-1}\big(\lambda(E)u'(c_i)\big) \qquad (i \in N). \tag{5.3}$$

Example 5.2. For an example of division under the SEC principle with homogeneous utilities, take the social norm utility function that is specified in terms of marginal utility by

$$u'(x) = \frac{1}{x}\, e^{-\alpha x} \tag{5.4}$$

where α is a positive constant. The associated Arrow–Pratt function is given by $r(x) = x^{-1} + \alpha$ which represents a mixture of constant absolute risk aversion and constant relative risk aversion. Suppose that there are three claimants, with claims c_i of sizes 300, 200, and 100. The two panels in Fig. 2 show the corresponding allocations as a function of estate value for $\alpha = 0.05$ and $\alpha = 0.005$ respectively. The plot in the right panel shows a nearly proportional allocation, while the left panel shows that increasing the parameter α leads to an allocation that becomes closer to Constrained Equal Losses.

5.2. Equivalence of utility functions

If two social norm utility functions generate the same SEC division rule, they will be said to be *equivalent*. It follows from Remark 4.5 that a positive affine transformation of social-norm utility does not affect the division rule. As is well known, two sufficiently smooth utility functions are connected by a positive affine transformation if and only if their associated Arrow–Pratt measures of risk aversion are the same. Proposition 4.9 shows that two social-norm utility functions still lead to the same SEC division rule when their corresponding Arrow–Pratt measures of risk aversion are related by multiplication by a positive constant. In fact, as shown by the following proposition, this property characterizes equivalence of utility functions in the sense just defined.

Proposition 5.3. *Two social-norm utility functions that satisfy the assumptions of Definition 3.3(i) generate the same SEC division rule if and only if their associated Arrow–Pratt measures of risk aversion are positively proportional.*

The proof of the proposition is in Appendix A.5. For sufficiently smooth utility functions, it is possible to define an associated function that captures exactly the transformations that are admitted in the proposition above, in the same way as the Arrow–Pratt measure of risk aversion captures exactly the positive affine transformations.

Definition 5.4. The *rate of decrease of risk aversion* that corresponds to a three times differentiable utility function $u(x)$ with Arrow–Pratt measure of risk aversion $r(x) > 0$ is the function $\rho(x)$ defined by

$$\rho(x) = -\frac{d}{dx}\log r(x) = \frac{u''(x)}{u'(x)} - \frac{u'''(x)}{u''(x)}\,. \tag{5.5}$$

It is straightforward to show that two sufficiently smooth social-norm utility functions are equivalent, in the sense that they generate the same SEC division rule, if and only if their associated rates of decrease of risk aversion are the same. Also, one can verify that two utility functions $u(x)$ and $\tilde{u}(x)$ have the same rate of decrease of risk aversion if and only if their corresponding marginal utilities are related by a power transformation (i.e. $\tilde{u}'(x) = (u'(x))^p$, with $p > 0$). The quantity $u'''(x)/u''(x)$ that appears in (5.5) is known as *prudence* (Kimball, 1990).

Trivially, a utility function belongs to the CARA class (constant absolute risk aversion) if and only if its rate of decrease of risk aversion is equal to 0. It is also not difficult to see that a utility function belongs to the CRRA class (constant relative risk aversion) if and only if its rate of decrease of risk aversion is equal to $1/x$. It can be verified directly (but see also Examples 6.7 and 6.8) that the SEC division rule corresponding to the CARA class of utilities is Constrained Equal Losses, and that the SEC rule corresponding to CRRA utilities is proportional allocation.

6. Rationalization under homogeneous utility

The relation (5.1) associates a parametric function to a given utility function. When a parametric function $f(c_0, \lambda)$ is obtained in this way from a social norm utility function $u(x)$, then it is said to be *rationalized* by the utility function $u(x)$ under the SEC principle. The division rule induced by $f(c_0, \lambda)$ is likewise said to be rationalized by $u(x)$. Rationalization of a given parametric function by a social norm utility function under the principle of absolute sacrifice has been discussed by Young (1988). Here we study rationalizability under the SEC principle. The short term "rationalization" will henceforth be used to mean "rationalization by a social norm utility function under the SEC principle". Two preliminary remarks are in order.

Remark 6.1. Under the condition $u'(0) = \infty$, which is satisfied for instance by the class of power utilities, the relation (5.1) implies that $f(c_0, \lambda) > 0$ for all $0 < \lambda \le 1$ and all $c_0 > 0$. This means that rationalization by a social norm utility function satisfying $u'(0) = \infty$ is only possible for rules that assign a

