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Dirac Induction for Loop Groups

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Abstract. Using a coset version of the cubic Dirac operators for affine Lie algebras, we give an algebraic construction of the Dirac induction homomorphism for loop group representations. With this, we prove a homogeneous generalization of the Weyl–Kac character formula and show compatibility with Dirac induction for compact Lie groups.

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0. Introduction

The aim of this note is to give a simple, algebraic construction of Dirac induction for loop groups. This construction induces finite dimensional representations of a compact Lie group H to positive energy representations of the loop group LG, where $\mathsf{H} \subseteq \mathsf{G}$ is an equal rank embedding. The main tool in this construction is a homogeneous generalization of the representation theoretic cubic Dirac operator of Landweber [12]. The same homomorphism can be constructed topologically in equivariant (twisted) K-theory, cf. [4], see also [3,8,14].

In the theory of loop groups, the Borel–Weil construction [17] and holomorphic induction [19] have clear interpretations in terms of the complex geometry (the $\bar{\partial}$-operator) of the corresponding homogeneous spaces. The geometric foundation, in terms of the spin-manifold $X = \mathsf{LG}/\mathsf{H}$, underlying the Dirac induction of this paper is less clear, although for $\mathsf{H} = \mathsf{T}$ and $\mathsf{H} = \mathsf{G}$ we do find back, up to the familiar ρ-shift, the above mentioned homomorphisms. Clearly, this is due to the fact that the homogeneous spaces are infinite dimensional, and the present algebraic construction cannot be made geometric by absence of a suitable version of the Peter–Weyl theorem for loop groups. For compact Lie groups, this is of course not a problem and in Section 1, we shows that the algebraic and geometric induction homomorphisms are the same.

The cubic Dirac operator is by now a well-studied object in the theory of loop groups: the operator has a long history in the physics literature of superconformal field theory as one of the supercharge operators. Mathematically, its use in representation theory was first demonstrated in the (unpublished) lecture notes of Wassermann [21]. In [12], Landweber constructs a homogeneous generalization corresponding to an embedding $LH \subset LG$, which is closer related to *string structures* rather than spin structures. A family version of the cubic Dirac operator was used in [5] to construct the isomorphism between the fusion ring of a loop group and the twisted equivariant K-theory of the underlying compact Lie group. Finally, a detailed treatment, including the homogeneous generalization considered in this paper, appeared in [15] (at the time of writing, the papers of Wassermann and Meinrenken were not available.)

This article makes no great claim to originality. It extends the techniques of [12] to a homogeneous setting not considered before, so the intellectual debt to that paper should be clear. One of the reasons that justifies publication in the author's opinion is the elegance of the machinery, and the fact that it leads to some interesting results. It sheds a different light on some geometric and topological aspects in representation theory and twisted K-theory, and in this way leads to short proofs of some otherwise well-known facts.

1. Dirac Induction for Compact Lie Groups

1.1. THE CUBIC DIRAC OPERATOR

Let G be a simple compact connected Lie group equipped with an Ad-invariant *basic* inner product $\langle,\rangle$ on its Lie algebra $\mathfrak{g}$. We fix an orthonormal basis $X_a, a = 1,\dots,\dim\mathfrak{g}$, and denote by f_{abc} the structure constants in this basis. The associated Clifford algebra $\mathrm{Cliff}(\mathfrak{g})$ over $\mathfrak{g}$ has a complex representation on $\mathbb{S}_{\mathfrak{g}}$, called the spin representation, which extends to a representation of $\mathfrak{g}$ given by $\mathrm{ad}_{X_a} = -\frac{1}{4}\sum_{b,c} f_{abc}\psi^b\psi^c$, where ψ^a denotes the Clifford algebra element corresponding to X_a. For any representation V of $\mathfrak{g}$, the cubic Dirac operator is defined as

$$\not{D}_{\mathfrak{g}} = \sum_{a=1}^{\dim\mathfrak{g}} \left(X_a \otimes \psi^a + 1 \otimes \frac{1}{3}\mathrm{ad}_{X_a}\psi^a \right). \tag{1}$$

As it stands, this is an operator $\not{D}_{\mathfrak{g}} \in \mathrm{End}(V \otimes \mathbb{S}_{\mathfrak{g}})$, but in [1] it was interpreted as a distinguished element of the so-called "noncommutative Weil algebra" $\mathcal{W}_G := U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$. The name "Dirac operator" comes from the fact that

$$\frac{1}{2}\not{D}_{\mathfrak{g}}^2 = \Delta + \frac{1}{24}\mathrm{tr}_{\mathfrak{g}}\Delta_{\mathrm{ad}}, \tag{2}$$

where Δ is the Casimir operator, and Δ_{ad} its action in the adjoint representation.

Choose a maximal torus $T \subset G$ and denote the associated weight lattice by Λ_G^*. Recall that in an irreducible representation V_λ with highest weight $\lambda \in \Lambda_G^*$ the value of the Casimir operator is given by $\Delta_\lambda = \frac{1}{2}\left(\|\lambda + \rho_{\mathfrak{g}}\|^2 - \|\rho_{\mathfrak{g}}\|^2\right)$, where $\rho_{\mathfrak{g}}$ is half the sum of all positive roots of $\mathfrak{g}$. Using the fact that $\mathrm{tr}_{\mathfrak{g}}\Delta_{\mathrm{ad}}/12 = \|\rho_{\mathfrak{g}}\|^2$, one finds in this case the following identity $\mathrm{End}(V_\lambda \otimes \mathbb{S}_{\mathfrak{g}})$:

$$\not{D}_{\mathfrak{g}}^2 = \|\lambda + \rho_{\mathfrak{g}}\|^2.$$

Next, we consider an inclusion $H \subset G$ of equal rank subgroups. Using the innerproduct we write $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ as H-representations and we assume our basis was chosen such that the first $X_a, a = 1, \ldots, \dim \mathfrak{p}$ form a basis of $\mathfrak{p}$. This decomposition induces an isomorphism $\mathrm{Cliff}(\mathfrak{g}) \cong \mathrm{Cliff}(\mathfrak{p}) \otimes \mathrm{Cliff}(\mathfrak{h})$, and the representation of H on $\mathbb{S}_{\mathfrak{g}}$ can be decomposed as $\mathbb{S}_{\mathfrak{g}} \cong \mathbb{S}_{\mathfrak{p}} \otimes \mathbb{S}_{\mathfrak{h}}$, where $\mathbb{S}_{\mathfrak{p}}$ and $\mathbb{S}_{\mathfrak{h}}$ are the irreducible modules of $\mathrm{Cliff}(\mathfrak{h})$ and $\mathrm{Cliff}(\mathfrak{p})$. Let $\not{D}_{\mathfrak{h}}' \in \mathcal{W}_G$ be the Dirac operator corresponding to the representation of $\mathfrak{h}$ on $V \otimes \mathbb{S}_{\mathfrak{g}} \cong (V \otimes \mathbb{S}_{\mathfrak{p}}) \otimes \mathbb{S}_{\mathfrak{h}}$. The difference $\not{D}_{\mathfrak{g}/\mathfrak{h}} := \not{D}_{\mathfrak{g}} - \not{D}_{\mathfrak{h}}'$ can be written out as

$$\not{D}_{\mathfrak{g}/\mathfrak{h}} = \sum_{a=1}^{\dim \mathfrak{p}} \left(X_a \otimes \psi^a + 1 \otimes \frac{1}{3} \mathrm{ad}_{X_a} \psi^a \right). \tag{3}$$

This is Kostant's Dirac operator [10], and its construction as the difference $\not{D}_{\mathfrak{g}} - \not{D}_{\mathfrak{h}}'$, cf. [12], is called the coset construction by physicists. A small computation shows that $\not{D}_{\mathfrak{g}/\mathfrak{h}}$ and $\not{D}_{\mathfrak{h}}'$ anticommute, and from this one easily deduces that for $V = V_\lambda, \lambda \in \Lambda_G^*$ irreducible, in the isotypical H-summand of $V_\lambda \otimes \mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$ labeled by the weight $\mu \in \Lambda_H^*$, one has

$$\not{D}_{\mathfrak{g}/\mathfrak{h}}^2 \Big|_{\mu} = \|\lambda + \rho_{\mathfrak{g}}\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2. \tag{4}$$

Since $\mathfrak{h} \subseteq \mathfrak{g}$ is of equal rank, $\dim \mathfrak{p} =$ even, and $\mathbb{S}_{\mathfrak{p}} = \mathbb{S}_{\mathfrak{p}}^+ \oplus \mathbb{S}_{\mathfrak{p}}^-$ is $\mathbb{Z}/2$-graded. The Dirac operator anticommutes with the grading and restricts to an operator $\not{D}_{\mathfrak{g}/\mathfrak{h}}^+ :$ $V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}^+ \to V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}^-$. By the Euler–Poincaré principle, its H-index equals $[V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}^+] - [V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}^-] \in R(H)$. On the other hand, the explicit formula for the square of $\not{D}_{\mathfrak{g}/\mathfrak{h}}$ yields the homogeneous Weyl formula of [6].

