UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

Some remarks on very-well-poised 8ϕ7 series

Stokman, J.V.

DOI
10.3842/SIGMA.2012.039
Publication date
2012
Document Version
Final published version
Published in
Symmetry, Integrability and Geometry : Methods and Applications (SIGMA)

Link to publication

Citation for published version (APA):
Stokman, J. V. (2012). Some remarks on very-well-poised 8ϕ7 series. *Symmetry, Integrability and Geometry : Methods and Applications (SIGMA)*, *8*, Article 039. https://doi.org/10.3842/SIGMA.2012.039

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, P.O. Box 19185, 1000 GD Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:16 Sept 2026

Some Remarks on Very-Well-Poised ${}_8\phi_7$ Series

Jasper V. STOKMAN

Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands
E-mail: *j.v.stokman@uva.nl*
URL: http://staff.science.uva.nl/~jstokman/

Received April 05, 2012, in final form June 18, 2012; Published online June 27, 2012
http://dx.doi.org/10.3842/SIGMA.2012.039

Abstract. Nonpolynomial basic hypergeometric eigenfunctions of the Askey–Wilson second order difference operator are known to be expressible as very-well-poised ${}_8\phi_7$ series. In this paper we use this fact to derive various basic hypergeometric and theta function identities. We relate most of them to identities from the existing literature on basic hypergeometric series. This leads for example to a new derivation of a known quadratic transformation formula for very-well-poised ${}_8\phi_7$ series. We also provide a link to Chalykh's theory on (rank one, BC type) Baker–Akhiezer functions.

Key words: very-well-poised basic hypergeometric series; Askey–Wilson functions; quadratic transformation formulas; theta functions

2010 Mathematics Subject Classification: 33D15; 33D45

1 Introduction

In this paper we present derivations of various basic hypergeometric and theta function identities using the interpretation of very-well-poised ${}_8\phi_7$ series as eigenfunctions of the Askey–Wilson second order difference operator $\mathcal{D}$. For instance, we reobtain a nonstandard type three term transformation formula for very-well-poised ${}_8\phi_7$ series [7] as the connection formula for the asymptotically free eigenfunctions of $\mathcal{D}$, we investigate the eigenfunctions of $\mathcal{D}$ with trivial quantum monodromy and relate them to (rank one, BC type) Baker–Akhiezer functions from [4, 5], we rederive the quadratic transformation formula [6, (3.5.10)] for very-well-poised ${}_8\phi_7$ series, and we obtain various theta function identities by translating symmetries of the Askey–Wilson function and of the asymptotically free eigenfunction of $\mathcal{D}$ in terms of the associated normalized c-functions.

The most general family of orthogonal polynomials satisfying a second order q-difference equation is the family of Askey–Wilson polynomials [1]. For our purposes it is convenient to view the associated difference equations as eigenvalue equations for the second order Askey–Wilson second order difference operator $\mathcal{D}$ already mentioned in the previous paragraph. The operator $\mathcal{D}$ depends, besides on the difference step-size and the deformation parameter q, on four additional free parameters.

Nonpolynomial basic hypergeometric eigenfunctions of $\mathcal{D}$ have been subject of study in various papers (see, e.g., [8, 9, 10, 19, 20, 24, 26]). Important examples are the Askey–Wilson function $\mathcal{E}(\cdot,z)$ and the asymptotically free eigenfunction $\Phi(\cdot,z)$ (the corresponding eigenvalue depends in an explicit way on $q^z+q^{-z}\in\mathbb{C}$). They are defined provided that $0<|q|<1$. They are selfdual eigenfunctions (the role of the argument and z is interchangeable). They naturally arise in harmonic analysis on the quantum $\mathrm{SU}(1,1)$ group and in the study of the double affine Hecke algebra of type $C^\vee C_1$ (the rank one Koornwinder case), see, e.g., [11] and [24, 25] respectively. From this representation theoretic viewpoint $\mathcal{E}(\cdot,z)$ plays the role of the spherical function and $\Phi(\cdot,z)$ the role of the Harish-Chandra series.

Ruijsenaars' R-function [19] is another nonpolynomial selfdual eigenfunction of $\mathcal{D}$ which is required to satisfy yet another second order difference equation of Askey–Wilson type. The step-direction of the two Askey–Wilson second order difference equations are allowed to be co-linear (which corresponds to deformation parameter q being of modulus one). The R-function arises as matrix coefficient of representations of the quantum double of the quantized universal enveloping algebra of $\mathfrak{sl}_2$, see [2].

The Askey–Wilson function $\mathcal{E}(\cdot,z)$ can be explicitly expressed in terms of the asymptotically free eigenfunctions $\Phi(\cdot,\pm z)$. The elliptic function $c(\cdot,z)$ governing the expansion coefficients,

$$\mathcal{E}(\cdot,z)=c(\cdot,z)\Phi(\cdot,z)+c(\cdot,-z)\Phi(\cdot,-z),$$

is called the (normalized) c-function [10]. It is explicitly given as quotient of theta functions. The selfduality of $\mathcal{E}$ and Φ and the fact that $\mathcal{E}(-x,z)=\mathcal{E}(x,z)$ then allow us to express $\Phi(-x,z)$ in terms of $\Phi(x,\pm z)$ (connection formula). The cases when the connection coefficient formula trivializes is particularly interesting since it directly relates to the theory of Baker–Akhiezer functions [4, 5]. We discuss this in Section 3.

Suitable two parameter specializations of the Askey–Wilson polynomials yield the continuous q-Jacobi polynomials. They have appeared in two guises (see [17] and [1]) which are interrelated by a quadratic transformation formula going back to Singh [22], see also [1, § 4] and [6, § 3.10]. This quadratic transformation formula was derived in [1] using the orthogonality relations of the Askey–Wilson polynomials. Ruijsenaars [20] stressed that for these parameter specializations the Askey–Wilson second order difference operator $\mathcal{D}$ factorizes up to an additive constant as a square of an Askey–Wilson type second order difference operator with step-size half of the step-size of $\mathcal{D}$. We use this observation to prove a quadratic transformation formula for a two parameter family of the asymptotically free eigenfunction $\Phi(\cdot,z)$ of $\mathcal{D}$. This complements Ruijsenaars' results [20], where he lifted the quadratic transformation formula for continuous q-Jacobi polynomials to a quadratic transformation formula for a two parameter subfamily of the R-function.

Using the known explicit expression of $\Phi(\cdot,z)$ in terms of basic hypergeometric series we link the above mentioned results to various known identities for very-well-poised ${}_8\phi_7$ series. For example, the quadratic transformation formula for $\Phi(\cdot,z)$ becomes the known quadratic transformation formula [6, (3.5.10)] for very-well-poised ${}_8\phi_7$ series after applying suitable transformation formulas to both sides of the identity, see Remark 5.3(i) (this was observed by M. Rahman).

Combining symmetries of the Askey–Wilson function with its c-function expansion yields nontrivial identities for the c-function, hence nontrivial theta function identities. For instance, the fact that $\mathcal{E}(\cdot,z)$ is invariant under negating the argument yields a theta function identity which is a six parameter dependent subcase of a theta function identity [6, Exercise 5.22] due to Slater [23]. The fact that $\Phi(\cdot,\pm z)$ satisfies a quadratic transformation formula but the Askey–Wilson function $\mathcal{E}(\cdot,z)$ does not, yields a nontrivial identity for the c-function and consequently a quadratic type theta function identity (7.4).

The results in the present paper play an important role in the study of the spectral problem of the trigonometric Macdonald–Ruijsenaars–Cherednik commuting family of difference operators associated to root systems. They are for instance needed in the asymptotic analysis of q-analogues of Harish-Chandra series associated to root systems (cf. [13, 14, 15, 25]) along codimension one facets of the Weyl chamber. I will return to this topic in a future work.

2 Eigenfunctions of the Askey–Wilson second order difference operator

We use, besides a deformation parameter $0<q=e^{2\pi\sqrt{-1}\tau}<1$ ($\tau\in\sqrt{-1}\mathbb{R}_{>0}$) and a choice of step-size $s\in\mathbb{Q}_{>0}$, four free parameters $\{\kappa,\lambda,\upsilon,\varsigma\}$ which we call Hecke parameters (this

name comes from their interpretation as multiplicity parameters in the Cherednik–Macdonald theory of the double affine Hecke algebra of type $C^\vee C_1$, see [16]). We assume that the Hecke parameters κ, λ, υ and ς are real. We set $q^x := e^{2\pi\sqrt{-1}\tau x}$ for $x \in \mathbb{C}$. In the paper q will be fixed throughout. Step-sizes s and $\frac{s}{2}$ will simultaneously appear in formulas. One can restrict without loss of generality to considering step-sizes 2 and 1, but formulas are more transparent when arbitrary values s of the step size are taken into account because it makes the s-dependence of the Askey–Wilson parameters

$$\{a_s, b_s, c_s, d_s\} := \{q^{\kappa+\lambda}, -q^{\kappa-\lambda}, q^{\frac{s}{2}+\upsilon+\varsigma}, -q^{\frac{s}{2}+\upsilon-\varsigma}\}$$

explicit. We define dual Askey–Wilson parameters by

$$\{\widetilde{a}_s, \widetilde{b}_s, \widetilde{c}_s, \widetilde{d}_s\} := \{q^{\kappa+\upsilon}, -q^{\kappa-\upsilon}, q^{\frac{s}{2}+\lambda+\varsigma}, -q^{\frac{s}{2}+\lambda-\varsigma}\},$$

i.e. the roles of the Hecke parameters λ and υ are interchanged. The parameters a_s, b_s, $\widetilde{a}_s$ and $\widetilde{b}_s$ do not depend on s, we therefore occasionally omit the subindex s for these Askey–Wilson parameters. Interchanging λ and υ defines an involution on the set of Hecke parameters, hence also on the associated set of Askey–Wilson parameters. In addition we have

$$\widetilde{a}_s^2 = q^{-s}abcd, \qquad \widetilde{a}_s\widetilde{b}_s = a_sb_s, \qquad \frac{q^s\widetilde{a}_s}{\widetilde{b}_s} = c_sd_s,$$
$$\widetilde{a}_s\widetilde{c}_s = a_sc_s, \qquad \frac{q^s\widetilde{a}_s}{\widetilde{c}_s} = b_sd_s, \qquad \widetilde{a}_s\widetilde{d}_s = a_sd_s, \qquad \frac{q^s\widetilde{a}_s}{\widetilde{d}_s} = b_sc_s.$$

