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Jourdain, B.; Vellekoop, M.H.

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Regularity of the Exercise Boundary for American Put Options on Assets with Discrete Dividends*

B. Jourdain† and M. H. Vellekoop‡

Abstract. We analyze the regularity of the optimal exercise boundary for the American Put option when the underlying asset pays a discrete dividend at a known time t_d during the lifetime of the option. The ex-dividend asset price process is assumed to follow Black–Scholes dynamics, and the dividend amount is a deterministic function of the ex-dividend asset price just before the dividend date. The solution to the associated optimal stopping problem can be characterized in terms of an optimal exercise boundary which, in contrast to the case when there are no dividends, may no longer be monotone. In this paper we prove that when the dividend function is positive and concave, then the boundary is nonincreasing in a left-hand neighborhood of t_d and tends to 0 as time tends to t_d^- with a speed that we can characterize. When the dividend function is linear in a neighborhood of zero, then we show continuity of the exercise boundary and a high contact principle in the left-hand neighborhood of t_d. When it is globally linear, then the right-continuity of the boundary and the high contact principle are proved to hold globally. Finally, we show how all the previous results can be extended to multiple dividend payment dates in that case.

Key words. optimal stopping, American options, dividends, early exercise boundary, smooth contact property

AMS subject classifications. 60G40, 91A60, 91G20

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Introduction. We consider the American Put option with strike $K > 0$ and maturity $T > 0$ on an underlying stock. We assume that the stochastic dynamics of the ex-dividend price process of this stock can be modeled by the Black–Scholes model and that at the $I \in \mathbb{N}$ given times $t_d^I < t_d^{I-1} < \cdots < t_d^1$ in the time interval $(0, T)$, discrete stock dividends are paid. The case without dividends is denoted by $I = 0$, and we will use the convention that $t_d^{I+1} = 0$ and $t_d^0 = T$ throughout the paper. The value of the dividend payments are functions $D^j : \mathbb{R}_+ \to \mathbb{R}_+$ $(1 \leq j \leq I)$ of the ex-dividend asset price. This means that the stock price process satisfies

$$dS_u = \sigma S_u dW_u + rS_u du - \sum_{j=1}^{I} D^j(S_{u-})d1_{\{u \geq t_d^j\}} \tag{0.1}$$

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†Université Paris-Est, CERMICS, Projet MathFi ENPC-INRIA-UMLV, 6 et 8 avenue Blaise Pascal, 77455 Marne La Vallée, Cedex 2, France (jourdain@cermics.enpc.fr). This author's research benefited from the support of the "Chair Risques Financiers," Fondation du Risque. This research was partially completed while this author was visiting the Institute for Mathematical Sciences, National University of Singapore, in 2009. The visit was supported by the institute.

‡Corresponding author. Department of Quantitative Economics, Faculty of Economics and Business, Section Actuarial Science, University of Amsterdam, Valckenierstraat 65-67, 1018 XE Amsterdam, The Netherlands (m.h.vellekoop@uva.nl).

for an initial price S_0, interest rate r, and volatility σ, which are assumed to be positive and with W a standard Brownian motion.

Throughout the paper we assume that the dividend functions D^j are nonnegative and nondecreasing for all $1 \leq j \leq I$ and such that $x \in \mathbb{R}_+ \mapsto x - D^j(x)$ is nonnegative and nondecreasing. We will pay particular attention to the following special cases:

- $D^j(x) = (1-\rho_j)x$, where $\rho_j \in (0,1)$, which we will call the *proportional* dividend case,
- $D^j(x) = D^j \wedge x$ with $D^j > 0$, which we will call the *constant* dividend case, and
- $D^j(x) = \min\{a_j + b_j x, c_j x\}$ with $a_j, b_j, c_j \geq 0$ and $c_j \leq 1$, which we call the *mixed* dividend case.

We will see that the behavior of D^j around zero determines the behavior of the exercise boundary at the dividend dates t_d^j, so the latter case will turn out to be very similar to that where we have proportional dividends.

For $t \in [0,T]$, let

$$U_t = \text{ess.}\sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}[e^{-r(\tau-t)}(K - S_\tau)^+ | \mathcal{F}_t] \tag{0.2}$$

denote the price at time t of the American Put option, where $\mathcal{T}_{[t,T]}$ is the set of stopping times w.r.t. the filtration $\mathcal{F}_t \stackrel{\text{def}}{=} \sigma(W_s, 0 \leq s \leq t)$ taking values in $[t,T]$. The solution to this optimal stopping problem for the case without dividends (i.e., $I = 0$) goes back to the work of McKean [16] and Van Moerbeke [21]. The optimal stopping time is the first time that the asset price process falls below a time-dependent value (the so-called exercise boundary which we will denote by c^0), and McKean derived a free-boundary problem involving both the pricing function u^0 such that $U_t = u^0(t, S_t)$ and c^0. Van Moerbeke derived an integral equation which involves both c^0 and its derivative, but in later work by Kim [14], Jacka [12], and Carr, Jarrow, and Myneni [3] an integral equation was derived which involves only c^0 itself. The regularity and uniqueness of solutions to this equation was left as an open problem in those papers. Uniqueness was proved by Peskir [19], using his change-of-variables formula with local time on curves [18]. It is known that the optimal exercise boundary is convex [5, 6] and its asymptotic behavior at maturity is given in [15]. But although it was claimed in several papers (for example, [17]) that it is C^1 at all points prior to maturity, a complete proof has been given only recently by Chen and Chadam [4]. In fact, in that paper it was actually shown that it is C^∞ in all those points, and a later paper by Bayraktar and Xing [2] shows that this remains true if the underlying asset pays continuous dividends at a fixed rate. In practice, continuous dividends are not a satisfying model since dividends are paid once a year or quarterly. That is why we are interested in dividends that are paid at a number of discrete points in time. To begin with, we deal in this paper with the simplest situation where there is only one dividend time t_d^1 before the maturity T of the Put option.[1] Afterwards we show how some results can be extended to the case of multiple dividends.

When we assume discrete dividend payments such as the proportional or fixed dividend payments mentioned above, the optimal exercise boundary will become discontinuous at the dividend dates, and before the dividend dates it may not be monotone (see Figure 1). Integral

[1] When there is only one dividend date, i.e., $I = 1$, we will often suppress the value $i = 1$ in our notation, so we will write t_d instead of t_d^1, D for D^1, and so on.

formulas for the exercise boundary which are similar to those in [3] have been derived under the assumption that the boundary is Lipschitz continuous (see Göttsche and Vellekoop [10]) or locally monotonic (Vellekoop and Nieuwenhuis [23]). In this paper we therefore study conditions under which such regularity properties of the optimal exercise boundary under discrete dividend payments can be proved.

In the first section, we introduce the pricing functions $u^i : [0,T]\times\mathbb{R}_+$ of the American Put option in the model (0.1) for $0 \le i \le I$ and the associated exercise boundaries c^i, where the i means for $i \ge 1$ that only at the times $t_d^i, t_d^{i-1}, \dots, t_d^1$ are dividends being paid while $i = 0$ means that no dividends are being paid. We then explain that for $I \ge i \ge 1$, on the time interval $[t_d^{i+1}, t_d^i)$, the American Put price u^i is equal to the price of an American option in the Black–Scholes model with no dividends if we take its maturity t_d^i and its payoff $x \mapsto (K-x)^+$ when exercised early and a modified payoff $x \mapsto u^{i-1}(t_d^i, x - D^i(x))$ when exercised at the maturity time t_d^i. Studying the properties of the single dividend case will then allow us to derive properties of the sequence of functions u^i and c^i in a recursive manner and prove our main result.

Theorem 0.1. *For all $1 \le j \le I$ let the dividend functions D^j be nonnegative and nondecreasing and such that $x \in \mathbb{R}_+ \mapsto x - D^j(x)$ is nonnegative and nondecreasing. Then for all $1 \le i \le I$ the exercise boundaries c^i are strictly positive and locally bounded away from zero on $[t_d^{i+1}, t_d^i)$. If D^i is positive on $(0,+\infty)$, then $\lim_{t\to t_d^i-} c^i(t) = 0$, and if D^i is also concave, then $c^i(t) \le rK(t_d^i - t)\inf_{x>0}\frac{x}{D^i(x)} + o(t_d^i - t)$ as $t \to t_d^i-$. Moreover, if for all $1 \le j \le i$ we have $D^j(x) = (1-\rho_j)x$ for some $\rho_j \in (0,1)$, then the following hold:*

- *for all $t \in [0,T]$ the value function $u^i(t,x)$ is convex in x;*
- *c^i is right-continuous on $[0,T]$ and for all $t \in [0,T)$, $\partial_x u^i(t, c^i(t)^+) = -1$; i.e., the smooth contact property holds; and*
- *there exist $\varepsilon^i > 0$ such that on $(t_d^i - \varepsilon^i, t_d^i)$, the function c^i is continuous and nonincreasing with $c^i(t) \sim rK(t_d^i - t)/(1-\rho_i)$ as $t \to t_d^i-$.*

The proof for this theorem can be found in the appendix. It is based on the stronger results that we obtain for the single dividend case.

In the second section, we therefore look at the single dividend case only and denote by $t_d \in (0,T)$ the dividend date. We prove that when the dividend function is positive and concave, then the boundary is nonincreasing in a left-hand neighborhood of t_d and tends to 0 as time tends to t_d^- with a speed that we can characterize. In the third section, we assume, moreover, that the dividend function is linear in a neighborhood of 0, a condition satisfied in the *proportional*, the *constant*, and the *mixed* dividend cases. Then we show that the exercise boundary is continuous and a high contact principle holds in a left-hand neighborhood of t_d. In the *proportional* dividend case, the right-continuity of the boundary and the high contact principle are proved to hold globally. Finally, we show how results for a single dividend date can be extended to multiple dividend dates in that case.

Notation and definitions.

- For $t \in [0,T]$ and $x \ge 0$, we use the notation $\bar{S}_t^x = xe^{\sigma W_t + (r-\frac{\sigma^2}{2})t}$ for the stock price at time t when the initial price is equal to x and when there is no dividend (i.e., $I = 0$). We also denote by $L_t^y(\bar{S}^x)$ the local time at level $y > 0$ and time t of the process $\bar{S}^x$ and by $p(t,y) = \frac{1_{\{y>0\}}}{\sigma y\sqrt{2\pi t}}\exp\left(-\frac{(\log(y/x)-(r-\frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right)$ the density of $\bar{S}_t^x$ w.r.t. the Lebesgue measure when $t, x > 0$.

- Let $\mathcal{A}$ denote the infinitesimal generator of the Black–Scholes model without dividends: $\mathcal{A}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + rxf'(x) - rf(x)$.
- If $(t, x) \in [0, T] \times \mathbb{R}_+$ and $1 \leq i \leq I$, we write $S_u^{x,t,i}$ for the solution to

$$dS_v^{x,t,i} = \sigma S_v^{x,t,i} dW_v + rS_v^{x,t,i} dv - \sum_{j=1}^{i} D^j(S_{v^-}^{x,t,i}) d1_{\{v \geq t_d^j\}} \tag{0.3}$$

  for $u \geq t$ under the initial condition that $S_t^{x,t,i} = x$. Note that we still retain the notation introduced in (0.1), so $S_u = S_u^{S_0,0,I}$.
- Let $N(y) = \int_{-\infty}^{y} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$ be the cumulative distribution function of the standard normal law.
- Let C denote a constant with may change from line to line.
- We say that $D : \mathbb{R}_+ \to \mathbb{R}_+$ is positive when for all $x > 0$, $D(x) > 0$.
- By a left-hand neighborhood of $x \in \mathbb{R}$, we mean an open interval $(x - \varepsilon, x)$ for some $\varepsilon > 0$.
- We will often denote the value function u^0 for the case without dividends by $\bar{u}$ and the value function u^1 for the case of one dividend by u.

1. Preliminary results.

The following results, which have been proven in [7, 8, 11, 20], provide an optimal stopping time in (0.2).

Proposition 1.1. *Let $\{G_t, t \in [0, T]\}$ be an $(\mathcal{F}_t)$-adapted, right-continuous, and upper-semicontinuous process with $\mathbb{E}(\sup_{t\in[0,T]} |G_t|) < \infty$. Then the càdlàg version of the Snell envelope $U_t = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}(G_\tau \mid \mathcal{F}_t)$ is continuous on $[0, T]$ and the stopping time $\tau = \inf\{s \geq t : U_s = G_s\}$ is optimal: $U_t = \mathbb{E}(G_\tau \mid \mathcal{F}_t)$.*

The conditions for this result are satisfied by $G_t^i = e^{-rt}(K - S_t^{S_0,0,i})^+$ since for all $t \in [0, T]$ we have $|G_t^i| \leq K$, and G_t^i is right-continuous and upper-semicontinuous for all $t \in [0, T]$ since the jump sizes of $S_t^{S_0,0,i}$ at $t = t_d^j$ are nonpositive for all $1 \leq j \leq i$ (for a Call option, $G_t^i = e^{-rt}(S_t^{S_0,0,i} - K)^+$ is no longer upper-semicontinuous, and, in the single dividend case, Battauz and Pratelli [1] artificially stretch the time interval $[0, T]$ by introducing a fictitious interval $[\underline{t}_d^1, \bar{t}_d^1]$, where $\underline{t}_d^1$ denotes the end of the cum-dividend date and $\bar{t}_d^1$ the beginning of the ex-dividend date to reduce the evaluation problem to the computation of the Snell envelope on stopping times taking values in $[0, \underline{t}_d^1] \cup [\bar{t}_d^1, T]$). According to [8], there thus exists pricing functions u^i defined as follows.