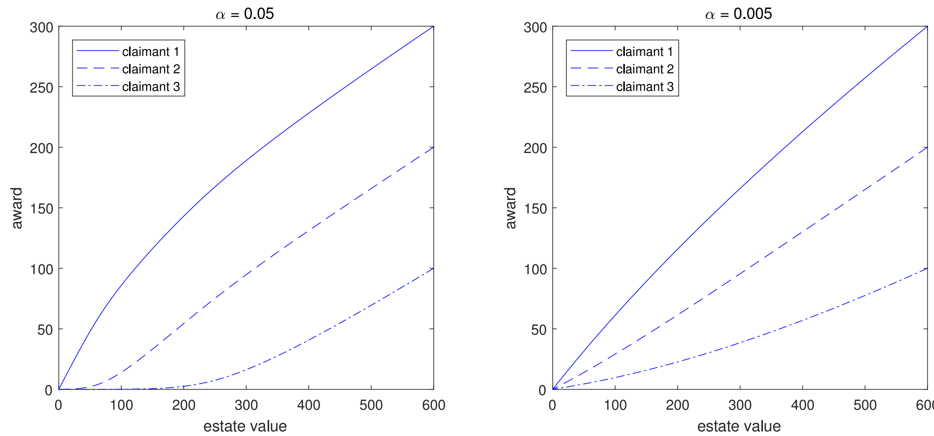

Fig. 2. Allocations for Example 5.2, for two values of the parameter α.

positive amount to every claimant whenever the size of the estate is positive. Under such a rule, no agent is excluded from receiving a positive payment, as long as there is anything to divide. A rule with this property will therefore be called *inclusive*. PRO and CEA are inclusive rules; CEL is not.

Remark 6.2. Under the smoothness assumptions of Definition 3.3(i), the condition (5.1) implies that only rules whose division functions are continuously differentiable and increasing on the domain where they take positive values can be rationalized. Moreover, the same relation also implies that agents who are awarded a nonzero allotment when the available amount is at some given level will receive a share in any further positive increment of the value of the estate. Consequently, the division function is loss-inclusive in the sense that, if there is a deficit, all agents will take a share in it. These properties are satisfied by PRO and CEL, but not by CEA. The restrictive nature of (5.1) is a consequence of the strong smoothness assumptions. Weakening these assumptions will not be undertaken in this paper.

It will be convenient to define the following domain of values (c_0, λ) for a given parametric function $f : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$:

$$D_f = \{(c_0, \lambda) \in \mathbb{R}_{++} \times [0, 1] \mid 0 < \lambda < 1,\ f(c_0, \lambda) > 0\}. \qquad (6.1)$$

The general definition of a parametric function does not impose smoothness constraints on the functions $\lambda \to f(c_0, \lambda)$. In fact, since a parametric function still defines the same division rule when the parameter λ is replaced by $h(\lambda)$ where h is a monotonic function form [0, 1] to itself, and since such a monotonic transformation need not be smooth, every division rule has parametric representations that are not smooth. In the analysis below it will be convenient, however, to use some amount of regularity as expressed in the following definition. The conditions that are stated in the definition are not restrictive in the sense that, if a division rule can be rationalized by a social-norm utility function that satisfies the regularity assumptions of Definition 3.3(i), then the division rule can be represented by a parametric function that satisfies these conditions.

Definition 6.3. A parametric function $f : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ is said to be *regular* if

- the restriction of f to D_f is continuously differentiable in both c_0 and λ
- on the domain D_f, the first partial derivative $f_{c_0}(c_0, \lambda)$ takes positive values and is continuously differentiable with respect to λ
- for $\lambda > 0$, there exists c_0 such that $f(c_0, \lambda) > 0$.

It follows from Remark 4.3 and Remark 6.2 that the regularity assumptions with respect to c_0 are in fact necessary conditions for a parametric function to be rationalizable by a regular social-norm utility function.

Proposition 5.1 shows that a parametric function $f(c_0, \lambda)$ is rationalized by a social norm utility function $u(x)$ via the SEC rule (5.1) if and only if there exists a surjective decreasing function $h : [0, 1] \to [1, u'(0)/u'(\infty)]$ such that, for all $\lambda \in [0, 1]$,

$$h(\lambda)u'(c_0) = u'\big(f(c_0, \lambda)\big) \qquad (c_0 > 0,\ f(c_0, \lambda) > 0) \qquad (6.2a)$$

$$h(\lambda)u'(c_0) \ge u'(0) \qquad (c_0 > 0,\ f(c_0, \lambda) = 0). \qquad (6.2b)$$

The following lemma indicates that it is sufficient to look for functions h that satisfy the equality (6.2a) for pairs (c_0, λ) with $0 < \lambda < 1$ and c_0 in the range of values for which the award $f(c_0, \lambda)$ is positive. The proof is in Appendix A.6.

Lemma 6.4. *A social norm utility function $u(x)$ satisfying the regularity assumptions of Definition 3.3(i) rationalizes a regular parametric function f if and only if there exists a surjective decreasing function $h : [0, 1] \to [1, u'(0)/u'(\infty)]$ such that (6.2a) holds for all $\lambda \in (0, 1)$.*

When a regular parametric function is chosen, the function h that appears in the lemma above is necessarily differentiable. Taking derivatives with respect to λ, one obtains from (6.2a), for $(c_0, \lambda) \in D_f$:

$$u''\big(f(c_0, \lambda)\big) f_\lambda(c_0, \lambda) = h'(\lambda)u'(c_0).$$

Making use of (6.2a) again, this can be rewritten as

$$\frac{u''\big(f(c_0, \lambda)\big)}{u'\big(f(c_0, \lambda)\big)} f_\lambda(c_0, \lambda) = \frac{h'(\lambda)}{h(\lambda)}. \qquad (6.3)$$