We now first recall the asymptotically free eigenfunction of the Askey–Wilson [1] second order q^s-difference operator, which we will regard here as a second order difference operator with step size s. Explicitly, the Askey–Wilson second order difference operator $\mathcal{D}$, acting on meromorphic functions on $\mathbb{C}$, is defined by

$$\big(\mathcal{D}f\big)(x) := A(x)\big(f(x+s) - f(x)\big) + A(-x)\big(f(x-s) - f(x)\big),$$
$$A(x) := \frac{(1-a_sq^x)(1-b_sq^x)(1-c_sq^x)(1-d_sq^x)}{\widetilde{a}_s(1-q^{2x})(1-q^{s+2x})}.$$

Sometimes it is important to write explicitly the dependence on the parameters, in which case we write $\mathcal{D}$ as $\mathcal{D}^{(s)}_{\kappa,\lambda,\upsilon,\varsigma;q}$.

Remark 2.1. The Askey–Wilson second order difference operator $\mathcal{D}$ can be interpreted as a second order q^s-difference operator when acting on τ^{-1}-translation invariant meromorphic functions on $\mathbb{C}$ (which are the meromorphic functions of the form $g(q^x)$ with g meromorphic on $\mathbb{C}^*$).

We now define the elementary function $W(x,z) = W(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ by

$$W(x,z) := q^{(\kappa+\lambda+x)(\kappa+\upsilon+z)/s}.$$

Note that $W(x+s,z) = q^{\kappa+\upsilon+z}W(x,z) = \widetilde{a}q^zW(x,z)$.

For generic b_j the ${}_{r+1}\phi_r$ basic hypergeometric series is defined by the convergent power series

$${}_{r+1}\phi_r\left(\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, z\right) := \sum_{j=0}^{\infty}\frac{\big(a_1, a_2, \ldots, a_{r+1}; q\big)_j}{\big(q, b_1, \ldots, b_r; q\big)_j}z^j, \qquad |z| < 1,$$

where $\big(a_1,\ldots,a_s;q\big)_j=\prod_{r=1}^s\prod_{i=0}^{j-1}(1-a_rq^i)$ for $j\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$ (empty products are equal to one by convention). The very-well-poised ${}_8\phi_7$ series is defined by

$$\begin{aligned}{}_8W_7\big(\alpha_0;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5;q,z\big)&={}_8\phi_7\left(\begin{matrix}\alpha_0,q\alpha_0^{\frac{1}{2}},-q\alpha_0^{\frac{1}{2}},\alpha_1,\ldots,\alpha_5\\ \alpha_0^{\frac{1}{2}},-\alpha_0^{-\frac{1}{2}},q\alpha_0/\alpha_1,\ldots,q\alpha_0/\alpha_5\end{matrix};q,z\right)\\&=\sum_{r=0}^{\infty}\frac{1-\alpha_0q^{2r}}{1-\alpha_0}z^r\prod_{j=0}^5\frac{\big(\alpha_j;q\big)_r}{\big(q\alpha_0/\alpha_j;q\big)_r}.\end{aligned}$$

In case $z=\alpha_0^2q^2/\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ it has a meromorphic continuation to $\big(\mathbb{C}^*\big)^6$ as function of $(\alpha_0,\alpha_1,\ldots,\alpha_5)$. This follows from the identity [6, (III.36)] expressing such very-well-poised ${}_8\phi_7$ series as a sum of two ${}_4\phi_3$ series. Define the holomorphic function $\mathrm{St}(x)=\mathrm{St}(x;\kappa,\lambda,\upsilon,\varsigma;q,s)$ ("St" is standing for singular term) in $x\in\mathbb{C}$ by

$$\mathrm{St}(x):=\big(q^{s+x}/a_s,q^{s+x}/b_s,q^{s+x}/c_s,q^{s+x}/d_s;q^s\big)_\infty.$$

We write $\mathrm{St}^d(z):=\mathrm{St}(z;\kappa,\upsilon,\lambda,\varsigma;q,s)$ for the singular term with respect to dual parameters,

$$\mathrm{St}^d(z)=\big(q^{s+z}/\widetilde{a}_s,q^{s+z}/\widetilde{b}_s,q^{s+z}/\widetilde{c}_s,q^{s+z}/\widetilde{d}_s;q^s\big)_\infty.$$

The following proposition combines and refines observations from [9, 10, 25].

Proposition 2.2. *There exist unique holomorphic functions Γ_r on $\mathbb{C}$ ($r\geq 0$) satisfying the following three conditions,*

(1) $\Gamma_0(z)=\big(q^{s+2z};q^s\big)_\infty$.

(2) *The power series $\Psi(x,z):=\sum_{r=0}^{\infty}\Gamma_r(z)q^{rx}$ is normally convergent on compacta of $(x,z)\in\mathbb{C}\times\mathbb{C}$ (consequently $\Psi(x,z)$ is a holomorphic function in $(x,z)\in\mathbb{C}\times\mathbb{C}$).*

(3) *The meromorphic function*

$$\Phi(x,z):=\frac{W(x,z)}{\mathrm{St}(x)\mathrm{St}^d(z)}\Psi(x,z)\tag{2.1}$$

satisfies

$$\big(\big(\mathcal{D}_x-q^z-q^{-z}+\widetilde{a}+\widetilde{a}^{-1}\big)\Phi\big)(x,z)=0,\tag{2.2}$$

where $\mathcal{D}_x$ stands for the Askey–Wilson second order difference operator $\mathcal{D}=\mathcal{D}^{(s)}_{\kappa,\lambda,\upsilon,\varsigma;q}$ acting on the x-variable.

Furthermore, Γ_r is τ^{-1}-translation invariant and

$$\begin{aligned}\Psi(x,z)=&\frac{\big(\frac{q^{s+x+z}a_s}{\widetilde{a}_s},\frac{q^{s+x+z}b_s}{\widetilde{a}_s},\frac{q^{s+x+z}c_s}{\widetilde{a}_s},\frac{q^{s+x+z}\widetilde{a}_s}{d_s},q^{s+2z},d_sq^x;q^s\big)_\infty}{\big(\frac{q^{2s+x+2z}}{d_s};q^s\big)_\infty}\\&\times{}_8W_7\left(\frac{q^{s+x+2z}}{d_s};\frac{q^{s+z}}{\widetilde{a}_s},\frac{q^{s+z}}{\widetilde{d}_s},\widetilde{b}_sq^z,\widetilde{c}_sq^z,\frac{q^{s+x}}{d_s};q^s,d_sq^x\right)\end{aligned}\tag{2.3}$$

if $|d_sq^x|<1$.

Proof. The explicit expression (2.3) is

$$\Psi(x,z)=\frac{\left(\frac{q^{s+x+z}a_s}{\widetilde{a}_s},\frac{q^{s+x+z}b_s}{\widetilde{a}_s},\frac{q^{s+x+z}c_s}{\widetilde{a}_s},\frac{q^{s+x+z}\widetilde{a}_s}{d_s},q^{s+2z},d_sq^x;q^s\right)_\infty}{\left(\frac{q^{s+x+2z}}{d_s};q^s\right)_\infty}$$
$$\times\sum_{r=0}^{\infty}\left(1-\frac{q^{s+2sr+x+2z}}{d_s}\right)\frac{\left(\frac{q^{s+z}}{\widetilde{a}_s},\frac{q^{s+z}}{\widetilde{d}_s},\widetilde{b}_sq^z,\widetilde{c}_sq^z,\frac{q^{s+x}}{d_s};q^s\right)_r d_s^rq^{rx}}{\left(\frac{q^{s+x+z}\widetilde{a}_s}{d_s},\frac{q^{s+x+z}a_s}{\widetilde{a}_s},\frac{q^{s+x+z}b_s}{\widetilde{a}_s},\frac{q^{s+x+z}c_s}{\widetilde{a}_s},q^{s+2z};q^s\right)_r}. \tag{2.4}$$