Proposition 1.2. *Take $1 \leq i \leq I$ and a constant $S_0 > 0$. The Snell envelope U^i of $\{G_t^i = e^{-rt}(K - S_t^{S_0,0,i})^+, t \in [0, T]\}$ is such that $U_t^i = e^{-rt} u^i(t, S_t^{S_0,0,i})$, where*

$$\forall (t, x) \in [0, T] \times \mathbb{R}_+,\ u^i(t, x) \overset{\text{def}}{=} \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}(e^{-r(\tau-t)}(K - S_\tau^{x,t,i})^+).$$

Moreover, the previous supremum is attained for $\tau = \inf\{s \geq t : u^i(s, S_s^{x,t,i}) = (K - S_s^{x,t,i})^+\}$.

Let us now derive some properties of the pricing functions u^i and define the exercise boundaries c^i.

Lemma 1.3. *For all $1 \leq j \leq I$ let the dividend functions D^j be nonnegative and nondecreasing and such that $x \in \mathbb{R}_+ \mapsto x - D^j(x)$ is nonnegative and nondecreasing. Then the* *functions* D^j *are continuous and we have*

$$\forall 0 \leq i \leq I,\ \forall t \in [0,T],\ \forall x > y \geq 0,\ 0 \leq u^i(t,y) - u^i(t,x) \leq x - y. \tag{1.1}$$

For $t \in [0,T]$, *let*

$$c^i(t) = \inf\{x > 0 : u^i(t,x) > (K-x)^+\}.$$

Then $c^i(t) < K$ *for* $t \in [0,T)$ *and we have that* $\{x \geq 0 : u^i(t,x) = (K-x)^+\} = [0, c^i(t)]$. *Last, the functions* c^i *cannot vanish on an interval.*

Figure 1 plots the exercise boundary $t \mapsto c^1(t)$ of the Put option with strike $K = 100$ and maturity $T = 4$ in the model (0.1) with $r = 0.04$, $\sigma = 0.3$, $t_d^1 = 3.5$ and *proportional* dividends with $\rho_1 = 0.05$. This exercise boundary was computed by a binomial tree method (see [22]).

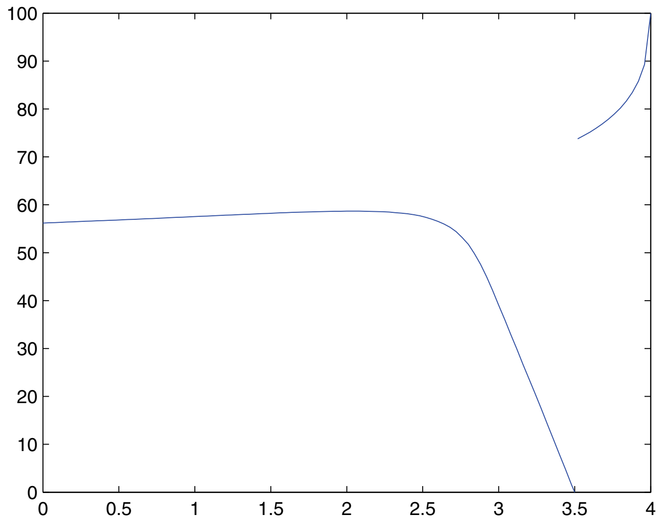

Figure 1. *Exercise boundary* $t \mapsto c^1(t)$ *(*$K = 100$, $T = 4$, $t_d^1 = 3.5$, $r = 0.04$, $\sigma = 0.3$, proportional *dividends:* $\rho_1 = 0.05$*) obtained by a binomial tree method.*

Proof. For the first part, we use a proof similar to that in [10]. We start by noting that our condition that both $x \mapsto D^j(x)$ and $x \mapsto x - D^j(x)$ be nondecreasing on their domains immediately gives continuity of D^j. For a fixed $t \in [0,T]$, take $x > y \geq 0$, which, with the monotonicity of $z \mapsto z - D^j(z)$ for all $1 \leq j \leq I$, implies that $S_v^{x,t,i} \geq S_v^{y,t,i}$ for all $v \in [t,T]$. Now fix the value of i with $0 \leq i \leq I$. For $\tau_x \in \mathcal{T}_{[t,T]}$ such that $u^i(t,x) = \mathbb{E}[e^{-r(\tau_x - t)}(K - S_{\tau_x}^{x,t,i})^+]$, since τ_x need not be optimal for the case where the stock price at time t equals y, we deduce

$$u^i(t,x) - u^i(t,y) \leq \mathbb{E}[e^{-r(\tau_x - t)}((K - S_{\tau_x}^{x,t,i})^+ - (K - S_{\tau_x}^{y,t,i})^+)] \leq 0.$$

For $\tau_y \in \mathcal{T}_{[t,T]}$ such that $u^i(t,y) = \mathbb{E}[e^{-r(\tau_y-t)}(K - S^{y,t,i}_{\tau_y})^+]$,

$$\begin{aligned} u^i(t,y) - u^i(t,x) &\leq \mathbb{E}[e^{-r(\tau_y-t)}(K - S^{y,t,i}_{\tau_y})^+] - \mathbb{E}[e^{-r(\tau_y-t)}(K - S^{x,t,i}_{\tau_y})^+] \\ &\leq \mathbb{E}[e^{-r(\tau_y-t)}(S^{x,t,i}_{\tau_y} - S^{y,t,i}_{\tau_y})] \\ &= x - y - \sum_{j=1}^{i} \mathbb{E}[e^{-r(\tau_y-t)} 1_{\{\tau_y \geq t^j_d > t\}} (D^j(S^{x,t,i}_{t^j_d-}) - D^j(S^{y,t,i}_{t^j_d-}))\, \bar{S}^1_{\tau_y - t^j_d}] \;\leq\; x - y \end{aligned}$$

because $S^{x,t,i}_{t^j_d-} \geq S^{y,t,i}_{t^j_d-}$ and the function D^j is nondecreasing.

Since $u^i(t,x) \geq (K-x)^+$ for all $t \in [0,T]$ and $x \geq 0$, the definition of $c^i(t)$ implies that $u^i(t,x) = (K-x)^+$ for $x \in [0, c^i(t))$ and by the continuity of $x \to u^i(t,x) - (K-x)^+$ this must then be true for $x = c^i(t)$ as well when $c^i(t) > 0$. When $c^i(t) = 0$, $u^i(t, c^i(t)) = K = (K - c^i(t))^+$. If $x > c^i(t)$, then, by the definition of $c^i(t)$, there exists $y \in (c^i(t), x]$ such that $u^i(t,y) > (K-y)^+$ and $u^i(t,x) \geq u^i(t,y) + y - x > K - x$. For $t \in [0,T)$, since $u^i(t,x) \geq \mathbb{E}(e^{-r(T-t)}(K - S^{x,t,i}_T)^+) > 0$, one deduces that $u^i(t,x) > (K-x)^+$ for $x > c^i(t)$ and that $c^i(t) < K$. Last, $c^i(T) = +\infty$.

Assume that there exists an interval $[t_1, t_2)$ with $0 \leq t_1 \leq t_2 \leq T$ such that c^i is zero in every point of this interval, and for $x > 0$, let $\tau_x \in \mathcal{T}_{[t_1,T]}$ be such that we have that $u^i(t_1,x) = \mathbb{E}[e^{-r(\tau_x - t_1)}(K - S^{x,t_1,i}_{\tau_x})^+]$. Then $\tau_x \geq t_2$ so $Ke^{-r(t_2-t_1)} \geq K\mathbb{E}[e^{-r(\tau_x-t_1)}] \geq u^i(t_1,x) \geq (K-x)^+$. Letting $x \to 0^+$, one deduces that $t_2 = t_1$. ■

Let us now prove some regularity properties of the pricing functions u^i.

Lemma 1.4. *Let $i \in \{1,\dots,I\}$. Under the assumptions of Lemma 1.3, the function u^i is continuous on the sets $[0,t^i_d) \times \mathbb{R}_+$, $[t^i_d, t^{i-1}_d) \times \mathbb{R}_+$, $[t^{i-1}_d, t^{i-2}_d) \times \mathbb{R}_+, \dots, [t^1_d, T] \times \mathbb{R}_+$ and for all $j \in \{1,\dots,i\}$ and all $x > 0$, the limit $\lim_{t \to t^j_d-} u^i(t,x)$ exists and is equal to $u^i(t^j_d, x - D^j(x))$. Moreover, the exercise boundary $t \mapsto c^i(t)$ is upper-semicontinuous on $[0,T]$.*

Last, for all points in the set $\{(t,x) : t \in [0,T) \setminus \{t^k_d, 1 \leq k \leq i\},\ x > c^i(t)\}$ the partial derivatives $\partial_t u^i(t,x)$, $\partial_x u^i(t,x)$, and $\partial_{xx} u^i(t,x)$ exist and satisfy $\mathcal{A}u^i(t,\cdot)(x) + \partial_t u^i(t,x) = 0$, and u^i is $C^{1,2}$ on this set.

Proof. Let us check the behavior of u^i as $t \to t^j_d-$ for $1 \leq j \leq i \leq I$; the continuity of u^i follows from a similar but easier argument.

Since $S_{t^j_d} = S_{t^j_d-} - D^j(S_{t^j_d-})$, one has, using (1.1) for the inequality,

$$\begin{aligned} |u^i(t, S_{t^j_d-}) - u^i(t^j_d, S_{t^j_d-} - D^j(S_{t^j_d-}))| &= |u^i(t, S_{t^j_d-}) - u^i(t^j_d, S_{t^j_d})| \\ &\leq |S_t - S_{t^j_d-}| + |u^i(t, S_t) - u^i(t^j_d, S_{t^j_d})|. \end{aligned}$$

By continuity of the process $(u^i(t,S_t))_{t\in[0,T]}$, ensured by Propositions 1.1 and 1.2, one deduces that a.s., $\lim_{t\to t^j_d-} u^i(t, S_{t^j_d-}) = u^i(t^j_d, S_{t^j_d-} - D^j(S_{t^j_d-}))$. Since $S_{t^j_d-}$ admits a positive density w.r.t. the Lebesgue measure on $(0,+\infty)$, dx a.e. $\lim_{t\to t^j_d-} u^i(t,x) = u^i(t^j_d, x - D^j(x))$. By (1.1), $u^i(t^j_d, x - D^j(x)) \geq u^i(t^j_d, x)$ and both $x \mapsto \liminf_{t\to t^j_d-} u^i(t,x)$ and $x \mapsto \limsup_{t\to t^j_d-} u^i(t,x)$ are Lipschitz continuous. By continuity of the functions $x \mapsto D^j(x)$ and $x \mapsto u^i(t^j_d, x)$, the function $x \mapsto u^i(t^j_d, x - D^j(x))$ is continuous. One deduces that for all x, $\lim_{t\to t^j_d-} u^i(t,x) =$

$u^i(t_d^j, x - D^j(x))$ and that for all $x > c^i(t_d^j)$, $\liminf_{t \to t_d^j -} u^i(t,x) \geq u^i(t_d^j, x) > (K-x)^+$, which ensures that $\limsup_{t \to t_d^j -} c^i(t) \leq c^i(t_d^j)$. Since, according to Lemma 1.3, for $t \in [0,T]$, $\{x \geq 0 : u^i(t,x) = (K-x)^+\} = [0, c^i(t)]$, the continuity properties of u^i imply that c^i is upper-semicontinuous on the sets $[0, t_d^i)$, $[t_d^i, t_d^{i-1})$, $[t_d^{i-1}, t_d^{i-2}), \dots, [t_d^1, T]$ and therefore on $[0,T]$.

Let $A_i = ([0,T) \setminus \{t_d^k, 1 \leq k \leq i\}) \times \mathbb{R}_+$. By the continuity of u^i on A_i, the set $\{(t,x) \in A_i :\ x > c^i(t)\}$ is an open subset of A_i. Let $(t,x) \in A_i$ and let B be an open neighborhood of (t,x) with regular boundary ∂B such that B is included in the connected component of A_i which contains (t,x). Define the stopping times $\tau = \inf\{v \geq t : S_v^{x,t,i} \leq c^i(v)\}$ and $\tau_{B^c} = \inf\{v \geq t : S_v^{x,t,i} \in B^c\} < \tau$. The flow property for the Black–Scholes model without dividends implies that for $v \geq \tau_{B^c}$, $S_v^{x,t,i} = S_v^{S_{\tau_{B^c}}^{x,t,i}, \tau_{B^c}, i}$ and $\tau = \inf\{v \geq \tau_{B^c} : S_v^{S_{\tau_{B^c}}^{x,t,i}, \tau_{B^c}, i} \leq c^i(v)\}$. Using the strong Markov property for the third equality, one deduces that

$$\begin{aligned} u^i(t,x) = \mathbb{E}[e^{-r(\tau - t)}(K - S_\tau^{x,t,i})^+] &= \mathbb{E}[e^{-r(\tau_{B^c} - t)}\mathbb{E}[e^{-r(\tau - \tau_{B^c})}(K - S_\tau^{S_{\tau_{B^c}}^{x,t,i}, \tau_{B^c}, i})^+ | \mathcal{F}_{\tau_{B^c}}]] \\ &= \mathbb{E}[e^{-r(\tau_{B^c} - t)} u^i(\tau_{B^c}, S_{\tau_{B^c}}^{x,t,i})]. \end{aligned} \tag{1.2}$$

Let $f(s,x)$ be a solution to the Dirichlet problem where $\partial_s f + \mathcal{A} f = 0$ on B and $f = u^i$ on ∂B. By Theorem 3.6.3 in [9], this function f is $C^{1,2}$ in B and continuous on $\bar{B}$. But then

$$\begin{aligned} u^i(t,x) = \mathbb{E}[e^{-r(\tau_{B^c} - t)} u^i(\tau_{B^c}, S_{\tau_{B^c}}^{x,t,i})] &= \mathbb{E}[e^{-r(\tau_{B^c} - t)} f(\tau_{B^c}, S_{\tau_{B^c}}^{x,t,i})] \\ &= f(t,x) + \mathbb{E}\int_t^{\tau_{B^c}} e^{-r(s-t)}(\partial_s f + \mathcal{A} f)(s, S_s^{x,t,i}) ds = f(t,x) \end{aligned}$$

by optional sampling, so $u^i = f$ on B, and therefore its partial derivatives exist in (t,x) and they satisfy $\partial_t u^i(t,x) + \mathcal{A} u^i(t,\cdot)(x) = 0$. ∎

The characterization of the restriction of u^i to $[0, t_d^i) \times \mathbb{R}_+$ as the pricing function of an American option in the Black–Scholes model without dividends, as stated in the next proposition, is the key to the study of the exercise boundaries $c^i(t)$ performed in the following sections.