Note that $h(\lambda) > 0$ and $h'(\lambda) < 0$ for all λ; consequently, (6.3) implies that $f_\lambda(c_0, \lambda) > 0$ for all $(c_0, \lambda) \in D_f$, in line with Remark 6.2. Taking logarithms of the negatives of both sides of (6.3) and differentiating with respect to c_0, we find, in terms of the rate of decrease of risk aversion $\rho(x)$ that is associated to $u(x)$:

$$\rho(f(c_0, \lambda))f_{c_0}(c_0, \lambda) - f_{c_0\lambda}(c_0, \lambda)/f_\lambda(c_0, \lambda) = 0. \qquad (6.4)$$

This provides part of the proof of the following result; the proof is completed in Appendix A.7. Recall the notation $f^\times$ that was introduced in Section 2.3 to denote the normalized cross derivative of a scalar function of two variables.

Theorem 6.5. *A necessary and sufficient condition for a regular parametric function $f(c_0, \lambda)$ to be rationalizable by a social* *norm utility function satisfying the regularity assumptions of Definition 3.3(i) under the SEC rule is that its normalized cross derivative $f^{\times}(c_0, \lambda)$, defined on D_f, depends on c_0 and λ only through $f(c_0, \lambda)$.*

Unfortunately, it does not seem easy to give an intuitive interpretation to the condition in the theorem. Another way of formulating the result given by the theorem is based on recognizing that, given the monotonic dependence of f on λ for any given fixed value of c_0, it is possible to convert from (c_0, λ) coordinates to (c_0, x) coordinates. Let $\lambda(c_0, x)$ denote the function that is defined implicitly, for $c_0 > 0$ and $0 < x \le c_0$, by the relation

$$f(c_0, \lambda(c_0, x)) = x. \tag{6.5}$$

The theorem may then be restated as follows.

Corollary 6.6. *Rationalizability of a given regular parametric function $f(c_0, \lambda)$ by a social norm utility function is possible under the SEC rule if and only if the mapping $(c_0, x) \mapsto f^{\times}(c_0, \lambda(c_0, x))$ does not depend on c_0, where $\lambda(c_0, x)$ is defined by (6.5). If this is satisfied, then the rationalizing rate of decrease of risk aversion is given by*

$$\rho(x) = f^{\times}(c_0, \lambda(c_0, x)) \qquad (0 < x \le c_0). \tag{6.6}$$

In specific cases, it may however be simpler to obtain the rationalizing rate of decrease of risk aversion directly from the equations $\rho(x) = f^{\times}(c_0, \lambda)$ and $f(c_0, \lambda) = x$.

Example 6.7. Consider the proportional division function, which is given by the regular parametric function $f(c_0, \lambda) = c_0\lambda$. One finds $f^{\times}(c_0, \lambda) = 1/(c_0\lambda) = 1/f(c_0, \lambda)$ so that the condition of Theorem 6.5 is satisfied. The rationalizing rate of decrease or risk aversion is given by $\rho(x) = 1/x$; this corresponds to the class of CRRA utilities.

Example 6.8. We can also reconsider the CEL rule, given by the regular parametric function $f(c_0, \lambda) = \max(c_0 - \frac{1}{\lambda}, 0)$. The normalized cross derivative, computed on the domain of positive awards, is equal to 0. Again, the condition of Theorem 6.5 is satisfied, with $\rho(x) = 0$. Consequently, the CEL rule is rationalized under the SEC principle by any member of the class of CARA utility functions.

Remark 6.9. The condition of Theorem 6.5 states that there should exist a scalar function $\rho(x)$ such that, for all c_0 and λ, one has $f^{\times}(c_0, \lambda) = \rho(f(c_0, \lambda))$. If this condition holds, and sufficient smoothness is present (i.e. the function f is three times continuously differentiable in both arguments), then for all $(c_0, \lambda) \in D_f$ we have[15] $\nabla f^{\times}(c_0, \lambda) = \rho'(f(c_0, \lambda))\nabla f(c_0, \lambda)$. Conversely, if there exists a scalar function $g(c_0, \lambda)$ such that $\nabla f^{\times}(c_0, \lambda) = g(c_0, \lambda)\nabla f(c_0, \lambda)$ for all $(c_0, \lambda) \in D_f$, then one can write, making use of the function $\lambda(c_0, x)$ defined implicitly by (6.5)[16]:

$$\begin{aligned}
\frac{\partial}{\partial c_0} f^{\times}(c_0, \lambda(c_0, x)) &= \nabla f^{\times}(c_0, \lambda(c_0, x))\begin{bmatrix} 1 & \lambda_{c_0}(c_0, x) \end{bmatrix}^T \\
&= g(c_0, \lambda(c_0, x))\nabla f(c_0, \lambda(c_0, x))\begin{bmatrix} 1 & \lambda_{c_0}(c_0, x) \end{bmatrix}^T \\
&= g(c_0, \lambda(c_0, x))\frac{\partial}{\partial c_0} f(c_0, \lambda(c_0, x)) = 0.
\end{aligned}$$

This shows that the condition of Corollary 6.6 is satisfied. Since the vector $\nabla f(c_0, \lambda)$ is nonzero for all $(c_0, \lambda) \in D_f$, it follows that, under sufficient smoothness, a necessary and sufficient for rationalizability of the given division function $f(c_0, \lambda)$ by a social norm utility function under the SEC rule is that the 2×2 matrix formed from $\nabla f(c_0, \lambda)$ and $\nabla f^{\times}(c_0, \lambda)$ has rank 1 for all $(c_0, \lambda) \in D_f$. In this way, a computational criterion for rationalizability under the SEC rule is obtained.

