Regularity properties and definability in the real number continuum: idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy

Khomskii, Y.D.

Citation for published version (APA):
Khomskii, Y. D. (2012). Regularity properties and definability in the real number continuum: idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy Amsterdam: Institute for Logic, Language and Computation
Chapter 2

Idealized regularity

Since the developments of forcing in the 1960s and Solovay’s celebrated result [Sol70] establishing the consistency of “ZF + DC+ all sets of reals are Lebesgue measurable, have the property of Baire and the perfect set property”, it gradually became commonplace to associate regularity properties with a notion of forcing. Random forcing was specifically designed by Solovay to prove the measurability result, Cohen forcing is naturally related to the Baire property, and we have already seen in Section 1.3.3 that Marczewski-style properties can be viewed in a forcing context. Frequently, a regularity property is isolated because of its significance for the combinatorics of certain forcings, and conversely, understanding a regularity property usually greatly benefits from finding a forcing that corresponds to it.

At first, one might have the hope that all regularity properties can be formulated in terms of forcing. Unfortunately, this seems over-ambitious and in subsequent chapters we will consider properties that do not seem to fall into this category. Nevertheless, a large number of regularity properties can be directly formulated as properties of a certain forcing, and it turns out that the framework of idealized forcing introduced by Jindřich Zapletal is very well suited for this purpose. The goal of this chapter is to develop a systematic theory of regularity properties in this framework.

An important inspiration for this chapter is the work of Daisuke Ikegami in [Ike10a, Ike10b], who considered a wide class of forcing notions called strongly arboreal forcings and showed that many regularity properties can be stated directly in terms of a forcing from this class. In Section 2.3 we pay special credit to these results and generalize the main theorem of [Ike10a].

It should be noted that despite the novel framework, most proofs in this chapter are not really new, but variations on, or generalizations of, arguments found in various other sources, such as the original result of Solovay, the work of Zapletal and Ikegami, and folklore results. In Sections 2.5 we present a slightly different point of view, raising interesting questions suitable for further research.
2.1 Idealized forcing

An ideal on $\omega$ is a set $I \subseteq \mathcal{P}(\omega)$ which is closed under subsets (if $B \in I$ and $A \subseteq B$ then $A \in I$) and unions (if $A, B \in I$ then $A \cup B \in I$). By standard convention, we also assume that all singletons $\{x\}$ are in the ideal and that the whole space $\omega$ is not. A $\sigma$-ideal is an ideal that is additionally closed under countable unions. For convenience we will usually talk of the Baire space when giving definitions, proving theorems etc., but in most cases this can easily be adapted to the Cantor space, the space $[\omega]^{\omega}$, or any other incarnation of the real numbers. Sets that lie in the ideal $I$ will be called $I$-small and those that do not will be called $I$-positive, following standard practice.

In [Zap04] and [Zap08], Jindřich Zapletal developed an extensive theory of idealized forcing, i.e., using the partial order of $I$-positive Borel sets of reals, ordered by inclusion, as a forcing notion. We start by reviewing a few of the basic concepts and results.

**Definition 2.1.1.** Let $I$ be a $\sigma$-ideal on $\omega$. Let $P_I := \mathcal{B}(\omega) \setminus I$ denote the partial order of all Borel $I$-positive subsets of $\omega$ ordered by inclusion.

**Fact 2.1.2** (Zapletal).

1. If $G$ is a $P_I$-generic filter, then there is a unique real $x_G \in V[G]$ such that for all Borel sets coded in $V$, $x_G \in B^{V[G]}$ iff $B \in G$. This is called the generic real, and the generic filter can be recovered from the generic real using the previous characterization, so $V[G] = V[x_G]$.

2. If $\dot{x}_G$ is the name for the generic real, then every $I$-positive $B$ forces $\dot{x}_G \in B$, and for every $B \in I$, $\models B \notin G$.

**Proof.** See [Zap08, Proposition 2.1.2].

In particular, a real $x$ is $P_I$-generic over a transitive model $M$ if $x \in \bigcup D$ for every dense $D \in M$. When $M$ is a non-transitive elementary submodel then, in accordance to common usage, we will say that a real $x$ is $(M, P_I)$-generic if $x \in \bigcup(D \cap M)$ for every dense $D \in M$. This is equivalent to saying that $x$ is the real derived from an $(M, P_I)$-generic filter (see Definition 1.2.16). We will often drop the reference to $P_I$ if it is clear from context.

Recall from Definition 1.2.16 that a forcing notion $\mathbb{P}$ is proper if for every countable elementary submodel $M \prec \mathcal{H}_\theta$ of a sufficiently large structure and every $p \in \mathbb{P} \cap M$, there is a master condition $q \leq p$, that is, a condition $q$ such that $q \models \text{“} \dot{G} \text{ is an } M\text{-generic filter} \text{”}$—or alternatively, $q \models \text{“} \dot{x}_G \text{ is an } M\text{-generic real} \text{”}$. 

2.1. Idealized forcing

**Fact 2.1.3** (Zapletal).

1. Let $I$ be a $\sigma$-ideal and $M \prec \mathcal{H}_\theta$ a countable elementary submodel of a sufficiently large structure. Then for every $B \in P_I \cap M$, the set

$$C := \{x \in B \mid x \text{ is } M\text{-generic}\}$$

is Borel.

2. The forcing $P_I$ is proper iff for every countable $M \prec \mathcal{H}_\theta$ and every $B \in P_I \cap M$, the set $C$ from above is $I$-positive.

**Proof.** See [Zap08, Proposition 2.2.2].

This set $C$ is the master condition, having the additional property that every real $x \in C$ is $M$-generic. This highly useful aspect of properness was used numerous times in [Zap04, Zap08] and we shall use it to good effect in many of our arguments, too.

Another essential feature of properness is the generation of new reals from the generic real, by Borel functions encoded in the ground model.

**Theorem 2.1.4** (Zapletal). Let $P_I$ be a proper forcing and $\dot{x}$ a name for a real. Then there is a Borel function $f$ and a condition $B$ such that $B \Vdash \dot{x} = f(\dot{x}_G)$.

**Proof.** See [Zap08, Proposition 2.3.1].

All the ideals we consider will have an absolute definition. To be precise, if $B$ is a Borel set, then the statement “$B \in I$” will be $\Sigma^1_2$ or $\Pi^1_2$ (formally this means that the sentence $\phi(x)$, saying that “the Borel set encoded by the real number $x$ is in $I$”, has complexity $\Sigma^1_2$ or $\Pi^1_2$). In particular, the membership of Borel sets in $I$ will be absolute between transitive models containing $\omega_1$, by Shoenfield absoluteness.

Idealized forcings are related to more standard forcings (using simple combinatorial objects) via dense embeddings. Suppose that $I$ is a $\sigma$-ideal and $Q$ a partial order consisting of simple sets (e.g., closed), also ordered by inclusion, and such that every $q \in Q$ is $I$-positive and every Borel $I$-positive set contains some $q \in Q$ as a subset. Then, clearly, there is a dense embedding from $Q$ to $P_I$ (denoted by $Q \rightarrow^d P_I$) so the two are forcing equivalent. At the heart of this embedding lies a dichotomy theorem: every Borel set is either in $I$ or contains a set $q \in Q$. Such theorems are typically hard to prove and require some familiarity with the specific combinatorics of the objects and the ideal. There are at least three different methods for this: the classical method, using direct combinatorial properties; the “forcing” method, using forcing with $Q$ and some absoluteness
Chapter 2. Idealized regularity

results; and the game-theoretic method, using the Borel determinacy of a corresponding game. Notice that this is exactly the same as saying that all Borel sets satisfy a certain regularity property, namely the property of either being \( I \)-small or containing a (large) object \( q \in Q \). We have already mentioned the perfect set property, \( K_\sigma \)-regularity and Laver-regularity (see Definition 1.3.7), which are all satisfied by analytic, and therefore Borel, sets. The following list shows some typical examples of this phenomenon (see Definition 1.2.17 for the forcing partial orders).

Example 2.1.5.

1. Let \( \text{ctbl} \) be the \( \sigma \)-ideal of countable sets, and recall the Sacks forcing partial order \( S \) consisting of perfect sets. Since Borel sets satisfy the perfect set property, it follows that there is a dense embedding \( S \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \text{ctbl} \).

2. Let \( K_\sigma \) be the ideal of \( \sigma \)-compact sets, i.e., sets \( A \) such that some \( x \) dominates all \( a \in A \), and recall the Miller forcing partial order \( M \) consisting of super-perfect trees. Since Borel sets satisfy \( K_\sigma \)-regularity, there is a dense embedding \( M \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus K_\sigma \).

3. Let \( \mathcal{L} \) be the Laver ideal, defined as the ideal of all sets \( A \) which are not strongly dominating (see Definition 1.3.7 (3)). Recall the Laver partial order \( L \). Since Borel sets satisfy the Laver-regularity, there is a dense embedding \( L \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathcal{L} \).

4. Consider the Lebesgue null ideal \( \mathcal{N} \). Random forcing is the algebra \( \mathcal{B}(\omega^\omega) \setminus \mathcal{N} \), but by classical results it is known that every Borel set of positive measure contains a closed set of positive measure. Therefore, the collection of all closed subsets of \( \omega^\omega \) with positive Lebesgue measure is forcing equivalent to \( \mathcal{B}(\omega^\omega) \setminus \mathcal{N} \).