Proposition 1.5. *Under the assumptions of Lemma 1.3, we have for all* $0 \leq i \leq I$,

$$\forall (t,x) \in [0, t_d^i) \times \mathbb{R}_+,\ u^i(t,x) = \sup_{\tau \in \mathcal{T}_{[0, t_d^i - t]}} \mathbb{E}[e^{-r\tau}(\,(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d^i - t\}} + g^i(\bar{S}_{t_d^i - t}^x) 1_{\{\tau = t_d^i - t\}}\,)],$$

where $g^0(x) \stackrel{\text{def}}{=} (K-x)^+$ *and* $g^i(x) \stackrel{\text{def}}{=} u^{i-1}(t_d^i, x - D^i(x))$ *for* $i \geq 1$, *and the supremum is attained for* $\tau = \inf\{s \in [0, t_d^i - t) : \bar{S}_s^x \leq c^i(t+s)\} \wedge (t_d^i - t)$ *(with the convention that* $\inf \emptyset = +\infty$*). Moreover, for all* $0 \leq j \leq i$ *and* $t \in [t_d^{j+1}, T]$, *we have that* $c^i(t) = c^j(t)$ *and* $u^i(t,x) = u^j(t,x)$ *for all positive* x.

Proof. For $i = 0$ the statement is trivial, so assume $i \geq 1$. The last statement of the proposition is obvious because when $0 \leq j \leq i$, the optimal stopping problems in Proposition 1.2 which define the values $u^i(t,x)$ and $u^j(t,x)$ and the values $c^i(t)$ and $c^j(t)$ are the same for $t \geq t_d^{j+1}$ and $x \geq 0$ because we then have that $S_v^{t,x,i} = S_v^{t,x,j}$ for $v \in [t,T]$. Take $t \in [0, t_d^i)$ and

$x \geq 0$, and define $\tau_x = \inf\{v \geq t : S_v^{x,t,i} \leq c^i(v)\}$. Arguing like in the derivation of (1.2), one easily checks that

$$\mathbb{E}\left[e^{-r(\tau_x-t)}(K - S_{\tau_x}^{x,t,i})^+ 1_{\{\tau_x \geq t_d^i\}}\right] = \mathbb{E}\left[e^{-r(t_d^i-t)} u^i(t_d^i, S_{t_d^i}^{x,t,i}) 1_{\{\tau_x \geq t_d^i\}}\right]$$
$$= \mathbb{E}\left[e^{-r(t_d^i-t)} u^{i-1}(t_d^i, S_{t_d^i}^{x,t,i}) 1_{\{\tau_x \geq t_d^i\}}\right] = \mathbb{E}\left[e^{-r(t_d^i-t)} g^i(S_{t_d^i-}^{x,t,i}) 1_{\{\tau_x \geq t_d^i\}}\right],$$

where we used the previous result for $j = i-1$ to obtain the second equality. We thus deduce that

$$u^i(t,x) = \mathbb{E}\left[e^{-r(\tau_x-t)}(K - S_{\tau_x}^{x,t,i})^+ 1_{\{\tau_x < t_d^i\}} + e^{-r(t_d^i-t)} g^i(S_{t_d^i-}^{x,t,i}) 1_{\{\tau_x \geq t_d^i\}}\right]$$
$$= \mathbb{E}\left[e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d^i - t\}} + e^{-r(t_d^i-t)} g^i(\bar{S}_{t_d^i-t}^x) 1_{\{\tau = t_d^i - t\}}\right]$$

when $\tau = \inf\{s \in [0, t_d^i - t) : \bar{S}_s^x \leq c^i(t+s)\} \wedge (t_d^i - t)$.

Now let τ be any stopping time in $\mathcal{T}_{[0,t_d^i-t]}$. For $f : C([0, t_d^i - t], \mathbb{R}) \to [0, t_d^i - t]$ such that $\tau = f(W_s, 0 \leq s \leq t_d^i - t)$, the random variable

$$\tau_x \stackrel{\text{def}}{=} \begin{cases} t + f(W_s - W_t, t \leq s \leq t_d^i) \text{ if } t + f(W_s - W_t, t \leq s \leq t_d^i) < t_d^i, \\ \inf\{s \geq t_d^i : S_s^{x,t,i} \leq c^i(s)\} \text{ otherwise} \end{cases}$$

belongs to $\mathcal{T}_{[t,T]}$ and is such that

$$\mathbb{E}\left[e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d^i - t\}} + e^{-r(t_d^i-t)} g^i(\bar{S}_{t_d^i-t}^x) 1_{\{\tau = t_d^i - t\}}\right]$$
$$= \mathbb{E}\left[e^{-r(\tau_x-t)}(K - S_{\tau_x}^{x,t,i})^+ 1_{\{\tau_x < t_d^i\}} + e^{-r(t_d^i-t)} u^{i-1}(t_d^i, S_{t_d^i}^{x,t,i}) 1_{\{\tau_x \geq t_d^i\}}\right]$$
$$= \mathbb{E}\left[e^{-r(\tau_x-t)}(K - S_{\tau_x}^{x,t,i})^+\right] \leq u^i(t,x). \qquad \blacksquare$$

This result shows that it is natural to consider the case with only one dividend date first and then use the results to generalize to multiple dividend dates. Remember that in the single dividend case we use the shorthand notation $u(t,x) = u^1(t,x)$, $g(x) = g^1(x)$, $D(x) = D^1(x)$, and $t_d = t_d^1$ and that $\bar{u}(t,x) = u^0(t,x)$ and $\bar{c}(t) = c^0(t)$ are used for the case when no dividends are present. We will also write $S^{x,t}$ for $S^{x,t,1}$ now that $I = 1$.

We first derive some properties of the function $g(x) = \bar{u}(t_d, x - D(x))$. When D is concave, to estimate the time derivative $\partial_t u(t,x)$ in Proposition 2.2 and Lemma 3.3, we will apply Tanaka's formula twice to compute $d(\bar{S}_t^x - D(\bar{S}_t^x))$ and then $d\bar{u}(t_d, \bar{S}_t^x - D(\bar{S}_t^x))$. In this computation we may use the convention $\partial_{22}\bar{u}(t_d, \bar{c}(t_d)) = 0$ since only the measure $\partial_{22}\bar{u}(t_d, x)dx$ plays some role. We obtain that $e^{-rt}g(\bar{S}_t^x)$ is equal to the sum of a stochastic integral w.r.t. W, of a nonnegative local time contribution, and of $g(x) + \int_0^t e^{-rv}\gamma(\bar{S}_v^x)dv$, where the function γ is defined and studied in the next lemma.

Lemma 1.6. *Assume that D is a nonnegative concave function such that $x - D(x)$ is nonnegative. Then D is continuous and nondecreasing and such that $x - D(x)$ is nondecreasing. Let $D'_-(x)$ and $D''(dx)$, respectively, denote the left-hand derivative of D and the nonpositive Radon measure equal to the second order distribution derivative of D on $(0, +\infty)$. The* *function g is continuous and nonincreasing, and $g(x) \geq (K-x)^+$ for all $x \geq 0$. The function*

$$\gamma(x) \stackrel{\text{def}}{=} \frac{\sigma^2 x^2}{2}(1-D'_-(x))^2 \partial_{22}\bar{u}(t_d, x-D(x))+rx(1-D'_-(x))\partial_2\bar{u}(t_d, x-D(x))-r\bar{u}(t_d, x-D(x)),$$

where, by convention, $\partial_{22}\bar{u}(t_d, \bar{c}(t_d)) = 0$, is not greater than $-rK$ on $(0, x^\star)$, where $x^\star \stackrel{\text{def}}{=} \sup\{x : x - D(x) < \bar{c}(t_d)\} > 0$, and is globally bounded.

If g is convex, then there is a constant $\rho \in [0,1]$ such that $g(x) = K - \rho x$ and $D(x) = (1-\rho)x$ for $x < x^\star$, the second order distribution derivative of g admits a density g'' w.r.t. the Lebesgue measure, $\mathcal{A}g(x)$ is equal to $-rK$ on $(0, x^\star)$, and dx a.e. on $(x^\star, +\infty)$, $\mathcal{A}g(x) \geq -rK$.

To prove this lemma, we need the following properties of the pricing function $\bar{u}$ in the model without dividends.

Lemma 1.7. *For the case without dividends we have that the partial derivatives $\partial_t\bar{u}(t,x)$, $\partial_x\bar{u}(t,x)$, and $\partial_{xx}\bar{u}(t,x)$ exist and $\partial_t\bar{u}(t,x) + \mathcal{A}\bar{u}(t,\cdot)(x) = 0$ for all $t \in [0,T)$ and $x > \bar{c}(t)$. Moreover, for all $t \in [0,T]$, $x \mapsto \bar{u}(t,x)$ is convex and C^1 on $\mathbb{R}_+$. Last,*

$$\forall t \in [0,T),\ \forall x > \bar{c}(t),\ \partial_t\bar{u}(t,x) \geq -\frac{e^{-r(T-t)}\sigma K}{2\sqrt{2\pi(T-t)}} \exp\left(-\frac{(\log(K/x) - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right).$$

Before proving these lemmas, let us give some examples of functions g obtained for different choices of the dividend function D.

Examples of functions g.

- In the *constant* dividend case, $x^\star = \bar{c}(t_d) + D$ and the function g is not convex since it is equal to K on $[0, D]$ and to $K + D - x$ for $x \in (D, x^\star)$. This function is C^1 on $[0,D) \cup (D, +\infty)$ with g' taking its values in $[-1, 0]$ and C^2 on $[0,D) \cup (D, x^\star) \cup (x^\star, +\infty)$ and such that $\mathcal{A}g(dx) = \gamma(x)dx - \frac{\sigma^2 D^2}{2}\delta_D(dx)$, where γ is equal to $-rK$ on $(0, D)$ and to $-r(K+D)$ on $(D, x^\star)$.
- In the *proportional* dividend case, $x^\star = \bar{c}(t_d)/\rho$ and $g(x) = \bar{u}(t_d, \rho x)$ is convex. This function g is C^1 with g' taking its values in $[-\rho, 0]$ and C^2 on $[0, x^\star) \cup (x^\star, +\infty)$.
- The *proportional* dividend case provides an example of a nonnegative concave function D such that $x - D(x)$ is nonnegative, which leads to a convex function g. This example is not unique. For instance, let $\rho \in (0,1)$. The function $y \mapsto \bar{u}(t_d, y)$ is convex positive nonincreasing and such that $\lim_{y \to +\infty} \bar{u}(t_d, y) = 0$. So it is continuous and decreasing and admits an inverse $V(t_d, .) : (0, K] \to [0, +\infty)$. For $x \in (\bar{c}(t_d)/\rho, K/\rho)$, we set $d(x) = x - V(t_d, K - \rho x)$. The continuous function $d'(x) = 1 + \rho/\partial_2\bar{u}(t_d, V(t_d, K - \rho x))$ is nonincreasing on $(\bar{c}(t_d)/\rho, K/\rho)$ by the nonincreasing property of both $V(t_d, .)$ and $-\partial_2\bar{u}(t_d, .)$ and the positivity of this last function. It tends, respectively, to $1-\rho$ and $-\infty$ as $x \to \bar{c}(t_d)/\rho$ and $x \to K/\rho$. Let $x_0 = \sup\{x \in (\bar{c}(t_d)/\rho, K/\rho) : d'(x) \geq 0\}$. One has $d'(x_0) = 0$, which also writes as $\partial_2\bar{u}(t_d, x_0 - d(x_0)) = -\rho$. The function

$$D(x) = \begin{cases} (1-\rho)x \text{ for } x \in [0, \bar{c}(t_d)/\rho], \\ d(x \wedge x_0) \text{ for } x > \bar{c}(t_d)/\rho \end{cases}$$

is nonnegative and concave and such that $x - D(x)$ is nonnegative. The convexity of

$x \mapsto \bar{u}(t_d, x)$ combined with the equality $\partial_2 \bar{u}(t_d, x_0 - d(x_0)) = -\rho$ implies that

$$g(x) = \begin{cases} K - \rho x \text{ for } x \in [0, x_0], \\ \bar{u}(t_d, x - d(x_0)) \text{ for } x > x_0 \end{cases}$$

is convex.