1.2. DIRAC INDUCTION

Given any finite dimensional representation W of H, the corresponding induced vector bundle over the homogeneous space $X := G/H$ is defined by $E_W := G \times_H W$. Since $H \subseteq G$ is of equal rank, X is a G-equivariant spin-manifold, and the associated spin bundle $\mathbb{S}_X$ can be identified as the induced bundle of the H-representation $\mathbb{S}_{\mathfrak{p}}$. By the Peter–Weyl theorem,

$$L^2(X, E_W \otimes \mathbb{S}_X) \cong \bigoplus_{\lambda \in \Lambda_G^*} V_\lambda \otimes \mathrm{Hom}_H(V_\lambda, W \otimes \mathbb{S}_{\mathfrak{p}}).$$

Applying the above construction of the Dirac operator above to the right G-action on $L^2(G)$, one finds an H-invariant operator on $L^2(G) \otimes \mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$, which also commutes with the left G-action. It therefore descends to a G-equivariant elliptic operator on X, which identifies with the geometric Dirac operator of the spin manifold X. In the same spirit, the construction applied to the tensor product $W \otimes \mathbb{S}_{\mathfrak{p}}$

yields the Dirac operator coupled to E_W, and with this Dirac induction is defined as

$$\mathrm{Ind}(W) := \mathrm{index}_G(\not{D}_W^+).$$

This defines a homomorphism $\mathrm{Ind}: R(H) \to R(G)$ of abelian groups. In [11], the Peter–Weyl decomposition together with Equation (4) for the square of the Dirac operator was used to deduce the following theorem of Slebarski:

THEOREM 1.1 [18]. For V_λ, $\lambda \in \Lambda_H^*$ irreducible, one has

$$\mathrm{Ind}(V_\lambda) = \begin{cases} (-1)^{\ell(w)} V_{w\cdot\lambda} & \exists w \in W_G,\ w(\lambda+\rho_{\mathfrak{h}}) - \rho_{\mathfrak{g}} \in \Lambda_G^* \\ 0 & \text{else.} \end{cases}$$

Functoriality of this induction procedure is expressed by the following theorem:

THEOREM 1.2 (Induction in stages) . *For* $H \subseteq K \subseteq G$ *of equal rank,*

$$\mathrm{Ind}_H^G = \mathrm{Ind}_K^G \circ \mathrm{Ind}_H^K.$$

Remark 1.3. The interpretation in terms of K-theory comes from the isomorphisms $K_G(\mathrm{pt.}) \cong R(G)$ and $K_G(X) \cong K_H(\mathrm{pt.}) \cong R(H)$. With this, the Dirac induction homomorphism is simply given by $\pi_*: K_G(X) \to K_G(\mathrm{pt.})$, where $\pi: X \to \mathrm{pt.}$ is the map that collapses X to a point, cf. [2].

2. The Cubic Dirac Operator on Loop Groups

2.1. LOOP GROUPS

A standard reference for loop groups is [17]. Let us briefly recall some facts. We now assume G to be simply connected as well, and denote the corresponding loop group by LG. A *positive energy representation* of LG is a projective unitary representation of LG on a Hilbert space E which extends to the semidirect product with the rotation group $\mathrm{Rot}(S^1)$, with energy bounded below. On the Lie algebra level this induces a projective representation of the Lie algebra $L\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}[z, z^{-1}]$ on the algebraic sum of the energy eigenspaces $\bigoplus_n E(n)$, in which $X^a(n) := X_a z^n$ acts by the commutation relations

$$[X_a(m), X_b(m)] = \sum_c f_{abc} X_c(m+n) + \ell m \delta_{ab} \delta_{m,-n},$$

for some positive integer ℓ called the *level* of the representation. The last term on the right corresponds to a Lie algebra cocycle defining a central extension of $L\mathfrak{g}_{\mathbb{C}}$, and if also add the generator of rotation action D, satisfying $[D,, X_a(n)] = nX^a(n)$, we arrive at the affine Lie algebra $\widehat{L\mathfrak{g}}_{\mathbb{C}} := L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}K \oplus \mathbb{C}D$, where K denotes the central element.

The Cartan subalgebra is given by $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ and weights are of the form $\boldsymbol{\lambda} = (n, \lambda, \ell)$, where $n \in \mathbb{N}$ gives the energy eigenspace, $\lambda \in \Lambda_G^*$ is a weight of G and $\ell \in \mathbb{N}$

is the level. The *affine* roots of LG are given by the weights $(n,\alpha,0)$, where $\alpha\in\Delta$ is a root of G, as well as $(n,0,0)$, with $n\neq 0$. Given a system of positive roots of G, the positive roots of LG are given by

$$\boldsymbol{\alpha}=\begin{cases}(0,\alpha,0) & \alpha>0\\ (n,\alpha,0) & n>0,\alpha\in\Delta\\ (n,0,0) & n>0,\end{cases}\tag{5}$$

where the last set of roots should be counted with multiplicity $\mathrm{rank}(G)=\dim(\mathfrak{t}_{\mathbb{C}})$, since the corresponding root spaces are not one dimensional. The simple affine roots are given by $\boldsymbol{\alpha}_i=(0,\alpha_i,0)$ with α_i a simple root of G, as well as $\boldsymbol{\alpha}=(1,-\alpha_{\max},0)$.

The *affine* Weyl group is defined as the semi-direct product $W_{\mathrm{aff}}:=W\ltimes\Lambda_G$, where $\Lambda_G\subseteq\mathfrak{t}$ is the coweight lattice. This group acts on the Cartan subalgebra $\mathbb{R}\oplus\mathfrak{t}^*\oplus\mathbb{R}$, where W acts as usual on $\mathfrak{t}^*$ and the action of $x\in\Lambda_G$ is given by

$$x\cdot(n,\lambda,\ell)=\left(n+\lambda(x)+\frac{1}{2}\ell\|x\|^2,\lambda+\ell x,\ell\right).$$

This action preserves the inner product on $\mathbb{R}\oplus\mathfrak{t}^*\oplus\mathbb{R}$ given by

$$\big\langle(n_1,\lambda_1,\ell_1),(n_2,\lambda_2,\ell_2)\big\rangle=\langle\lambda_1,\lambda_2\rangle-n_1\ell_2-n_2\ell_1.$$

he affine Weyl group W_{aff} is generated by reflections $s_{\boldsymbol{\alpha}}$ in the affine hyperplanes corresponding to the simple roots $\boldsymbol{\alpha}$. This determines the length $\ell(w)$ of an element $w\in W_{\mathrm{aff}}$ as the minimal k for which $w=s_{\boldsymbol{\alpha}_1}\ldots s_{\boldsymbol{\alpha}_k}$, with all $\boldsymbol{\alpha}_i$, $i=1,\ldots,k$ simple (for $k=\ell(w)$ such an expression is called reduced).