5. Recall the Mathias forcing partial order \( \mathbb{R} \). Each condition \( (s, S) \in \mathbb{R} \) gives rise to the closed set \( [s, S] := \{ s \cup a \mid a \in [S]^\omega \} \). Such sets generate the Ellentuck topology, due to Ellentuck \[Ell74\], and we can consider the \( \sigma \)-ideal \( I_{\mathcal{R}} \) of sets meager in this topology, equal to the \( \sigma \)-ideal of nowhere dense sets in this topology, also called the Ramsey-null ideal. By Ellentuck’s original proof \[Ell74\], every Borel set is either in \( I_{\mathcal{R}} \), or contains a set of the form \( [s, S] \). Therefore there is a dense embedding \( \mathbb{R} \hookrightarrow_d \mathcal{B}(\mathcal{L}) \setminus \mathcal{R} \).

6. The last two examples we exhibit are somewhat more involved. Let \( E_0 \) be the equivalence relation on \( 2^\omega \) given by \( x E_0 y \) iff \( \forall n \, (x(n) = y(n)) \). A partial \( E_0 \)-transversal is a set \( A \) which contains at most one element from each \( E_0 \)-equivalence class, in other words, \( \forall x, y \in A : \text{if } x \neq y \text{ then } \exists n \, (x(n) \neq y(n)) \). Let \( I_{E_0} \) be the \( \sigma \)-ideal generated by Borel partial \( E_0 \)-transversals.
The Borel equivalence relation $E_0$ is well-known among descriptive set theorists and it played a key role in the study of the Glimm-Effros dichotomy in [HKL90]. The ideal $I_{E_0}$ was investigated by Zapletal who, among other things, isolated the notion of an $E_0$-tree.

**Definition 2.1.6. (Zapletal)** An $E_0$-tree is a perfect tree $T \subseteq 2^{<\omega}$ such that

(a) there is a stem $s_0$ with $|s_0| = k_0$, and

(b) there are numbers $k_0 < k_1 < k_2 < \ldots$ and for each $i$ exactly two sequences $s_i^0, s_i^1 \in [k_i, k_{i+1})2$, such that

$$[T] = \{ s_0 \circ s_{z(0)}^0 \circ s_{z(1)}^1 \circ s_{z(2)}^2 \circ \cdots | z \in 2^{\omega}\}.$$  

Let $E_0$ denote the forcing partial order of $E_0$-trees ordered by inclusion. A standard fusion argument can be used to show the properness of this forcing. Zapletal [Zap04, Lemma 2.3.29] proved the corresponding dichotomy: every Borel (even analytic) set is either in $I_{E_0}$ or contains $[T]$ for some $T \in E_0$. It follows that $E_0 \rightarrow_{d} B(2^{\omega}) \setminus I_{E_0}$.

7. Similarly to the above, let $G$ be the relation on $2^{\omega}$ given by $xGy$ iff there is exactly one $n$ such that $(x(n) \neq y(n))$. A Borel set $B$ is $G$-independent if any two distinct elements $x, y \in B$ are not $G$-related, i.e., if $x \neq y$ then $x$ and $y$ differ in at least two digits. Let $I_G$ be the $\sigma$-ideal generated by Borel $G$-independent sets. This definition is due to [Zap04, Section 2.3.11].

**Definition 2.1.7.** A perfect tree $T \subseteq 2^{<\omega}$ is called a Silver tree if for every $s, t \in T$, if $|s| = |t|$ then $\{ i \in 2 \mid s \circ \langle i \rangle \in T \} = \{ i \in 2 \mid t \circ \langle i \rangle \in T \}$, i.e., the branching at each node depends only on the length of that node. Another way to put this is: $[T] = \prod_{n \in \omega} Z_n$ for some sequence $\{ Z_n \mid n \in \omega \}$ such that each $Z_n$ is either $\{0\}$, $\{1\}$, or $\{0, 1\}$, and such that $\exists^\infty n (Z_n = \{0, 1\})$. The partial order of Silver trees ordered by inclusion is called Silver forcing and typically abbreviated by $\forall$.

By [Zap04, Lemma 2.3.37], every Borel (even analytic) set is either in the ideal $I_G$ or contains $[T]$ for some Silver tree $T$. Therefore $\forall \rightarrow_{d} B(2^{\omega}) \setminus I_G$.

There are other situations (often when the forcing $P_I$ satisfies the c.c.c.) when there is no strict dichotomy in the above sense, but rather one in the sense of “modulo $I$”. Suppose that for every $I$-positive Borel set $B$ there is a $q \in Q$ such that $(q \setminus B) \in I$. In this case there is no dense embedding from $Q$ to $P_I$ but only to the algebra $B(\omega^\omega)/I$ of Borel sets modulo $I$. Since there is always a dense embedding from $P_I$ to $B(\omega^\omega)/I$, the two notions $Q$ and $P_I$ are still forcing equivalent.

**Example 2.1.8.**
1. Consider the meager ideal $\mathcal{M}$ and Cohen forcing $\mathbb{C}$. From the fact that Borel sets satisfy the Baire property, it follows that for any Borel non-meager set there is a basic open set $[s]$ contained in $B$ modulo meager. Therefore $\mathbb{C} \hookrightarrow_{d} B(\omega^\omega)/\mathcal{M}$ so $\mathbb{C}$ and $B(\omega^\omega) \setminus \mathcal{M}$ are forcing equivalent.

2. Recall the Hechler partial order $\mathbb{D}$. Each condition $(s, f) \in \mathbb{D}$ gives rise to the closed set $[s, f] := \{ x \in \omega^\omega \mid s \subseteq x \text{ and } \forall n (f(n) < x(n)) \}$, and these sets generate the dominating topology, a non-second-countable topology refining the standard topology on $\omega^\omega$. Let $M_D$ be the ideal of sets meager in the dominating topology. For the same reason as with Cohen forcing, $\mathbb{D}$ densely embeds into $B(\omega^\omega)/M_D$, and hence is forcing equivalent to $B(\omega^\omega) \setminus M_D$.

Many other interesting examples can be found in [Zap04, Zap08] to which we refer the reader for further study.

We will mostly be interested in ideals that are Borel generated, in the sense that every $A \in I$ is contained in some Borel set $B \in I$. Even when an ideal $I$ does not have this property by nature, for our purposes it will be sufficient (and often, more interesting—cf. Section 2.4) to consider its Borelized version $I_B := \{ A \in P(\omega^\omega) \mid A \subseteq B \text{ for some Borel set } B \in I \}$.

If $I$ is a Borel generated $\sigma$-ideal, we can use it to derive a “Marczewski-null-style” ideal.

**Definition 2.1.9.** Let $I$ be a Borel generated $\sigma$-ideal. Define $N_I$ by stipulating

$$A \in N_I \iff \forall B \in P_I \exists C \leq B (C \cap A = \emptyset).$$

**Lemma 2.1.10.** Let $I$ be a Borel generated $\sigma$-ideal such that $P_I$ is proper.

1. $N_I$ is a $\sigma$-ideal extending $I$ and coinciding with $I$ on Borel sets.

2. If $P_I$ satisfies the c.c.c., then $N_I = I$.

**Proof.**

1. It is clear that if $A \in I$ then there is a $B \supseteq A$ is such that $B \in I$, and any $C \in P_I$ can be extended to $C \setminus B \in P_I$ which is disjoint from $A$, so $A \in N_I$.

Also it is clear that if $B$ is Borel and $I$-positive, then $B$ itself is a witness to the fact that $B \notin N_I$.

To prove that $N_I$ is a $\sigma$-ideal we use properness: let $A_n$ be sets in $N_I$ and let $A = \bigcup_n A_n$. For each $n$, let $D_n := \{ B \in P_I \mid B \cap A_n = \emptyset \}$. By definition all $D_n$ are dense. Fix some $B$ and let $M$ be a countable model containing all the $D_n$ and $B$. Let $C := \{ x \in B \mid x \text{ is } M\text{-generic} \}$. By properness and Fact 2.1.3 $C$ is I-positive, and for every $x \in C$ and every $n$ we have $x \in \bigcup(D_n \cap M)$, implying that $x \notin A_n$. Therefore $C \cap A = \emptyset$ as had to be shown.
2.2. From ideals to regularity

2. Let $A \in \mathcal{N}_I$ and define $D := \{ B \in \mathbb{P}_I \mid B \cap A = \emptyset \}$. Since $D$ is dense, let $E \subseteq D$ be a maximal antichain. By the c.c.c. it is countable, so $C := \omega^\omega \setminus \bigcup E$ is a Borel set and $A \subseteq C$. Since $C$ is disjoint from all $B \in E$ it must be $I$-small, otherwise it would contradict $E$’s maximality. $\square$

The $\sigma$-ideal $\mathcal{N}_I$ is usually not Borel generated when $\mathbb{P}_I$ is not c.c.c. In the presence of a dense embedding $Q \hookrightarrow_{\text{d}} \mathbb{P}_I$, it is equivalent to the standard Marczewski null ideal for partial orders. For example, $\mathcal{N}_{\text{ctbl}}$ is the classical Marczewski-null ideal, i.e., the ideal of sets $A$ such that for every perfect set $p$ there is a perfect subset $q \subseteq p$ with $q \cap A = \emptyset$. The same holds for $\mathcal{N}_{K_\sigma}$ and $\mathcal{N}_{I_L}$ with perfect sets replaced by super-perfect resp. Laver trees.

One of the reasons for introducing the derived ideal $\mathcal{N}_I$ is that it is closely related to density in the forcing-theoretic sense. It will be useful in several places in the subsequent sections.