Figure 2 illustrates the construction of the function g from $x \mapsto \bar{u}(t_d, x)$ on the three previous examples of dividend functions.

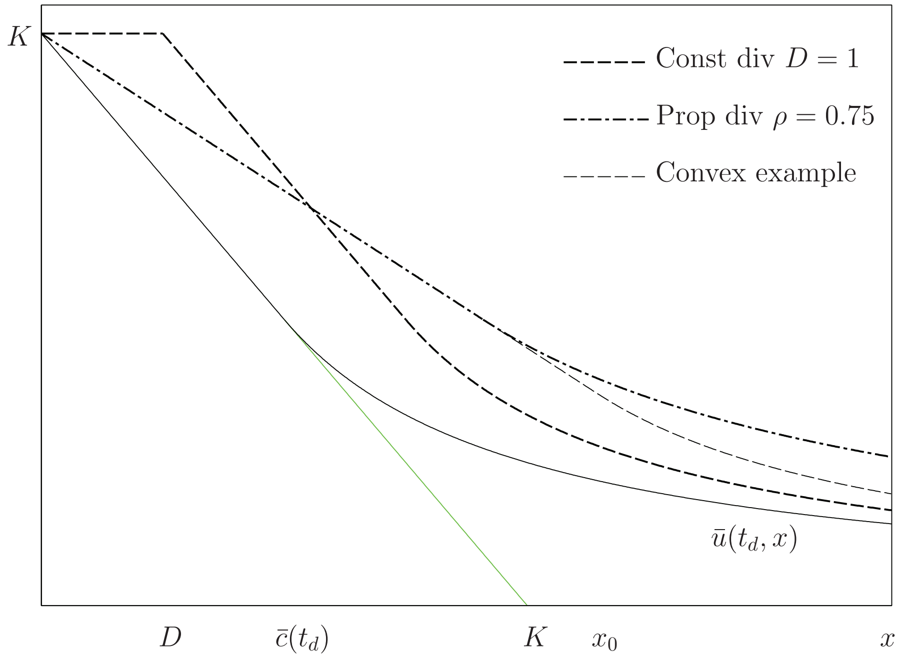

Figure 2. *Examples of functions g.*

Remark 1.8. *We have just constructed an example of a nonproportional dividend function such that the corresponding function g is convex. Nevertheless, this mathematical example is not of much use in practice. Moreover, according to Lemma 1.6, the convexity of g implies that D is linear below $x^\star$. So only the proportional dividend case is considered to ensure the convexity of g.*

Proof of Lemma 1.6. Since the concave function D is nonnegative, it is continuous and nondecreasing. And since $x - D(x)$ is nonnegative, $D(0) = 0$. The convex function $x - D(x)$ being nonnegative and equal to 0 for $x = 0$ is nondecreasing. Since $x \mapsto \bar{u}(t_d, x)$ is continuous, nonincreasing, and not smaller than $(K - x)^+$, the same properties hold for g.

For $x \in (0, x^\star)$, $\gamma(x) = rx(D'_-(x) - 1) - r(K - x + D(x)) = -rK - r(D(x) - xD'_-(x))$.

By concavity of D,

$$\forall x > 0,\ D(x) - xD'_-(x) \geq D(0) = 0. \tag{1.3}$$

So γ is not greater than $-rK$ on $(0, x^\star)$. The constant $x^\star$ is infinite if and only if D is the identity function, and then γ is constant and equal to $-rK$. When $x^\star < +\infty$, γ is bounded from below by $-r(K + D(x^\star))$ on $(0, x^\star)$. Moreover, since D is concave and continuous and $D(0) = 0$,

$$\forall x > x^\star,\ \frac{D(x)}{x} \leq \frac{D(x^\star)}{x^\star} = \frac{x^\star - \bar{c}(t_d)}{x^\star} \quad \text{and} \quad x - D(x) \geq \frac{x\bar{c}(t_d)}{x^\star} > \bar{c}(t_d). \tag{1.4}$$

One has

$$\begin{aligned}\gamma(x) - \mathcal{A}\bar{u}(t_d, .)(x - D(x)) = {} & \frac{\sigma^2}{2}\partial_{22}\bar{u}(t_d, x - D(x))[x^2(1 - D'_-(x))^2 - (x - D(x))^2] \\ & + r(D(x) - xD'_-(x))\partial_2\bar{u}(t_d, x - D(x)),\end{aligned} \tag{1.5}$$

where the last term is nonpositive by (1.3) and since $\partial_2\bar{u} \leq 0$. Define $M = \sup_{x > \bar{c}(t_d)} \mathcal{A}\bar{u}(t_d, .)(x)$, which is finite by Lemma 1.7. Since $\bar{u}(t_d, x) - x\partial_2\bar{u}(t_d, x)$ is nonincreasing by the convexity of $x \mapsto \bar{u}(t_d, x)$ and equal to K on $[0, \bar{c}(t_d))$, one deduces that

$$\forall x > \bar{c}(t_d),\ \partial_{22}\bar{u}(t_d, x) \leq \frac{2(M + rK)}{\sigma^2 x^2}. \tag{1.6}$$

With $x - D(x)$, which is larger than $\bar{c}(t_d)$, substituted in (1.6), and using (1.4) and $D'_-(x) \in [0, 1]$, one concludes that when $x^\star < +\infty$,

$$\forall x > x^\star,\ \gamma(x) \leq M + (M + rK)\frac{x^{\star 2} - \bar{c}(t_d)^2}{\bar{c}(t_d)^2}.$$

For $x > x^\star$, since $-xD'_-(x)\partial_2\bar{u}(t_d, x - D(x))$ and

$$\partial_{22}\bar{u}(t_d, x - D(x))[x^2(1 - D'_-(x))^2 - (x - D(x))^2]$$

are nonnegative and $\mathcal{A}\bar{u}(t_d, .)(x - D(x)) = -\partial_t\bar{u}(t_d, x - D(x)) > 0$, we have, by (1.5),

$$\begin{aligned}\gamma(x) &\geq rD(x)\partial_2\bar{u}(t_d, x - D(x)) \geq r\frac{x^\star - \bar{c}(t_d)}{\bar{c}(t_d)}(x - D(x))\partial_2\bar{u}(t_d, x - D(x)) \\ &= r\frac{x^\star - \bar{c}(t_d)}{\bar{c}(t_d)}\left(-K + \int_{\bar{c}(t_d)}^{x - D(x)} y\partial_{22}\bar{u}(t_d, y)dy + \bar{u}(t_d, x - D(x))\right) \geq -rK\frac{x^\star - \bar{c}(t_d)}{\bar{c}(t_d)},\end{aligned}$$

where we used that $D(x) \leq (x - D(x))(x^* - \bar{c}(t_d))/\bar{c}(t_d)$ by (1.4) for the second inequality and the smooth fit property $\partial_2\bar{u}(t_d, \bar{c}(t_d)) = -1$ and a partial integration for the equality.

Last, assume that g is convex. If g'_+ and D'_+, respectively, denote the right-hand derivatives of g and D, one has $g'_+(x) - g'_-(x) = -\partial_2\bar{u}(t_d, x - D(x))(D'_+(x) - D'_-(x))$, and since $\partial_2\bar{u}$ is negative and $D'_+ - D'_-$ nonpositive, the right-hand side of this equality is nonpositive and the left-hand side is nonnegative. So both are zero and the functions g and D are C^1 with

$g'(x) = \partial_2 \bar{u}(t_d, x - D(x))(1 - D'(x))$. The first factor in the right-hand side being globally continuous and C^1 on $(0, x^\star) \cup (x^\star, +\infty)$, one deduces that the distribution derivative of g' is equal to $\partial_{22}\bar{u}(t_d, x - D(x))(1 - D'(x))^2 dx - \partial_2 \bar{u}(t_d, x - D(x))D''(dx)$. This measure being nonnegative by the convexity of g, D'' is absolutely continuous w.r.t. the Lebesgue measure and so is the second order distribution derivative of g. For $x < x^\star$, $g'(x) = D'(x) - 1$, where the left-hand side is nondecreasing and the right-hand side nonincreasing. So there is a constant $\rho \in [0, 1]$ such that $g(x) = K - \rho x$ and $D(x) = (1 - \rho)x$ for $x < x^\star$. As a consequence $x^\star = \bar{c}(t_d)/\rho$ and $\mathcal{A}g(x) = rxg'(x) - rg(x) = -rK$ on $(0, x^\star)$. The convexity of g implies that $rxg'(x) - rg(x)$ is nondecreasing and therefore that dx a.e. on $(x^\star, +\infty)$, $\mathcal{A}g(x) = \frac{\sigma^2 x^2}{2} g''(x) + rxg'(x) - rg(x) \geq -rK$. ■

Proof of Lemma 1.7. The proof of the first statement is similar to that of the last statement in Lemma 1.4. Moreover, $x \mapsto \bar{u}(t, x) = \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\left(e^{-r\tau}(K - xe^{\sigma W_\tau + (r - \frac{\sigma^2}{2})\tau})^+\right)$ is convex as the supremum of convex functions. We refer the reader, for instance, to Lemma 7.8 in section 2.6 of [13] for the continuous differentiability property of this function.

Let $0 \leq s \leq t \leq T$, $x > 0$, and take $\tau \in \mathcal{T}_{[0,T-s]}$ such that $\bar{u}(s, x) = \mathbb{E}(e^{-r\tau}(K - \bar{S}^x_\tau)^+)$ and $\tilde{\tau} = \tau \wedge (T - t)$. One has

$$\begin{aligned}\bar{u}(t, x) &\geq \mathbb{E}\left(e^{-r\tilde{\tau}}(K - \bar{S}^x_{\tilde{\tau}})^+\right) \\ &= \bar{u}(s, x) - \mathbb{E}\left(1_{\{\tau > T-t\}}\left(e^{-r\tau}(K - \bar{S}^x_\tau)^+ - e^{-r(T-t)}(K - \bar{S}^x_{T-t})^+\right)\right).\end{aligned}$$

By Tanaka's formula, when $\tau > T - t$,

$$(K - \bar{S}^x_\tau)^+ = (K - \bar{S}^x_{T-t})^+ - \int_{T-t}^{\tau} 1_{\{\bar{S}^x_v \leq K\}}(\sigma \bar{S}^x_v dW_v + r\bar{S}^x_v dv) + \frac{1}{2}(L^K_\tau(\bar{S}^x) - L^K_{T-t}(\bar{S}^x)).$$

One deduces that

$$\bar{u}(t, x) \geq \bar{u}(s, x) - \frac{e^{-r(T-t)}}{2}\mathbb{E}(L^K_{T-s}(\bar{S}^x) - L^K_{T-t}(\bar{S}^x)).$$

By the occupation times formula, dy-a.e. $\mathbb{E}(L^y_{T-s}(\bar{S}^x) - L^y_{T-t}(\bar{S}^x)) = \sigma^2 y^2 \int_s^t p(T - v, y)dv$. Since the left-hand side is equal to $2\mathbb{E}\left((y - \bar{S}^x_{T-s})^+ - (y - \bar{S}^x_{T-t})^+ + r\int_{T-t}^{T-s} \bar{S}^x_v 1_{\{\bar{S}^x_v \leq y\}} dv\right)$ by Tanaka's formula, it is continuous in y. The right-hand side is also continuous in y by definition of the density p, and the equality holds for all $y > 0$. Therefore

$$\bar{u}(t, x) \geq \bar{u}(s, x) - \frac{e^{-r(T-t)}\sigma^2 K^2}{2}\int_s^t p(T - v, K)dv. \qquad ■$$

2. Limit behavior and monotonicity of the exercise boundary as $t \to t_d^-$.

Using the results in the previous section, we first check that $c(t)$ tends to 0 as $t \to t_d^-$ if D is positive (i.e., for all $x > 0$, $D(x) > 0$).

Lemma 2.1. *Let D be a nonnegative and nondecreasing function such that $x \mapsto x - D(x)$ is nonnegative and nondecreasing.*

Assume, moreover, that there exists a $d_0 \geq 0$ such that D is zero on $[0, d_0]$ and positive on $]d_0, \infty[$; then we have $\limsup_{t \to t_d^-} c(t) \leq d_0 \wedge \bar{c}(t_d)$. When $d_0 = 0$, i.e., D is positive, then $\lim_{t \to t_d-} c(t) = 0$ and the following hold:

- *if D is such that $\frac{x}{D(x)}$ admits a finite limit as $x \to 0^+$, then $c(t) \leq rK(t_d - t)\lim_{x\to 0^+}\frac{x}{D(x)} + o(t_d - t)$ as $t \to t_d^-$;*
- *if D is concave, g is convex, and the constant ρ such that, according to Lemma 1.6, for all $x \in (0, x^\star)$, $D(x) = (1-\rho)x$ belongs to $(0,1)$, then for all $t \in [0, t_d)$, $c(t) < \frac{1-e^{-r(t_d-t)}}{1-\rho}K$. When $\rho = 0$, i.e., D is the identity function, then for all $t \in [0, t_d)$, $c(t) \leq (1 - e^{-r(t_d-t)})K$.*

Note that when D is positive and concave, then $\frac{x}{D(x)}$ admits a finite limit as $x \to 0^+$ which is equal to $\inf_{x>0}\frac{x}{D(x)}$.