The decomposition into positive and negative roots introduces the triangular decomposition $L\mathfrak{g}_{\mathbb{C}}=\overline{\mathfrak{N}}\oplus\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{N}$, with $\mathfrak{N}:=\bigoplus_{\alpha>0}\mathfrak{g}_\alpha\oplus\bigoplus_{k>0}\mathfrak{g}z^k$ and $\overline{\mathfrak{N}}:=\bigoplus_{\alpha<0}\mathfrak{g}_\alpha\oplus\bigoplus_{k<0}\mathfrak{g}z^k$. A positive energy representation is generated by a lowest weight vector annihilated by $\mathfrak{N}$. The irreducible positive energy representations are uniquely classified by their lowest weight $\boldsymbol{\lambda}=(n,\lambda,\ell)$: n labels the lowest nonzero energy eigenspace, ℓ is the level, and the lowest weight λ of the G-module $E(n)$ satisfies the constraint $\langle\lambda,\alpha_{\max}\rangle\leq\ell$. We denote by $R_\ell(G)$ the abelian group generated by the irreducible representations at level ℓ. It is well-known that it has a finite basis. In fact, it has a ring structure, see e.g. [19], known as the *fusion product*, a structure we will ignore in this paper.

2.2. THE SPIN REPRESENTATION

The natural L^2-innerproduct on $L\mathfrak{g}_{\mathbb{C}}$ given by $\langle X_a(m),X_b(n)\rangle=2\delta_{a,b}\delta_{m,-n}$ defines the real Clifford algebra Cliff($L\mathfrak{g}$) as the algebra generated by odd elements $\psi^a(n)$ satisfying $\psi^a(m)\psi^b(n)+\psi^b(n)\psi^a(m)=2\delta_{a,b}\delta_{m,-n}$. Write $L\mathfrak{g}_{\mathbb{C}}=L\mathfrak{g}_{\mathbb{C}}^+\oplus\mathfrak{g}_{\mathbb{C}}\oplus L\mathfrak{g}_{\mathbb{C}}^-$, where $L\mathfrak{g}_{\mathbb{C}}^-=\bigoplus_{k<0}\mathfrak{g}_{\mathbb{C}}z^k$ and $L\mathfrak{g}_{\mathbb{C}}^+=\bigoplus_{k>0}\mathfrak{g}_{\mathbb{C}}z^k$ are subspaces of negative and non-negative Fourier frequencies. The spin representation of this algebra acts on

$$\mathbb{S}_{L\mathfrak{g}} := \mathbb{S}_{\mathfrak{g}} \otimes \bigotimes_{k>0} \bigwedge^{\bullet}(\mathfrak{g}_{\mathbb{C}} z^k). \tag{6}$$

Explicitly, the action is given by

$$\psi(X) = \begin{cases} 1 \otimes \epsilon_X & X \in L\mathfrak{g}_{\mathbb{C}}^+, \\ 1 \otimes \iota_X & X \in L\mathfrak{g}_{\mathbb{C}}^-, \\ \psi(X) \otimes (-1)^F & X \in \mathfrak{g}_{\mathbb{C}}. \end{cases}$$

Here ϵ and ι denote exterior and interior multiplication, F is the degree operator and $\psi(X)$, $X \in \mathfrak{g}_{\mathbb{C}}$ denotes the finite dimensional Clifford action on $\mathbb{S}_{\mathfrak{g}}$.

The adjoint representation of G on its Lie algebra $\mathfrak{g}$ defines a real orthogonal representation $G \to SO(\mathfrak{g})$, and this lifts to a positive energy representation of LG on $\mathbb{S}_{L\mathfrak{g}}$. On the Lie algebra level this representation is given by

$$\mathrm{ad}_{X_a(k)} = -\frac{1}{4} \sum_{p+q=k} \sum_{b,c} f_{abc} \psi^b(p) \psi^c(q)$$

and this intertwines with the action of the Clifford algebra: $[\mathrm{ad}_{X_a(m)}, \psi^b(n)] = \sum_c f_{abc} \psi^c(m+n)$. The corresponding semi-direct product $\mathrm{Cliff}(L\mathfrak{g}) \ltimes L\mathfrak{g}$ is called the super Kac–Moody algebra, see [7] complete description of this representation. The level of the positive energy representation of LG on $\mathbb{S}_{L\mathfrak{g}}$ is given by the dual Coxeter number $h_{\mathfrak{g}}^{\vee}$ of G. The lowest weight of $\mathbb{S}_{L\mathfrak{g}}$ is easily seen to be given by $-\boldsymbol{\rho}_{\mathfrak{g}}$, where $\boldsymbol{\rho}_{\mathfrak{g}} = (0, \rho_{\mathfrak{g}}, -h_{\mathfrak{g}}^{\vee})$. The lowest weight vector is called the vacuum $\Omega_{L\mathfrak{g}}$ of the representation. As for $\mathbb{S}_{\mathfrak{g}}$, the spin representation $\mathbb{S}_{L\mathfrak{g}}$ carries a $\mathbb{Z}/2$ grading given by the operator $(-1)^F$. Therefore, we have the decomposition $\mathbb{S}_{L\mathfrak{g}} = \mathbb{S}_{L\mathfrak{g}}^+ \oplus \mathbb{S}_{L\mathfrak{g}}^-$, and this is in fact a decomposition of positive energy representations of LG.

2.3. THE CUBIC DIRAC OPERATOR

Let E be a positive energy representation of LG at level ℓ. The cubic Dirac operator is given by the following unbounded operator acting on $E \otimes \mathbb{S}_{L\mathfrak{g}}$:

$$\not{D}_{L\mathfrak{g}} := \sum_{n \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} \left(X_a(n) \otimes \psi^a(-n) + 1 \otimes \frac{1}{3} \mathrm{ad}_{X_a(n)} \psi^a(-n) \right), \tag{7}$$

In the physics literature this operator is known as the "supercharge", cf. [7, Section 2], see also [21]. In [12], an alternative derivation (without the use of the superconformal algebra) of this operators is given, see also [5] for a family version representing classes in twisted equivariant K-theory. Restricting to the "zero modes" $n = 0$, one finds the finite dimensional Dirac operator $\not{D}_{\mathfrak{g}}$ given by (1), including the "cubic term". The commutation relations in [7] readily imply:

PROPOSITION 2.1. *The Dirac operator* $\not{D}_{L\mathfrak{g}}$ *satisfies the following relations*:

$$\not{D}^2_{L\mathfrak{g}} = 2(\ell + h^\vee_{\mathfrak{g}})\left(L_0 - \frac{c}{24}\right), \quad c = \frac{1}{2}\dim\mathfrak{g} + \frac{\ell \dim \mathfrak{g}}{(\ell + h^\vee_{\mathfrak{g}})}$$

$$[\not{D}_{L\mathfrak{g}}, X] = 0, \quad \text{for all } X \in \mathfrak{g}$$

$$[\not{D}_{L\mathfrak{g}}, L_0] = 0$$

Here L_0 is the operator in the Virasoro algebra associated to the rotation action $\partial/\partial\theta$ given by Sugawara's formula

$$L_0 = -\frac{1}{\ell + h^\vee}\left(\frac{1}{2}\sum_a X_a(0)X_a(0) + \sum_{n\neq 0}\sum_a X_a(n)X_a(-n)\right),$$

and therefore closely related to the energy operator D. Of course, the third equation follows immediately from the first. Notice that $\not{D}_{L\mathfrak{g}}$ only commutes with $\mathfrak{g}$, not with the whole Lie algebra $L\mathfrak{g}$ as its finite dimensional counterpart. In special cases, the square of the Dirac operator can be worked out further to give:

COROLLARY 2.2 (cf. [12]) . *For* $E = E_\lambda$ *irreducible, one has*

$$\not{D}^2_\lambda = 2(\ell + h^\vee_{\mathfrak{g}})d + ||\lambda - \rho_{\mathfrak{g}}||^2.$$

Proof. This follows from the previous lemma and the fact that L_0 differs from the energy operator d by a constant which can be computed explicitly, see [12]. □