2.2 From ideals to regularity

Recall the notion of Marczewski measurability, and the generalized $X$-Marczewski measurability, defined in Section 1.3.1. Adapting it to the idealized forcing context, we get the following regularity property, which we will call $I$-regularity.

**Definition 2.2.1.** Let $I$ be a $\sigma$-ideal, assume that $\mathbb{P}_I$ is proper, and let $A$ be an arbitrary subset of $\omega^\omega$ (or a similar space). We say that $A$ is $I$-regular, denoted by $\text{Reg}(I)$, if

$$\forall B \in \mathbb{P}_I \exists C \in \mathbb{P}_I \text{ s.t. } C \leq B \text{ and } (C \subseteq A \text{ or } C \cap A = \emptyset).$$

**Lemma 2.2.2.** The collection of $I$-regular sets forms a $\sigma$-algebra.

**Proof.** By definition, this collection is closed under complements. Let $A_n$ be $I$-regular sets, and let $B$ be an $I$-positive Borel set. It is clear that $B \cap A_n$ is still $I$-regular. Now, if $(A_n \cap B) \in \mathcal{N}_I$ for all $n$, then by Lemma 2.1.10 we have $\bigcup_n (A_n \cap B) \in \mathcal{N}_I$ so there exists $C \leq B$ disjoint from $\bigcup_n A_n$. On the other hand, if some $(A_n \cap B)$ is not in $\mathcal{N}_I$, then by $I$-regularity there must be a $C \leq B$ such that $C \subseteq A_n$. $\square$

So the property of being $I$-regular is equivalent to being $\mathbb{P}_I$-Marczewski measurable (according to the defined in Section 1.3.1), and in the presence of a dense embedding $Q \hookrightarrow_{\text{d}} \mathbb{P}_I$ it is equivalent to being $Q$-Marczewski measurable. In the case of a dense embedding modulo $I$, it is only equivalent to an analogous statement with “inclusion” replaced by “inclusion modulo $I$”.

Furthermore, in many cases when $Q$ generates a topology and $I$ is the $\sigma$-ideal of sets meager in that topology, $I$-regularity is equivalent to the Baire property (in the respective topology). This applies to Cohen, Hechler and Mathias forcing,
the latter fact being the essence of Ellentuck’s proof that all analytic sets are Ramsey [Ell74]. For the σ-ideal of Lebesgue null sets \( \mathcal{N} \), our notion of regularity coincides with Lebesgue measurability.

Note that when dealing with the regularity of all sets within a given projective pointclass \( \Gamma \), the first quantifier from Definition 2.2.1 can usually be dropped. In most cases the σ-ideal is homogeneous, meaning that for every Borel \( I \)-positive set \( B \) there is a Borel function \( f \) which is a bijection between \( B \) and \( \omega^\omega \) and preserves membership in \( I \). Such a function can easily be used to transform the set \( A \cap B \) to another set \( A' \) without increasing its complexity. Thus the first clause "∀\( B \)" in the definition can be eliminated. This is important in such situations as Mathias forcing: formally, \( I_{\text{RN}} \)-regularity is the property usually called being completely Ramsey, but on the level of projective pointclasses \( \Gamma \), it is equivalent to the Ramsey property (as given in Definition 1.3.5). The same may be said about Silver forcing: here, the homogeneity of the corresponding ideal \( I_G \) together with the dense embedding implies that \( I_G \)-regularity is equivalent to the property of containing or being disjoint from a set \( [T] \) where \( T \) is a Silver tree. By identifying infinite subsets of \( \omega \) with their characteristic functions, it is easy to see that this property is equivalent to the doughnut property (see Definition 1.3.6).

So, we see that a fairly wide class of regularity properties can be captured by a single definition in the idealized forcing context. Table 2.1 sums up a few of the standard forcing notions with their corresponding ideals and regularity properties.

<table>
<thead>
<tr>
<th>Forcing</th>
<th>σ-ideal</th>
<th>( I )-regularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>c.c.c.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cohen (( \mathbb{C} ))</td>
<td>( \mathcal{M} )</td>
<td>Baire property</td>
</tr>
<tr>
<td>random (( \mathbb{B} ))</td>
<td>( \mathcal{N} )</td>
<td>Lebesgue measurable</td>
</tr>
<tr>
<td>Hechler (( \mathbb{D} ))</td>
<td>( \mathcal{M}_D )</td>
<td>( \mathbb{D} )-Baire property</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>non-c.c.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sacks (( \mathbb{S} ))</td>
<td>ctbl</td>
<td>Marczewski measurable; not a Bernstein set</td>
</tr>
<tr>
<td>Miller (( \mathbb{M} ))</td>
<td>( K_\sigma )</td>
<td>M-Marczewski measurable</td>
</tr>
<tr>
<td>Laver (( \mathbb{L} ))</td>
<td>( I_\sigma )</td>
<td>L-Marczewski measurable</td>
</tr>
<tr>
<td>Mathias (( \mathbb{R} ))</td>
<td>( I_{\text{RN}} )</td>
<td>Ramsey; Baire property in Ellentuck topology</td>
</tr>
<tr>
<td>E₀-tree (( \mathbb{E}_0 ))</td>
<td>( I_{E_0} )</td>
<td>( \mathbb{E}_0 )-Marczewski measurable</td>
</tr>
<tr>
<td>Silver (( \mathbb{V} ))</td>
<td>( I_G )</td>
<td>doughnut property</td>
</tr>
</tbody>
</table>

Table 2.1: Some standard regularity properties

We will now prove a number of general results about \( I \)-regularity and definability, following the pattern described in Section 1.3. Since our definition was based on forcing, it is not surprising that all the proofs proceed via forcing-theoretic
methods (even when the result is a ZFC-theorem). From now on, we will always assume that \( I \) is a \( \sigma \)-ideal such that \( \mathbb{P}_I \) is a proper forcing partial order. As usual, \( \Gamma(\text{Reg}(I)) \) is shorthand for the statement “all sets in \( \Gamma \) are \( I \)-regular.”

**Proposition 2.2.3.** All analytic sets are \( I \)-regular.

**Proof.** Let \( A \) be an analytic set, defined by a \( \Sigma^1_1 \) formula \( \phi \) with parameter \( r \) (we will suppress \( r \) for convenience of notation). Let \( B \) be any \( I \)-positive Borel set. Let \( M \) be a countable elementary submodel containing \( B \) and \( r \). In \( M \), there is a stronger condition \( B' \leq B \) such that \( B' \Vdash \phi(x_G) \) or \( B' \Vdash \neg \phi(x_G) \). Assume the former, and let \( C := \{ x \in B' \mid x \text{ is } M\text{-generic} \} \) be the Master-condition. By properness it is \( I \)-positive, and since \( C \Vdash x_G \in B' \), for every \( x \in C \) we have \( M[x] \Vdash \phi(x) \). Since \( \phi \) is \( \Sigma^1_1 \) it is absolute between \( M[x] \) and \( V \) and hence \( \phi(x) \) holds in \( V \). Therefore \( C \subseteq A \). The case when \( B' \Vdash \neg \phi(x_G) \) proceeds analogously noting that \( \Pi^1_1 \)-absoluteness between \( M[x] \) and \( V \) also holds. In that case we get \( C \cap A = \emptyset \). \( \square \)

**Proposition 2.2.4.** If \( V = L \) then there is a \( \Delta^1_2 \) non-\( I \)-regular set.

**Proof.** By Fact 1.3.8 we know that if \( V = L \) then there is a \( \Delta^1_2 \) Bernstein set, i.e., a set \( A \) such that there is no perfect set completely contained in \( A \) or completely disjoint from \( A \). By our assumption that all singletons, and hence all countable sets, are \( I \)-small, it follows that every \( I \)-positive Borel set must be uncountable, so by the perfect set property it must contain a perfect set. Therefore the Bernstein set cannot be \( I \)-regular. \( \square \)

**Proposition 2.2.5.** If for every \( r \in \omega^\omega \) there is a \( \mathbb{P}_1 \)-generic real over \( L[r] \) then all \( \Delta^1_2 \) sets are \( I \)-regular.

**Proof.** Let \( A \) be a \( \Delta^1_2 \) set, defined by \( \Sigma^1_1 \) formulas \( \phi \) and \( \psi \) with parameter \( r \). Let \( B \) be any \( I \)-positive Borel set, and let \( c \) be its code. Let \( x \) be \( \mathbb{P}_1 \)-generic over \( L[r,c] \). We know that either \( \phi(x) \) or \( \psi(x) \) is true, and by Shoenfield absoluteness, the same holds in \( L[r,c,x] \). Hence, by the forcing theorem, we can find a \( B' \leq B \) in \( L[r,c] \) such that \( B' \Vdash \phi(x_G) \) or \( B' \Vdash \psi(x_G) \). Since both \( \phi \) and \( \psi \) are \( \Sigma^1_2 \), the situation is symmetrical, so without loss of generality we may assume the former. Let \( M \) be a countable elementary submodel containing \( B', r \) and \( c \). Let \( C := \{ x \in B' \mid x \text{ is } M\text{-generic} \} \) be the master condition. Now in \( V \), for every \( x \in C \), \( M[x] \Vdash \phi(x) \), and by upwards \( \Sigma^1_1 \)-absoluteness we get \( \phi(x) \) in \( V \). Therefore \( C \subseteq A \). In the situation where \( B' \Vdash \psi(x_G) \) we would get \( C \cap A = \emptyset \) by an identical argument. \( \square \)

**Proposition 2.2.6.** If for every \( r \in \omega^\omega \) the set \( \{ x \mid x \text{ is not } \mathbb{P}_1 \text{-generic over } L[r] \} \) is in \( N_I \), then all \( \Sigma^1_2 \) sets are \( I \)-regular.
**Proposition 2.2.8.** In the Solovay model, all sets are $I$-regular.