Proof. Suppose that $\limsup_{t\to t_d^-} c(t) > d_0 \wedge \bar{c}(t_d)$; then there exist a $y > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \uparrow t_d$ with $c(t_n) > y > d_0 \wedge \bar{c}(t_d)$ and since $c(t_n) \leq K$ we have $y < K$. Then $K - y = u(t_n, y)$ for all t_n and taking the limit and applying Lemma 1.4 give that $K - y = \bar{u}(t_d, y - D(y))$, but either $y > d_0$ and then $\bar{u}(t_d, y - D(y)) \geq (K - y + D(y))^+ = K - y + D(y) > K - y$ or $y > \bar{c}(t_d)$ and then $\bar{u}(t_d, y - D(y)) \geq \bar{u}(t_d, y) > K - y$, so in both cases we get a contradiction. Assume that D is such that $\mu \overset{\text{def}}{=} \lim_{x\to 0^+}\frac{x}{D(x)}$ exists and is finite. Since both $D(x)$ and $x - D(x)$ are nonnegative, necessarily $\mu \geq 1$. For $(t, x) \in [0, t_d) \times \mathbb{R}_+^*$,

$$\begin{aligned}
u(t,x) &\geq \mathbb{E}(e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})) \geq \mathbb{E}\left(e^{-r(t_d-t)}\left(K - \bar{S}^x_{t_d-t} + \frac{D(\bar{S}^x_{t_d-t})}{\bar{S}^x_{t_d-t}}\bar{S}^x_{t_d-t}\right)\right) \\
&\geq e^{-r(t_d-t)}K - x + \inf_{0<y\leq 4rK\mu(t_d-t)}\frac{D(y)}{y}\left(x - \mathbb{E}\left(e^{-r(t_d-t)}\bar{S}^x_{t_d-t}1_{\{\bar{S}^x_{t_d-t} > 4rK\mu(t_d-t)\}}\right)\right) \\
&\geq e^{-r(t_d-t)}K + \left(\inf_{0<y\leq 4rK\mu(t_d-t)y}\frac{D(y)}{y} - 1\right)x \\
&\quad - xN\left(\frac{\log(\frac{x}{4rK\mu(t_d-t)}) + (r + \frac{\sigma^2}{2})(t_d - t)}{\sigma\sqrt{t_d - t}}\right).
\end{aligned}$$

For $x \leq 2rK\mu(t_d - t)$ and $(t_d - t) \leq \frac{\log(2)}{2r+\sigma^2}$, $\frac{\log(\frac{x}{4rK\mu(t_d-t)})+(r+\frac{\sigma^2}{2})(t_d-t)}{\sigma\sqrt{t_d-t}} \leq -\frac{\log(2)}{2\sigma\sqrt{t_d-t}}$, which implies that

$$N\left(\frac{\log(\frac{x}{4rK\mu(t_d-t)}) + (r + \frac{\sigma^2}{2})(t_d - t)}{\sigma\sqrt{t_d - t}}\right) \leq \frac{2\sigma\sqrt{t_d - t}}{\sqrt{2\pi}\log(2)}e^{-\frac{\log^2(2)}{8\sigma^2(t_d-t)}}.$$

With $\lim_{t\to t_d^-}\inf_{0<y\leq 4rK\mu(t_d-t)}\frac{D(y)}{y} = \frac{1}{\mu}$, one deduces that, as $t \to t_d^-$, for $x \leq 2rK\mu(t_d - t)$, $u(t, x) \geq K - x + \left(\frac{x}{\mu} - rK(t_d - t)\right) + o(t_d - t)$, where the $o(t_d - t)$ does not depend on x. One easily deduces the desired upper bound for $c(t)$.

When g is also convex, according to Lemma 1.6, either D is the identity function and g is constant and equal to K or there is a constant $\rho \in (0,1)$ such that $D(x) = (1-\rho)x$ for $x \in (0, \bar{c}(t_d)/\rho]$. In the latter case, one has $g(x) = K - \rho x$ for $x \in (0, \bar{c}(t_d)/\rho]$ and $g(x) \geq (K - \rho x)^+$ for $x > \bar{c}(t_d)/\rho$. As a consequence,

$$\mathbb{E}(e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})) > \mathbb{E}(e^{-r(t_d-t)}(K - \rho\bar{S}^x_{t_d-t})) = e^{-r(t_d-t)}K - \rho x.$$

One deduces that when $x \geq \frac{1-e^{-r(t_d-t)}}{1-\rho}K$, $u(t,x) > K - x$, which implies that $c(t) < \frac{1-e^{-r(t_d-t)}}{1-\rho}K$. When D is the identity function, the inequality is obvious. ■

We now obtain the monotonicity of the exercise boundary in a left-hand neighborhood of the dividend date t_d.

Proposition 2.2. *If D is a positive concave function such that $x - D(x)$ is nonnegative, there exists a constant $\varepsilon > 0$ such that for $x \in (0, \varepsilon)$, $t \mapsto u(t,x)$ is nondecreasing on $(t_d - \varepsilon, t_d)$. Moreover, we have for all $t \in [0, t_d)$ and all $x > c(t)$ that*

$$\partial_t u(t,x) \geq -e^{-r(t_d-t)} \sup_{y>0} \gamma^+(y), \tag{2.1}$$

$$\frac{\sigma^2 x^2}{2} \partial_{xx} u(t,x) \leq e^{-r(t_d-t)} \sup_{y>0} \gamma^+(y) + r(x+K). \tag{2.2}$$

Last, for any $t \in [0, t_d)$ such that $c(t) > 0$, for all $x > c(t)$, $\int_{c(t)}^x |\partial_{xx} u(t,y)| dy < +\infty$ and $x \mapsto \partial_x u(t,x)$ admits a right-hand limit $\partial_x u(t, c(t)^+) \in [-1, 0]$ as $x \to c(t)^+$.

One easily deduces the following corollary.

Corollary 2.3. *If the dividend function D is nonnegative and nondecreasing and such that $x \in \mathbb{R}_+ \mapsto x - D(x)$ is nonnegative and nondecreasing, then the exercise boundary does not vanish on $[0,T]$. Moreover, for all $t \in [0, t_d)$, $\inf_{s \in [0,t]} c(s) > 0$.*

If D is a positive concave function such that $x - D(x)$ is nonnegative, then $t \mapsto c(t)$ is nonincreasing and left-continuous on $(t_d - \varepsilon, t_d)$. Moreover, $c(t) \sim rK(t_d - t) \inf_{x>0} \frac{x}{D(x)}$ as $t \to t_d^-$.

Remark 2.4. *In contrast to the result of Corollary 2.3, we notice that in the alternative model formulation known as the Escrowed model,*

$$S_t = (S_0 - De^{-rt_d}) e^{\sigma W_t + (r - \frac{\sigma^2}{2})t} + De^{-r(t_d-t)} 1_{\{t<t_d\}},$$

where D is a positive constant, the boundary is actually equal to 0 on a left-hand neighborhood of t_d. Indeed, reasoning like in the proof of Proposition 1.5, one can check that for $(t,x) \in (0,t_d) \times \mathbb{R}_+$, the value function in this model is

$$u(t,x) = \sup_{\tau \in \mathcal{T}_{[0,t_d-t]}} \mathbb{E}\left[e^{-r\tau} ((K - De^{-r(t_d-\tau)} - \bar{S}_\tau^y)^+ 1_{\{\tau < t_d - t\}} + \bar{u}(t_d, \bar{S}_{t_d-t}^y) 1_{\{\tau = t_d - t\}}) \right],$$

where $y = x - De^{-r(t_d-t)}$. Since

$$\mathbb{E}\left(e^{-r(t_d-t)} \bar{u}(t_d, \bar{S}_{t_d-t}^y) \right) \geq \mathbb{E}\left(e^{-r(t_d-t)} (K - \bar{S}_{t_d-t}^y)^+ \right) \geq (Ke^{-r(t_d-t)} - y)^+,$$

early exercise is never optimal when $K - De^{-r(t_d-t)} < Ke^{-r(t_d-t)}$, i.e., $t_d - t < \frac{1}{r} \log\left(\frac{K+D}{K}\right)$.

Proof of Corollary 2.3. For $t \in [t_d, T]$, $c(t)$ is larger than the exercise boundary $\frac{2rK}{\sigma^2+2r}$ of the perpetual Put in the Black–Scholes model without dividends. For $(t,x) \in [0,t_d) \times \mathbb{R}_+$, by Proposition 1.5, the pricing function $u(t,x)$ is smaller than that corresponding to the identity dividend function. Therefore, for $t \in [0, c(t_d))$, $c(t)$ is larger than the associated boundary. For the identity dividend function, (2.1) holds with the function γ constant and equal to $-rK$. Hence the exercise boundary corresponding to the identity dividend function is nonincreasing on $[0, t_d)$ and therefore does not vanish by Lemma 1.3.

Let us now assume that D is a positive concave function such that $x - D(x)$ is nonnegative. The constant $\mu = \inf_{x>0} \frac{x}{D(x)} = \lim_{x\to 0^+} \frac{x}{D(x)}$ is finite and not smaller than 1. The monotonicity of c is a consequence of Proposition 2.2, and the left-continuity then follows from the upper-semicontinuity. Let us now assume that $c(t)$ is not equivalent to $rK\mu(t_d - t)$ as $t \to t_d^-$ and obtain a contradiction. Because of the upper bound stated in Lemma 2.1, this implies that $\liminf_{t\to t_d^-} \frac{c(t)}{rK(t_d-t)} < \mu$. Hence there exist a constant $\tilde{\mu} \in (0, \mu)$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $(t_d - \varepsilon, t_d)$ such that $\lim_{n\to\infty} t_n = t_d$ and for all $n \in \mathbb{N}$, $c(t_n) \le rK\tilde{\mu}(t_d - t_n)$. For $n \in \mathbb{N}$, let $x_n = \frac{\mu+\tilde{\mu}}{2} rK(t_d - t_n)$ and $\tau_n = \inf\{s \in [0, t_d - t_n) : \bar{S}_s^{x_n} \le c(t_n + s)\} \wedge (t_d - t_n)$ denote the optimal stopping time starting from x_n at time t_n. One has

$$\begin{aligned}
u(t_n, x_n) &\le K\mathbb{P}\left(\exists s \in [0, t_d - t_n) : \bar{S}_s^{x_n} \le c(t_n)\right) + \mathbb{E}\left(e^{-r(t_d-t_n)} g(\bar{S}_{t_d-t_n}^{x_n})\right) \\
&\le K\mathbb{P}\left(\sigma \inf_{s\in[0,t_d-t_n]} W_s \le \log\left(\frac{2\tilde{\mu}}{\mu+\tilde{\mu}}\right) + \left(\frac{\sigma^2}{2} - r\right)^+ (t_d - t_n)\right) \\
&\quad + \mathbb{E}(e^{-r(t_d-t_n)} g(\bar{S}_{t_d-t_n}^{x_n})) \\
&= K\mathbb{P}\left(|W_{t_d-t_n}| \ge \frac{1}{\sigma}\left(\log\left(\frac{\mu+\tilde{\mu}}{2\tilde{\mu}}\right) - \left(\frac{\sigma^2}{2} - r\right)^+ (t_d - t_n)\right)\right) \\
&\quad + \mathbb{E}(e^{-r(t_d-t_n)} g(\bar{S}_{t_d-t_n}^{x_n})) \\
&= \mathbb{E}(e^{-r(t_d-t_n)} g(\bar{S}_{t_d-t_n}^{x_n})) + o(t_d - t_n), \qquad (2.3)
\end{aligned}$$

where we used the monotonicity of c on $(t_d - \varepsilon, t_d)$ for the first inequality and a reasoning similar to that made when D is concave in the proof of Lemma 2.1 for the last equality.

Assume that D is not the identity function which implies that $x^\star < +\infty$. Using the monotonicity of both g and $\frac{D(x)}{x}$, one gets that for $(t, x) \in [0, t_d) \times \mathbb{R}_+^*$, $\mathbb{E}(e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x))$ is not greater than

$$\begin{aligned}
&\mathbb{E}\left(e^{-r(t_d-t)}(K - \bar{S}_{t_d-t}^x + D(\bar{S}_{t_d-t}^x))1_{\{\bar{S}_{t_d-t}^x \le x^\star\}}\right) + e^{-r(t_d-t)} g(x^\star)\mathbb{P}(\bar{S}_{t_d-t}^x > x^\star) \\
&\le e^{-r(t_d-t)}K - x + \mathbb{E}\left(e^{-r(t_d-t)} \bar{S}_{t_d-t}^x 1_{\{\bar{S}_{t_d-t}^x > x^\star\}}\right) + \frac{x}{\mu} + e^{-r(t_d-t)} g(x^\star)\mathbb{P}(\bar{S}_{t_d-t}^x > x^\star).
\end{aligned}$$

Hence, for $x \in (0, x^\star/2)$, $\mathbb{E}(e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x)) \le K - x + \frac{x}{\mu} - rK(t_d - t) + o(t_d - t)$ with the $o(t_d - t)$ not depending on $x \le x^\star/2$. This inequality still holds when D is the identity function since then $\mu = 1$ and $\mathbb{E}(e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x)) = e^{-r(t_d-t)}K$.