2.4. THE HOMOGENEOUS DIRAC OPERATOR

Next, we again consider the inclusion $H \subseteq G$ of an equal rank closed subgroup. The decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$, induces a decomposition $L\mathfrak{g} = L\mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$, and the Clifford algebra factors as $\mathrm{Cliff}(L\mathfrak{g}) = \mathrm{Cliff}(L\mathfrak{g}/\mathfrak{h}) \otimes \mathrm{Cliff}(\mathfrak{h})$. Eventually, one finds an H-equivariant decomposition of the Spin representation $\mathbb{S}_{L\mathfrak{g}} = \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}} \otimes \mathbb{S}_{\mathfrak{h}}$, where

$$\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}} = \mathbb{S}_{\mathfrak{g}/\mathfrak{h}} \otimes \bigotimes_{k>0} \bigwedge^{\bullet} \mathfrak{g}_{\mathbb{C}} z^k.$$

Next, consider the difference operator $\not{D}_{L\mathfrak{g}/\mathfrak{h}} := \not{D}_{L\mathfrak{g}} - \not{D}'_{\mathfrak{h}}$. Using the Equations (1), (3), and (7) one finds the explicit expression

$$\not{D}_{L\mathfrak{g}/\mathfrak{h}} = \not{D}_{\mathfrak{g}/\mathfrak{h}} + \sum_{n\neq 0}\sum_{a=1}^{\dim\mathfrak{g}}\left(X_a(n)\otimes\psi^a(-n) + 1\otimes\frac{1}{3}\mathrm{ad}_{X_a(n)}\psi^a(-n)\right).$$

This formula clearly shows that $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$ acts on any space of the type $E \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$, where E is a positive energy representation of LG at level ℓ. Again restricting to "zero modes", one finds (3), the finite dimensional coset Dirac operator.

LEMMA 2.3. $[\not{D}_{L\mathfrak{g}/\mathfrak{h}}, X] = 0,\ \forall X \in \mathfrak{h}$.

Proof. As remarked, $\not{D}_{L\mathfrak{g}} - \not{D}_{L\mathfrak{g}/\mathfrak{h}} = \not{D}'_{\mathfrak{h}}$, the Dirac operator acting on $E \otimes \mathbb{S}_{L\mathfrak{g}} \cong (E \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}) \otimes \mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$, associated to the $\mathfrak{h}$-representation on $E \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$. Therefore, this operator is $\mathfrak{h}$-invariant by construction. The result now follows from the fact that $\not{D}_{L\mathfrak{g}}$ commutes with the representation of $\mathfrak{g}$, cf. Proposition 2.1. □

For the sake of future index calculations it is convenient to compute the square of this operator:

PROPOSITION 2.4. *The square of the Dirac operator is given by*

$$\frac{1}{2}\not{D}^2_{L\mathfrak{g}/\mathfrak{h}} = (\ell + h^\vee_{\mathfrak{g}})\left(L_0 - \frac{c}{24}\right) - \Delta_{\mathfrak{h}} - \frac{1}{24}\mathrm{tr}_{\mathfrak{h}}\Delta_{\mathfrak{h}}.$$

Proof. This is proved by direct computation. □

Remark 2.5. Lemma 2.3 and Proposition 2.4, are direct analogues of properties of the Dirac operator associated to the inclusion $L\mathfrak{h} \subset L\mathfrak{g}$ proved in [12]. Also, Proposition 2.4, in the case $L\mathfrak{g}/\mathfrak{g}$, appeared in [16, Section 2], in a slightly different language.

Clearly, the Dirac operator $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$ commutes with the energy operator D, so we can decompose $E \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ with respect to the $\mathrm{Rot}(S^1) \times H$-action, to exhibit the structure of $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$ in more detail. We label the irreducible representations of $\mathrm{Rot}(S^1) \times H$ by elements in $\mathbb{Z} \times \Lambda^*_H$. For an irreducible positive energy representation we find:

COROLLARY 2.6. *For* E_λ, $\boldsymbol{\lambda} \in \Lambda^*_\ell$ *irreducible, one has, restricted to the isotypical summand labeled by* $(\mu, n) \in \Lambda^*_H \times \mathbb{Z}_+$,

$$\begin{aligned}\not{D}^2_{L\mathfrak{g}/\mathfrak{h}}\Big|_{(\mu,n)} &= 2(\ell + h^\vee_{\mathfrak{g}})n + \|\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}}\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 \\ &= \|\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}}\|^2 - \|\boldsymbol{\mu} - \boldsymbol{\rho}_{\mathfrak{h}}\|^2,\end{aligned}$$

where $\boldsymbol{\mu} = (n, \mu, \ell + h^\vee_{\mathfrak{g}})$ *and* $\boldsymbol{\rho}_{\mathfrak{h}} = (0, -\rho_{\mathfrak{h}}, 0)$.

Proof. This follows from Proposition 2.4, and the fact that the value of the Casimir in the irreducible representation V_λ is given by $(\|\lambda - \rho_{\mathfrak{g}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)/2$. □

3. The Kernel and a Generalized Weyl–Kac Formula

3.1. THE KERNEL OF THE DIRAC OPERATOR

For an equal rank pair $\mathfrak{h} \subseteq \mathfrak{g}$ we have $W_H \subseteq W_{\mathrm{aff}}$, where W_{aff} denotes, as before, the affine Weyl group associated to G. Introduce the following subset $C_{\mathfrak{h},\ell} \subseteq W_{\mathrm{aff}}$; $C_{\mathfrak{h},\ell}$ is the subset of elements of W_{aff} that map the affine Weyl alcove of $\mathfrak{g}$ at level ℓ into the negative Weyl chamber of $\mathfrak{h}$. Notice that by using the negative instead of the positive Weyl chamber, we have $1\in C_{\mathfrak{h},\ell}$. This subset defines a section of the projection $W_{\mathrm{aff}}\to W_{\mathrm{aff}}/W_{\mathrm{H}}$ to the cosets of W_{H} in W_{aff}. In other words, the obvious map $W_{\mathrm{H}}\times C_{\mathfrak{h},\ell}\to W_{\mathrm{aff}}$ is bijective. Notice that when $\mathfrak{h}=\mathfrak{t}$, this set is simply the whole affine Weyl group. Let Δ^+ be the system of positive roots for LG given in (5). For $w\in W_{\mathrm{aff}}$, define

$$\Phi_w := w(-\Delta^+)\cap\Delta^+.$$

In the following we will need the following properties of this set, which are a direct generalization of the finite dimensional case:

LEMMA 3.1. *The set* $\Phi_w\subseteq\Delta^+$ *has the following properties*:

(i) Φ_w *is a finite set, and* $|\Phi_w|=\ell(w)$.
(ii) $\Phi_w\subseteq\Delta^+\backslash\Delta^+(\mathfrak{h})$, *if and only if* $w\in C_{\mathfrak{h},\ell}$.
(iii) *One has*

$$\sum_{\alpha_i\in\Phi_w}\alpha_i=\rho_{\mathfrak{g}}-w\rho_{\mathfrak{g}}.$$

Proof. Notice that for $w\in W\subseteq W_{\mathrm{aff}}$, Φ_w is the usual subset of the positive roots of $\mathfrak{g}\subseteq L\mathfrak{g}$, for which the properties above are well known [9]. The proof is a straightforward adaption to the affine case:

(i) Let α be a simple affine root, and denote by s_α the associated reflection in the hyperplane determined by α. Since $s_\alpha(\alpha)=-\alpha$ and s_α permutes $\Delta^+\backslash\{\alpha\}$, one finds $\Phi_{s_\alpha}=\{\alpha\}$. Next, observe that for a reduced expression $w=s_{\alpha_1}\dots s_{\alpha_{\ell(w)}}$ one has $\alpha_i\in\Phi_w$, $\forall i=1,\dots,\ell(w)$, by the following argument: Suppose that some $\alpha_k\notin\Phi_w$. Then there is a $0<j\leq k$ such that $w_j:=s_{\alpha_{j+1}}\dots s_{\alpha_{k-1}}$ satisfies $w_j\alpha_k\in\Delta^+$, $s_{\alpha_j}w_j\alpha_k\in-\Delta^+$. Therefore $w_j\alpha_k=\alpha_j$ and it follows from the general relation $ws_\alpha w^{-1}=s_{w\alpha}$, $\forall w\in W_{\mathrm{aff}}$ that $w_js_{\alpha_k}w_j^{-1}=s_{\alpha_j}$. Writing out this relation one finds $w=s_{\alpha_1}\dots s_{\alpha_{j-1}}s_{\alpha_{j+1}}\dots s_{\alpha_{\ell(w)-1}}$, contradicting the fact that the expression for w was reduced.