It is a well defined meromorphic function in $(x,z)\in\mathbb{C}\times\mathbb{C}$ provided that $|d_sq^x|<1$, with possible poles at $q^{s+sr+x+2z}=d_s$ ($r\in\mathbb{Z}_{\geq 0}$). It can be expressed as sum of two ${}_4\phi_3$ series using [6, (III.36)] with parameters (a,b,c,d,e,f,q) in [6, (III.36)] specialized to

$$(q^{s+x+2z}/d_s,q^{s+z}/\widetilde{a}_s,q^{s+z}/\widetilde{d}_s,\widetilde{b}_sq^z,\widetilde{c}_sq^z,q^{s+x}/d_s,q^s),$$

leading to the expression

$$\Psi(x,z)=\frac{\left(\frac{q^{s+x}}{a_s},d_sq^x,\frac{q^{s+z}}{\widetilde{b}_s},\frac{q^{s+z}}{\widetilde{c}_s},\frac{q^{s+x+z}a_s}{\widetilde{a}_s},\frac{q^{s+x+z}\widetilde{a}_s}{d_s};q^s\right)_\infty}{\left(\frac{d_s}{a_s};q^s\right)_\infty}$$
$$\times\,{}_4\phi_3\left(\begin{matrix}a_sq^x,\frac{q^{s+x}}{d_s},\widetilde{b}_sq^z,\widetilde{c}_sq^z\\ \frac{q^{s+x+z}a_s}{\widetilde{a}_s},\frac{q^{s+x+z}\widetilde{a}_s}{d_s},\frac{q^sa_s}{d_s}\end{matrix};q^s,q^s\right)$$
$$+\frac{\left(a_sq^x,\frac{q^{s+x}}{d_s},\widetilde{b}_sq^z,\widetilde{c}_sq^z,\frac{q^{s+x+z}\widetilde{a}_s}{a_s},\frac{q^{s+x+z}\widetilde{d}_s}{a_s};q^s\right)_\infty}{\left(\frac{a_s}{d_s};q^s\right)_\infty}$$
$$\times\,{}_4\phi_3\left(\begin{matrix}\frac{q^{s+x}}{a_s},d_sq^x,\frac{q^{s+z}}{\widetilde{b}_s},\frac{q^{s+z}}{\widetilde{c}_s}\\ \frac{q^{s+x+z}\widetilde{a}_s}{a_s},\frac{q^{s+x+z}\widetilde{d}_s}{a_s},\frac{q^sd_s}{a_s}\end{matrix};q^s,q^s\right). \tag{2.5}$$

This alternative expression provides the meromorphic continuation and shows that $\Psi(x,z)$ is holomorphic in $(x,z)\in\mathbb{C}\times\mathbb{C}$. The expression (2.4) of $\Psi(x,z)$ shows that $\Psi(x,z)$ satisfies (1) and (2). By [9] (see also [10] for notations close to the present one), the resulting meromorphic function $\Phi(x,z)$ (see (2.1)) indeed satisfies the difference equation (2.2).

It remains to prove uniqueness. If a series of the form

$$\Phi(x,z)=\frac{W(x,z)}{\mathrm{St}(x)\mathrm{St}^d(z)}\sum_{r\geq 0}\Gamma_r(z)q^{rx}$$

is a formal solution of (2.2) then the $\Gamma_r(z)$ ($r\geq 0$) satisfy recursion relations of the form

$$\widetilde{a}(q^{sr}-1)(q^z-q^{-sr-z})\Gamma_r(z)=\sum_{t=0}^{r-1}v_t^r(q^z)\Gamma_t(z),\qquad r\geq 1$$

for some Laurent polynomials v_t^r. This shows that the $\Gamma_r(z)$ ($r\geq 1$) are uniquely determined by $\Gamma_0(z)$. ■

If confusion may arise about the parameter dependencies then we write the asymptotically free solution $\Phi(x,z)$ as $\Phi(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ and $\Psi(x,z)$ as $\Psi(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$.

Remark 2.3. The characterization of $\Phi(x,z)$ as eigenfunction of $\mathcal{D}$ is equivalent to a characterization of $\Psi(x,z)=\sum_{r=0}^{\infty}\Gamma_r(z)q^{rx}$ as eigenfunction of the second order difference operator obtained from $\mathcal{D}$ by gauging it with gauge factor $W(\cdot,z)/\mathrm{St}(\cdot)$. The gauged difference operator and the

relevant eigenvalue are symmetric under arbitrary permutations of the Askey–Wilson parameters a_s, b_s, c_s, d_s. Since the normalization $\Gamma_0(z)=\big(q^{s+2z};q^s\big)_\infty$ of $\Psi(x,z)$ is independent of the Askey–Wilson parameters, it follows that $\Psi(x,z)$ is symmetric under arbitrary permutations of the Askey–Wilson parameters a_s, b_s, c_s, d_s.

Let $\theta(u;q):=\big(u,q/u;q\big)_\infty$ be the modified Jacobi theta function and write

$$\theta(u_1,\dots,u_r;q)=\prod_{i=1}^r\theta(u_i;q)$$

for products of theta functions. Define the (normalized) c-function $c(x,z)=c(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ by

$$c(x,z):=\frac{\theta\big(\widetilde{a}_sq^{-z},\widetilde{b}_sq^{-z},\widetilde{c}_sq^{-z},\frac{d_sq^{x-z}}{\widetilde{a}_s};q^s\big)}{W(x,z)\theta\big(q^{-2z},d_sq^x;q^s\big)}. \tag{2.6}$$

Using $\theta(qu;q)=-u^{-1}\theta(u;q)$ it follows that $c(x+s,z)=c(x,z)$ and $c(x,z+s)=c(x,z)$. We write $c^d(x,z)=c(x,z;\kappa,\upsilon,\lambda,\varsigma;q,s)$ for the c-function with respect to dual parameters.

The Askey–Wilson function $\mathcal{E}(x,z)=\mathcal{E}(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ is defined by

$$\begin{aligned}\mathcal{E}(x,z):=&\frac{\big(\frac{\widetilde{a}_sq^{s+z-x}}{d_s},\frac{\widetilde{a}_sq^{s+z+x}}{d_s},a_sb_s,a_sc_s,\frac{q^sa_s}{d_s};q^s\big)_\infty}{\big(\frac{q^{s+x}}{d_s},\frac{q^{s-x}}{d_s},\frac{q^{s+z}}{\widetilde{d}_s},\widetilde{a}_s\widetilde{b}_s\widetilde{c}_sq^z;q^s\big)_\infty}\\
&\times{}_8W_7\left(\widetilde{a}_s\widetilde{b}_s\widetilde{c}_sq^{-s+z};a_sq^x,a_sq^{-x},\widetilde{a}_sq^z,\widetilde{b}_sq^z,\widetilde{c}_sq^z;q^s,\frac{q^{s-z}}{\widetilde{d}_s}\right)\end{aligned} \tag{2.7}$$

for $|q^{s-z}/\widetilde{d}_s|<1$ (see [10]). Using [6, (III.36)] to express $\mathcal{E}(x,z)$ as sum of two ${}_4\phi_3$ series it follows that $\mathcal{E}(\cdot,\cdot)$ has a meromorphic extension to $\mathbb{C}\times\mathbb{C}$. We write $\mathcal{E}^d$ and Φ^d for the meromorphic functions $\mathcal{E}$ and Φ with respect to dual parameters. The following properties of $\mathcal{E}$ and Φ are known from [10, 25] (cf. also [8, 9, 26]).

Proposition 2.4.

(*i*) $\mathcal{E}(x,z)=\mathcal{E}^d(z,x)$ (*selfduality*).

(*ii*) $\mathcal{E}(-x,z)=\mathcal{E}(x,z)$ *and* $\mathcal{E}(x,-z)=\mathcal{E}(x,z)$.

(*iii*) $\Phi(x,z)=\Phi^d(z,x)$ (*selfduality*).

(*iv*) $\mathcal{E}(x,z)=c(x,z)\Phi(x,z)+c(x,-z)\Phi(x,-z)$ (*c-function expansion*).

Proof. (*i*) This follows from the transformation formula [6, (III.23)] for very-well-poised ${}_8\phi_7$ series.

(*ii*) By the explicit expression (2.7) it is clear that $\mathcal{E}(-x,z)=\mathcal{E}(x,z)$. By (*i*) it then also follows that $\mathcal{E}(x,-z)=\mathcal{E}(x,z)$.

(*iii*) This follows again by application of the transformation formula [6, (III.23)] for very-well-poised ${}_8\phi_7$ series.

(*iv*) Use Bailey's three term transformation formula [6, (III.37)] for very-well-poised ${}_8\phi_7$ series. ■

Remark 2.5. The selfduality of $\Phi(x,z)$ and $\mathcal{E}(x,z)$ ensures that $\Phi(x,\cdot)$ and $\mathcal{E}(x,\cdot)$ are eigenfunctions of the Askey–Wilson second order difference operator with respect to dual parameters.

Note that the Askey–Wilson second order difference operator $\mathcal{D}$ is invariant under $x\mapsto -x$, hence $\Phi(-x,z)$ is again an eigenfunction of $\mathcal{D}_x$ with eigenvalue $q^z+q^{-z}+\widetilde{a}+\widetilde{a}^{-1}$. It can be expressed in terms of the eigenfunctions $\Phi(x,z)$ and $\Phi(x,-z)$ as follows.

Corollary 2.6.

$$\Phi(-x,z)=\left(\frac{c(x,z)-c^d(z,x)}{c^d(z,-x)}\right)\Phi(x,z)+\frac{c(x,-z)}{c^d(z,-x)}\Phi(x,-z) \tag{2.8}$$

(*connection formula*).

Proof. By the previous proposition we have

$$\begin{aligned}\mathcal{E}(x,z)&=\mathcal{E}^d(z,-x)=c^d(z,-x)\Phi^d(z,-x)+c^d(z,x)\Phi^d(z,x)\\&=c^d(z,x)\Phi(x,z)+c^d(z,-x)\Phi(-x,z).\end{aligned}$$

Compared with the c-function expansion for $\mathcal{E}(x,z)$ (see Proposition 2.4(iv)) we get the desired result. ■

Remark 2.7. The connection formula (2.8) is not a direct consequence of Bailey's three term transformation formula [6, (III.37)] for very-well-poised ${}_8\phi_7$ series. This is reflected by the fact that the coefficient of $\Phi(x,z)$ in (2.8) does not admit an explicit expression as a single product of theta functions. The connection formula (2.8) is though directly related to the three term transformation formula [7, (5.8)].