**Proof.** Let $V$ be a model with an inaccessible cardinal $\kappa$ and $V[G]$ the corresponding Lévy collapse of $\kappa$. In $V[G]$, let $A$ be a set of reals definable from a countable sequence of ordinals. Let $s \in \text{Ord}^s$ be such that $A$ is definable by $\varphi(s,x)$. By standard properties of the Lévy collapse (Lemma 1.2.20), there is a formula $\check{\varphi}$ such that for all $x$, $V[G] \models \varphi(s,x)$ iff $V[s][x] \models \check{\varphi}(s,x)$.

Let $B$ be an $I$-positive Borel set in $V[G]$. Without loss of generality we may assume that the code of $B$ is contained in $s$. Now consider the forcing $P_I$ in $V[s]$. There is $B' \leq B$ in $V[s]$ such that $B' \Vdash \check{\varphi}(\check{x}_G)$ or $B' \Vdash \neg\check{\varphi}(\check{x}_G)$. Assume the former. Since $\kappa = \aleph_1^{V[G]}$ is inaccessible in $V[s]$, the collection of all dense sets in $V[s]$.
2.3. Quasi-generic reals and characterization results

Propositions 2.2.5 and 2.2.6 tell us that sufficient transcendence over $L$ implies $I$-regularity for $\Delta^1_2$ or $\Sigma^1_2$ sets, but they are not yet characterization theorems, i.e., their converse does not necessarily hold. In Section 1.3.2 we saw that the converse is true for Cohen and random forcing. However, we also saw that for Sacks, Miller and Laver forcing the characterization involved new reals, unbounded reals and dominating reals rather than Sacks-, Miller- and Laver-generic reals, respectively. The difference can be explained by introducing the following definition:

Definition 2.3.1. Let $I$ be a $\sigma$-ideal on $\omega^\omega$ and $M$ a transitive model of set theory. A real $x$ is called $I$-quasi-generic over $M$ if for every Borel set $B \in I$ whose Borel code lies in $M$, $x \notin B$. We will also use the term $Q$-quasi-generic if we know that $Q \hookrightarrow d B(\omega^\omega) \setminus I$.

An obvious generalization of Cohen and random reals, the concept of quasi-generic reals was first explicitly introduced in [BHL05, Section 1.5] in the context of Silver forcing. In [Ike10a] the notion was fully exploited in order to prove a characterization theorem for arboreal forcings in an abstract setting. Note that by Fact 2.1.2 every real $x$ which is $P_I$-generic over $M$ is also $I$-quasi-generic over $M$. The converse is true for all c.c.c. forcings.

Lemma 2.3.2 (Ikegami). Let $I$ be a $\sigma$-ideal such that $P_I$ is c.c.c., and let $M$ a transitive model. Then every real $x$ is $P_I$-generic over $M$ iff it is $I$-quasi-generic over $M$.

Proof. Let $x$ be $I$-quasi-generic over $M$. For every maximal antichain $E$ in $M$, let $B_E := \omega^\omega \setminus \bigcup E$. By the c.c.c., this is a Borel set, it is coded in $M$, and is in $\mathcal{N}_I$, hence in $I$. By $I$-quasi-genericity $x \notin B_E$, hence $x \in \bigcup E$. Therefore $x$ is $P_I$-generic. □

For non-c.c.c. forcings, the notion of $I$-quasi-genericity is usually different from $P_I$-genericity. For example, the following is easy to show (see Lemma 2.5.3 for a proof):

$\mathbb{P}^V_s$ is countable in $V[G]$. Just as in the proof of Corollary 2.2.7, the collection \{ $x \mid x$ is not $P_I$-generic over $V[s]$ \} is in $\mathcal{N}_I$. Therefore, in $V[G]$, there is a Borel $I$-positive set $C \subseteq B'$ such that every $x \in C$ is $P_I$-generic over $V[s]$. Hence, for every such $x$ we have $V[s][x] \models \tilde{\varphi}(s,x)$, which implies that $V[G] \models \varphi(s,x)$, i.e., $x \in A$. The case that $B' \Vdash \neg \tilde{\varphi}(x_G)$ is analogous.

Corollary 2.2.9. Con($\text{ZFC} + \text{“all projective sets are } I\text{-regular”}$) and Con($\text{ZF} + \text{DC} + \text{“all sets are } I\text{-regular”}$).
• a real is $S$-quasi-generic (i.e., $ctbl$-quasi-generic) over $M$ iff it is not in $M$,
• a real is $M$-quasi-generic (i.e., $K_\sigma$-quasi-generic) over $M$ iff it is unbounded over $M$,
• a real is $L$-quasi-generic (i.e., $I_L$-quasi-generic) over $M$ iff it is strongly dominating over $M$ (see Definition 1.3.7 (3)).

Concerning the last point: if $x$ is strongly dominating over $M$ then it is also dominating over $M$. The converse is false; however, it is not hard to see that if there is a dominating real over $M$ then there is also a strongly dominating real over $M$. Therefore, as far as statements about transcendence over a model go, dominating and strongly dominating reals amount to the same thing.

We can extend Ikegami’s characterization theorems to the idealized forcing context. First, we show that quasi-generics are sufficient to ensure $I$-regularity on the second projective level, giving stronger versions of Propositions 2.2.5 and 2.2.6.

**Proposition 2.3.3.** If for every $r \in \omega^\omega$, for every $I$-positive set $B$, there is an $x \in B$ which is $I$-quasi-generic over $L[r]$, then all $\Delta^1_2$ sets are $I$-regular.

If we assume sufficient homogeneity of the ideal then the additional clause “for every $I$-positive set $B” may be omitted.

**Proof.** Let $A$ be a $\Delta^1_2(r)$ set and let $B$ be any $I$-positive set. We may assume that $\exists s \in \omega^\omega$ ($\mathfrak{N}^{L[s]}_1 = \mathfrak{N}_1$) since otherwise the result trivially follows from Corollary 2.2.7. Also we may assume that $\mathfrak{N}^{L[r]}_1 = \mathfrak{N}_1$ without loss of generality (otherwise consider $L[r, s]$). By Shoenfield’s classical analysis of $\Sigma^1_2$ sets (Fact 1.2.15) we may write $(A \cap B) = \bigcup_{\alpha < \mathfrak{N}_1} C_\alpha$ and $(B \setminus A) = \bigcup_{\alpha < \mathfrak{N}_1} D_\alpha$, with $C_\alpha$ and $D_\alpha$ Borel sets coded in $L[r]$. Let $x \in B$ be $I$-quasi-generic. Then $x \in C_\alpha$ or $x \in D_\alpha$ for some $\alpha$, so either $C_\alpha$ or $D_\alpha$ is the required $I$-positive subset of $B$.

If $I$ is homogeneous we can transform $A \cap B$ to some $A'$ without increasing the complexity, and run the same argument with $A'$.

**Proposition 2.3.4.** If for every $r \in \omega^\omega$, $\{x \mid \text{x is not } I\text{-quasi-generic over } L[r]\} \in \mathcal{N}_I$, then all $\Sigma^1_2$ sets are $I$-regular.

**Proof.** Let $A$ be $\Sigma^1_2(r)$ and again assume that $\mathfrak{N}^{L[r]}_1 = \mathfrak{N}_1$. Let $B$ be an $I$-positive set. If there are no $I$-quasi-generics in $A \cap B$, then by assumption there is $C \preceq B$ disjoint from $A$ so we are done. So assume $x \in A \cap B$ is $I$-quasi-generic, and, as before, write $A \cap B = \bigcup_{\alpha < \mathfrak{N}_1} C_\alpha$ with $C_\alpha$ Borel and coded in $L[r]$. Since $x$ is in some $C_\alpha$, this $C_\alpha$ will be the $I$-positive subset of $B$ contained in $A$. 

\qed
2.3. Quasi-generic reals and characterization results

With the stronger notion of quasi-genericity instead of genericity, a suitable converse of Propositions 2.3.3 and 2.3.4 can indeed be proved, but only if we assume that the ideal \( I \) (i.e., the membership of Borel sets in \( I \)) is \( \Sigma^1_3 \), as shown by the results of Ikegami [Ike10a]. For its statement we require two additional definitions.

**Definition 2.3.5.** A forcing \( P \) is \( \Sigma^1_3 \)-absolute if every \( \Sigma^1_3 \) formula is absolute between \( V \) and \( V[G] \), for any \( P \)-generic \( G \).

Since upwards \( \Sigma^1_3 \)-absoluteness holds between every model and an extension of it by a forcing preserving \( \omega_1 \), it is only downwards \( \Sigma^1_3 \)-absoluteness which matters.

**Definition 2.3.6.** For a projective pointclass \( \Gamma \), we say that \( \Gamma \)-I-uniformization holds if for every \( I \)-positive \( B \) and every \( A \subseteq B \times \omega^\omega \) such that \( A \in \Gamma \) and \( \forall x \in B \exists y ((x,y) \in A) \), there is an \( I \)-positive Borel set \( C \subseteq B \) and a Borel function \( g : C \to \omega^\omega \) uniformizing \( A \), i.e., such that \( \forall x \in C ((x,g(x)) \in A) \).

Zapletal already showed that analytic \( I \)-uniformization holds [Zap08, Proposition 2.3.4], and that \( \Pi^1_1 \)-I-uniformization fails in \( L \) [Zap08, Example 2.3.5]. The following theorem shows that it is another transcendence property over \( L \). Despite our more general framework, the proof of this theorem is essentially due to Ikegami [Ike10a].