Since $\alpha_i\in\Phi_w$, $ws_{\alpha_i}(\alpha_i)=-w\alpha_i\in\Delta^+$, so $\alpha_i\notin\Phi_{ws_{\alpha_i}}$. In this case one easily checks that $\Phi_{ws_{\alpha_i}}=s_{\alpha_i}\Phi_w\cup\{\alpha_i\}$. It follows that $\left|\Phi_{ws_{\alpha_i}}\right|=|\Phi_w|+1$. This completes the proof of i).

(ii) This follows from the definitions: As usual, we embed $\mathfrak{t}^*\hookrightarrow\mathbb{R}\oplus\mathfrak{t}^*\oplus\mathbb{R}$ by $\lambda\mapsto(0,\lambda,\ell)$. The negative Weyl chamber of $\mathfrak{h}$ is defined by

$$S_{\mathfrak{h}}:=\{\lambda\in\mathfrak{t}^*,\ \ \langle\lambda,\alpha\rangle\leq 0,\ \ \forall\alpha\in\Delta^+(\mathfrak{h})\}.$$

Now suppose that $\Phi_w\subseteq\Delta^+\backslash\Delta^+(\mathfrak{h})$ for some $w\in W_{\mathrm{aff}}$. Then $w^{-1}(\alpha)\in\Delta^+$, $\forall\alpha\in\Delta^+(\mathfrak{h})$. Let $\boldsymbol{\lambda}=(0,\lambda,\ell)\in\mathfrak{A}_\ell$. It therefore follows that $\langle w\boldsymbol{\lambda},\alpha\rangle=\langle\boldsymbol{\lambda},w^{-1}\alpha\rangle\leq 0$, $\forall\alpha\in\Delta^+(\mathfrak{h})$. In other words $w(\mathfrak{A}_\ell)\subseteq S_{\mathfrak{h}}$, i.e., $w\in C_{\mathfrak{h},\ell}$. The implication in the other direction follows similarly since the Weyl alcove $\mathfrak{A}_\ell$ determines the positive roots Δ^+ as those for which $\langle\boldsymbol{\lambda},\alpha\rangle\leq 0$ for all $\boldsymbol{\lambda}\in\mathfrak{A}_\ell$.

(iii) This is [17, Prop. 14.3.3.]. □

Now, let λ be a dominant weight at level ℓ. We see from Corollary 2.6 that we are actually interested in the affine Weyl group action at the shifted level $\ell + h_{\mathfrak{g}}^{\vee}$. Therefore we put $C := C_{\mathfrak{h}, \ell + h_{\mathfrak{g}}^{\vee}}$. The shifted weight $\lambda - \rho_{\mathfrak{g}}$ lies in the fundamental alcove $\mathfrak{A}_{\ell + h_{\mathfrak{g}}^{\vee}}$ of $\mathfrak{g}$. Therefore, the weight

$$w \bullet \lambda := w(\lambda - \rho_{\mathfrak{g}}) + \rho_{\mathfrak{h}},$$

where $\rho_{\mathfrak{h}} = (0, -\rho_{\mathfrak{h}}, 0)$ and $w \in C$, is dominant for $H \times \mathbb{T}$. Before stating the main theorem about the kernel of the Dirac operator, we need two results concerning the weight decomposition of $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$:

PROPOSITION 3.2. *For $w \in C$, the multiplicity of the irreducible representation $V_{w \bullet \lambda}$ of $H \times \mathbb{T}$ in $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ is one.*

Proof. As is well known, the lowest weight of the tensor product $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$ is given by $\lambda - \rho_{\mathfrak{g}}$. Since the multiplicity of the lowest weight is always one, one finds the weights $w(\lambda - \rho_{\mathfrak{g}})$ appearing in $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$ with multiplicity 1. This proves the assertion in the case $\mathfrak{h} = \mathfrak{t}$; it follows alternatively from the Weyl–Kac formula.

For $\mathfrak{t} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$, one has $\mathbb{S}_{L\mathfrak{g}/\mathfrak{t}} \cong \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}} \otimes \mathbb{S}_{\mathfrak{h}/\mathfrak{t}}$, and the lowest weight of $\mathbb{S}_{\mathfrak{h}/\mathfrak{t}}$ is $-\rho_{\mathfrak{h}}$. Therefore, the multiplicity of the weight $w \bullet \lambda$ in $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ is at most one, since such a weight contributes a weight of the form $w(\lambda - \rho_{\mathfrak{g}})$ in $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$ using the factorization of the spin representation above.

To show that the multiplicity is actually one, we construct, following [10] in the finite dimensional case, an element in $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ with weight $w \bullet \lambda$. Choose coroots $e_\alpha \in \mathfrak{g}_\alpha \subseteq L\mathfrak{g}$, and define the element

$$\left(\prod_{\alpha_i \in \Phi_w} \psi(e_{\alpha_i}) \right) \Omega_{L\mathfrak{g}/\mathfrak{h}} \in \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}},$$

where $\Omega_{L\mathfrak{g}/\mathfrak{h}}$ is the vacuum of $\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$. Notice that this is well defined because of Lemma 3.1 (ii), and is nonzero because the α_i are positive roots and therefore do not annihilate Ω (that is, they act as "creation operators"). The ray determined by this vector is independent of the choice ordering of Φ_w in the product above, as well as the choice of coroots. Using the commutation relations in $L\mathfrak{g} \ltimes \mathrm{Cliff}(L\mathfrak{g})$, and the fact that the weight of $\Omega_{L\mathfrak{g}/\mathfrak{h}}$ with respect to $\mathrm{Rot}(S^1) \times H$ is given by $(0, \rho_{\mathfrak{g}} - \rho_{\mathfrak{h}})$, the weight of e_w may be computed as

$$-\sum_{\alpha_i \in \Phi_w} \alpha_i + \rho_{\mathfrak{g}} - \rho_{\mathfrak{h}} = w\rho_{\mathfrak{g}} - \rho_{\mathfrak{h}}.$$

Therefore, the tensor product of e_w with we_λ, where $e_\lambda \in E_\lambda$ is the lowest weight vector, has weight $w \bullet \lambda$, as required. □

LEMMA 3.3. *For any weight μ of $E_\lambda \otimes \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ with $\|\mu - \rho_{\mathfrak{h}}\|^2 = \|\lambda - \rho_{\mathfrak{g}}\|^2$, there exists a unique $w \in W_{aff}$ such that $\mu - \rho_{\mathfrak{h}} = w(\lambda - \rho_{\mathfrak{g}})$.*

Proof. The proof is exactly the same as in the finite dimensional case [10,12]: First notice that any weight of $\mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$ can be obtained from $-\rho_{\mathfrak{g}}$ by adding a sum of positive roots. Therefore, the same applies to the weight $\lambda-\rho_{\mathfrak{g}}$ of $E_\lambda\otimes\mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$. Now let μ be a weight of $E_\lambda\otimes\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$. Then $\mu-\rho_{\mathfrak{h}}$ is a weight of $E_\lambda\otimes\mathbb{S}_{L\mathfrak{g}/\mathfrak{t}}$, and therefore

$$||\mu-\rho_{\mathfrak{h}}||^2\geq||\lambda-\rho_{\mathfrak{g}}||^2.$$

Now pick a $w\in W_{\mathrm{aff}}$ for which the weight $w^{-1}(\mu-\rho_{\mathfrak{h}})$ lies in the fundamental alcove at level $\ell+h^\vee_{\mathfrak{g}}$. The transformed weight satisfies the same inequality as above, with equality only when $w^{-1}(\mu-\rho_{\mathfrak{h}})=\lambda-\rho_{\mathfrak{g}}$ (cf. [17, Lemma 14.4.7]). But in this case, since $\lambda-\rho_{\mathfrak{g}}$ lies in the interior of the fundamental alcove, $w\in W_{\mathrm{aff}}$ is unique. □

After these preparations, we are ready to state the fundamental result about the kernel of the Dirac operator $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$. Let $R(H)[[q]]$ be the representation ring of representations of $H\times\mathrm{Rot}(S^1)$, where we allow infinite sums, with finite multiplicity, of irreducibles of $\mathrm{Rot}(S^1)$.