Corollary 2.8. *The c-function satisfies*

$$\begin{aligned}&c(x,z)c^d(z,-x)c^d(-z,-x)+c(-x,z)c^d(z,x)c^d(-z,-x)\\&\quad=c(x,z)c(-x,z)c^d(-z,-x)+c(x,z)c(-x,-z)c^d(z,-x).\end{aligned} \tag{2.9}$$

Proof. Since $\mathcal{E}(-x,z)=\mathcal{E}(x,z)$ we have

$$c(x,z)\Phi(x,z)+c(x,-z)\Phi(x,-z)=c(-x,z)\Phi(-x,z)+c(-x,-z)\Phi(-x,-z).$$

Applying (2.8) twice to the right hand side of this formula implies

$$\alpha(x,z)\Phi(x,z)=-\alpha(x,-z)\Phi(x,-z) \tag{2.10}$$

with the function $\alpha(x,z)$ given by

$$\alpha(x,z)=\frac{c(-x,z)c(x,z)-c(-x,z)c^d(z,x)}{c^d(z,-x)}+\frac{c(-x,-z)c(x,z)}{c^d(-z,-x)}-c(x,z).$$

Replace in (2.10) the variable x by $x+ms$ and consider the asymptotic behaviour as $m\to\infty$ of both sides, using the fact that $c(x,z)$ is s-translation invariant in both x and z and using that

$$\widetilde{a}^{-m}q^{-mz}\Phi(x+ms,z)=\Gamma_0(z)\big(1+\mathcal{O}(q^{sm})\big)$$

as $m\to\infty$. It gives $\alpha(x,z)=0$. This is equivalent to (2.9). ■

Remark 2.9. Substituting in (2.9) the explicit expression (2.6) of the c-function $c(x,z)$ gives the theta function identity

$$\begin{aligned}&\theta\left(\frac{d}{\widetilde{a}}q^{x+z},\frac{d}{\widetilde{a}}q^{x-z},aq^x,bq^x,cq^x,dq^{-x},q^{2z};q\right)-\theta\left(\frac{d}{\widetilde{a}}q^{-x+z},\frac{d}{\widetilde{a}}q^{x+z},\widetilde{a}q^z,\widetilde{b}q^z,\widetilde{c}q^z,\widetilde{d}q^{-z},q^{2x};q\right)\\&\quad=q^{2x}\theta\left(\frac{d}{\widetilde{a}}q^{-x-z},\frac{d}{\widetilde{a}}q^{-x+z},aq^{-x},bq^{-x},cq^{-x},dq^x,q^{2z};q\right)\\&\qquad-q^{2z}\theta\left(\frac{d}{\widetilde{a}}q^{-x-z},\frac{d}{\widetilde{a}}q^{x-z},\widetilde{a}q^{-z},\widetilde{b}q^{-z},\widetilde{c}q^{-z},\widetilde{d}q^z,q^{2x};q\right),\end{aligned}$$

where we have taken $s=1$ and have written $a=a_1,\ldots,d=d_1$ (and similarly for the dual parameters). This is a special case of Slater's theta function identity [6, Exercise 5.22], with the parameters (a,b,c,d,e,f,g,h) in [6, Exercise 5.22] specialized to

$$\Big(1,\frac{d}{\widetilde{a}}q^{-x+z},\widetilde{a}q^{-z},\widetilde{b}q^{-z},\widetilde{c}q^{-z},\frac{q^{1-z}}{\widetilde{d}},\frac{d}{\widetilde{a}}q^{x+z},q^{2z}\Big).$$

Note that Slater's formula [6, Exercise 5.22] is more general since it has, besides q, seven free parameters, while the formula (2.9) only has six.

3 The case of trivial quantum monodromy

Determining the monodromy representation for Gauss' second order hypergeometric differential equation is equivalent to deriving its connection coefficient formulas. The connection coefficient formulas explicitly relate the fundamental series expansion solutions around the three regular singularities of the hypergeometric differential equation. These formulas turn out to be directly related to well known three term transformation formulas for the Gauss' hypergeometric function ${}_2F_1$. From the above notion of monodromy only its incarnation in terms of connection coefficient formulas generalizes to the difference setup (see Sauloy [21] and references therein for a detailed discussion of this issue). We therefore say that the explicit connection coefficient formula (2.8) solves the quantum monodromy problem of $\mathcal{D}$. In addition we say that the quantum monodromy is trivial if the coefficient of $\Phi(x,z)$ in (2.8) vanishes. The latter terminology is motivated as follows. Firstly, for trivial quantum monodromy the connection coefficient formula (2.8) reduces to a simple equivariance property of the following renormalization $\widetilde{\Phi}(x,z)=\widetilde{\Phi}(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ of the asymptotically free eigenfunction $\Phi(x,z)$,

$$\widetilde{\Phi}(x,z):=c(x,z)\Phi(x,z),$$

see Proposition 3.1(iii) below. Secondly, for an important subclass with trivial quantum monodromy, a suitable renormalization of the asymptotically free eigenfunction $\Phi(x,z)$ has a terminating series expansion, see Remark 3.2(i) for further discussions on this issue.

Note that the renormalization $\widetilde{\Phi}(x,z)$ of $\Phi(x,z)$ still satisfies the eigenvalue equation

$$\mathcal{D}\big(\widetilde{\Phi}(\cdot,z)\big)=\big(q^z+q^{-z}-\widetilde{a}-\widetilde{a}^{-1}\big)\widetilde{\Phi}(\cdot,z)$$

since $c(x,z)$ is s-translation invariant in x. Write $\widetilde{\Phi}^d(z,x):=\widetilde{\Phi}(z,x;\kappa,\upsilon,\lambda,\varsigma;q,s)$.

Proposition 3.1.

(i) *If* $\frac{\kappa}{s},\frac{\lambda}{s},\frac{\upsilon}{s},\frac{\varsigma}{s}\in\frac{1}{2}\mathbb{Z}$ *with an even number of them being integers, then* $c(x,z)=c^d(z,x)$.

In the remaining items of the proposition we assume that $(\kappa,\lambda,\upsilon,\varsigma)$ *is a four-tuple of real parameters such that* $c(x,z)=c^d(z,x)$.

(ii) $\widetilde{\Phi}(x,z)=\widetilde{\Phi}^d(z,x)$ (*selfduality*).

(iii) $\widetilde{\Phi}(-x,z)=\widetilde{\Phi}(x,-z)$.

(iv) $\mathcal{E}(x,z)=\widetilde{\Phi}(x,z)+\widetilde{\Phi}(-x,z)$.

Proof. (i) Consider the quotient $\alpha:=c(x,z)/c^d(z,x)$ (we suppress the dependence on the variables and parameters). Using the explicit expression of the normalized c-function (2.6) in terms of theta functions it follows that $\alpha=1$ if $(\frac{\kappa}{s},\frac{\lambda}{s},\frac{\upsilon}{s},\frac{\varsigma}{s})$ is taken from the set

$$\Big\{(0,0,0,0),\Big(\frac{1}{2},\frac{1}{2},0,0\Big),\Big(\frac{1}{2},0,\frac{1}{2},0\Big),\Big(\frac{1}{2},0,0,\frac{1}{2}\Big),$$
$$\Big(0,\frac{1}{2},\frac{1}{2},0\Big),\Big(0,\frac{1}{2},0,\frac{1}{2}\Big),\Big(0,0,\frac{1}{2},\frac{1}{2}\Big),\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)\Big\}.$$

In addition, it is easy to check that α is invariant under integral shifts of the rescaled Hecke parameters $\frac{\kappa}{s}$, $\frac{\lambda}{s}$, $\frac{\upsilon}{s}$, $\frac{\varsigma}{s}$. This gives the result.

(ii) This follows from Proposition 2.4(iii).

(iii) This follows from the connection formula (2.8) for $\Phi(x,z)$.

(iv) This is immediate from the c-function expansion of the Askey–Wilson function, see Proposition 2.4(iv). ■

Remark 3.2.

(i) Due to [13, § 4] the function $\widetilde{\Phi}(x,z)$ for Hecke parameters $(\kappa,\lambda,\upsilon,\varsigma)$ satisfying, besides the integrality conditions from Proposition 3.1(i), suitable additional (positivity) conditions, is up to normalization Chalykh's BC_1 type normalized Baker–Akhiezer function [4, § 6]. An important property of the Baker–Akhiezer function is the fact that it has a terminating series expansion. Surprisingly this is not evident from the explicit expressions of $\widetilde{\Phi}(x,z)$ in terms of basic hypergeometric series. The properties from Proposition 3.1 can be easily matched with the properties (see [4, 5]) of the Baker–Akhiezer functions. For instance, the Askey–Wilson function $\mathcal{E}(x,z)$ relates to the symmetrized normalized Baker–Akhiezer function Φ_+ from [5, (3.20)]. In particular, formula [5, Theorem 3.9] relating the symmetrized Baker–Akhiezer function to Askey–Wilson polynomials matches with the known fact that the Askey–Wilson function $\mathcal{E}(x,z)$ reduces to the normalized Askey–Wilson polynomial for suitable discrete values of z, see (6.2) below.

(ii) The relation between the A_1 type Baker–Akhiezer function (see [4, § 4.1]) and Heine's basic hypergeometric series ${}_2\phi_1$ was stressed in an informal note of Koornwinder [12].

4 The factorization of the Askey–Wilson second order difference operator

Ruijsenaars [20] analyzed when the square $\bigl(\mathcal{D}+\widetilde{a}+\widetilde{a}^{-1}\bigr)^2$ is an Askey–Wilson second order difference operator again (with doubled step-size). An important special case turns out to be when the Hecke parameters are of the form $(\kappa,\lambda,0,0)$. In our notations the resulting formula [20, (3.11)] reads as

$$\begin{aligned}&\bigl(\mathcal{D}^{(s/2)}_{\kappa,\lambda,0,0;q}-q^{\frac{z}{2}}-q^{-\frac{z}{2}}+q^{\kappa}+q^{-\kappa}\bigr)\bigl(\mathcal{D}^{(s/2)}_{\kappa,\lambda,0,0;q}+q^{\frac{z}{2}}+q^{-\frac{z}{2}}+q^{\kappa}+q^{-\kappa}\bigr)\\&\quad=\mathcal{D}^{(s)}_{\kappa,\lambda,\kappa,\lambda;q}-q^{z}-q^{-z}+q^{2\kappa}+q^{-2\kappa}.\end{aligned}\tag{4.1}$$

Remark 4.1. Our operator $\mathcal{D}+\widetilde{a}+\widetilde{a}^{-1}=\mathcal{D}+q^{2\kappa}+q^{-2\kappa}$ for arbitrary Hecke parameters $(\kappa,\lambda,\upsilon,\varsigma)$ is essentially the second order difference operator $A_\delta(\mathbf{c};\cdot)$ in Ruijsenaars' paper [20, (2.1)] with the parameters (c_0,c_1,c_2,c_3) in [20] related to our Hecke parameters by $\kappa+\lambda=-\sqrt{-1}c_0$, $\kappa-\lambda=-\sqrt{-1}c_1$, $\upsilon+\varsigma=-\sqrt{-1}c_2$ and $\upsilon-\varsigma=-\sqrt{-1}c_3$.