**Theorem 2.3.7 (Ikegami).** Let \( I \) be a \( \sigma \)-ideal such that \( P_I \) is proper. The following are equivalent:

1. All \( \Delta^1_2 \) sets are \( I \)-regular,
2. \( \Pi^1_1 \)-I-uniformization holds, and
3. \( P_I \) is \( \Sigma^1_3 \)-absolute.

If \( I \) is \( \Sigma^1_2 \) then it is also equivalent to

4. \( \forall r \in \omega^\omega, \forall B \in P_I, \) there is an \( x \in B \) which is \( I \)-quasi-generic over \( L[r] \).

**Proof.**

- (1 \( \Rightarrow \) 2). Let \( B \) be \( I \)-positive and let \( A \subseteq B \times \omega^\omega \) be \( \Pi^1_1 \). By Kondô’s uniformization theorem, let \( f \) be a \( \Pi^1_1 \) function with \( \text{dom}(f) = B \) uniformizing \( A \). Let \( D_{n,m} := \{ B' \leq B \mid \forall x \in B' \ (f(x)(n) = m) \} \), and let \( D_n := \bigcup_m D_{n,m} \).

**Claim.** \( D_n \) is dense below \( B \).

**Proof.** Fix \( n \), let \( B' \leq B \), and consider \( A_m := \{ x \in B' \mid f(x)(n) = m \} \). Since (the graph of) \( f \) is \( \Pi^1_1 \), each \( A_m \) is a \( \Delta^1_2 \) set, therefore \( I \)-regular. Now,
if \( A_m \in \mathcal{N}_I \) for every \( m \), then \( B' = \bigcup_m A_m \in \mathcal{N}_I \) by Lemma 2.1.10, hence \( B' \in I \), contradicting the assumption. Therefore at least one \( A_m \) must be \( \mathcal{N}_I \)-positive, and since it is \( I \)-regular, there must be a \( B'' \leq B' \) such that \( B'' \subseteq A_m \). Then \( B'' \in D_{n,m} \). \( \square \)(Claim.)

Now let \( M \) be a countable elementary submodel, containing \( B \) and all the \( D_n \). Let \( C := \{ x \in B \mid x \text{ is } M\text{-generic} \} \) be the \( I \)-positive \( M \)-master condition. Define \( g : C \to \omega^\omega \) by \( g(x)(n) = m \) iff \( x \in \bigcup(D_{n,m} \cap M) \). This is a Borel function since \( (D_{n,m} \cap M) \) is countable. As \( x \) is \( M \)-generic and \( D_n \) is dense, \( x \in \bigcup(D_n \cap M) \) so there is an \( m \) such that \( x \in \bigcup(D_{n,m} \cap M) \).

Also, it is clear that there can be at most one \( m \) such that \( x \in \bigcup(D_{n,m} \cap M) \) since \( \bigcup D_{n,m} \) and \( \bigcup D_{n,m'} \) are disjoint for all \( m \neq m' \). By definition of \( D_{n,m} \) it follows that \( g(x)(n) = m \) iff \( f(x)(n) = m \), so \( g \upharpoonright C = f \upharpoonright C \) and the result follows.

- (2 \( \Rightarrow \) 3). Let \( \phi \) be a \( \Sigma^1_3 \) formula (with parameter \( r \), which we suppress for ease of notation) and let \( \exists x \forall y \phi(x,y) \) be the \( \Sigma^1_3 \) formula in question. We must show that it is downwards absolute; so assume \( V[G] \models \exists x \forall y \phi(x,y) \).

Let \( B \) be a condition and \( x \) a name for a real, such that \( B \models \forall y \phi(x,y) \). By Fact 2.1.4, there is a Borel function \( f \) such that \( B \models \forall y \phi(f(x),y) \).

We now claim that the statement \( \exists x \forall y \phi(x,y) \) must hold in \( V \) as well. Towards contradiction, suppose it does not. Then for every \( x \in B \) there is a \( y \) such that \( \neg \phi(f(x),y) \). Using \( \Pi^1_1 \)-uniformization let \( C \subseteq B \) be \( I \)-positive, and let \( g \) be a Borel function uniformizing the \( \Pi^1_1 \) set \( \{ (x,y) \in B \times \omega^\omega \mid \neg \phi(f(x),y) \} \). Then for every \( x \in C \), \( \neg \phi(f(x),g(x)) \) holds. Since both \( f \) and \( g \) are Borel functions, the statement \( \forall x \in C \neg \phi(f(x),g(x)) \) is \( \Pi^1_1 \), hence absolute. Therefore \( C \models \neg \phi(f(x),g(x)) \). But this contradicts \( B \models \forall y \phi(f(x),y) \), so we are forced to conclude that \( \exists x \forall y \phi(x,y) \) is downwards absolute.

- (3 \( \Rightarrow \) 1). Let \( A \) be a \( \Delta^1_1 \) set, defined by \( \Sigma^1_2 \) formulas \( \phi \) and \( \psi \). The statement \( \forall x (\phi(x) \iff \neg \psi(x)) \) is \( \Pi^1_3 \), so by assumption it is true in the extension. Now proceed as in the proof of Proposition 2.2.5. For every \( I \)-positive \( B \) there is an \( I \)-positive \( B' \leq B \) such that \( B' \models \phi(\dot{x}_G) \) or \( B' \models \neg \psi(\dot{x}_G) \). Assume the former, let \( M \) be a countable elementary submodel containing \( B' \) and let \( C := \{ x \in B' \mid x \text{ is } M\text{-generic} \} \). For every \( x \in C \), \( M[x] \models \phi(x) \), and by upwards \( \Sigma^1_2 \)-absoluteness \( \phi(x) \) holds. The situation with \( B' \models \neg \psi(\dot{x}_G) \) is analogous because \( \psi \) is also \( \Sigma^1_2 \).

For the equivalence with (4), suppose \( I \) is \( \Sigma^1_2 \). Let \( B \) be \( I \)-positive. Then the statement \( \exists x (x \in B \text{ and } x \text{ is } I\text{-quasi-generic over } L[r]) \) is \( \Sigma^1_3(r,B) \), and is clearly forced by \( B \) to be true in the extension \( V[G] \) (it is true of the generic real \( x_G \), which is \( I \)-quasi-generic over \( V \) and hence also over \( L[r] \)). So by \( \Sigma^1_3 \)-absoluteness it is true in \( V \). The direction (4) \( \Rightarrow \) (1) is Proposition 2.3.3. \( \square \)
Corollary 2.3.8. Assume that $I$ is $\Sigma^1_2$. Then the following are equivalent:

1. For every $r \in \omega^\omega$, $\{ x \mid x$ is not $I$-quasi-generic over $L[r] \} \in \mathcal{N}_1$, and

2. All $\Sigma^1_2$ sets are $I$-regular.

Proof. Assume that all $\Sigma^1_2$ sets are $I$-regular. If $I$ is $\Sigma^1_2$, then $\{ x \mid x$ is not $I$-quasi-generic over $L[r] \}$ is also $\Sigma^1_2$, hence $I$-regular. If it is not in $\mathcal{N}_1$, then there must be an $I$-positive Borel set $B$ contained in it, i.e., such that for every $x \in B$, $x$ is not $I$-quasi-generic over $L[r]$. But then by Theorem 2.3.7 $\Delta^1_2(\text{Reg}(I))$ would be false, which certainly contradicts the assumption.

We can now also be somewhat more precise about the relationship between regularity hypotheses and cardinal invariants which we mentioned in Section 1.3.3. Recall that the covering number $\text{cov}(I)$ is the least number of $I$-small sets needed to cover the whole space $\omega^\omega$, and the additivity number $\text{add}(I)$ is the least number of $I$-small sets whose union is not $I$-small. A variation of the covering number is $\text{cov}^*(I)$ defined as the least number of $I$-small sets needed to cover some $I$-positive Borel set. In the case of homogeneity of $I$, $\text{cov}^*(I) = \text{cov}(I)$.

Lemma 2.3.9.

1. $\text{cov}^*(I) > \aleph_1 \implies \Delta^1_2(\text{Reg}(I))$,

2. $\text{add}(\mathcal{N}_I) > \aleph_1 \implies \Sigma^1_2(\text{Reg}(I)).$

Proof. This follows immediately from Propositions 2.3.3 and 2.3.4, noting that $\{ x \mid x$ is not $I$-quasi-generic over $L[r] \}$ can be written as $\bigcup \{ B \in \mathcal{B} \mid B \in I$ and $B$ is coded by a real in $L[r] \}$ and that there are only $\aleph_1$ reals in $L[r]$.

As before, the converse is clearly false, for instance in $\aleph_1$-iteration of $\mathbb{P}_I$ starting from $L$, or in models of CH with a measurable cardinal (the latter due to Corollary 2.2.7). Nevertheless, we can say a little more if we look at the details of cardinal inequality proofs. Assuming that $I$ and $J$ are $\Sigma^1_2$ ideals, we can say the following:

1. if $\text{cov}^*(I) \leq \text{cov}^*(J)$ is provable in ZFC, then, most likely, $\Delta^1_2(\text{Reg}(I)) \Rightarrow \Delta^1_2(\text{Reg}(J))$ is also provable in ZFC, and

2. if it is consistent that $\text{cov}^*(I) < \text{cov}^*(J)$, then, most likely, it is consistent that $\Delta^1_2(\text{Reg}(J)) \not\Rightarrow \Delta^1_2(\text{Reg}(I))$.