THEOREM 3.4. *Let $E_\lambda,\lambda\in\Lambda^*_\ell$ be the irreducible positive energy representation of* LG *with lowest weight λ. Then the kernel of the Dirac operator $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$ on $E_\lambda\otimes\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ is given by*

$$\ker(\not{D}_{L\mathfrak{g}/\mathfrak{h}})=\bigoplus_{w\in C}V_{w\bullet\lambda}\in R(H)[[q]].$$

Proof. Clearly, one has

$$||w\bullet\lambda-\rho_{\mathfrak{h}}||^2=||w(\lambda-\rho_{\mathfrak{g}})||^2=||\lambda-\rho_{\mathfrak{g}}||^2,$$

since the affine Weyl group acts by isometries, and therefore by Corollary 2.6 the isotypical summands of type $w\bullet\lambda$ belong to the kernel of the Dirac operator. By the previous two lemmas, this proves the theorem. □

Since $\mathfrak{h}\subseteq\mathfrak{g}$ is of equal rank, $\mathfrak{p}=\mathfrak{g}/\mathfrak{h}$ is even dimensional and therefore $\mathbb{S}_{\mathfrak{p}}$ is $\mathbb{Z}_2$-graded, and the grading commutes with the action of $\mathfrak{h}$. Likewise, the spin representation $\mathbb{S}_{L\mathfrak{g}}$ has a $\mathbb{Z}/2$-grading as well, as has $\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$, and we write $\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}=\mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}}\oplus\mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}$. The Dirac operator $\not{D}_{L\mathfrak{g}/\mathfrak{h}}$ anticommutes with this grading and we write

$$\not{D}_{L\mathfrak{g}/\mathfrak{h}}=\begin{pmatrix}0 & \not{D}^-_{L\mathfrak{g}/\mathfrak{h}}\\ \not{D}^+_{L\mathfrak{g}/\mathfrak{h}} & 0\end{pmatrix}, \tag{8}$$

where $\not{D}^\pm_{L\mathfrak{g}/\mathfrak{h}}:E\otimes\mathbb{S}^\pm_{L\mathfrak{g}/\mathfrak{h}}\to E\otimes\mathbb{S}^\mp_{L\mathfrak{g}/\mathfrak{h}}$. Notice that the formal adjoint of $\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}$ is given by $\not{D}^-_{L\mathfrak{g}/\mathfrak{h}}$, and therefore the cokernel of $\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}$ is the kernel of $\not{D}^-_{L\mathfrak{g}/\mathfrak{h}}$ and vice versa. Clearly, the $H\times\mathbb{T}$-equivariant index of $\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}$ does not exists in the representation ring of $H \times \mathbb{T}$ since E and $\mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ are infinite dimensional representations. However, by the positive energy condition, its *formal* H-*index*

$$\widehat{\mathrm{index}}_H(\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}) \in R(H)[[q]]$$

defined by taking the H-index in each energy degree, is well defined.

COROLLARY 3.5. *The formal* H-*index is given by*

$$\widehat{\mathrm{index}}_H(\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}) = \bigoplus_{w \in C} (-1)^{\ell(w)} V_{w \bullet \lambda}.$$

Proof. By the previous theorem, we only have to determine whether $V_{w \cdot \lambda}$ is in the kernel of $\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}$ or $\not{D}^-_{L\mathfrak{g}/\mathfrak{h}}$, i.e., whether the vector $e_w \in \mathbb{S}_{L\mathfrak{g}/\mathfrak{h}}$ lies inside $\mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}}$ or $\mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}$. But since one has, by Lemma 3.1, that $\ell(w) = |\Phi_w|$, it follows that for $\ell(w) =$ even, $e_w \in \mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}}$, and for $\ell(w) =$ odd, $e_w \in \mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}$. Separating the terms in Theorem 3.4 according to the sign $(-1)^{\ell(w)}$ now gives the result. $\square$

3.2. A GENERALIZATION OF THE WEYL–KAC FORMULA

We now come to a generalization of the homogeneous Weyl formula of [6] to loop groups. It generalizes the Weyl–Kac formula (see below) in the same manner as the homogeneous Weyl formula generalizes the Weyl formula. As before, $H \subseteq G$ is an equal rank subgroup.

COROLLARY 3.6 (Homogeneous Weyl–Kac formula) . *The following identity holds in in the formal representation ring* $R(H)[[q]]$:

$$\left[E_\lambda \otimes \mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}}\right] - \left[E_\lambda \otimes \mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}\right] = \bigoplus_{w \in C} (-1)^{\ell(w)} V_{w \bullet \lambda}$$

Proof. Consider the Dirac operator $\not{D}^+_{L\mathfrak{g}/\mathfrak{h}} : E_\lambda \otimes \mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}} \to E_\lambda \otimes \mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}$. Since the Dirac operator preserves the energy grading and we have, by the positive energy assumption, a finite dimensional H-representation in each degree, its formal H-index is given by

$$\widehat{\mathrm{index}}_H(\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}) = \left[E \otimes \mathbb{S}^+_{L\mathfrak{g}/\mathfrak{h}}\right] - \left[E \otimes \mathbb{S}^-_{L\mathfrak{g}/\mathfrak{h}}\right] \in R(H)[[q]].$$

On the other hand, we have the explicit expression for the index given by Corollary 3.5. Putting the two together gives the result. $\square$

As a further corollary, we have the Weyl–Kac formula:

COROLLARY 3.7 (Weyl–Kac character formula) . *The character in* $R(T)[[q]]$ *of an irreducible positive energy representation* E_λ *is given by*

$$\chi(E_\lambda) = \frac{\sum_{w \in W_{\mathrm{aff}}} (-1)^{\ell(w)} e^{iw(\lambda - \rho)}}{e^{i\rho} \prod_{\alpha > 0} (1 - e^{i\alpha})}.$$

Proof. Notice that for $H = T$, the maximal torus, the subset C is simply the whole affine Weyl group W_{aff}. Since the super-character of the spin representation is given by

$$\chi(\mathbb{S}^{+}_{L\mathfrak{g}/\mathfrak{t}}) - \chi(\mathbb{S}^{-}_{L\mathfrak{g}/\mathfrak{t}}) = e^{i\rho} \prod_{\alpha>0} (1 - e^{i\alpha}),$$

the result follows from the previous corollary. □

Remark 3.8. A similar proof of the Weyl–Kac formula was given in [21].

4. Dirac Induction for Loop Groups

4.1. THE LG-EQUIVARIANT INDEX

Let E and F be Hilbert spaces carrying positive energy representations of LG at level ℓ, not necessarily of finite type, and let $T: E \to F$ be an intertwining operator. We say that T is LG-Fredholm when both $\ker(T)$ and $\operatorname{coker} T$ are positive energy representations of finite type and define its LG-equivariant index as

$$\operatorname{index}_{\widetilde{LG}}(T) = [\ker(T)] - [\operatorname{coker}(T)] \in R_\ell(G).$$

Notice that when both E and F are already of finite type (this does not happen in our case below), then one proves, using the energy grading,

$$\operatorname{index}_{\widetilde{LG}}(T) = [E] - [F],$$

see [12, Lemma 13].