A simple derivation of (4.1) is as follows. Consider the second order difference operator $\mathcal{L}=\mathcal{L}^{(s/2)}_{\kappa,\lambda;q}$ defined by

$$(\mathcal{L}f)(x):=\frac{(1-q^{\kappa+\lambda+x})(1+q^{\kappa-\lambda+x})}{q^{\kappa}(1-q^{2x})}f\left(x+\frac{s}{2}\right)+\frac{(1-q^{\kappa+\lambda-x})(1+q^{\kappa-\lambda-x})}{q^{\kappa}(1-q^{-2x})}f\left(x-\frac{s}{2}\right).$$

Then

$$\begin{aligned}\big((\mathcal{L}^{(s/2)}_{\kappa,\lambda,q}-q^{\kappa}-q^{-\kappa})f\big)(x)&=\frac{(1-q^{\kappa+\lambda+x})(1+q^{\kappa-\lambda+x})}{q^{\kappa}(1-q^{2x})}\Big(f\Big(x+\frac{s}{2}\Big)-f(x)\Big)\\&\quad+\frac{(1-q^{\kappa+\lambda-x})(1+q^{\kappa-\lambda-x})}{q^{\kappa}(1-q^{-2x})}\Big(f\Big(x-\frac{s}{2}\Big)-f(x)\Big)\\&=\big(\mathcal{D}^{(s/2)}_{\kappa,\lambda,0,0;q}f\big)(x).\end{aligned}$$

Formula (4.1) thus is equivalent to

$$\big(\mathcal{L}^{(s/2)}_{\kappa,\lambda;q}-q^{\frac{z}{2}}-q^{-\frac{z}{2}}\big)\big(\mathcal{L}^{(s/2)}_{\kappa,\lambda;q}+q^{\frac{z}{2}}+q^{-\frac{z}{2}}\big)=\mathcal{D}^{(s)}_{\kappa,\lambda,\kappa,\lambda;q}-q^{z}-q^{-z}+q^{2\kappa}+q^{-2\kappa},\tag{4.2}$$

which is an easy check (use that the Askey–Wilson parameters associated to Hecke parameters $(\kappa,\lambda,\kappa,\lambda)$ satisfy $c_s=q^{\frac{s}{2}}a_s=q^{\frac{s}{2}+\kappa+\lambda}$ and $d_s=q^{\frac{s}{2}}b_s=-q^{\frac{s}{2}+\kappa-\lambda}$).

The eigenfunctions of $\mathcal{D}_x$ of the form $p_n(q^x+q^{-x})$ ($n\in\mathbb{Z}_{\geq 0}$) with p_n a polynomial of degree n, are the well known Askey–Wilson polynomials [1] (see also Section 6). In the special case that the associated four Hecke parameters are taken to be $(\kappa,\lambda,\kappa,\lambda)$ (corresponding to the condition that the Askey–Wilson parameters satisfy $c_s=q^{\frac{s}{2}}a_s$ and $d_s=q^{\frac{s}{2}}b_s$) the Askey–Wilson polynomials are called the continuous q-Jacobi polynomials by Askey and Wilson [1, § 4]. In [1, (4.22)] a quadratic transformation formula for balanced ${}_4\phi_3$ series, going back to Singh [22], is used to relate the above continuous q-Jacobi polynomials to Rahman's definition [17] of the continuous q-Jacobi polynomials (see also Section 6).

Ruijsenaars [20] used the factorization (4.1) to motivate and analyze quadratic transformation formulas for the hyperbolic nonpolynomial generalization of the continuous q-Jacobi polynomial, which is Ruijsenaars' R-function with the continuous q-Jacobi specialization $(\kappa,\lambda,\kappa,\lambda)$ of the associated Hecke algebra parameters (see Remark 4.1 for the relation with Ruijsenaars' notations [20]). In the following section we use the factorization (4.1) to prove quadratic transformation formulas for the asymptotically free eigenfunction $\Phi(\cdot,z;\kappa,\lambda,\kappa,\lambda;q,s)$.

5 Quadratic transformation formulas

For a function $\Xi(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ we write

$$\Xi_R(x,z):=\Xi\Big(x,\frac{z}{2};\kappa,\lambda,0,0;q,\frac{s}{2}\Big).$$

This is the parameter specialization which reduces the Askey–Wilson polynomials to Rahman's version of the continuous q-Jacobi polynomials [17] (in base $q^{\frac{s}{2}}$). Furthermore we set $\Xi^d_R(x,z):=\Xi(x,\frac{z}{2};\kappa,0,\lambda,0;q,\frac{s}{2})$ for its dual version. Since

$$\mathcal{L}^{(s/2)}_{\kappa,\lambda,q}=\mathcal{D}^{(s/2)}_{\kappa,\lambda,0,0;q}+q^{\kappa}+q^{-\kappa},$$

we know from the previous section that the meromorphic function $\Phi_R(x,z)$ satisfies

$$\big(\mathcal{L}_x-q^{\frac{z}{2}}-q^{-\frac{z}{2}}\big)\Phi_R(\cdot,z)=0.\tag{5.1}$$

In fact, $\Phi_R(x,z)$ is the unique solution to (5.1) which is of the form

$$\Phi_R(x,z)=\frac{W_R(x,z)}{\mathrm{St}_R(x)\mathrm{St}^d_R(z)}\sum_{r=0}^{\infty}\Gamma_{R,r}(z)q^{rx}$$

with $\Gamma_{R,r}(z)$ holomorphic in $z\in\mathbb{C}$, with $\Gamma_{R,0}(z)=\big(q^{\frac{s}{2}+z};q^{\frac{s}{2}}\big)_{\infty}$ and with the power series $\Psi_R(x,z):=\sum_{r=0}^{\infty}\Gamma_{R,r}(z)q^{rx}$ converging normally on compacta of $(x,z)\in\mathbb{C}\times\mathbb{C}$.

For a function $\Xi(x,z;\kappa,\lambda,\upsilon,\varsigma;q,s)$ we write

$$\Xi_J(x,z) := \Xi(x,z;\kappa,\lambda,\kappa,\lambda;q,s).$$

This is the parameter specialization which reduces the Askey–Wilson polynomials to Askey's and Wilson's version of the continuous q-Jacobi polynomials [1, § 4]. In addition we write $\Xi_J^d(x,z) := \Xi(x,z;\kappa,\kappa,\lambda,\lambda;q,s)$ for its dual version.

Theorem 5.1. $\Phi_J(x,z) = \Phi_R(x,z)$.

Proof. First of all note that the Askey–Wilson parameters associated to the Hecke parameters $(\kappa,\lambda,0,0)$, deformation parameter q and step-size $\frac{s}{2}$ are given by

$$\big(q^{\kappa+\lambda}, -q^{\kappa-\lambda}, q^{\frac{s}{4}}, -q^{\frac{s}{4}}\big).$$

The expression of $\Psi(x,z)$ as a sum of two ${}_4\phi_3$'s, see (2.5), shows that

$$\big(-q^{\frac{s}{4}+x};q^{\frac{s}{2}}\big)_\infty^{-1}\Psi_R(x,z)$$

is holomorphic in $(x,z)\in\mathbb{C}\times\mathbb{C}$. By Remark 2.3 also

$$\big(q^{\frac{s}{4}+x}, -q^{\frac{s}{4}+x};q^{\frac{s}{2}}\big)_\infty^{-1}\Psi_R(x,z) = \big(q^{\frac{s}{2}+2x};q^s\big)_\infty^{-1}\Psi_R(x,z)$$

is holomorphic in $(x,z)\in\mathbb{C}\times\mathbb{C}$. We write its power series expansion in q^x as

$$\big(q^{\frac{s}{2}+2x};q^s\big)_\infty^{-1}\Psi_R(x,z) = \sum_{r=0}^\infty \widetilde{H}_r(z)q^{rx}.$$

The $\widetilde{H}_r(z)$ are holomorphic in $z\in\mathbb{C}$, the series converges normally on compacta of $(x,z)\in\mathbb{C}\times\mathbb{C}$, and $\widetilde{H}_0(z) = \big(q^{\frac{s}{2}+z};q^{\frac{s}{2}}\big)_\infty$. Define new holomorphic functions by

$$H_r(z) := \big(-q^{\frac{s}{2}+z};q^{\frac{s}{2}}\big)_\infty\widetilde{H}_r(z)$$

for $r\in\mathbb{Z}_{\geq 0}$. Then $H_0(z) = \big(q^{s+2z};q^s\big)_\infty$ and

$$\Phi_R(x,z) = \frac{W_R(x,z)\big(q^{\frac{s}{2}+2x};q^s\big)_\infty}{\mathrm{St}_R(x)\mathrm{St}_R^d(z)\big(-q^{\frac{s}{2}+z};q^{\frac{s}{2}}\big)_\infty}\sum_{r=0}^\infty H_r(z)q^{rx}$$

with the series converging normally on compacta of $(x,z)\in\mathbb{C}\times\mathbb{C}$. A direct computation now shows that

$$\Phi_R(x,z) = \frac{W_J(x,z)}{\mathrm{St}_J(x)\mathrm{St}_J^d(z)}\sum_{r=0}^\infty H_r(z)q^{rx}.$$

By (5.1) and (4.2) we furthermore have

$$\big(\mathcal{D}^{(s)}_{\kappa,\lambda,\kappa,\lambda;q} - q^z - q^{-z} + q^{2\kappa} + q^{-2\kappa}\big)\Phi_R(\cdot,z) = 0.$$

Hence $\Phi_R(x,z)$ satisfies the characterizing properties of $\Phi_J(x,z)$. ■

By the proof of the theorem we in particular have

$$\Psi_R(x,z) = \frac{\big(q^{\frac{s}{2}+2x};q^s\big)_\infty}{\big(-q^{\frac{s}{2}+z};q^{\frac{s}{2}}\big)_\infty}\Psi_J(x,z).$$

Substituting the explicit expressions of Ψ_R and Ψ_J in this formula gives the following quadratic transformation formula for very-well-poised ${}_8\phi_7$ series.