7. Conclusions

The starting point in this paper has been the observation that estate division problems only arise in situations where claimants have not agreed in advance on a division rule for the case in which the estate value is less than the sum of the claims. Situations of this nature are not unusual, neither in business nor in daily life; it is often impractical to negotiate in advance about every possible situation that might arise. Arbitration may be needed to deal appropriately with the consequences of the unforeseen circumstance. A court of arbitration may be guided by the principle that the rule should be followed that the agents would have agreed upon if, in contrast to the actual fact, they *would* have negotiated beforehand. Such a rule needs to be constructed hypothetically, but the court may be able to make some reasonable assumptions regarding the preferences of the agents. The court then needs to solve an *ex-ante* version of the estate division problem.

When looked at in this way, the estate division problem is similar to risk sharing problems that have been studied in particular in actuarial science, with the added feature that claim values need to be taken into account. The closest analogy is with risk sharing problems in which a "financial fairness" constraint plays a role, as studied for instance in Bühlmann and Jewell (1979) and Pazdera et al. (2017). On the basis of results from the actuarial science literature, with some modifications, it has been shown that a strong notion of Pareto efficiency in combination with the requirement that all claims should be fulfilled when possible is sufficient to define uniquely a division rule when agents' utilities are given, under the assumption that agents' preferences can be described by the classical expected utility model with homogeneous beliefs. The rule, called the SEC rule, can be given a "utilitarian" as well as an "equal sacrifice" interpretation.

If claimants are essentially of the same nature (i.e. they do not belong to different categories, such as employees and bond holders in case of a firm bankruptcy), a court of arbitration may use the same utility function for all of them. It has been shown that taking such a "social norm" utility function from the CRRA class leads to the proportional division rule, whereas taking it from the CARA class leads to the rule of Constrained Equal Losses. A criterion has been given by means of which it can be verified, for a division rule described in terms of a sufficiently smooth parametric function, whether or not it is possible to derive this rule from a social norm utility function.

The analysis in this paper has several limitations. In the literature on the bankruptcy problem, many properties are discussed that division rules may or may not satisfy; see Thomson (2019) for an extensive survey. Relatively little attention has been given in the paper to determining which axioms are satisfied by SEC rules. Claims Boundedness and monotonicity properties have been discussed in Remark 4.3. The satisfaction of symmetry and consistency properties can be inferred from the fact that SEC rules form a subclass of Young–Kaminski rules (Young rules in the case of homogeneous utility), using the theorems that have been proven for parametric families (Young, 1987, Thm. 1; Kaminski, 2006, Thm. 1; Thomson, 2019, Ch. 10). It would be of interest to find further axioms that characterize SEC rules with homogeneous utility within the class of all Young rules. The condition given in Theorem 6.5 is necessary and sufficient and can effectively be verified as indicated in Remark 6.9, but in the stated

[15] The symbol ∇ denotes gradient, written as a row vector.

[16] Superscript T is used to indicate transposition.

form it is of a purely mathematical nature, without an axiomatic interpretation.

The smoothness assumptions that have been made in the paper are fairly strong (in fact as strong as needed to avoid all difficulties relating to nonsmoothness). The assumptions could be weakened by making use of the techniques of convex analysis. The Arrow–Pratt measure of risk aversion, which has been used for instance in Proposition 4.9, can still be given a meaning for utility functions that are only supposed to be nondecreasing, concave, and upper semicontinuous (Würth and Schumacher, 2011). The strict concavity assumption may be weakened as well; it may then no longer be true, however, that the axioms of strong efficiency and compliance lead to a uniquely defined division rule when agents' utility functions are given.

The preference model that is expressed by social norm utility may be too strictly rational in many situations. The importance of benchmarks has often been emphasized in the literature on behavioral economics; see for instance Kahneman and Tversky (1979). It may be reasonable at least to admit claim-dependent utility functions, so that agents with different claims have different utilities. SEC rules based on claim-dependent utilities are still based on claim size only, so that rationalization by means of claim-based utility functions is a meaningful question. It is to be expected that, to obtain a degree of uniqueness, fairly strong restrictions must be imposed on the way in which agents' utilities can depend on claims.

While the framework of expected utility has been followed in this paper, there are of course many other models for modeling preferences that have been proposed in the literature. The axioms of strong Pareto efficiency and compliance can be used in combination with such alternative preferences as well. In the context of bankruptcy, it may in particular be of interest to use other-regarding preferences, which express that agents take not only their own awards into account, but also those received by others. A special question of interest is whether it would be possible to rationalize the Constrained Equal Awards rule by making use of such preferences and/or by weakening of the regularity assumptions.