4.2. INDUCTION

We now follow the same path as in the finite dimensional case to construct the induction homomorphism. First we need a Hilbert space which can be considered as the L^2- space of the loop group LG, or rather of section of the line bundle $\mathcal{L}$ determined by the central extension of LG, i.e., the level. Notice that literally this space does not exist, since the smooth loop group does not support an invariant measure. However, we can consider the following projective representation of $LG \times LG$ on the Hilbert space of the Wess–Zumino–Witten (WZW) model:

$$H^{\ell}_{\mathrm{WZW}} := \bigoplus_{\lambda \in \Lambda^*_\ell} E_{*\lambda} \otimes E_\lambda. \tag{9}$$

Here $*: \Lambda^*_\ell \to \Lambda^*_\ell$ is the involution given by $-w_0$, where $w_0 \in W_G$ is the longest Weyl group element. Therefore, $E_{*\lambda}$ has the lowest energy G-module $E_{*\lambda}(0) \cong V^*_{-\lambda} \cong V_\lambda$.

Remark 4.1. It is not the aim of this paper to justify this choice of Hilbert space, but morally it comes from conformal field theory: recall the Peter–Weyl decomposition of $L^2(G)$, as well the fact that from the point of view of physics the Hilbert space $L^2(G)$ describes the quantum theory of particles on G. By contrast, the Wess–Zumino–Witten model in conformal field theory describes strings moving on G, i.e., the loop group analogue of particles on G. As far as I know, a rigorous justification, that is, a Peter–Weyl decomposition for loop groups is not available. In [20], an algebraic version is proved identifying H^{ℓ}_{wzw}, or rather its completion, as the space of algebraic sections of a line bundle over a homogeneous space $Y(\mathbb{P}^1, 0, \infty)$ for a product of two Laurent completions $G_{\mathbb{C}}((z))$ of the loop group, canonically associated to $\mathbb{P}^1$ with the marked points $z=0,\infty$, see also [5, Section 17.3]. In connection with Section 4.3, this opens the possibility to make contact with the geometric representation theory of loop groups.

Again consider an equal rank pair $H\subseteq G$, and a representation V of H. By analogy, we now consider the Hilbert space

$$H^{\pm}_V := \left(H^{\ell - h^\vee_{\mathfrak{g}}}_{\mathrm{wzw}} \otimes \mathbb{S}^{\pm}_{L\mathfrak{g}/\mathfrak{h}} \otimes V \right)^H.$$

It is most natural to introduce the above shift by the dual Coxeter number at this stage. Geometrically, it can be understood from the fact that the spin bundle over LG only carries an action of the level $h^\vee_{\mathfrak{g}}$ central extension of LG: to trivialize it, one has to tensor with dual of the fundamental line bundle over LG at level $h^\vee_{\mathfrak{g}}$. Next, the construction of Section 2.4 yields an LG equivariant operator

$$\not{D}^+_V = H^+_V \to H^-_V.$$

Therefore, we define the Dirac induction of V to be the virtual positive energy representation

$$\mathrm{Ind}(V) := \mathrm{index}_{\widetilde{LG}}(\not{D}^+_V). \tag{10}$$

PROPOSITION 4.2. *Dirac induction defines an additive homomorphism*

$$\mathrm{Ind} : R(H) \to R_{\ell - h^\vee}(G).$$

Proof. Evidently, the induced space carries a positive energy representation of LG. Therefore the only thing that remains to be proved is that this representation is of finite type. For this, observe that the formal H-index of Corollary 3.5, by “forgetting the $\mathbb{T}$-action”, in fact defines an element in the formal completion $\hat{R}(H)$ of $R(H)$,

$$\widehat{\mathrm{index}}_H(\not{D}^+_{L\mathfrak{g}/\mathfrak{h}}) \in \hat{R}(H),$$

since every irreducible representation occurs with finite multiplicity. Using the decomposition (9) and taking H-invariants, one finds

$$\mathrm{index}_{\widetilde{LG}}(\not{D}_V^+) = \sum_{\lambda\in\Lambda_\ell^*} \left\langle \widehat{\mathrm{index}}_H(\not{D}_\lambda^+), V\right\rangle E_\lambda, \tag{11}$$

where $\langle,\rangle : \hat{R}(H)\times R(H)\to\mathbb{Z}$ is the pairing $\langle V,W\rangle := \dim \mathrm{Hom}_H(V,W)$. This clearly defines an element of $R_{\ell-h^\vee}(G)$, and the result follows. □

We can even, as in the finite dimensional case, explicitly compute the value of the generators of $R(H)$, i.e., the irreducible representations of H:

THEOREM 4.3. *Let* $V_\lambda\in R(H)$ *be irreducible, then one has*

$$\mathrm{Ind}(V_\lambda) = \begin{cases} (-1)^{\ell(w)} E_{w(\boldsymbol{\lambda}-\rho_{\mathfrak{h}})+\rho_{\mathfrak{g}}} & \exists w\in W_{aff},\ w(\boldsymbol{\lambda}-\rho_{\mathfrak{h}})+\rho_{\mathfrak{g}}\in\Lambda^*_{\ell-h^\vee} \\ 0 & \text{else}, \end{cases}$$

where, as before, $\boldsymbol{\lambda}=(0,-\lambda,\ell)$.

Proof. It follows from Corollary 3.5 that the coefficients in the expansion (11) in this case are given by

$$\left\langle \widehat{\mathrm{index}}_H(\not{D}_\mu^+), V_\lambda\right\rangle = \begin{cases} (-1)^{\ell(w)} & \exists w\in W_{\mathrm{aff}},\ w\bullet\boldsymbol{\mu} = *\boldsymbol{\lambda} \\ 0 & \text{else}. \end{cases}$$

Since at most one $\boldsymbol{\mu}\in\Lambda^*_{\ell-h^\vee}$ lies in the W_{aff}-orbit of $\boldsymbol{\lambda}$, the result follows. □

In fact, looking at the form (8) of the Dirac operator $\not{D}_V$, we see that the induced representation is realised as the kernel (instead of the index) $\ker(\not{D}_V)$, where the $\mathbb{Z}/2$ grading comes from the grading $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$;

$$\mathrm{Ind}(V) = \ker(\not{D}_V^+) - \ker(\not{D}_V^-).$$

4.3. LIE ALGEBRA COHOMOLOGY

There are two important special cases of the induction homomorphism of the previous section, both of which reduce to holomorphic induction: first, when $H=T$, the maximal torus, the theorem above shows that one finds back the Borel–Weil theorem for loop groups as in [17]. And secondly, in contrast to the finite dimensional Dirac induction procedure, the statement of the theorem above is also interesting for $H=G$, in which case one finds a homomorphism $\mathrm{Ind}: R(G)\to R_{\ell-h^\vee}(G)$. In this case, the theorem shows that it is essentially the holomorphic induction map of [19], whose definition is attributed to R. Bott. In fact, in these cases we can give a precise relation between Dirac induction and holomorphic induction as follows. we have two decompositions of the Lie algebra: first $L\mathfrak{g}_{\mathbb{C}} = L\mathfrak{g}_{\mathbb{C}}^-\oplus\mathfrak{g}_{\mathbb{C}}\oplus L\mathfrak{g}_{\mathbb{C}}^+$, and second the triangular decomposition $L\mathfrak{g}_{\mathbb{C}} = \bar{\mathfrak{N}}\oplus\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{N}$, where