3. The same holds regarding $\text{add}(\mathcal{N}_I)$ and $\Sigma^1_2(\text{Reg}(I)).$

To say a bit more about point 1, if $\text{cov}^*(I) \leq \text{cov}^*(J)$ is a theorem, then most likely the proof is as follows: “Given a collection $\{ B_\alpha \in J \mid \alpha < \kappa \}$ for some $\kappa < \text{cov}^*(I)$, transform it to another collection $\{ B'_\alpha \in I \mid \alpha < \kappa \}$, find a real
$x$ outside $\bigcup_{\alpha<\kappa} B'_\alpha$, and then transform it again to get another real $y$ outside $\bigcup_{\alpha<\kappa} B_\alpha$, proving that $\kappa < \text{cov}^*(J)$.” If the transformation process in this proof is recursive, then $B'_\alpha$ will have a code in $L[r]$ whenever $B_\alpha$ has a code in $L[r]$. As a result, the same proof will show that $\Delta_1^2(\text{Reg}(I)) \Rightarrow \Delta_1^2(\text{Reg}(J))$, via Proposition 2.3.3 and Theorem 2.3.7. For $\text{add}(\mathcal{N}_J)$ and $\Sigma_2^1(\text{Reg}(I))$, the same applies, using Proposition 2.3.4 and Corollary 2.3.8.

Concerning point 2, the idea is that if $\text{cov}^*(I) < \text{cov}^*(J)$ is consistent, then most likely the proof involves a forcing iteration with some $\mathbb{P}$ which adds $J$-quasi-generic reals but no $I$-quasi-generic reals (for example, $\mathbb{P}_J$ itself). In this case, it is clear that an iteration of this forcing will also yield a model in which $\Delta_1^2(\text{Reg}(J))$ is true but $\Delta_1^2(\text{Reg}(I))$ is false. For the additivity and the $\Sigma_2^1$ level, we have to consider a $\mathbb{P}$ which adds a co-$\mathcal{N}_J$ set of $J$-quasi-generics but does not add a co-$\mathcal{N}_I$ set of $I$-quasi-generics.

However, the above is merely heuristics: for example, if the cardinal inequality is proved by some other means than described above, the conclusion may well fail to hold, and conversely, relationships between regularity hypotheses can be established without a direct link to cardinal invariants.

Finally, we would like to mention that Ikegami’s equivalence theorems depend heavily on the complexity of the ideal $I$, and it is still open whether a suitable characterization theorem can be proved in case $I$ is not $\Sigma_2^1$. It so happens that many naturally occurring ideals (for example, most of the ideals in [Zap04]) are $\Sigma_2^1$, and for some time it was considered unlikely that a natural counterexample would exist at all. However, in a recent development, Marcin Sabok [Sab10] proved that the Ramsey-null ideal $I_{RN}$ is one such counterexample, i.e., it is $\Pi_2^1$-complete. This naturally leads to the following question.

**Question 2.3.10.** Does $\Delta_1^2(\text{Ramsey})$ imply that $\forall r \in \omega\omega$ there is a Ramsey-null-quasi-generic real over $L[r]$?

### 2.4 Dichotomies

The notion of $I$-regularity was introduced as a generalization of many well-known regularity properties. There is another class of properties, however, which do not (and cannot) fall into this framework, and these are the dichotomy properties such as the perfect set property and its relatives $K_\sigma$-regularity, $u$-regularity and Laver-regularity (see Definition 1.3.7).

Proving this dichotomy for Borel sets is a necessary requirement for an embedding $Q \hookrightarrow \mathcal{B}(\omega\omega) \setminus I$. Often the proof can be extended to cover analytic sets, and a natural question arises as to what happens at higher complexity levels. By Theorem 1.3.13 we know that “all $\Sigma_2^1$ sets satisfy the perfect set property” is equivalent to “all $\Pi_1^1$ sets satisfy the perfect set property”, and equivalent to $\forall r \ (\mathcal{N}_I \setminus [r] < \aleph_1)$: the strongest regularity hypothesis. Notice that this is very
2.4. Dichotomies

different from $\Sigma^1_2(S)$, which is equivalent to $\forall r \ (L[r] \cap \omega^\omega \neq \omega^\omega)$: the weakest regularity hypothesis. Though both properties involve the ideal $\mathfrak{ctbl}$ and the partial order $S$ of perfect sets, the resulting hypotheses could hardly be more different.

To be able to study such dichotomies in our idealized framework, we introduce a new definition.

**Definition 2.4.1.** Let $I$ be a Borel generated $\sigma$-ideal. A set $A$ satisfies the $I$-dichotomy, denoted by $\text{Dich}(I)$, if it is either in $I$ or there is an $I$-positive Borel set $B$ such that $B \subseteq A$.

Notice that in the presence of a dense embedding $Q \hookrightarrow_{d} P_I$, the $I$-dichotomy, as we defined it above, is exactly the original dichotomy, involved in the proof of the embedding.

Notice also that this definition is really only interesting when the ideal $I$ is Borel generated. For example, if we would replace $I$ by $\mathcal{N}_I$ and define $\text{"N}_I$-dichotomy" analogously, it would be equivalent to $I$-regularity and would not give us anything interesting. It is also clear that $\Gamma(\text{Dich}(I))$ implies $\Gamma(\text{Reg}(I))$ for projective pointclasses $\Gamma$, and that the two notions coincide if $I = \mathcal{N}_I$, which is the case for all c.c.c.-ideals. In particular, $\text{Dich}(\mathcal{N})$ and $\text{Dich}(\mathcal{M})$ are Lebesgue measurability and the Baire property, respectively.

Originally, it was our intention to prove abstract results about the $I$-dichotomy similar to the ones we proved about $I$-regularity. It turns out, however, that $I$-dichotomy is a much more mysterious property. For example, we were not able to prove that all analytic sets satisfy the $I$-dichotomy, although we don’t know of any counterexamples. Strangely enough, on the $\Sigma^1_2$ level results similar to Propositions 2.2.6 and 2.3.4 can be proved.

**Proposition 2.4.2.** If for every $r \in \omega^\omega$, $\{x \mid x$ is not $P_I$-generic over $L[r]\} \in I$, then $\Sigma^1_2(\text{Dich}(I))$ holds.

**Proof.** Let $A$ be a $\Sigma^1_2$ set, defined by a formula $\phi$ with parameter $r$. If there are no reals in $A$ which are $P_I$-generic over $L[r]$ then we are done since $A \in I$.

So suppose there is an $x \in A$ which is $P_I$-generic over $L[r]$. Since by Shoenfield absoluteness $L[r, x] \models \phi(x)$, by the forcing theorem there is a $B \in L[r]$ such that $B \models \phi(x_G)$. Now let $M$ be a countable elementary submodel containing $r$ and $B$, and let $C := \{x \in B \mid x$ is $M$-generic$\}$ be the master condition. For every $x \in C$ we have $M[x] \models \phi(x)$ and by upwards $\Sigma^1_2$-absoluteness $\phi(x)$ holds in $V$, hence $C \subseteq A$. \qed

Currently we do not know of any applications of this theorem, except where $P_I$ has the c.c.c., in which case it says the same as Proposition 2.2.6. Also, note that, unlike Proposition 2.2.6, here we cannot conclude that $\Sigma^1_2(\text{Dich}(I))$ follows from the strongest hypothesis $\forall r \ (\aleph_1^{L[r]} < \aleph_1)$ (the reason will soon become clear).

Also, if we wish to prove analogous propositions with *quasi-generics* instead of *generics*, we must assume that $\aleph_1$ is not inaccessible in $L$. 

\textbf{Proposition 2.4.3.} If \( \forall r \{ x \mid x \text{ is not } I\text{-quasi-generic over } L[r] \} \in I \) and \( \exists r (\mathbb{N}_1^{L[r]} = \mathbb{N}_1) \), then \( \Sigma^1_2 (\text{Dich}(I)) \) holds. If additionally \( I \) is \( \Sigma^1_2 \), then, conversely, \( \Sigma^1_2 (\text{Dich}(I)) \) implies that \( \forall r \{ x \mid x \text{ is not } I\text{-quasi-generic over } L[r] \} \in I \).

\textbf{Proof.} Let \( A \) be \( \Sigma^1_2 [r] \) and without loss of generality assume \( \mathbb{N}_1^{L[r]} = \mathbb{N}_1 \). Then \( A \) can be written as \( \bigcup_{\alpha < \aleph_1} B_\alpha \) with \( B_\alpha \) coded in \( L[r] \). If there are no reals in \( A \) which are \( I\)-quasi-generic over \( L[r] \) then \( A \in I \), and if there is, then it is in some \( B_\alpha \) which must then be \( I\)-positive.