Corollary 5.2.

$$\frac{\left(q\beta xz,-\frac{qxz}{\beta},\frac{q^{\frac{3}{2}}xz}{\alpha},-q^{\frac{1}{2}}\alpha xz;q\right)_\infty}{\left(q^{\frac{1}{2}}x,-q^{\frac{3}{2}}xz^2,\frac{q^2\beta xz^2}{\alpha};q\right)_\infty}\,{}_8W_7\left(-q^{\frac{1}{2}}xz^2;\frac{qz}{\alpha},-\frac{q^{\frac{1}{2}}z}{\beta},-\alpha z,q^{\frac{1}{2}}\beta z,-q^{\frac{1}{2}}x;q,-q^{\frac{1}{2}}x\right)$$
$$=\frac{\left(-\frac{q^2xz^2}{\alpha\beta},-q\alpha\beta xz^2,-\frac{q\alpha x}{\beta};q^2\right)_\infty}{\left(-\frac{q^3\beta xz^4}{\alpha};q^2\right)_\infty}$$
$$\times\,{}_8W_7\left(-\frac{q\beta xz^4}{\alpha};\frac{q^2z^2}{\alpha^2},-qz^2,-z^2,q\beta^2z^2,-\frac{q\beta x}{\alpha};q^2,-\frac{q\alpha x}{\beta}\right) \tag{5.2}$$

if both $|q^{\frac{1}{2}}x|<1$ *and* $|\frac{q\alpha x}{\beta}|<1$.

Remark 5.3.

(*i*) As observed by Mizan Rahman (private communication), the quadratic transformation formula (5.2) for ${}_8W_7$ is equivalent to the quadratic transformation formula [6, (3.5.10)] by applying to the left hand side of (5.2) the transformation formula [6, (III.23)] with parameters (a,b,c,d,e,f) in [6, (III.23)] specialized to

$$\left(-q^{\frac{1}{2}}xz^2,-\frac{q^{\frac{1}{2}}z}{\beta},-\alpha z,-q^{\frac{1}{2}}x,\frac{qz}{\alpha},q^{\frac{1}{2}}\beta z\right)$$

and to the right hand side of (5.2) the transformation formula [6, (III.23)] with parameters (a,b,c,d,e,f,q) in [6, (III.23)] specialized to

$$\left(-\frac{q\beta xz^4}{\alpha},-qz^2,-z^2,q\beta^2z^2,\frac{q^2z^2}{\alpha^2},-\frac{q\beta x}{\alpha},q^2\right).$$

(*ii*) The formal classical limit $q\uparrow 1$ of (5.2) can be computed after replacing the parameters (x,z,α,β) in (5.2) by $(-x,q^z,q^\alpha,q^\beta)$ and moving all the infinite q-shifted factorials in (5.2) to one side. The resulting classical limit turns out to be trivial, since both sides reduce to

$${}_2F_1\left(\begin{matrix}1+2z-2\alpha,\frac{1}{2}+2z+2\beta\\ 1+4z\end{matrix};\frac{4x}{(1+x)^2}\right),$$

where ${}_2F_1$ is Gauss' hypergeometric series (in contrast to the classical limits of the quadratic transformations of very-well-poised ${}_8\phi_7$'s from [18, § 5], which reduce to nontrivial quadratic transformations for ${}_2F_1$). On the polynomial level (5.2) reduces to the quadratic transformation formula for the continuous q-Jacobi polynomials, see Section 6, which is known to reduce to nontrivial quadratic transformations (see [1, (4.24)]) on the classical level when taking the limit $q\uparrow 1$ after replacing the parameters (x,z,α,β) by $(-q^x,q^z,q^\alpha,q^\beta)$.

There is a dual version of (5.2), which can be obtained by using the selfduality of $\Phi(x,z)$ to both sides of the quadratic transformation formula $\Phi_J(x,z)=\Phi_R(x,z)$ before substituting the explicit expression as a ${}_8W_7$ series. It leads to the following quadratic transformation formula.

Corollary 5.4.

$$\frac{\left(-q^{\frac12}\alpha xz,-q^{\frac12}\beta z,\frac{q^{\frac32}xz}{\alpha};q\right)_\infty}{\left(-\frac{q^{\frac32}x^2z}{\beta};q\right)_\infty}\,{}_8W_7\left(-\frac{q^{\frac12}x^2z}{\beta};\frac{qx}{\alpha\beta},-q^{\frac12}x,-\frac{\alpha x}{\beta},q^{\frac12}x,-\frac{q^{\frac12}z}{\beta};q,-q^{\frac12}\beta z\right)$$
$$=\frac{\left(-q\alpha\beta xz^2,\frac{q^2\alpha xz^2}{\beta},-\frac{q^2xz^2}{\alpha\beta},\frac{q^3\beta xz^2}{\alpha};q^2\right)_\infty}{\left(-q^3x^2z^2,-q^2z^2,\frac{q^2x^2z^2}{\beta^2};q^2\right)_\infty}$$
$$\times\,{}_8W_7\left(-qx^2z^2;\frac{q^2x}{\alpha\beta},-\frac{q\beta x}{\alpha},-\frac{\alpha x}{\beta},q\alpha\beta x,-\frac{q\beta z^2}{\alpha};q^2,-qx\right). \tag{5.3}$$

Also (5.3) can be related to [6, (3.5.10)] by applying the transformation formula [6, (III.23)] to both sides, cf. Remark 5.3(i).

6 Polynomial reduction

Both the asymptotically free eigenfunctions $\Phi(x,z)$ of the Askey–Wilson second order difference operator, and the Askey–Wilson function $\mathcal{E}(x,z)$, reduce to the Askey–Wilson polynomials when z is specialized appropriately (see, e.g., [10, 25]). Concretely, for $\Phi(x,z)$ we have for $n\in\mathbb{Z}_{\geq 0}$,

$$\Phi(x,-\kappa-\upsilon-ns)=\frac{a_s^{-2n}\left(a_sb_s,a_sc_s,a_sd_s;q^s\right)_n\left(\frac{q^{2(1-n)s}}{a_sb_sc_sd_s};q^s\right)_\infty}{\left(q^{(n-1)s}a_sb_sc_sd_s;q^s\right)_n\mathrm{St}^d(-\kappa-\upsilon-ns)}P_n(x) \tag{6.1}$$

with $P_n(x)=P_n(x;\kappa,\lambda,\upsilon,\varsigma;q,s)$ the normalized Askey–Wilson polynomial in q^x+q^{-x} of degree n,

$$P_n(x):={}_4\phi_3\left(\begin{matrix}q^{-ns},q^{(n-1)s}a_sb_sc_sd_s,a_sq^x,a_sq^{-x}\\ a_sb_s,a_sc_s,a_sd_s\end{matrix};q^s,q^s\right).$$

This can be proved directly from the expression of $\Phi(x,-\kappa-\upsilon-ns)$ as a very-well-poised ${}_8W_7$ series (see Proposition 2.2) by first applying Watson's transformation [6, (III.17)] with parameters (a,b,c,d,e,f,q) in [6, (III.17)] specialized to

$$\left(\frac{q^{2(1-n)s+x}}{a_sb_sc_sd_s^2},\frac{q^{s+x}}{d_s},\frac{q^{(2-n)s}}{a_sb_sc_sd_s},\frac{q^{(1-n)s}}{a_sd_s},\frac{q^{(1-n)s}}{b_sd_s},\frac{q^{(1-n)s}}{c_sd_s},q^s\right),$$

followed by Sear's transformation [6, (III.16)] with parameters (a,b,c,d,e,f,q) in [6, (III.16)] specialized to

$$\left(\frac{q^{(1-n)s}}{a_sd_s},\frac{q^{(1-n)s}}{b_sd_s},\frac{q^{(1-n)s}}{c_sd_s},\frac{q^{2(1-n)s}}{a_sb_sc_sd_s},\frac{q^{(1-n)s+x}}{d_s},\frac{q^{(1-n)s-x}}{d_s},q^s\right).$$

Formula (6.1) can also be proved by observing that $P_n(x)$ is an eigenfunction of $\mathcal{D}_x$ with eigenvalue $\widetilde{a}(q^n+1)+\widetilde{a}^{-1}(q^{-n}+1)$ (see [1]) of the form

$$P_n(x)=\frac{\left(q^{-ns},q^{(n-1)s}a_sb_sc_sd_s;q^s\right)_n(-a_s)^nq^{\frac{s}{2}n(n+1)}}{\left(a_sb_s,a_sc_s,a_sd_s;q^s\right)_n}q^{nx}+\sum_{m<n}c_mq^{mx},$$

hence up to a multiplicative constant it must be equal to $\Phi(x,-\kappa-\upsilon-ns)$ (this argument generalizes to the setting of Macdonald–Koornwinder polynomials, see [13, 25]). Also the Askey–Wilson function $\mathcal{E}(x,-\kappa-\upsilon-ns)$ ($n\in\mathbb{Z}_{\geq 0}$) is a multiple of the Askey–Wilson polynomial,

$$\mathcal{E}(x,-\kappa-\upsilon-ns)=\frac{\big(a_sb_s,a_sc_s;q^s\big)_\infty}{\big(\frac{q^s}{a_sd_s};q^s\big)_\infty}P_n(x). \tag{6.2}$$

This can again be proved directly using transformation formulas and the expression of $\mathcal{E}(x,z)$ as a ${}_8W_7$ series, see [10]. It can also be proved using the c-function expansion of $\mathcal{E}(x,z)$ (see Proposition 2.4(iv)) and (6.1), since $c(x;\kappa+\upsilon+ns)=0$ for $n\in\mathbb{Z}_{\geq 0}$.