Acknowledgment

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Appendix. Proofs

A.1. Proof of Theorem 4.1

Theorem 4.1 is a direct consequence of the following two lemmas. First, some notation: for a given finite nonempty index set N, write

$$\mathcal{S}_N = \{s : [0, M] \to \mathbb{R}^N_+ \mid s(\cdot) \text{ piecewise continuous}, \sum_{i\in N} s_i(E) = E \text{ for all } 0 \le E \le M\}.$$

Lemma A.1. *Let $(\alpha_i)_{i\in N}$ be a set of positive weights indexed by a finite nonempty set N, and let μ be a measure on $[0, M]$ that has positive density with respect to the Lebesgue measure. Furthermore, for each $i \in N$, let $u_i(\cdot)$ be a utility function satisfying the regularity assumption of Definition 3.3(i). The optimization problem*

$$\sum_{i\in N} \alpha_i \int_0^M u_i(s_i(E))\, d\mu(E) \to \max \tag{A.1}$$

has a unique solution in $\mathcal{S}_N$. This solution is given by

$$s_i(E) = (u')^{-1}\big(\lambda(E)/\alpha_i\big) \tag{A.2a}$$

where, for each $E \ge 0$, $\lambda(E) \in (0, \infty]$ is chosen such that

$$\sum_{i\in N} (u')^{-1}\big(\lambda(E)/\alpha_i\big) = E. \tag{A.2b}$$

The functions $s_i(\cdot)$ defined in (A.2) are continuous.

Proof. To verify the claim that λ is uniquely defined by (A.2b), note that the function φ defined by $\varphi(\lambda) = \sum_{i\in N}(u')^{-1}(\lambda/\alpha_i)$ is a surjective decreasing mapping from $[\max_{i\in N} \alpha_i u_i'(\infty), \max_{i\in N} \alpha_i u_i'(0)]$ to $[0, \infty]$. Therefore there is a unique solution $\lambda(E) \in [\max_{i\in N} \alpha_i u_i'(\infty), \max_{i\in N} \alpha_i u_i'(0)]$ to Eq. (A.2b), which depends continuously on E. Since $\varphi(\lambda) = \infty$ for $0 \le \lambda \le \max_{i\in N} \alpha_i u_i'(\infty)$ and $\varphi(\lambda) = 0$ for $\lambda \ge \max_{i\in N} \alpha_i u_i'(0)$, the functions $s_i(E)$ are still uniquely defined even if the range of values from which $\lambda(E)$ can be chosen is extended to $(0, \infty]$. It follows that (A.2) uniquely defines a continuous function $s(E) = (s_i(E))_{i\in N}$. Now, for each fixed $E \in [0, M]$, one can consider the optimization problem

$$\sum_{i\in N} \alpha_i u_i(x_i) \to \max \quad \text{subject to } x_i \ge 0, \quad \sum_{i\in N} x_i \le E. \tag{A.3}$$

A standard application of the Karush–Kuhn–Tucker theorem shows that the solution $(x_i)_{i\in N}$ of this optimization problem is given by $x_i = s_i(E)$ as defined in (A.2).

To show that the function s is the unique optimizer, take a function $\tilde{s} \in \mathcal{S}_N$ that is not equal to s. Define $f(E)$ for $E \in [0, M]$ by

$$f(E) = \sum_{i\in N} \alpha_i \big(u_i(s_i(E)) - u_i(\tilde{s}_i(E))\big).$$

Due to the pointwise optimality of $s(E)$, we have $f(E) \ge 0$ for all E, and since we assumed that $\tilde{s} \ne s$, we must have $f(E) > 0$ for some E. Because the function f is piecewise continuous, this implies that there exists an interval $(a, b) \subset [0, M]$ such that $f(E) > 0$ for all $a < E < b$. Since the measure μ has positive density with respect to Lebesgue measure on $[0, M]$, it follows that

$$\sum_{i\in N} \alpha_i \int_0^M u_i(\tilde{s}_i(E))\, d\mu(E) \;<\; \sum_{i\in N} \alpha_i \int_0^M u_i(s_i(E))\, d\mu(E).$$

Consequently, any element of $\mathcal{S}_N$ that is different from s as defined by (A.2) is strictly suboptimal. □

Lemma A.2. *If $s \in \mathcal{S}_N$ is Pareto efficient for the multicriteria optimization problem with objectives (3.1), $i \in N$, then there exist nonnegative weights $(\alpha_i)_{i\in N}$, not all equal to 0, such that s is optimal for the weighted-sum problem*

$$\sum_{i\in N} \alpha_i \int_0^M u_i(s_i(E))\, d\mu(E) \to \max$$
$$\text{subject to } \sum_{i\in N} s_i(E) = E, \quad s_i(E) \ge 0. \tag{A.4}$$

Proof. The statement follows from standard convexity arguments; for the reader's convenience, a brief summary follows. Let $U : \mathcal{S}_N \to (\mathbb{R} \cup \{-\infty\})^N$ denote the mapping that takes a division function s to the corresponding vector of agents' felicities as determined by (3.1), and let $\mathcal{U} \subset (\mathbb{R} \cup \{-\infty\})^N$ denote the range of U. From the concavity of the utility functions, it follows that the set $\mathcal{U} + \mathbb{R}^N_-$, where $\mathbb{R}^N_-$ denotes the nonpositive cone $\{z \in \mathbb{R}^N \mid z \le 0\}$, is convex; see for instance Aubin (1993, Prop. 2.6), or compare the use of "Condition C" in Gerber (1978). Let $s \in \mathcal{S}_N$ be Pareto efficient; then, by definition, we have