For the converse direction, suppose \( I \) is \( \Sigma^1_2 \). Since \( \Sigma^1_2 (\text{Dich}(I)) \) implies \( \Sigma^1_2 (\text{Reg}(I)) \) which implies \( \Delta^1_2 (\text{Reg}(I)) \), by Theorem 2.3.7 there exists an \( I\)-quasi-generic real in every \( I\)-positive set \( B \). Thus \( \{ x \mid x \text{ is not } I\text{-quasi-generic over } L[r] \} \in \Sigma^1_2 \) set which does not contain an \( I\)-positive Borel set so by the dichotomy it is in \( I \).  \( \square \)

Despite these results, the trouble with \( \Sigma^1_2 (\text{Dich}(I)) \) is that it can be inconsistent! Zapletal [Zap08, Proposition 3.9.2] essentially proved the following strong result: if \( \mathbb{P}_I \) is a forcing notion (satisfying additional requirements which are true in all natural cases), and \( \Sigma^1_2 (\text{Dich}(I)) \) holds, then any intermediate extension \( N \) of the forcing extension \( V^{\mathbb{P}_I} \) is either a c.c.c. extension of \( V \) or \( N = V^{\mathbb{P}_I} \). This immediately implies that for several ideals \( I \), among them the Borelized version of the Ramsey-null ideal \( I_{I_{\text{BN}}} \) and the ideals \( I_{E_0} \) and \( I_G \) (see Example 2.1.5), the statement \( \Sigma^1_2 (\text{Dich}(I)) \) is simply false. In the case of the \( E_0 \)-ideal, there is even a direct diagonalization proof, also an (unpublished) result of Zapletal.

\textbf{Proposition 2.4.4 (Zapletal).} There is a \( \Sigma^1_2 \) set not satisfying the \( I_{E_0} \)-dichotomy.

\textbf{Proof.} First, construct a two-dimensional perfect tree \( T \), inductively generated by the following clauses:

- \((\varnothing, \varnothing) \in T \),
- if \((s, t) \in T \) then the following pairs of extending sequences are in \( T \):
  - \((s \upharpoonright 0, t \upharpoonright 0) \upharpoonright (0))\)
  - \((s \upharpoonright 0, t \upharpoonright 0) \upharpoonright (1))\)
  - \((s \upharpoonright 1, t \upharpoonright 1) \upharpoonright (0))\)
  - \((s \upharpoonright 1, t \upharpoonright 1) \upharpoonright (1))\)

It is clear that \( T \) generates a perfect tree, with \([T] \subseteq 2^\omega \times 2^\omega \) the set of branches through \( T \). For \( x \in 2^\omega \), let \( T_x := \{ t \mid \exists s \subseteq x ((s, t) \in T) \} \). We claim that the following two conditions are satisfied:

(a) each \( T_x \) is an \( E_0 \)-tree, and
(b) \( \forall x \neq x', \forall y \in [T_x], \forall y' \in [T_{x'}], \neg(yE_0 y') \).
Part (a) follows immediately from the construction. For part (b), note that the construction of $T$ guarantees that if $s \subseteq x$ and $(x, y) \in [T]$, then the sequence $s$ appears infinitely often in $y$. Therefore, if $x \neq x'$, there exists some $n$ such that $s := x|n$ differs from $s' := x'|n$. Therefore, if $y \in [T_x]$ then $s$ will appear infinitely often in $y$ and if $y' \in [T_{x'}]$ then $s'$ will appear infinitely often in $y'$; moreover, this will happen on the same digits of the respective reals $y$ and $y'$. Hence, there will be infinitely many digits on which $y$ and $y'$ disagree, i.e., $\neg(yE_0y')$.

Now let $Y$ be a universal analytic subset of $2^\omega \times 2^\omega$, i.e., a set which is itself analytic and such that every analytic subset of $2^\omega$ is equal to some vertical section $([T] \setminus Y)_x := \{y \upharpoonright (x, y) \in Y\}$. Then $[T] \setminus Y$ is $\Pi^1_1$ so by Kondô’s uniformization theorem (see [Jec03, Theorem 25.36]) we can find a $\Pi^1_1$ function $g$ uniformizing it, i.e., such that for every $x$, if $([T] \setminus Y)_x \neq \emptyset$ then $g(x) \in ([T] \setminus Y)_x$. Let $A := \text{ran}(g)$.

We claim that $A$ is the required counterexample. It is clear that $A$ is $\Sigma^1_2$ because it is the range of a $\Pi^1_1$ function. To show that $A \notin \text{Dich}(I_{E_0})$, first suppose there is a Borel $I_{E_0}$-positive set $B \subseteq A$. Then there are at least two $y, y' \in A$ such that $yE_0y'$. Let $x, x'$ be such that $y = g(x)$ and $y' = g(x')$. Then $y \in ([T]_x) = [T_x]$ and $y' \in ([T]_{x'}) = [T_{x'}]$ are related by $E_0$, contradicting assumption (b) above. Now, suppose $A \in I_{E_0}$. Then there exists a Borel, hence analytic, set $B \in I_{E_0}$ such that $A \subseteq B$. By the universal property of $Y$, there must be some $x$ so that $B = ([Y]_x)$. Moreover, since by condition (a), $[T_x]$ is an $E_0$-tree, hence $I_{E_0}$-positive, the set $([T] \setminus Y)_x = [T_x] \setminus (Y)_x$ is non-empty, hence $x \in \text{dom}(g)$. Therefore $g(x) \notin B$, but that contradicts $g(x) \in A$.

A similar proof works for the ideal $I_G$. Combining this fact with Propositions 2.4.2 and 2.4.3, we see that for the ideals $I_{E_0}$, $I_{G}$ and $I_{B_{\text{RN}}}$, the antecedents in these propositions must themselves be inconsistent.

We will not be able to say much more about $I$-dichotomy. As counterexamples exist in ZF we cannot prove Dich($I$) for all sets of reals in the Solovay model, and this also seems to form an obstacle to proving $\Sigma^1_1(\text{Dich}(I))$. The following questions seem very interesting in this context:

1. Is there a general proof of $\Sigma^1_1(\text{Dich}(I))$, or are there natural counterexamples?
2. Under what conditions is $\Sigma^1_2(\text{Dich}(I))$ consistent?
3. Under what conditions does Dich($I$) hold for all sets of reals in the Solovay model?
2.5 From transcendence to quasi-generics

In the previous sections we have taken a $\sigma$-ideal on the reals as a starting point and proved results connecting regularity to transcendence over $L$. We can take the reverse approach and consider a natural transcendence property as the starting point. We will assume this property to be simple enough, in any case given by a Borel relation between reals (in practice it is usually of low complexity in the arithmetic hierarchy).

**Definition 2.5.1.** Let $R$ be a Borel relation on $\omega^\omega$ (or a similar space). We say:

1. $y$ is $R$-transcendent over $A$ if $\forall x \in A (x R y)$, and
2. $\{y_n \mid n \in \omega\}$ is $R$-$\sigma$-transcendent over $A$ if $\forall x \in A \exists n (x R y_n)$.

For a model $M$ we say that $y$ is $R$-transcendent over $M$ iff it is $R$-transcendent over $\omega^\omega \cap M$, and similarly for $R$-$\sigma$-transcendent.

Intended examples of relations “$x R y$” are: $y$ dominates $x$, $y$ is not dominated by $x$, $y$ is eventually different from $x$, $y$ splits $x$, etc.

**Definition 2.5.2.** Let $R$ be a Borel relation. For $x \in \omega^\omega$, let

$$K^R_x := \{y \in \omega^\omega \mid \neg(x R y)\}.$$  

Let $I^R$ be the $\sigma$-ideal generated by the sets $K^R_x$.

The $\sigma$-ideal $I^R$ is Borel generated, and it is easily seen to be $\Sigma^1_2$. Given $R$, it is useful to think about the dual relation $\bar{R}$, defined by $x \bar{R} y$ iff $\neg(y R x)$. Note that $A \in I^R$ iff there exists $\{x_n \mid n < \omega\}$ which $\bar{R}$-$\sigma$-transcends $A$.

**Lemma 2.5.3.** Let $R, K^R_x$ and $I^R$ be as above, and let $M$ be a model such that $\omega_1 \subseteq M$. Then $y$ is $R$-transcendent over $M$ iff it is $I^R$-quasi-generic over $M$.

**Proof.** First, suppose $y$ is $I^R$-quasi-generic over $M$. For any $x \in \omega^\omega \cap M$, $K^R_x$ is a Borel set in $I$, coded in $M$. Therefore $y \notin K^R_x$, hence $x R y$.

Conversely, suppose $y$ is $R$-transcendent over $M$. Let $B \in I^R$ be a Borel set coded in $M$. By $\Sigma^1_2$-absoluteness $M \models B \in I^R$, so there are $x_n$ in $M$ such that $M \models B \subseteq \bigcup_n K^R_{x_n}$, which by $\Pi^1_1$-absoluteness is also true in $V$. Now, if $y \in B$ then $y \in K^R_{x_n}$ for some $x_n$, i.e., $\neg(x_n R y)$ for some $x_n$. But this contradicts the $R$-transcendence of $y$ over $M$. Therefore $y \notin B$, which proves that $y$ is $I^R$-quasi-generic.

So forcing with $\mathbb{P}_I$ is, in a sense, a very canonical way to add a $R$-transcendent real. The biggest problem is to verify whether $\mathbb{P}_I$ is proper, but if it is, then by Proposition 2.3.3, Corollary 2.3.8 and Proposition 2.4.3 it immediately follows that
2.5. From transcendence to quasi-generics

- \( \Delta^1_2(\text{Reg}(I^R)) \) iff for all \( r \) there is a \( R \)-transcendent real over \( L[r] \),
- \( \Sigma^1_2(\text{Reg}(I^R)) \) iff for all \( r, \{ x \mid x \text{ not } R \text{-transcendent over } L[r] \} \in \mathcal{N}_r \), and
- If \( \exists r (\aleph_{L[r]}^1 = \aleph_1) \), then \( \Sigma^1_2(\text{Dich}(I^R)) \) iff for all \( r, \{ x \mid x \text{ not } R \text{-transcendent over } L[r] \} \in I^R \).

A little more can be proved assuming that \( R \) is a transitive relation.