Specializing now $z=-2\kappa-ns$ ($n\in\mathbb{Z}_{\geq 0}$) in the identity $\Phi_J(x,z)=\Phi_R(x,z)$ and using (6.1) twice gives, after straightforward simplifications, the following result (we take $s=2$ without loss of generality),

$${}_4\phi_3\left(\begin{matrix}q^{-2n},aq^x,aq^{-x},q^{2n}a^2b^2\\ ab,qa^2,qab\end{matrix};q^2,q^2\right)={}_4\phi_3\left(\begin{matrix}q^{-n},aq^x,aq^{-x},-q^nab\\ ab,q^{\frac{1}{2}}a,-q^{\frac{1}{2}}a\end{matrix};q,q\right)$$

for $n\in\mathbb{Z}_+$. This quadratic transformation formula was first proved by Singh [22]. It was reobtained by Askey and Wilson [1, (4.22)] and interpreted as quadratic transformation formula [1, (4.22)] for the continuous q-Jacobi polynomial (to get it in the same form one has to apply Sear's transformation formula [6, (III.15)]), see also [6, § 3.10] for further discussions. It was also obtained in [20, (3.20)] ($-a^2q$ in [20, (3.20)] should be $-abq$) by specialization of the quadratic transformation formula [20, (3.16)] for the R-function. A similar polynomial reduction can be done with the dual version (5.3) of the quadratic transformation formula, in which case it reduces to the quadratic transformation formula [1, (3.1)].

7 A theta function identity

By the c-function expansion of the Askey–Wilson function (see Proposition 2.4(iv)) and by Theorem 5.1 we have

$$\begin{aligned}\mathcal{E}_J(x,z)&=c_J(x,z)\Phi_J(x,z)+c_J(x,-z)\Phi_J(x,-z)\\&=c_J(x,z)\Phi_R(x,z)+c_J(x,-z)\Phi_R(x,-z)\end{aligned}$$

while on the other hand,

$$\mathcal{E}_R(x,z)=c_R(x,z)\Phi_R(x,z)+c_R(x,-z)\Phi_R(x,-z).$$

The c-functions $c_J(x,z)$ and $c_R(x,z)$ are explicitly given by

$$\begin{aligned}c_J(x,z)&=\frac{\theta\big(q^{2\kappa-z},-q^{-z},q^{\frac{s}{2}+2\lambda-z},-q^{\frac{s}{2}-\kappa-\lambda+x-z};q^s\big)}{W_J(x,z)\theta\big(q^{-2z},-q^{\frac{s}{2}+\kappa-\lambda+x};q^s\big)},\\ c_R(x,z)&=\frac{\theta\big(q^{2\kappa-z};q^s\big)\theta\big(q^{\frac{s}{4}+\lambda-\frac{z}{2}},-q^{\frac{s}{4}-\kappa+x-\frac{z}{2}};q^{\frac{s}{2}}\big)}{W_R(x,z)\theta\big(q^{-z},-q^{\frac{s}{4}+x};q^{\frac{s}{2}}\big)}.\end{aligned} \tag{7.1}$$

Note that $W_J(x,z)=q^{(\kappa+\lambda+x)(2\kappa+z)/s}=W_R(x,z)$ and that

$$\frac{c_J(x,z)}{c_R(x,z)}=\frac{\theta\big(-q^{\frac{s}{2}-\kappa-\lambda+x-z};q^s\big)\theta\big(-q^{\frac{s}{4}+\lambda-\frac{z}{2}},-q^{\frac{3}{4}+x};q^{\frac{s}{2}}\big)}{\theta\big(-q^{\frac{s}{2}-z},-q^{\frac{s}{2}+\kappa-\lambda+x};q^s\big)\theta\big(-q^{\frac{s}{4}-\kappa+x-\frac{z}{2}};q^{\frac{s}{2}}\big)}$$

is not invariant under $z\to -z$ for generic Hecke parameters, since it is zero at $z=2\lambda+\frac{s}{2}+2\pi\sqrt{-1}\tau$ but nonzero at $z=-2\lambda-\frac{s}{2}-2\pi\sqrt{-1}\tau$. Hence $\mathcal{E}_J(x,z)$ cannot be of the form $\alpha(x,z)\mathcal{E}_R(x,z)$ with $\alpha(x,z)$ meromorphic and s-invariant in the variable x.

Remark 7.1. Both $\mathcal{E}_J(\cdot,z)$ and $\mathcal{E}_R(\cdot,z)$ are eigenfunctions of $\mathcal{D}^{(s)}_{\kappa,\lambda,\kappa,\lambda;q}$ with eigenvalue $q^z+q^{-z}-q^{2\kappa}-q^{-2\kappa}$ (for $\mathcal{E}_R(\cdot,z)$ this follows from (4.1) and the fact that $c_R(\cdot,z)$ is $\frac{s}{2}$-translation invariant). On the other hand, $\mathcal{E}_R(\cdot,z)$ is an eigenfunction of $\mathcal{L}^{(s/2)}_{\kappa,\lambda,q}$ with eigenvalue $q^{\frac{z}{2}}+q^{-\frac{z}{2}}$, but this is not true for $\mathcal{E}_J(\cdot,z)$ since $c_J(\cdot,z)$ is not $\frac{s}{2}$-translation invariant.

Remark 7.2. Ruijsenaars' R-function $\mathcal{R}(\cdot,z)$ [20] is a hyperbolic eigenfunction of the Askey–Wilson second order difference operator (implying in particular that it admits an analytic continuation to the regime $|q|=1$) satisfying $\mathcal{R}(-x,z)=\mathcal{R}(x,z)$. For the R-function $\mathcal{R}$ a quadratic transformation formula of the form $\mathcal{R}_J(x,z)=\mathcal{R}_R(x,z)$ holds true, see [20, (3.16)]. The discrepancy with the fact that $\mathcal{E}_J(x,z)\neq\mathcal{E}_R(x,z)$ is not unexpected since the R-function $\mathcal{R}$ in the trigonometric regime $|q|<1$ has a nontrivial factorization in Askey–Wilson functions: it expands as a sum of two terms, each term being essentially the product of two Askey–Wilson functions, see [3, Theorem 6.5].

The quadratic transformation formula $\Phi_J(x,z)=\Phi_R(x,z)$ implies the following result for the connection coefficients in (2.8).

Proposition 7.3. *We have*

$$\frac{c_J(x,-z)}{c^d_J(z,-x)}=\frac{c_R(x,-z)}{c^d_R(\frac{z}{2},-2x)},\qquad \frac{c_J(x,z)-c^d_J(z,x)}{c^d_J(z,-x)}=\frac{c_R(x,z)-c^d_R(\frac{z}{2},2x)}{c^d_R(\frac{z}{2},-2x)}. \tag{7.2}$$

Proof. Using (2.8) and Theorem 5.1 we have

$$\begin{aligned}&\left(\frac{c_J(x,z)-c^d_J(z,x)}{c^d_J(z,-x)}\right)\Phi_J(x,z)+\frac{c_J(x,-z)}{c^d_J(z,-x)}\Phi_J(x,-z)\\&\quad=\left(\frac{c_R(x,z)-c^d_R(\frac{z}{2},2x)}{c^d_R(\frac{z}{2},-2x)}\right)\Phi_J(x,z)+\frac{c_R(x,-z)}{c^d_R(\frac{z}{2},-2x)}\Phi_J(x,-z),\end{aligned} \tag{7.3}$$

since both sides are equal to $\Phi_J(-x,z)=\Phi_R(-x,z)$. By a straightforward asymptotic argument (compare with the proof of Corollary 2.8) it follows that the coefficients of $\Phi_J(x,z)$ (resp. of $\Phi_J(x,-z)$) on both sides of this equation should be the same.

Alternatively, one verifies by a direct computation that

$$\frac{c_J(x,-z)}{c^d_J(z,-x)}=q^{-\frac{4\kappa x}{s}}q^{\frac{2(\kappa+\lambda)z}{s}}\frac{\theta\big(q^{2x},q^{\frac{s}{2}+2\lambda+z},q^{2\kappa+z};q^s\big)}{\theta\big(q^z,q^{\kappa+\lambda+x},-q^{\kappa-\lambda+x};q^{\frac{s}{2}}\big)}=\frac{c_R(x,-z)}{c^d_R(\frac{z}{2},-2x)},$$

yielding the first equality of (7.2). Combined with (7.3) this implies the second equality of (7.2). ∎

Since $c(x,z)$ is s-translation invariant in both x and z, it follows from the right hand sides of (7.2) that the quotients

$$\frac{c_J(x,-z)}{c^d_J(z,-x)},\qquad \frac{c_J(x,z)-c^d_J(z,x)}{c^d_J(z,-x)}$$

are $\frac{s}{2}$-translation invariant in x (although $c_J(x,z)$ is not, cf. Remark 7.1).