$(U(s) + \mathbb{R}^N_+) \cap (\mathcal{U} + \mathbb{R}^N_-) = \{U(s)\}$. Since the relative interiors of the two convex sets $U(s) + \mathbb{R}^N_+$ and $\mathcal{U} + \mathbb{R}^N_-$ do not intersect, the separation theorem of finite-dimensional convex analysis (see for instance Rockafellar, 1997, Thm. 11.3) implies that there exists a nonzero vector $\alpha \in \mathbb{R}^N$ such that $\langle \alpha, U(s)\rangle \geq \langle \alpha, U(\tilde{s})\rangle$ for all $\tilde{s} \in \mathcal{S}_N$, and $\langle \alpha, U(s)\rangle \leq \langle \alpha, U(s) + z\rangle$ for all $z \geq 0$. The latter condition implies that $\alpha \geq 0$, and the former that s is a maximizer for the weighted-sum criterion stated in (A.4). □

Proof of Theorem 4.1. For the "if" part, assume that $s(\cdot)$ is a division function satisfying the conditions of the theorem with index set N' and weights $(\alpha_i)_{i\in N'}$. Note that the function $\lambda(\cdot)$ that appears in the statement of the theorem is related to the division function $s(\cdot)$ via (A.2a) (see (4.2)), and hence (A.2b) holds. Suppose that a Pareto improvement is possible, i.e. there exists $\tilde{s} \in \mathcal{S}_{N'}$ such that

$$\int_0^M u_i(\tilde{s}_i(E))\,d\mu(E) \;\geq\; \int_0^M u_i(s_i(E))\,d\mu(E) \qquad \text{for all } i \in N \tag{A.5}$$

with strict inequality for at least one $i \in N$. If $(\tilde{s}_i(\cdot))_{i\in N'} \neq (s_i(\cdot))_{i\in N'}$, then the strict inequality

$$\sum_{i\in N'} \alpha_i \int_0^M u_i(\tilde{s}_i(E))\,d\mu(E) \;<\; \sum_{i\in N'} \alpha_i \int_0^M u_i(s_i(E))\,d\mu(E) \tag{A.6}$$

must hold since the expression on the right is uniquely optimized by $(s_i(\cdot))_{i\in N'}$ by Lemma A.1. Because all $(\alpha_i)_{i\in N'}$ are positive, this leads to a contradiction with (A.5). If $(\tilde{s}_i(\cdot))_{i\in N'} = (s_i(\cdot))_{i\in N'}$, then equality between $s_i(E)$ and $\tilde{s}_i(E)$ for all i and E is forced by the balance condition. Consequently, for no $i \in N$ can there be a strict inequality in (A.5), and again there is a contradiction.

To show the necessity of the condition, let $(\alpha_i)_{i\in N}$ denote the nonnegative weights that are provided by Lemma A.2. Define N' as the set of indices i such that $\alpha_i > 0$. The optimizer $s \in \mathcal{S}_N$ for problem (A.4) is such that $s_i(\cdot) = 0$ for $i \notin N'$, and $(s_i)_{i\in N'}$ is an optimizer for the problem (A.1). Lemma A.1 then implies that s is of the form indicated in the theorem. □

A.2. Proof of Proposition 4.7

To verify that the triple $(T, \max, f)$ as specified is indeed a Young–Kaminski parametrization, note that it follows from the definition of the generalized functional inverse that its value at ∞ is equal to 0. Therefore, $f(t, 0) = t^{-1}(h(0)) = t^{-1}(\infty) = 0$. The condition $t(0) > 1 > t(\infty)$ guarantees that $\max(t) = t^{-1}(1)$ is well-defined as a finite positive number. Furthermore, $f(t, 1) = \max(t)$ by definition. Finally, since t^{-1} is a continuous and nonincreasing function from $[0,\infty]$ to $[0,\infty]$, it follows that $f(t,\cdot)$ is a continuous and nondecreasing function from $[0, 1]$ to $[0, \max(t)]$ for every fixed value of t.

The claim that the functions defined in (4.8) belong to the space T follows from the regularity assumptions. In particular, because the marginal utilities $u_i'(x)$ are decreasing, we have $u_i'(0) > u_i'(c_i) > u_i'(\infty)$ for all $i \in N$, so that $t_i(0) > 1 > t_i(\infty)$.

The division method defined by (2.2) in Proposition 2.7 gives the division corresponding to an estate value E as the uniquely determined vector $(x_1, \ldots, x_n)$ that satisfies $\sum_{i\in N} x_i = E$ and $x_i = f(t_i, \lambda)$ for some constant λ. Given the definitions (4.7) and (4.8) and the definition of the generalized functional inverse, the latter condition is equivalent to the existence of a constant λ such that $u_i'(x_i)/u_i'(c_i) = \lambda$ for all i with $x_i > 0$, and $u_i'(x_i)/u_i'(c_i) \leq \lambda$ for all i with $x_i = 0$.