**Proposition 2.5.4.** Suppose that \( R \) is transitive. Then

1. all analytic sets satisfy the \( I^R \)-dichotomy, and
2. the following are equivalent:
   (a) \( \Delta^1_2(\text{Reg}(I^R)) \),
   (b) \( \Sigma^1_2(\text{Reg}(I^R)) \),
   (c) \( \Sigma^1_2(\text{Dich}(I^R)) \).

**Proof.**

1. Let \( A \) be analytic, and let \( A = \bigcup_{\alpha < \aleph_1} B_\alpha \) be its Borel decomposition. If one of the \( B_\alpha \)'s is \( I^R \)-positive we are done, so suppose they are all in \( I^R \). Then for every \( \alpha \) there are \( \{ x^\alpha_n \mid n < \omega \} \) s.t. for all \( y \in B_\alpha \), \( \exists n \) s.t. \( \neg(x^\alpha_n \mathrel{R} y) \). Consider \( V[G] \), the forcing extension with \( P_{I^R} \). There the generic real \( x_G \) is \( R \)-transcendent over \( V \), hence \( x^\alpha_n \mathrel{R} x_G \) holds for all \( \alpha, n \). By Shoenfield absoluteness \( A = \bigcup_{\alpha < \aleph_1} B_\alpha \) still holds in the extension. Therefore, in \( V[G] \), for all \( y \in A \) there are \( \alpha, n \) such that \( \neg(x^\alpha_n \mathrel{R} y) \), and by transitivity of \( R \), \( \neg(x_G \mathrel{R} y) \) (otherwise \( x^\alpha_n \mathrel{R} x_G \mathrel{R} y \)). Therefore \( V[G] \models A \in I^R \), and since this statement is \( \Sigma^1_2 \) it is true in \( V \), too.

2. The implications (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) are trivial, so we have to show (a) \( \Rightarrow \) (c). If \( \Delta^1_2(\text{Reg}(I^R)) \) holds then by Theorem 2.3.7 \( \Sigma^1_3 \)-absoluteness holds between \( V \) and the forcing extension \( V[G] \) by \( P_{I^R} \). Now apply exactly the same argument as in part 1, using \( \Sigma^1_3 \)-absoluteness instead of Shoenfield absoluteness. \( \square \)

Below are some examples of familiar transcendence properties.

**Example 2.5.5.**

1. Let \( R \) be defined by \( x \mathrel{R} y \text{ iff } x \neq y \). Then \( I^R = \text{ctbl} \), the corresponding forcing is Sacks forcing, and a real is \( I^R \)-quasi-generic over \( M \) iff it is not in \( M \). Moreover, \( I^R \)-dichotomy is the perfect set property.

2. Let \( R \) be defined by \( x \mathrel{R} y \text{ iff } x \not\leq^* y \). Then \( I^R = K_\sigma \), the corresponding forcing is Miller forcing, and a real is \( K_\sigma \)-quasi-generic over \( M \) iff it is unbounded over \( M \). Moreover, \( I^R \)-dichotomy is \( K_\sigma \)-regularity.
3. Let $R$ be defined by $x \, R \, y$ iff $y$ dominates $x$. The corresponding ideal consist of all sets $A$ which are not dominating. There are two ways to look at this: considering $x, y$ as members of $\omega^\omega$, or of $\omega^{<\omega}$ (strictly increasing sequences). The latter case is easier because by Borel u-regularity (see Definition 1.3.7 (2)), there is a dense embedding from the set of Spinas trees into $B(\omega^{<\omega})\setminus I^R$. This is a proper forcing notion which, by [BHS95, Theorem 5.1], is equivalent to Laver forcing. If we consider the $\omega^\omega$ situation, then there is a dense embedding consisting of so-called nice sets (see [BHS95, p. 294]), which are combinatorially more complicated.

4. Let $R$ be defined by $x \, R \, y$ iff $y$ strongly dominates $x$ (Definition 1.3.7 (3)). Now the corresponding ideal is the Laver ideal $I_L$, and a real $x$ is strongly dominating iff it is Laver-quasi-generic.

Next we consider several examples of transcendence properties which are well-known, but have not yet been studied from the point of view of regularity. As always, the main problem is determining whether the corresponding forcing is proper.

**Example 2.5.6.**

1. For $x, y \in [\omega]^\omega$, let $R$ be defined by $x \, R \, y$ iff $y$ splits $x$, i.e., $x \cap y$ and $x \setminus y$ are infinite. The corresponding ideal $I^R$ has been studied by Spinas in [Spi04, Spi08]. In his terminology, a set $A$ was called countably splitting iff $A \notin I^R$. Spinas isolated the following notion of a splitting tree:

**Definition 2.5.7.** A tree $T$ on $2^{<\omega}$ is called a splitting tree if for every $s$ in $T$ there exists $N \in \omega$ such that $\forall n \geq N$ there exist two extensions $t_0$ and $t_1$ in $T$ such that $t_0(n) = 0$ and $t_1(n) = 1$. Let $\text{SPL}$ denote the partial order of splitting trees ordered by inclusion.

By [Spi04, Theorem 1.2], there is a dense embedding $\text{SPL} \hookrightarrow \text{P}_{I^R}$. Since splitting trees easily allow a standard fusion construction, it follows that $\text{P}_{I^R}$ is proper, so all the preceding theorems apply. In particular, all $\Delta^1_2$ sets are $I^R$-regular iff for every $r$ there exists a splitting real over $L[r]$. However, $R$ is not transitive and it is not clear whether $\Sigma^1_2(\text{Dich}(I^R))$ is consistent and whether $\Delta^1_2(\text{Reg}(I^R))$ and $\Sigma^1_2(\text{Reg}(I^R))$ are equivalent.

2. Consider the dual relation, i.e., $x \, R \, y$ iff $x$ does not split $y$. In [Spi08], Spinas studied the corresponding ideal and attempted to find a dense combinatorial object, but without success. Until this has been achieved, it is not clear whether the corresponding forcing $\text{P}_{I^R}$ is proper, and whether any of the preceding theorems hold.

3. Let $R$ be defined by: $x \, R \, y$ iff $x$ and $y$ are infinitely often equal, i.e., $\exists^\infty n (x(n) = y(n))$. In [Spi08] Spinas has isolated the notion of an i.o.e.-tree:
**Definition 2.5.8.** A tree $T$ on $\omega$ is an i.o.e.-tree if every $s \in T$ has an extension $t \in T$ such that $|\text{Succ}_T(t)| = \omega$.

Spinias [Spi08, Theorem 3.3] showed that the partial order of i.o.e.-trees densely embeds into $P_{IR}$, which, again, implies that $P_{IR}$ is proper, hence the equivalence theorems hold. In particular, all $\Delta^1_2$ sets are $I^R$-regular if for every $r$ there exists an infinitely often equal real over $L[r]$. It is open whether $\Sigma^1_2(\text{Dich}(I^R))$ is consistent and whether $\Delta^1_2(\text{Reg}(I^R))$ and $\Sigma^1_2(\text{Reg}(I^R))$ are equivalent.

4. Now consider the dual notion again, i.e., $x R y$ iff $x$ and $y$ are **eventually different**. The corresponding ideal has not received much attention so far, and it is currently unknown whether there is a dense partial order consisting of trees or other simple objects, and whether the forcing is proper.

### 2.6 Questions and further research

We would like to conclude this chapter by pointing out a number of open questions, or ideas for further research, that have come up in the development of this general theory.

**Question 2.6.1.** Concerning $I$-regularity:

1. Does the direction “$(3) \Rightarrow (4)$” in Ikegami’s theorem (Theorem 2.3.7) work without the assumption on the complexity of the ideal $I$?

2. A test-case for the above: does $\Sigma^1_2(\text{Ramsey})$ imply the existence of Ramsey-null-quasi-generics over $L[r]$?

3. Under which conditions is $\Delta^1_2(\text{Reg}(I))$ and $\Sigma^1_2(\text{Reg}(I))$ equivalent?

**Question 2.6.2.** Concerning $I$-dichotomy:

1. Is $\Sigma^1_1(\text{Dich}(I))$ true?

2. Under which assumptions is $\Sigma^1_2(\text{Dich}(I))$ consistent?

3. Under which assumptions is Dich($I$) true for all sets in the Solovay model?

4. What can we say about $\Pi^1_1(\text{Dich}(I))$? When is it equivalent to $\Sigma^1_2(\text{Dich}(I))$?

**Question 2.6.3.** Concerning $R$-transcendence:

1. Are the forcings related to the unsplit reals and the eventually different reals (Example 2.5.6 (2) and (4)) proper?
In this chapter, we have not really talked about infinite games or the axiom of determinacy. The main reason is that the framework of idealized forcing seems too general to allow us to prove any results. Of course, we know that AD (the axiom of determinacy) implies the Baire property and Lebesgue measurability for all sets of reals, as well as many of the $I$-dichotomy properties. One might ask whether the following holds:

**Question 2.6.4.** Does AD imply that all sets are $I$-regular?

Although no counterexample to this implication is currently known, it is also known to be a very difficult open problem for many regularity properties. Most notably, it is still open whether AD implies that all sets are Ramsey. On the other hand, the Ramsey property does follow from the axiom of real determinacy $\text{AD}_R$, by [Pri76, Kas83]. The same holds for many other Marczewski-style properties, as shown in [Löw98]. In a sense, it is more natural to use games with real moves when talking about more complicated regularity properties. So, one might at least wonder whether the following is true:

**Question 2.6.5.** Does $\text{AD}_R$ imply that all sets are $I$-regular?

Unfortunately we do not have a proof of this in general.