With (2.3), one deduces that $u(t_n, x_n) \le K - x_n + rK\frac{\tilde{\mu}-\mu}{2\mu}(t_d - t_n) + o(t_d - t_n)$. Hence, for n large enough, $u(t_n, x_n) < K - x_n$, which provides the desired contradiction. ■

Proof of Proposition 2.2. Let $0 \le t \le s < t_d$, $x > 0$, and $\tau \in \mathcal{T}_{[0,t_d-t]}$ be such that $u(t, x) = \mathbb{E}\left(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau<t_d-t\}} + e^{-r(t_d-t)} g(\bar{S}_{t_d-t}^x)1_{\{\tau=t_d-t\}}\right)$. Since, by Lemma 1.6, for all $x > 0$, $g(x) \ge (K - x)^+$,

$$\begin{aligned}
u(t, x) &\le \mathbb{E}\left(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau<t_d-s\}} + e^{-r\tau} g(\bar{S}_\tau^x)1_{\{\tau\ge t_d-s\}}\right) \\
&= \mathbb{E}\left(e^{-r\tau}(K - \bar{S}_\tau^x)^+ 1_{\{\tau<t_d-s\}} + e^{-r(t_d-s)} g(\bar{S}_{t_d-s}^x)1_{\{\tau\ge t_d-s\}}\right) \\
&\quad + \mathbb{E}\left(1_{\{\tau>t_d-s\}}\left(e^{-r\tau} g(\bar{S}_\tau^x) - e^{-r(t_d-s)} g(\bar{S}_{t_d-s}^x)\right)\right). \qquad (2.4)
\end{aligned}$$

By Tanaka's formula,

$$d(\bar{S}_v^x - D(\bar{S}_v^x)) = (1 - D'_-(\bar{S}_v^x))d\bar{S}_v^x - \frac{1}{2}\int_0^{+\infty} D''(da)dL_v^a(\bar{S}^x).$$

In particular, $d\left\langle \bar{S}^x - D(\bar{S}^x)\right\rangle_v = (\sigma\bar{S}_v^x(1 - D'_-(\bar{S}_v^x)))^2 dv$. The function $x \mapsto \bar{u}(t_d, x)$ is convex and C^1 on $[0, +\infty)$ and C^2 on $[0, \bar{c}(t_d))$ and $(\bar{c}(t_d), +\infty)$. Hence its second order distribution derivative is equal to $\partial_{22}\bar{u}(t_d, x)dx$, where, by convention, $\partial_{22}\bar{u}(t_d, \bar{c}(t_d)) = 0$. Applying again Tanaka's formula and the occupation times formula, one deduces that

$$dg(\bar{S}_v^x) = \partial_2\bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))d(\bar{S}_v^x - D(\bar{S}_v^x)) + \frac{\sigma^2}{2}\partial_{22}\bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))((1 - D'_-(\bar{S}_v^x))\bar{S}_v^x)^2 dv.$$

One deduces that for γ defined in Lemma 1.6,

$$d(e^{-rv}g(\bar{S}_v^x)) = e^{-rv}\Bigg(\partial_2\bar{u}(t_d, \bar{S}_v^x - D(\bar{S}_v^x))\left[(1 - D'_-(\bar{S}_v^x))\sigma\bar{S}_v^x dW_v - \frac{1}{2}\int_0^{+\infty} D''(da)dL_v^a(\bar{S}^x)\right] + \gamma(\bar{S}_v^x)dv\Bigg). \tag{2.5}$$

The process $(\int_0^v e^{-rw}\sigma\bar{S}_w^x\partial_2\bar{u}(t_d, \bar{S}_w^x - D(\bar{S}_w^x))(1 - D'_-(\bar{S}_w^x))dW_w)_v$ is a martingale since $\partial_2\bar{u} \in [-1, 0]$ by (1.1) and $(1 - D'_-) \in [0, 1]$ according to Lemma 1.6. With (2.4), one deduces that

$$u(s, x) - u(t, x) \geq -\mathbb{E}\left(1_{\{\tau > t_d - s\}}\int_{t_d - s}^{\tau} e^{-rv}\gamma(\bar{S}_v^x)dv\right) = -\mathbb{E}\left(\int_{t_d - s}^{t_d - t} 1_{\{\tau > v\}}e^{-rv}\gamma(\bar{S}_v^x)dv\right). \tag{2.6}$$

One easily deduces (2.1), and since, by Lemma 1.6, $C \stackrel{\text{def}}{=} \sup_{x>0}\gamma(x) < +\infty$ and $\gamma(x)$ is not greater than $-rK$ for $x < x^\star$,

$$u(s, x) \geq u(t, x) + \int_{t_d - s}^{t_d - t} e^{-rv}\left(rK\mathbb{P}(\tau > v, \bar{S}_v^x < x^\star) - C\mathbb{P}(\tau > v, \bar{S}_v^x \geq x^\star)\right)dv. \tag{2.7}$$

Define $\hat{c}(s) = \sup_{v\in[t_d - s, t_d)} c(v)$, and let $\alpha \in (0, t_d]$ be such that $\hat{c}(\alpha) < x^\star$. The existence of α is ensured by Lemma 2.1, which applies since, according to the proof of Lemma 1.6, the function D is continuous and both D and $x - D(x)$ are nondecreasing. We now choose $t \in [t_d - \alpha, t_d)$ and $x \in (c(t), y)$, where $y \in (\hat{c}(\alpha), x^\star)$. One has

$$\tau = \inf\{v \in [0, t_d - t) : \bar{S}_v^x \leq c(t + v)\}$$

with convention $\inf\emptyset = t_d - t$. Let $\tau_y = \inf\{v \geq 0 : \bar{S}_v^x = y\}$. For $v \in [0, t_d - t)$, by the Markov property, one has

$$\mathbb{P}(\tau > v, \bar{S}_v^x \geq x^\star) = \mathbb{P}(\tau > v, \tau_y \leq v, \bar{S}_v^x \geq x^\star) \leq \mathbb{P}(\tau_y \leq v, \tau > \tau_y)\mathbb{P}\left(\max_{w\in[0,v]}\bar{S}_w^1 \geq x^\star/y\right).$$

At the same time,

$$\mathbb{P}(\tau > v) \geq \mathbb{P}(\tau_y \leq v, \tau > v) \geq \mathbb{P}(\tau_y \leq v, \tau > \tau_y)\mathbb{P}\left(\min_{w\in[0,v]} \bar{S}^1_w > \hat{c}(\alpha)/y\right).$$

Combining both inequalities, one obtains

$$\mathbb{P}(\tau > v, \bar{S}^x_v \geq x^\star) \leq \mathbb{P}(\tau > v)\frac{\mathbb{P}\left(\max_{w\in[0,\alpha]} \bar{S}^1_w \geq x^\star/y\right)}{\mathbb{P}\left(\min_{w\in[0,\alpha]} \bar{S}^1_w > \hat{c}(\alpha)/y\right)}.$$

The ratio $\frac{\mathbb{P}\left(\max_{w\in[0,\beta]} \bar{S}^1_w \geq z\right)}{\mathbb{P}\left(\min_{w\in[0,\beta]} \bar{S}^1_w > \eta\right)}$ equals

$$\frac{N((\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\beta}-\frac{\log z}{\sigma\sqrt{\beta}}) + e^{\frac{2\log z}{\sigma}(\frac{r}{\sigma}-\frac{\sigma}{2})}N(-(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\beta}-\frac{\log z}{\sigma\sqrt{\beta}})}{1-N(\frac{\log\eta}{\sigma\sqrt{\beta}}-(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\beta}) - e^{\frac{2\log\eta}{\sigma}(\frac{r}{\sigma}-\frac{\sigma}{2})}N(\frac{\log\eta}{\sigma\sqrt{\beta}}+(\frac{r}{\sigma}-\frac{\sigma}{2})\sqrt{\beta})},$$

and for $\beta > 0$ and $z > 1 > \eta > 0$ this converges to 0 as β and η go to 0^+ while z goes to $+\infty$. Since, by Lemma 2.1, $\hat{c}(\alpha)$ converges to 0 as α goes to 0^+, one may choose positive constants y, α such that $y \in (\hat{c}(\alpha), x^\star)$ and

$$\frac{\mathbb{P}\left(\max_{w\in[0,\alpha]} \bar{S}^1_w \geq x^\star/y\right)}{\mathbb{P}\left(\min_{w\in[0,\alpha]} \bar{S}^1_w > \hat{c}(\alpha)/y\right)} \leq \frac{rK}{rK+C}.$$

With $\mathbb{P}(\tau > v, \bar{S}^x_v < x^\star) = \mathbb{P}(\tau > v) - \mathbb{P}(\tau > v, \bar{S}^x_v \geq x^\star)$ and (2.7), we conclude that

$$\forall t_d - \alpha \leq t \leq s < t_d,\ \forall x \in (0, y),\ u(t,x) \leq u(s,x).$$

Since, for $t \in (0, t_d)$ and $x > c(t)$, $\frac{\sigma^2 x^2}{2}\partial_{xx}u(t,x) = -\partial_t u(t,x) - rx\partial_x u(t,x) + ru(t,x)$ with $\partial_x u \in [-1, 0]$ according to (1.1) and $u \leq K$, (2.2) easily follows from (2.1). Let $t \in [0, t_d)$ be such that $c(t) > 0$. For $z \geq x > c(t)$, one has $\partial_x u(t,x) = \partial_x u(t,z) - \int_x^z \partial_{xx}u(t,y)dy$. By (1.1), $\partial_x u(t,x) \in [-1, 0]$. With (2.2), one deduces that $y \mapsto \partial_{xx}u(t,y)$ is integrable on $[c(t), z]$ and the right-hand limit $\partial_x u(t, c(t)^+)$ makes sense. ■

Remark 2.5. *When $T = +\infty$, i.e., when the Put option is perpetual,*

$$u(t_d, x) = \begin{cases} K - x \text{ if } x < \bar{c}(t_d) = \frac{-K\alpha}{1-\alpha}, \\ (K - \bar{c}(t_d))(x/\bar{c}(t_d))^\alpha \text{ otherwise,} \end{cases} \qquad \text{where } \alpha = -\frac{2r}{\sigma^2}.$$

In the proportional *dividend case, $\gamma(x) = -rK1_{\{x<\bar{c}(t_d)/\rho\}}$ since $\mathcal{A}f(x) = 0$ for $f(x) = x^\alpha$. With (2.6), one deduces that for any $x > 0$, $t \mapsto u(t,x)$ is nondecreasing on $[0, t_d)$.*

In the constant *dividend case,*

$$\gamma(x) = \begin{cases} -rK \text{ if } x \in (0, D), \\ -r(K+D) \text{ if } x \in (D, \bar{c}(t_d)+D), \\ -\alpha(K-\bar{c}(t_d))\bar{c}(t_d)^{-\alpha}D(rx + \frac{\sigma^2}{2}(2x-D))(x-D)^{\alpha-2} \text{ if } x > \bar{c}(t_d)+D \end{cases}$$

is positive on $(\bar{c}(t_d)+D, +\infty)$.

3. Continuity of the exercise boundary and high contact principle. We can now state our main result concerning the continuity of the exercise boundary $c(t)$ for the single dividend case. Note that it applies to the *proportional*, the *constant*, and the more general *mixed* dividend cases.

Proposition 3.1. *Assume that D is a positive concave function such that $x - D(x)$ is nonnegative. Then, for $t \in [0, t_d)$ such that c is right-continuous at t, the smooth contact property holds $\partial_x u(t, c(t)^+) = -1$ and $\lim_{s \to t^+} \partial_x u(s, c(s)^+) = -1$.*

If g is convex, then $t \mapsto c(t)$ is right-continuous on $[0, t_d)$. More generally, if D is such that

$$\exists x_0 > 0, \exists \rho \in [0, 1),\ \forall x \in (0, x_0),\ D(x) = (1 - \rho)x, \tag{3.1}$$

then there exists an $\varepsilon \in (0, t_d]$ such that $t \mapsto c(t)$ is continuous on $(t_d - \varepsilon, t_d)$.

Remark 3.2. *On any open interval on which c is nondecreasing, it is right-continuous by upper-semicontinuity, and therefore the smooth contact property holds.*

In order to prove the proposition, we will need the following estimations of the first order time derivative and the second order spatial derivative of the pricing function u in the continuation region.

Lemma 3.3. *Assume that D is a nonnegative concave function such that $x - D(x)$ is nonnegative. Then*

$$\forall t \in [0, t_d),\ \forall x > c(t),\ \partial_t u(t, x) \leq -e^{-r(t_d - t)} \inf_{y>0} \gamma(y) + \frac{\sigma x}{2\sqrt{2\pi(t_d - t)}} \tag{3.2}$$

and

$$\frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) \geq e^{-r(t_d - t)} \inf_{y>0} \gamma(y) - \frac{\sigma x}{2\sqrt{2\pi(t_d - t)}} + r(K - x)^+. \tag{3.3}$$

If g is convex, then for $t \in [0, t_d)$, $x \mapsto u(t, x)$ is convex and for $x > c(t)$, $\partial_t u(t, x) \leq rKe^{-r(t_d - t)}$ and $\partial_{xx} u(t, x) \geq 0$.

More generally, under (3.1), there exists $\varepsilon \in (0, t_d]$ such that for all $t \in (t_d - \varepsilon, t_d)$ and for all $x \in (c(t), c(t) + \varepsilon)$ we have $\partial_t u(t, x) \leq rK \frac{1 + e^{-r(t_d - t)}}{2}$.