A.3. Proof of Theorem 4.8

The optimization problem (4.9) calls for maximization of a strictly concave function on a convex and bounded domain; hence there is a unique solution. The standard Karush–Kuhn–Tucker conditions are necessary and sufficient for optimality, due to the regularity assumptions that have been made. These conditions state that there exist Lagrange multipliers $\lambda \in \mathbb{R}$ and $\nu_i \leq 0$ such that, for all $i \in N$,

$$\frac{u_i'(x_i)}{u_i'(c_i)} = \lambda + \nu_i, \qquad \nu_i = 0 \text{ if } x_i > 0. \tag{A.7}$$

These conditions are the same as the ones stated in (4.6). It follows that the divisions induced by the optimization problem and by the SEC rule are the same.

A.4. Proof of Proposition 4.9

Since the Arrow–Pratt measure of risk aversion is minus the derivative of log marginal utility, we have

$$\int_{x_i}^{c_i} r_i(x)\,dx = \log u_i'(x_i) - \log u_i'(c_i) \qquad \text{for all } i \in N.$$

Therefore, the conditions in (4.16) can be rewritten in the form (A.7). Consequently, the division given by (4.16) in combination with the requirement $\sum_{i\in N} x_i = E$ coincides with the division given by the SEC rule.

A.5. Proof of Proposition 5.3

Given the statements in Remark 4.5 and Proposition 4.9, it only remains to show the necessity of the condition. Suppose that u_1 and u_2 are utility functions that satisfy the conditions of Definition 3.3(i) and that generate the same SEC division rule. Let f_1 and f_2 denote the corresponding parametric functions defined via (5.1). Since these parametric functions generate the same division rule, they specify the same family of functions of claim size c_0, even though these families may be differently parametrized. Consequently, there exists an increasing function g from $[1, u_1'(0)/u_1'(\infty)]$ onto $[1, u_2'(0)/u_2'(\infty)]$ such that, for all $c_0 > 0$ and $0 < x \leq c_0$,

$$u_1'(x) = \lambda u_1'(c_0) \quad \Leftrightarrow \quad u_2'(x) = g(\lambda) u_2'(c_0). \tag{A.8}$$

For any fixed $c_0 > 0$, one can write $g(\lambda) = u_2'\big((u_1')^{-1}(\lambda u_1'(c_0))\big)/u_2'(c_0)$. This shows that the mapping g is continuously differentiable on $[1, u_1'(0)/u_1'(\infty)]$. Define the function $\lambda(c_0, x)$ for $c_0 > 0$ and $0 < x \leq c_0$ by $\lambda(c_0, x) = u_1'(x)/u_1'(c_0)$; then

$$u_2'(x) = g(\lambda(x, c_0)) u_2'(c_0). \tag{A.9}$$

Taking partial derivatives of both sides of the equality (A.9) with respect to x, and using the definition of $\lambda(c_0, x)$, one finds that $g'(\lambda(c_0, x))u_1''(x)/u_1'(c_0) = u_2''(x)/u_2'(c_0)$. This, in combination with the definition of $\lambda(c_0, x)$ and the relation (A.9), leads to the equality

$$\frac{u_2''(x)}{u_2'(x)} = \frac{g'(\lambda(c_0, x))}{g(\lambda(c_0, x))}\,\lambda(c_0, x)\,\frac{u_1''(x)}{u_1'(x)} \tag{A.10}$$

which holds for all $c_0 > 0$ and $0 < x \leq c_0$. Define a function $G(\lambda)$ for $\lambda \in (1, u_1'(0)/u_1'(\infty))$ by

$$G(\lambda) = \frac{g'(\lambda)}{g(\lambda)}\,\lambda.$$

From (A.10), it follows that the function $(c_0, x) \mapsto G(\lambda(c_0, x))$ actually depends on x only. Take λ_1 and λ_2 in $(1, u_1'(0)/u_1'(\infty))$, with $\lambda_1 < \lambda_2$. Choose $c_0^1 > 0$, and define $x := f_1(c_0^1, \lambda_1) > 0$.

Because $\lim_{c_0\downarrow 0} f_1(c_0, \lambda_2) = 0$ and $f_1(c_0^1, \lambda_2) > f_1(c_0^1, \lambda_1) = x$, there exists $c_0^2 \in (0, c_0^1)$ such that $f_1(c_0^2, \lambda_2) = x$. Then $\lambda_1 = \lambda(c_0^1, x)$ and $\lambda_2 = \lambda(c_0^2, x)$. Since $G(\lambda(c_0, x))$ depends on x only, this implies that $G(\lambda_1) = G(\lambda_2)$. It follows that the function $G(\lambda)$ is in fact constant. The relation (A.10) then shows that the measures of risk aversion corresponding to u_1 and u_2 are related by a multiplicative constant, which must be positive since both utility functions have been assumed to be increasing and concave.