Using (7.1) and using the explicit expressions

$$\begin{aligned}c^d_J(z,x)&=\frac{\theta\big(q^{\kappa+\lambda-x},-q^{\kappa-\lambda-x},q^{\frac{s}{2}+\kappa+\lambda-x},-q^{\frac{s}{2}-\kappa-\lambda+z-x};q^s\big)}{W^d_J(z,x)\theta\big(q^{-2x},-q^{\frac{s}{2}+z};q^s\big)},\\ c^d_R(z,x)&=\frac{\theta\big(q^{\kappa+\lambda-\frac{x}{2}},-q^{\kappa-\lambda-\frac{x}{2}},q^{\frac{s}{4}-\frac{x}{2}},-q^{\frac{s}{4}-\kappa+z-\frac{x}{2}};q^{\frac{s}{2}}\big)}{W^d_R(z,x)\theta\big(q^{-x},-q^{\frac{s}{4}+\lambda+z};q^{\frac{s}{2}}\big)}\end{aligned}$$

with $W_J^d(z,x)=q^{(\kappa+\lambda+x)(2\kappa+z)/s}$ and $W_R^d(z,x)=q^{2(\kappa+z)(\kappa+\lambda+\frac{x}{2})/s}$, we have explicitly,

$$\frac{c_J(x,z)-c_J^d(z,x)}{c_J^d(z,-x)}=q^{-2x(2\kappa+z)/s}\frac{\theta\big(q^{2x};q^s\big)}{\theta\big(q^{\kappa+\lambda+x},-q^{\kappa-\lambda+x},q^{\frac{s}{2}+\kappa+\lambda+x},-q^{\frac{s}{2}-\kappa-\lambda+x+z};q^s\big)}\times\Bigg\{\frac{\theta\big(q^{2\kappa-z},-q^{-z},q^{\frac{s}{2}+2\lambda-z},-q^{\frac{s}{2}-\kappa-\lambda+x-z},-q^{\frac{s}{2}+z};q^s\big)}{\theta\big(q^{-2z},-q^{\frac{s}{2}+\kappa-\lambda+x};q^s\big)}-\frac{\theta\big(q^{\kappa+\lambda-x},-q^{\kappa-\lambda-x},q^{\frac{s}{2}+\kappa+\lambda-x},-q^{\frac{s}{2}-\kappa-\lambda-x+z};q^s\big)}{\theta\big(q^{-2x};q^s\big)}\Bigg\}$$

and

$$\frac{c_R(x,z)-c_R^d(\frac{z}{2},2x)}{c_R^d(\frac{z}{2},-2x)}=q^{-2x(2\kappa+z)/s}\frac{\theta\big(q^{2x};q^{\frac{s}{2}}\big)}{\theta\big(q^{\kappa+\lambda+x},-q^{\kappa-\lambda+x},q^{\frac{s}{4}+x},-q^{\frac{s}{4}-\kappa+x+\frac{z}{2}};q^{\frac{s}{2}}\big)}\times\Bigg\{\frac{\theta\big(q^{2\kappa-z};q^s\big)\theta\big(q^{\frac{s}{4}+\lambda-\frac{z}{2}},-q^{\frac{s}{4}-\kappa+x-\frac{z}{2}},-q^{\frac{s}{4}+\lambda+\frac{z}{2}};q^{\frac{s}{2}}\big)}{\theta\big(q^{-z},-q^{\frac{s}{4}+x};q^{\frac{s}{2}}\big)}-\frac{\theta\big(q^{\kappa+\lambda-x},-q^{\kappa-\lambda-x},q^{\frac{s}{4}-x},-q^{\frac{s}{4}-\kappa-x+\frac{z}{2}};q^{\frac{s}{2}}\big)}{\theta\big(q^{-2x};q^{\frac{s}{2}}\big)}\Bigg\}.$$

Hence (7.2) is equivalent to the following theta function identity.

Corollary 7.4.

$$\theta\left(a^2,b^2c^2,-b^2,qb^2d^2,-\frac{qb^2}{acd},-qb^2;q^2\right)-\theta\left(b^4,-\frac{qc}{ad},acd,-\frac{ac}{d},qacd,-\frac{qa}{b^2cd};q^2\right)$$
$$=\frac{\theta\big(-qab^2cd;q^2\big)\theta\big(-b^2;q\big)}{\theta\big(qa^2;q^2\big)\theta\big(-q^{\frac{1}{2}}abc;q\big)}\left\{\theta\left(bc,-bc,a^2,q^{\frac{1}{2}}bd,-\frac{q^{\frac{1}{2}}b}{ac},-\frac{q^{\frac{1}{2}}d}{b};q\right)-\theta\left(b^2,-q^{\frac{1}{2}}a,acd,-\frac{ac}{d},q^{\frac{1}{2}}a,-\frac{q^{\frac{1}{2}}a}{bc};q\right)\right\}. \tag{7.4}$$

Acknowledgements

I thank Tom Koornwinder for drawing my attention to the quadratic transformation formula for continuous q-Jacobi polynomials. I thank Mizan Rahman for pointing out to me how the quadratic transformations (5.2) and (5.3) for very-well-poised ${}_8\phi_7$ series are related to the known quadratic transformation formula [6, (3.5.10)] (see Remark 5.3(i)).

References

[1] Askey R., Wilson J., Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **54** (1985), no. 319.

[2] van de Bult F.J., Ruijsenaars' hypergeometric function and the modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$, *Adv. Math.* **204** (2006), 539–571, math.QA/0501405.

[3] van de Bult F.J., Rains E.M., Stokman J.V., Properties of generalized univariate hypergeometric functions, *Comm. Math. Phys.* **275** (2007), 37–95, math.CA/0607250.

[4] Chalykh O.A., Macdonald polynomials and algebraic integrability, *Adv. Math.* **166** (2002), 193–259, math.QA/0212313.

[5] Chalykh O.A., Etingof P., Orthogonality relations and Cherednik identities for multivariable Baker–Akhiezer functions, arXiv:1111.0515.

[6] Gasper G., Rahman M., Basic hypergeometric series, *Encyclopedia of Mathematics and its Applications*, Vol. 96, 2nd ed., Cambridge University Press, Cambridge, 2004.

[7] Gupta D.P., Masson D.R., Contiguous relations, continued fractions and orthogonality, *Trans. Amer. Math. Soc.* **350** (1998), 769–808, math.CA/9511218.

[8] Haine L., Iliev P., Askey–Wilson type functions with bound states, *Ramanujan J.* **11** (2006), 285–329, math.QA/0203136.

[9] Ismail M.E.H., Rahman M., The associated Askey–Wilson polynomials, *Trans. Amer. Math. Soc.* **328** (1991), 201–237.

[10] Koelink E., Stokman J.V., The Askey–Wilson function transform, *Int. Math. Res. Not.* (2001), no. 22, 1203–1227, math.CA/0004053.

[11] Koelink E., Stokman J.V., Fourier transforms on the quantum SU(1, 1) group, *Publ. Res. Inst. Math. Sci.* **37** (2001), 621–715, math.QA/9911163.

[12] Koornwinder T., Comment on the paper "Macdonald polynomials and algebraic integrability" by O.A. Chalykh, available at `http://staff.science.uva.nl/~thk/art/comment/ChalykhComment.pdf`.

[13] Letzter G., Stokman J.V., Macdonald difference operators and Harish-Chandra series, *Proc. Lond. Math. Soc. (3)* **97** (2008), 60–96, math.QA/0701218.

[14] van Meer M., Bispectral quantum Knizhnik–Zamolodchikov equations for arbitrary root systems, *Selecta Math. (N.S.)* **17** (2011), 183–221, arXiv:0912.3784.

[15] van Meer M., Stokman J., Double affine Hecke algebras and bispectral quantum Knizhnik–Zamolodchikov equations, *Int. Math. Res. Not.* (2010), no. 6, 969–1040, arXiv:0812.1005.

[16] Noumi M., Stokman J.V., Askey–Wilson polynomials: an affine Hecke algebra approach, in Laredo Lectures on Orthogonal Polynomials and Special Functions, *Adv. Theory Spec. Funct. Orthogonal Polynomials*, Nova Sci. Publ., Hauppauge, NY, 2004, 111–144, math.QA/0001033.

[17] Rahman M., The linearization of the product of continuous q-Jacobi polynomials, *Canad. J. Math.* **33** (1981), 961–987.

[18] Rahman M., Verma A., Quadratic transformation formulas for basic hypergeometric series, *Trans. Amer. Math. Soc.* **335** (1993), 277–302.

[19] Ruijsenaars S.N.M., A generalized hypergeometric function satisfying four analytic difference equations of Askey–Wilson type, *Comm. Math. Phys.* **206** (1999), 639–690.

[20] Ruijsenaars S.N.M., Quadratic transformations for a function that generalizes ${}_2F_1$ and the Askey–Wilson polynomials, *Ramanujan J.* **13** (2007), 339–364.

[21] Sauloy J., Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie, *Ann. Inst. Fourier (Grenoble)* **50** (2000), 1021–1071.

[22] Singh V.N., The basic analogues of identities of the Cayley–Orr type, *J. London Math. Soc.* **34** (1959), 15–22.

[23] Slater L.J., A note on equivalent product theorems, *Math. Gaz.* **38** (1954), 127–128.

[24] Stokman J.V., An expansion formula for the Askey–Wilson function, *J. Approx. Theory* **114** (2002), 308–342, math.CA/0105093.

[25] Stokman J.V., The c-function expansion of a basic hypergeometric function associated to root systems, arXiv:1109.0613.

[26] Suslov S.K., Some orthogonal very-well-poised ${}_8\phi_7$-functions that generalize Askey–Wilson polynomials, *Ramanujan J.* **5** (2001), 183–218, math.CA/9707213.