Proof of Proposition 3.1. For $t \in [0, t_d)$, $c(t) > 0$ by Corollary 2.3, and by Proposition 2.2, the following Taylor expansion makes sense:

$$\forall x \geq c(t),\ u(t, x) = (K - c(t)) + (x - c(t))\partial_x u(t, c(t)^+) + \int_{c(t)}^{x} (x - y)\partial_{xx} u(t, y) dy. \tag{3.4}$$

Substituting z for x in (3.4) and subtracting the result from (3.4) itself give for $x > z \geq c(t)$

$$\partial_x u(t, c(t)^+) = \frac{u(t, x) - u(t, z)}{x - z} - \int_{c(t)}^{z} \partial_{xx} u(t, y) dy - \frac{1}{x - z} \int_{z}^{x} (x - y)\partial_{xx} u(t, y) dy. \tag{3.5}$$

If $s \in [0, t_d)$ is such that $c(s) \geq c(t)$, choosing $z = c(s)$ and computing $\partial_x u(s, c(s)^+)$ from (3.5) written with s replacing t, one deduces that for $x > c(s)$,

$$\begin{aligned}\partial_x u(s,c(s)^+) - \partial_x u(t,c(t)^+) &= \frac{1}{x-c(s)}\Big(u(s,x)-u(t,x)+u(t,c(s))-u(s,c(s))\Big)\\ &\quad+\frac{1}{x-c(s)}\int_{c(s)}^{x}(x-y)(\partial_{xx}u(t,y)-\partial_{xx}u(s,y))dy\\ &\quad+\int_{c(t)}^{c(s)}\partial_{xx}u(t,y)dy. \end{aligned}\tag{3.6}$$

We decompose the proof in three steps using the above expansions. First, we check that when $t_0 \in [0, t_d)$ is such that c is right-continuous at t_0, then $\lim_{t\to t_0^+} \partial_x u(t, c(t)^+) = \partial_x u(t_0, c(t_0)^+)$. In the second step, we check that when c is right-continuous at t_0, then the smooth contact property holds at t_0. In the last step, we prove that c is right-continuous at t_0 for t_0 close to t_d under (3.1) and with no restriction in the convex case.

Step 1. Let $t_0 \in [0, t_d)$ be such that c is right-continuous at t_0 and $x > c(t_0)$. For $t > t_0$ such that $c(t) < x$, by (3.6), $|\partial_x u(t_0, c(t_0)^+) - \partial_x u(t, c(t)^+)|$ is smaller than

$$\begin{aligned}&\frac{1}{x-c(t)\vee c(t_0)}|u(t_0,x)-u(t,x)+u(t,c(t)\vee c(t_0))-u(t_0,c(t)\vee c(t_0))|\\ &+\frac{1}{x-c(t)\vee c(t_0)}\int_{c(t)\vee c(t_0)}^{x}(x-y)|\partial_{xx}u(t,y)-\partial_{xx}u(t_0,y)|dy\\ &+\int_{c(t)\wedge c(t_0)}^{c(t)\vee c(t_0)}|\partial_{xx}u(t,y)|+|\partial_{xx}u(t_0,y)|dy.\end{aligned}$$

By the continuity of u, the first term converges to 0 as $t \to t_0^+$. Moreover, (2.2) and (3.3) ensure that the second term is arbitrarily small uniformly for $t < (t_0 + t_d)/2$ when x is close enough to $c(t_0)$. Last, with the right-continuity of c at t_0, the third term converges to 0 as $t \to t_0^+$, which ensures the desired right-continuity property.

Step 2. Let us now assume that for $t_0 \in [0, t_d)$ such that c is right-continuous at t_0, $\partial_x u(t_0, c(t_0)^+) > -1$, and let us obtain a contradiction. Let $t \in (t_0, \frac{t_0+t_d}{2})$ be such that $c(t) \le c(t_0)$. According to (3.2) and (3.3), there exists a constant $C \in (0, +\infty)$ such that $u(t, c(t_0)) \le K - c(t_0) + C(t - t_0)$ and $\int_{c(t)}^{c(t_0)}(c(t_0)-y)\partial_{xx}u(t,y)dy \ge -C\frac{(c(t_0)-c(t))^2}{c(t)^2}$. Writing (3.4) for $x = c(t_0)$, one deduces that

$$\left(1+\partial_x u(t,c(t)^+)-C\frac{c(t_0)-c(t)}{c(t)^2}\right)(c(t_0)-c(t)) \le C(t-t_0).$$

Since $\partial_x u(t, c(t)^+)$ tends to $\partial_x u(t_0, c(t_0)^+) > -1$ as $t \to t_0^+$ and c is right-continuous at t_0, one deduces the existence of $\varepsilon \in (0, t_d - t_0)$ such that

$$\forall t \in [t_0, t_0+\varepsilon],\ c(t)-c(t_0) \ge -\frac{2C(t-t_0)}{1+\partial_x u(t_0,c(t_0)^+)}.\tag{3.7}$$

For $x > c(t_0)$, let $\tau_x = \inf\{s > 0 : \bar S^x_s \le c(t_0 + s)\} \wedge (t_d - t_0)$ denote the stopping time such that

$$u(t_0,x) = \mathbb{E}\left(e^{-r\tau_x}(K-\bar S^x_{\tau_x})^+1_{\{\tau_x<t_d-t_0\}}+e^{-r(t_d-t_0)}g(\bar S^x_{\tau_x})1_{\{\tau_x=t_d-t_0\}}\right).$$

One has $u(t_0, c(t_0)) \geq \mathbb{E}\left(e^{-r\tau_x}(K - \bar{S}^{c(t_0)}_{\tau_x})^+ 1_{\{\tau_x < t_d - t_0\}} + e^{-r(t_d - t_0)} g(\bar{S}^{c(t_0)}_{\tau_x}) 1_{\{\tau_x = t_d - t_0\}}\right)$. Computing the difference, using the monotonicity of g and the Lipschitz continuity of $y \mapsto (K-y)^+$, one deduces that

$$\frac{u(t_0, x) - u(t_0, c(t_0))}{x - c(t_0)} \leq -\mathbb{E}\left(e^{-r\tau_x} \bar{S}^1_{\tau_x} 1_{\{\tau_x < t_d - t_0\}}\right). \tag{3.8}$$

By (3.7), $\tau_x \leq \tilde{\tau}_x \overset{\text{def}}{=} \inf\{s \in (0, \varepsilon] : \bar{S}^x_s \leq c(t_0) - 2Cs/(1 + \partial_x u(t_0, c(t_0)^+))\} \wedge (t_d - t_0)$. When x tends to $c(t_0)^+$, $\tilde{\tau}_x$ converges a.s. to $\inf\{s \in (0, \varepsilon] : \bar{S}^1_s < 1 - 2Cs/(c(t_0)(1 + \partial_x u(t_0, c(t_0)^+)))\} \wedge (t_d - t_0)$, which is equal to 0 according to the iterated logarithm law satisfied by the Brownian motion W. Hence τ_x converges a.s. to 0 as $x \to c(t_0)^+$. Since $\mathbb{E}(\sup_{s \in [0, t_d - t_0]} \bar{S}^1_s) < +\infty$, by Lebesgue's theorem, the right-hand side of (3.8) converges to -1 as $x \to c(t_0)^+$, which implies the desired contradiction: $\partial_x u(t_0, c(t_0)^+) \leq -1$.

Step 3. Let $t_0 \in [0, t_d)$ be such that c is not right-continuous at t_0. We are going to derive a contradiction when g is convex or t_0 close to t_d under (3.1). The continuity of c on a left-hand neighborhood of t_d then follows from the left-continuity stated in Corollary 2.3. By the upper-semicontinuity of c and the positivity of $\inf_{t \in [0, \frac{t_0 + t_d}{2}]} c(t)$ stated in Corollary 2.3, there exists a sequence $(s_k)_{k \in \mathbb{N}}$ in (t_0, t_d) converging to t_0 as $k \to \infty$ and such that $\lim_{k \to \infty} c(s_k) \in (0, c(t_0))$. Let $x, z \in (\lim_{k \to \infty} c(s_k), c(t_0))$ with $x > z$. For k large enough, $c(s_k) < z$ and we may use (3.5) for $t = s_k$. The left-hand side is not smaller than -1. When k tends to ∞, by the continuity of u, the first term in the right-hand side tends to $\frac{K - x - (K - z)}{x - z} = -1$. Moreover, by (3.3), there is a constant $C \in (0, +\infty)$ not depending on k such that

$$\int_{c(s_k)}^{z} \partial_{xx} u(s_k, y) dy + \frac{1}{x - z} \int_z^x (x - y) \partial_{xx} u(s_k, y) dy \geq -\frac{C}{c^2(s_k)} \left(2(z - c(s_k)) + (x - z)\right).$$

Hence $\limsup_{k \to \infty} \partial_x u(s_k, c(s_k)^+) \leq -1 + \frac{C}{\lim_{k \to \infty} c^2(s_k)} (x + z - 2 \lim_{k \to \infty} c(s_k))$ and one deduces that

$$\lim_{k \to \infty} \partial_x u(s_k, c(s_k)^+) = -1 \tag{3.9}$$

by letting x and z go to $\lim_{k \to \infty} c(s_k)$. By (3.4) and Proposition 2.2,

$$\begin{aligned} \forall x > c(t), \quad \int_{c(t)}^{x} y \partial_{xx} u(t, y) dy &= x \partial_x u(t, x) - c(t) \partial_x u(t, c(t)^+) - u(t, x) + u(t, c(t)) \\ &= x \partial_x u(t, x) - u(t, x) + K - c(t) \left(1 + \partial_x u(t, c(t)^+)\right). \end{aligned}$$

With the equality $\partial_t u(t, x) + \mathcal{A} u(t, x) = 0$ and Lemma 3.3, one deduces that for t and t_0 close to t_d under (3.1) and with no restriction in the convex case,

$$\begin{aligned} \forall x \in (c(t), c(t_0)), \quad \frac{\sigma^2 x^2}{2} \partial_{xx} u(t, x) + r \int_{c(t)}^{x} y \partial_{xx} u(t, y) dy \\ &= rK - \partial_t u(t, x) - rc(t) \left(1 + \partial_x u(t, c(t)^+)\right) \\ &\geq \frac{rK(1 - e^{-r(t_d - t)})}{2} - rc(t) \left(1 + \partial_x u(t, c(t)^+)\right). \end{aligned} \tag{3.10}$$

According to (2.2), there is a finite constant C such that for all $t \in [0,t_d)$, for all $x \in (c(t),c(t_0)]$, $\partial_{xx}u(t,y) \leq \frac{C}{y}$ so $r\int_{c(t)}^x y\partial_{xx}u(t,y)dy \leq rK(1-e^{-r(t_d-t)})/8$ if we take $x \leq c(t_0)\wedge c(t)e^{\frac{K(1-e^{-r(t_d-t)})}{8C}}$. With (3.9) and (3.10), one deduces that for t_0 close to t_d under (3.1) and with no restriction in the convex case, for k large enough,

- $\forall y \in \left(c(s_k), c(t_0)\wedge c(s_k)e^{\frac{K(1-e^{-r(t_d-s_k)})}{8C}}\right)$, $\frac{\sigma^2 y^2}{2}\partial_{xx}u(s_k,y) \geq \frac{rK(1-e^{-r(t_d-s_k)})}{4}$,
- and therefore for $x,z \in \left(\lim_{l\to\infty} c(s_l), c(t_0)\wedge \lim_{l\to\infty} c(s_l)e^{\frac{K(1-e^{-r(t_d-t_0)})}{16C}}\right)$ with $x>z$,

$$\int_{c(s_k)}^z \partial_{xx}u(s_k,y)dy + \frac{1}{x-z}\int_z^x (x-y)\partial_{xx}u(s_k,y)dy \geq \frac{rK(1-e^{-r(t_d-s_k)})}{4\sigma^2x^2}(x+z-2c(s_k)).$$

Taking the limit $k\to\infty$ in (3.5) written for $t=s_k$, we obtain $\limsup_{k\to\infty}\partial_x u(s_k,c(s_k)^+) < -1$, which contradicts (3.9). ■

Proof of Lemma 3.3. Let $t\in[0,t_d)$. When g is convex, since $x\mapsto (K-x)^+$ is also convex, for each stopping time $\tau\in\mathcal{T}_{[0,t_d-t]}$, $x\mapsto \mathbb{E}(e^{-r\tau}(K-\bar{S}^x_\tau)^+1_{\{\tau<t_d-t\}}+e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})1_{\{\tau=t_d-t\}})$ is convex. So $x\mapsto u(t,x)$, which is equal to the supremum over τ of the previous functions, is convex.