A.6. Proof of Lemma 6.4

Only the "if" part requires proof. Let h be as in the statement of the lemma. It needs to be verified that the condition (6.2) is satisfied for $\lambda = 0$, for $\lambda = 1$, and in all other cases (if any) in which $f(c_0, \lambda) = 0$. If $\lambda = 0$, we have $f(c_0, \lambda) = 0$ by definition of a parametric function, and condition (6.2b) is satisfied since $h(0) = u'(0)/u'(\infty)$ and u' is decreasing. For $\lambda = 1$, we have $f(c_0, \lambda) = c_0$ for all $c_0 > 0$; in this case, condition (6.2a) must be checked, and it is indeed satisfied since $h(1) = 1$. Finally, assume that (c_0, λ) is such that $\lambda > 0$ and $f(c_0, \lambda) = 0$. Due to items (ii) and (iii) in Definition 6.3, there exists c_0^* such that $f(c_0, \lambda) = 0$ for all $c_0 \le c_0^*$ and $f(c_0, \lambda) > 0$ for all $c_0 > c_0^*$. Taking the limit as c_0 tends to c_0^* from above, and making use of the continuity of u', one derives from (6.2a) that $h(\lambda)u'(c_0^*) = u'(0)$. For $c_0 \le c_0^*$, the fact that u' is decreasing implies that $h(\lambda)u'(c_0) \ge h(\lambda)u'(c_0^*) = u'(0)$. In other words, (6.2b) holds.

A.7. Proof of Theorem 6.5

The necessity part follows directly from (6.4). Conversely, if the condition of the theorem is satisfied, one can construct a rationalizing utility function as follows. Given $x > 0$, take c_0 with $c_0 > x$ and let $\lambda \in (0, 1)$ be such that $f(c_0, \lambda) = x$ (such λ must exist because $f(c_0, 0) = 0$, $f(c_0, 1) = c_0$, and $f(c_0, \cdot)$ is continuous). Now define the function $\rho(x)$ by $\rho(x) = f^{\times}(c_0, \lambda)$ for c_0 and λ such that $f(c_0, \lambda) = x$. The condition of the theorem guarantees that this definition is unambiguous. Take any utility function $u(x)$ that has the function $\rho(x)$ as its rate of decrease of risk aversion. By construction, the function $\rho(x)$ satisfies the relation (6.4). As a consequence, we have

$$\frac{\partial}{\partial\lambda}\log\big(r(f(c_0,\lambda))f_{c_0}(c_0,\lambda)\big) = -\rho(f(c_0,\lambda))f_\lambda(c_0,\lambda) + \frac{f_{c_0\lambda}(c_0,\lambda)}{f_{c_0}(c_0,\lambda)} = 0$$

where $r(\cdot)$ denotes the Arrow–Pratt measure of risk aversion corresponding to the utility function $u(\cdot)$. It follows from this that the function

$$(c_0,\lambda) \mapsto \frac{u''(f(c_0,\lambda))}{u'(f(c_0,\lambda))} f_{c_0}(c_0,\lambda) \tag{A.11}$$

is constant as a function of λ. Given also the continuity of (A.11) as a function of λ, it can be concluded that, for every $c_0 > 0$, the expression given by (A.11) takes the same value in (c_0, λ) as it does in $(c_0, 1)$. Given that $f(c_0, 1) = c_0$ for all $c_0 > 0$ and hence $f_{c_0}(c_0, 1) = 1$, we find that

$$\frac{u''(f(c_0,\lambda))}{u'(f(c_0,\lambda))} f_{c_0}(c_0,\lambda) = \frac{u''(c_0)}{u'(c_0)}.$$

This implies that

$$\frac{\partial}{\partial c_0}\frac{u'(f(c_0,\lambda))}{u'(c_0)} = \frac{u''(f(c_0,\lambda))f_{c_0}(c_0,\lambda)u'(c_0) - u'(f(c_0,\lambda))u''(c_0)}{(u'(c_0))^2} = 0. \tag{A.12}$$

Now, define a function $h : [0, 1] \to [1, u'(0)/u'(\infty)]$ as follows. For $\lambda \in (0, 1)$, there exists $c_0 > 0$ such that $f(c_0, \lambda) > 0$. Define

$$h(\lambda) = \frac{u'(f(c_0,\lambda))}{u'(c_0)}.$$

It is guaranteed by (A.12) that this definition is unambiguous (i.e. the right hand side does not depend on the choice of c_0). Furthermore, define $h(1) = 1$ and $h(0) = u'(0)/u'(\infty)$. It follows from the monotonicity and continuity assumptions on f and u' that the function h is continuous and decreasing on $(0, 1)$. Moreover, from the equality $f(c_0, 1) = c_0$ and the continuity of f as a function of λ, it follows that h is continuous at 1. Since $f(c_0, 0) = 0$ for all c_0, and again using the continuity of f as a function of λ, we have that $f(c_0, \lambda)$ can be arbitrarily close to 0 even when c_0 is arbitrarily large, so that

$$\sup\{u'(f(c_0,\lambda))/u'(c_0) \mid (c_0,\lambda) \in D_f\} = u'(0)/u'(\infty).$$

This proves that the function h as defined above is continuous at 0. This function is therefore a continuous decreasing mapping from $[0, 1]$ to $[1, u'(0)/u'(\infty)]$, taking the value $u'(0)/u'(\infty)$ at 0 and the value 1 at 1. It follows that h is surjective. Moreover, the relation (6.2a) is satisfied. The result now follows from Lemma 6.4.

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