$\mathfrak{N} := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \oplus L\mathfrak{g}_{\mathbb{C}}^+$. Since $L\mathfrak{g}_{\mathbb{C}}^+$ and $\mathfrak{N}$ are isotropic with respect to the cocycle defining the central extension by which $L\mathfrak{g}_{\mathbb{C}}$ acts on a positive energy representation on E, these representations restrict to honest, i.e., non-projective, representations of these Lie subalgebras. We can therefore consider the Lie algebra cohomology of $L\mathfrak{g}_{\mathbb{C}}^+$ and $\mathfrak{N}$ with values in E. For $L\mathfrak{g}_{\mathbb{C}}^+$, this cohomology is computed by the Chevalley differential

$$\bar{\partial}_{L\mathfrak{g}_{\mathbb{C}}^+} := \sum_{k=1}^{\infty} \sum_{a=1}^{\dim(\mathfrak{g})} \left(\psi_k^a \otimes X_{-k}^a + \frac{1}{2} \psi_k^a \cdot \mathrm{ad}_{X_{-k}^a} \otimes 1 \right),$$

acting on $E \otimes \bigwedge^\bullet (z\mathfrak{g}_{\mathbb{C}}[z])$. In case of $\mathfrak{N}$, we choose a root basis X^a of $\mathfrak{n} := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ to define

$$\bar{\partial}_{\mathfrak{n}} := \sum_a \left(\psi^a \otimes X^a + \frac{1}{2} \psi^a \cdot \mathrm{ad}_{X^a} \otimes 1 \right),$$

and finally put $\bar{\partial}_{\mathfrak{N}} := \bar{\partial}_{L\mathfrak{g}_{\mathbb{C}}^+} + \bar{\partial}_{\mathfrak{n}}$, which acts on $E \otimes \bigwedge^\bullet \mathfrak{N} \cong E \otimes \bigoplus_{p+q=\bullet} \bigwedge^p \mathfrak{n} \otimes \bigwedge^q (z\mathfrak{g}_{\mathbb{C}}[z])$. Now both exterior algebras of $L\mathfrak{g}_{\mathbb{C}}^+$ and $\mathfrak{N}$ appear as a factor in $\mathbb{S}_{L\mathfrak{g}} \cong \mathbb{S}_{\mathfrak{t}} \otimes \bigwedge^\bullet \mathfrak{n} \otimes \bigwedge^\bullet (z\mathfrak{g}_{\mathbb{C}}[z])$. Denote by $\bar{\partial}^*$ the formal adjoint of the Chevalley differential and with this we have

PROPOSITION 4.4. *The following equalities hold true*:

(i) $\not{D}_{L\mathfrak{g}/\mathfrak{t}} = \bar{\partial}_{\mathfrak{N}} + \bar{\partial}_{\mathfrak{N}}^*$,

(ii) $\not{D}_{L\mathfrak{g}/\mathfrak{g}} = \bar{\partial}_{L\mathfrak{g}_{\mathbb{C}}^+} + \bar{\partial}_{L\mathfrak{g}_{\mathbb{C}}^+}^*$.

Proof. (i) This is Proposition 14.3 of [5], stated in a slightly different notation. ii) follows from i) by the definition of $\not{D}_{L\mathfrak{g}/\mathfrak{g}} := \not{D}_{L\mathfrak{g}} - \not{D}_{\mathfrak{g}}$, together with the fact that $\not{D}_{\mathfrak{g}/\mathfrak{t}} = \bar{\partial}_{\mathfrak{n}} + \bar{\partial}_{\mathfrak{n}}^*$. □

In both cases the kernel of the Dirac operator can be identified as harmonic forms representing classes in Lie algebra cohomology. In the second case, induction from G to LG, this identifies Dirac induction as defined in Equation (10) with Teleman's definition, cf. [19, Prop. 3.6.1], of holomorphic induction in terms of Lie algebra cohomology:

$$\mathrm{Ind}(V) := \bigoplus_{\lambda \in \Lambda^*_{\ell - h^\vee_{\mathfrak{g}}}} \chi\left(L^+\mathfrak{g}_{\mathbb{C}}; E_{*\lambda} \otimes V\right) E_\lambda \in R_{\ell - h^\vee}(G),$$

where $\chi\left(L^+\mathfrak{g}_{\mathbb{C}}; -\right)$ is the Euler characteristic of the Lie algebra cohomology groups $H^\bullet_{Lie}(L\mathfrak{g}_{\mathbb{C}}^+; -)$. In fact, he also proves that these cohomology groups are concentrated in degree zero. The main point of [19] was to prove that holomorphic induction is a *ring* homomorphism with respect to the fusion product of loop group representations. This was achieved by using the Lie algebra cohomology of a general evaluation module $\vec{V}(\vec{z})$ of $\mathfrak{g}_{\mathbb{C}}[z]$, instead of V, a single $\mathfrak{g}_{\mathbb{C}}$-module at the origin of the unit disk. Geometrically, this defines a holomorphic vector bundle over the fundamental homogeneous space LG/G. It would be interesting to consider the Dirac operator coupled to such more general vector bundles.

4.4. INDUCTION IN STAGES

It should be clear that our construction is a natural generalization of Dirac induction for compact Lie groups. In fact, given the Dirac operator $\not{D}_{L\mathfrak{g}}$, the proof of Theorem 4.3 is almost literary the same as that of Theorem 1.1. To stress this point, we combine both induction homomorphisms to prove the following induction in stages:

THEOREM 4.5 (Induction in stages). *For* $H \subseteq K \subseteq G$ *of equal rank*

$$\mathrm{Ind}_H^{LG} = \mathrm{Ind}_K^{LG} \circ \mathrm{Ind}_H^K.$$

Proof. We evaluate both sides on generators $V_\lambda \in R(H)$ irreducible. Suppose there exists a $w \in W_{\mathrm{aff}}$ such that $w(\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{h}}) + \boldsymbol{\rho}_{\mathfrak{g}} \in \Lambda^*_{\ell-h^\vee}$. In other words, there is a $\boldsymbol{\mu} \in \Lambda^*_{\ell-h^\vee}$ such that $c \bullet \boldsymbol{\mu} = \boldsymbol{\lambda}$, with $c = w^{-1} \in C$, where this time elements of C map the Weyl alcove at level ℓ into the negative Weyl chamber of $\mathfrak{h}$. As for W_{aff}, there is a cros-section $C_{\mathfrak{h}} = \{w \in W_K,\ w(S_{\mathfrak{k}}) \subseteq S_{\mathfrak{h}}\}$ of the fibration $W_K \to W_K/W_H$ [10]. Denote by $D \subseteq W_{\mathrm{aff}}$ the subset of elements that map the Weyl alcove of G at level ℓ into the negative Weyl chamber of K. Then one finds a bijection

$$W_H \times C_{\mathfrak{h}} \times D \to W_{\mathrm{aff}},\ (w, c, s) \mapsto wcs.$$

Combining with the map $W_H \times C \to W_{\mathrm{aff}}$, one finds a bijection $C_{\mathfrak{h}} \times D \to C$ induced by multiplication in W_{aff}. In other words, every $c \in C$ has a unique factorization $c = ws,\ s \in D$ and $w \in C_{\mathfrak{h}}$. By definition then $s \bullet \boldsymbol{\mu} \in \Lambda^*_K$ and $w \bullet s \bullet \boldsymbol{\mu} = (ws) \bullet \boldsymbol{\mu} = c \bullet \boldsymbol{\mu} = \boldsymbol{\lambda} \in \Lambda^*_H$. Since the factorization of $c \in C$ is unique, this proves that both homomorphisms have the same value on V_λ, and therefore coincide. □

COROLLARY 4.6. *For any equal rank inclusion* $H \subset G$, *Dirac induction factors as:*

$$\mathrm{Ind}_H^{LG} = \mathrm{Ind}_G^{LG} \circ \mathrm{Ind}_H^G.$$

Proof. Apply the previous theorem to the chain of inclusions $H \subset G \subset LG$. □

Remark 4.7. As for finite dimensional case, these results are perhaps better understood in (twisted) K-theory. Indeed, recall that Freed, Hopkins and Teleman proved, cf. [4] that there is a natural isomorphism of $R(G)$-modules $K^{[\ell+h^\vee]+\dim(G)}(G) \cong R_\ell(G)$, where the level $\ell + h^\vee$ is now interpreted as a twising for equivariant K-theory. Together with the isomorphism $R(H) \cong K^0_G(G/H)$, the result above shows that the induction map as defined in this paper, can be identified in K-theory as the push-forward along the map obtained by composing $G/H \to e \hookrightarrow G$. For this side of the story, see also [8,14].

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