Now let $0\leq t\leq s<t_d$, $x>0$, and $\tau\in\mathcal{T}_{[0,t_d-s]}$ be such that

$$u(s,x)=\mathbb{E}\left(e^{-r\tau}(K-\bar{S}^x_\tau)^+1_{\{\tau<t_d-s\}}+e^{-r(t_d-s)}g(\bar{S}^x_{t_d-s})1_{\{\tau=t_d-s\}}\right).$$

For every $\tilde{\tau}\in\mathcal{T}_{[0,t_d-t]}$ we have that

$$u(t,x)\geq\mathbb{E}\left(e^{-r\tilde{\tau}}(K-\bar{S}^x_{\tilde{\tau}})^+1_{\{\tilde{\tau}<t_d-t\}}+e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})1_{\{\tilde{\tau}=t_d-t\}}\right),$$

so choosing $\tilde{\tau}=\tau 1_{\{\tau<t_d-s\}}+(t_d-t)1_{\{\tau=t_d-s\}}$ we find that

$$u(t,x)\geq\mathbb{E}\left(e^{-r\tau}(K-\bar{S}^x_\tau)^+1_{\{\tau<t_d-s\}}+e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})1_{\{\tau=t_d-s\}}\right).$$

Combining this with the equality for $u(s,x)$ above, one has

$$u(t,x)-u(s,x)\geq\mathbb{E}\left(1_{\{\tau=t_d-s\}}\left(e^{-r(t_d-t)}g(\bar{S}^x_{t_d-t})-e^{-r(t_d-s)}g(\bar{S}^x_{t_d-s})\right)\right).$$

When g is convex, according to Lemma 1.6, $\mathcal{A}g$ is a function bounded from below by $-rK$, and the right-hand side is equal to $\mathbb{E}\left(1_{\{\tau=t_d-s\}}\int_{t_d-s}^{t_d-t}e^{-rv}\mathcal{A}g(\bar{S}^x_v)dv\right)$, so one easily derives the statement for convex functions g. In general, by (2.5) and the martingale property of the process $(\int_0^v e^{-rw}\sigma\bar{S}^x_w\partial_2\bar{u}(t_d,\bar{S}^x_w-D(\bar{S}^x_w))(1-D'_-(\bar{S}^x_w))dW_w)_v$, the previous inequality writes as

$$u(t,x)-u(s,x) \geq\mathbb{E}\left(1_{\{\tau=t_d-s\}}\int_{t_d-s}^{t_d-t}e^{-rv}\left[\gamma(\bar{S}^x_v)dv-\frac{\partial_2\bar{u}(t_d,\bar{S}^x_v-D(\bar{S}^x_v))}{2}\int_0^\infty D''(da)dL^a_v(\bar{S}^x)\right]\right). \quad (3.11)$$

Since $\partial_2\bar{u}(t_d,y)\geq -1$, using the occupation times formula, one deduces that

$$u(s,x)-u(t,x)\leq \int_{t_d-s}^{t_d-t} e^{-rv}\left(-\inf_{y>0}\gamma(y)-\int_0^{+\infty}\frac{\sigma^2a^2}{2}p(v,a)D''(da)\right)dv.$$

Since $D(x)$ and $x-D(x)$ are both nondecreasing, $D''((0,+\infty))\geq -1$. Using, moreover,

$$\forall v\in[0,t_d-t],\ \forall a>0,\ a^2p(v,a)=\frac{xe^{rv}}{\sigma\sqrt{2\pi v}}e^{-\frac{(\log(a/x)-(r+\frac{\sigma^2}{2})v)^2}{2\sigma^2 v}}\leq \frac{xe^{rv}}{\sigma\sqrt{2\pi v}},$$

one deduces (3.2). The inequality (3.3) follows since for $x>c(t)$ we have $\frac{\sigma^2x^2}{2}\partial_{xx}u(t,x)=-\partial_tu(t,x)-rx\partial_xu(t,x)+ru(t,x)\geq -\partial_tu(t,x)+r(K-x)^+$.

Assume (3.1). Then γ is equal to $-rK$ on $(0,x_0\wedge x^\star)$, $D''((0,x_0))=0$, and (3.11) implies that

$$\begin{aligned}u(s,x)-u(t,x)\leq \int_{t_d-s}^{t_d-t} e^{-rv}\Bigg(&rK-(\inf_{y>0}\gamma(y)+rK)\mathbb{P}(\bar{S}_v^x\geq x_0\wedge x^\star)\\ &-\int_{x_0}^{+\infty}\frac{\sigma^2a^2}{2}p(v,a)D''(da)\Bigg)dv.\end{aligned}$$

For $x\in(0,x_0e^{-(r+\frac{\sigma^2}{2})(t_d-t)}]$, one has for all $v\in[0,t_d-t]$, for all $a\geq x_0$, $a^2p(v,a)\leq \frac{xe^{rv}}{\sigma\sqrt{2\pi v}}e^{-\frac{(\log(x_0/x)-(r+\frac{\sigma^2}{2})v)^2}{2\sigma^2 v}}$. For t close enough to t_d, we have that $c(t)<x_0e^{-(r+\frac{\sigma^2}{2})(t_d-t)}$ by Lemma 2.1 and for $x\in(c(t),x_0e^{-(r+\frac{\sigma^2}{2})(t_d-t)})$,

$$\begin{aligned}\partial_tu(t,x)\leq e^{-r(t_d-t)}&\left(rK-(\inf_{y>0}\gamma(y)+rK)N\left(\frac{\log(x/(x_0\wedge x^\star))+(r-\frac{\sigma^2}{2})(t_d-t)}{\sigma\sqrt{t_d-t}}\right)\right)\\ &+\frac{\sigma x}{2\sqrt{2\pi(t_d-t)}}e^{-\frac{(\log(x_0/x)-(r+\frac{\sigma^2}{2})(t_d-t))^2}{2\sigma^2(t_d-t)}}.\end{aligned}$$

Bounding from above the two last terms like in the derivation of the upper bound for $c(t)$ in the proof of Lemma 2.1, one deduces the last assertion. ■

4. Conclusions and further research. We have proven local results concerning the regularity of the exercise boundary for a dividend-paying asset. Even in the simplest case of *proportional dividends*, it would be of great interest to prove the following feature observed in numerical calculations: for a single dividend payment, when t_d is large, the exercise boundary is nondecreasing for small times and monotonicity seems to change only once before t_d. We also would like to extend the results that we have obtained for multiple dividend payments in the *proportional* case to more general functions D^i. The key issue in this perspective is to derive global estimates on the derivatives of the value function u^1 before t_d^1 to replace those which follow from the convexity in the variable x in the *proportional* case.

Another interesting matter to investigate would be the optimal exercise boundary for the alternative model for dividends known as the Escrowed model. As we have shown in Remark

2.4, this boundary is zero on an interval with strictly positive length before every dividend date, but other properties of this boundary have yet to be established.

Appendix A. Proof of Theorem 0.1. The two first statements can easily be deduced by adapting, respectively, the comparison argument given at the beginning of the proof of Corollary 2.3 and the proof of Lemma 2.1.

Let us now consider the case of multiple proportional dividends. We will prove by induction on i that the statement holds together with the following lemma.

Lemma A.1. *If for all $1 \leq j \leq i$ we have $D^j(x) = (1-\rho_j)x$ for some $\rho_j \in (0,1)$, then g^i is convex and C^1 on $\mathbb{R}_+$ and C^2 on $[0, x_i^\star) \cup (x_i^\star, +\infty)$ for $x_i^\star \overset{\text{def}}{=} \frac{c^{i-1}(t_d^i)}{\rho_i}$. Moreover, the function $\gamma_i(x) \overset{\text{def}}{=} \mathcal{A}g^i(x)$ is equal to $-rK$ on $[0, x_i^\star)$, not smaller than $-rK$, and bounded on $(x_i^\star, +\infty)$ and satisfies*

$$\forall t \in [0, t_d^i),\ \forall x > c^i(t),\ -e^{-r(t_d^i - t)} \sup_{y>0} \gamma_i^+(y) \leq \partial_t u^i(t,x) \leq e^{-r(t_d^i - t)} rK \tag{A.1}$$

and

$$0 \leq \frac{\sigma^2 x^2}{2} \partial_{xx} u^i(t,x) \leq e^{-r(t_d^i - t)} \sup_{y>0} \gamma_i^+(y) + rK. \tag{A.2}$$

For $i = 1$, the result is a consequence of Propositions 2.2 and 3.1, Corollary 2.3, and Lemma 3.3, the refinement over (2.2) in the last inequality in (A.2) following from the monotonicity of $x \mapsto x\partial_x u^1(t,x) - u^1(t,x)$ which is a consequence of the convexity of $x \mapsto u^1(t,x)$.

Assume the induction hypothesis to be true for a certain $i \geq 1$. Then $x \mapsto g^{i+1}(x) = u^i(t_d^{i+1}, \rho_{i+1}x)$ is convex and arguing like in the beginning of the proof of Lemma 3.3, one obtains that for $t \in [0, t_d^{i+1})$,

$$x \mapsto u^{i+1}(t,x) = \sup_{\tau \in \mathcal{T}_{[0, t_d^{i+1} - t]}} \mathbb{E}[e^{-r\tau}((K - \bar{S}_\tau^x)^+ 1_{\{\tau < t_d^{i+1} - t\}} + g^{i+1}(\bar{S}^x_{t_d^{i+1} - t}) 1_{\{\tau = t_d^{i+1} - t\}})]$$

is convex and nonincreasing. The function g^{i+1} is C^1 on $\mathbb{R}_+$ by the smooth contact property for u^i at time t_d^{i+1} and C^2 on $[0, x_{i+1}^\star) \cup (x_{i+1}^\star, +\infty)$ by the regularity properties of u^i stated in Lemma 1.4. Moreover, the function $\gamma_{i+1}(x) = \mathcal{A}u^i(t_d^{i+1}, .)(\rho_{i+1}x)$ is equal to $-rK$ on $[0, x_{i+1}^\star)$, not smaller than $-rK$, and bounded on $(x_{i+1}^\star, +\infty)$, respectively, by the convexity of $x \mapsto u^i(t_d^{i+1}, x)$ and by the lower bound in (A.1) combined with the equality $\partial_t u^i(t_d^{i+1}, x) + \mathcal{A}u^i(t_d^{i+1}, x) = 0$ which is satisfied for $x > c^i(t_d^{i+1})$. One may now adapt the proofs of Proposition 2.2, Lemma 3.3, and Corollary 2.3 to check that the exercise boundary $c^{i+1}(t)$ is nonincreasing and equivalent to $\frac{rK(t_d^{i+1} - t)}{1-\rho_{i+1}}$ in a left-hand neighborhood of t_d^{i+1} and that (A.1) and (A.2) hold with $i+1$ replacing i. Next, with these bounds on the derivatives of u^{i+1}, one adapts the proof of Proposition 3.1 to obtain the right-continuity of the exercise boundary c^{i+1} on $[0, t_d^{i+1})$ and smooth contact: for all $t \in [0, t_d^{i+1})$, $\partial_x u^{i+1}(t, c^{i+1}(t)^+) = -1$. This proves the statement for $i+1$ and concludes the proof.

REFERENCES

[1] A. Battauz and M. Pratelli, *Optimal stopping and American options with discrete dividends and exogenous risk*, Insurance Math. Econom., 35 (2004), pp. 255–265.

[2] E. Bayraktar and H. Xing, *Analysis of the optimal exercise boundary of American options for jump diffusions*, SIAM J. Math. Anal., 41 (2009), pp. 825–860.

[3] P. Carr, R. Jarrow, and R. Myneni, *Alternative characterizations of American puts*, Math. Finance, 2 (1992), pp. 87–106.

[4] X. Chen and J. Chadam, *A mathematical analysis of the optimal exercise boundary for American put options*, SIAM J. Math. Anal., 38 (2007), pp. 1613–1641.

[5] X. Chen, J. Chadam, L. Jiang, and W. Zheng, *Convexity of the exercise boundary of the American put option on a zero dividend asset*, Math. Finance, 18 (2008), pp. 185–197.

[6] E. Ekström, *Convexity of the optimal stopping boundary for the American put option*, J. Math. Anal. Appl., 229 (2004), pp. 147–156.

[7] N. El Karoui, *Les aspects probabilistes du contrôle stochastique*, Lecture Notes in Math. 876, Springer-Verlag, Berlin, 1981.

[8] N. El Karoui, J.-P. Lepeltier, and A. Millet, *A probabilistic approach of the reduite*, Probab. Math. Statist., 13 (1992), pp. 97–121.

[9] A. Friedman, *Stochastic Differential Equations and Applications*: I, Academic Press, New York, London, 1975.

[10] O. Göttsche and M. H. Vellekoop, *The early exercise premium for the American put under discrete dividends*, Math. Finance, 21 (2011), pp. 335–354.

[11] S. Hamadéne and J.-P. Lepeltier, *Reflected BSDEs and mixed game problem*, Stochastic Process. Appl., 85 (2000), pp. 177–188.

[12] S. D. Jacka, *Local times, optimal stopping and semimartingales*, Ann. Probab., 21 (1993), pp. 329–339.

[13] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Appl. Math. (N. Y.) 39, Springer-Verlag, New York, 1998.

[14] I. J. Kim, *The analytical valuation of American options*, Rev. Financ. Stud., 3 (1990), pp. 547–572.

[15] D. Lamberton, *Critical price for an American option near maturity*, in Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), Progr. Probab. 36, Birkhäuser, Basel, 1995, pp. 353–358.

[16] H. P. McKean, *Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics*, Ind. Management Rev., 6 (1965), pp. 32–39.

[17] R. Myneni, *The pricing of the American option*, Ann. Appl. Probab., 2 (1992), pp. 1–28.

[18] G. Peskir, *A change-of-variable formula with local times on curves*, J. Theoret. Probab., 18 (2005), pp. 499–535.

[19] G. Peskir, *On the American option problem*, Math. Finance, 15 (2005), pp. 169–181.

[20] G. Peskir and A. Shiryaev, *Optimal Stopping and Free-Boundary Problems*, Birkhäuser, Basel, 2006.

[21] P. Van Moerbeke, *On optimal stopping and free boundary problems*, Arch. Rational Mech. Anal., 60 (1975/76), pp. 101–148.

[22] M. H. Vellekoop and J. W. Nieuwenhuis, *Efficient pricing of derivatives on assets with discrete dividends*, Appl. Math. Finance, 13 (2006), pp. 265–284.

[23] M. H. Vellekoop and J. W. Nieuwenhuis, *An integral equation for American put options on assets with general dividend processes*, Stochastics, to appear.