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Currency option pricing in a credible exchange rate target zone*

Dirk Veestraeten
University of Amsterdam
Department of Economics
Roetersstraat 11
1018 WB Amsterdam
The Netherlands

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Abstract

This article examines currency option pricing within a credible target zone arrangement where interventions at the boundaries push the exchange rate back into its fluctuation band. Valuation of such options is complicated by the requirement that the reflection mechanism should prevent the arbitrage opportunities that would arise if the exchange rate were to spend finite time on the boundaries. To prevent the latter, we superimpose instantaneously reflecting boundaries upon the familiar Geometric Brownian Motion (GBM) framework. We derive closed-form expressions for European call and put option prices and show that prices for the GBM model of Garman and Kohlhagen (1983) arise as the limit case for infinitely wide bands. We also illustrate that taking account of boundaries is of considerable economic value as erroneously using the unbounded-domain model of Garman and Kohlhagen (1983) easily overprices options by more than 100%.

Keywords: currency option pricing, exchange rate target zones, geometric Brownian motion, reflecting boundaries, Brownian motion, risk-neutral valuation

JEL Classification: F31; G13

*Address correspondence to Dirk Veestraeten, Department of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands; e-mail: dirk.veestraeten@uva.nl. We are indebted to Hans Dewachter, Henk Jager, Franc Klaassen, Piet Sercu and Koen Vermeylen for stimulating discussions. The usual disclaimer applies.

I Introduction

In exchange rate target zones, monetary authorities keep the price of their currency between lower and upper boundaries via foreign-exchange market interventions. The Bretton Woods system and the Exchange Rate Mechanism (ERM) are well-known historical examples of such regimes. The ERM II arrangement for the prospective new members of the Euro zone continues the presence of target zones within the European Union. Switzerland installed a one-sided target zone vis-à-vis the Euro in September 2011 to stop the appreciation of the Swiss Franc caused by the European sovereign debt crisis. Moreover, a number of emerging market economies and transition countries implemented target zones or are moving towards such regimes. For instance, the fluctuation range of the Chinese Renminbi (RMB) towards the US dollar widened from $\pm 0.3\%$ around the central parity in 2005 over $\pm 0.5\%$ in 2007 to $\pm 1\%$ since April 2012 (People's Bank of China, 2012). Chinese authorities intend to further increase flexibility for the RMB causing the foreign-exchange options markets, that at end-March 2012 had 28 members (People's Bank of China, 2012), to further grow in relevance.

Target zone arrangements have intensively been studied following the publication of Krugman (1991). Option pricing techniques for such regimes, however, are still not well-developed. The latter is primarily due to the requirement that interventions at the boundaries should not yield arbitrage opportunities (see for instance Larsen and Sørensen (2007) for a recent discussion). More in particular, the intervention mechanism within the valuation model must ensure that the exchange rate cannot spend finite time on the target zone boundaries. Indeed, allowing the exchange rate to spend finite time on the upper (lower) boundary implies that subsequently it can only decrease (increase) again and investment strategies of no initial outlay but with certain gains would be enabled. This would then violate the no-arbitrage condition of rational option pricing. The boundaries of the target zone thus should possess no probability mass, i.e. the speed of reflection away from them should be infinite, which will be guaranteed in this article by choosing for instantaneous and infinitesimal reflection in the

definition of Skorokhod (1961). This reflection mechanism subsequently will be superimposed upon the familiar stochastic set-up of Geometric Brownian Motion (GBM) such that we can discuss the resulting stochastic process as Reflected Geometric Brownian Motion (RGBM).

Employing RGBM for the valuation of currency options in credible target zones, i.e. in zones of which sustainability is not questioned by markets, has two desirable characteristics. First, the familiar GBM-based currency option model of Garman and Kohlhagen (1983) emerges within our valuation strategy as its unbounded-domain limit. Second, the valuation equations are analytic closed-form formulas that consist of infinite sums but for which convergence is extremely fast such that both accuracy and workability are guaranteed. Taking account of target zone boundaries within the RGBM valuation model also has strong economic implications as prices generally differ considerably from option prices under GBM. Or, applying the unbounded-domain model of Garman and Kohlhagen (1983) also to target zone exchange rates typically results in severe mispricing. In fact, in most cases RGBM prices fall well below GBM prices as the upper boundary caps the upward fluctuation potential of the exchange rate. Depending on the actual position of the exchange rate and the width of the target zone, GBM prices can then easily surpass RGBM prices by more than 100%.

The remainder of this article is organized as follows. Section 2 develops our stochastic framework of RGBM in which due attention will be given to the required absence of arbitrage opportunities. The transition probability density function (pdf), i.e. the conditional density function, of RGBM will be obtained in Section 3. Section 4 employs the latter density to derive European call and put option prices when two-sided target zones exist and also specifies the resulting hedge ratios. Section 5 then specializes option prices and hedge ratios for one-sided target zones, i.e. for set-ups where monetary authorities only defend an upper or lower boundary. Section 6 concludes.

II Reflected Geometric Brownian Motion (RGBM)

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space where Ω is the outcome space containing all events ω . $\mathcal{F}$ is a right-continuous increasing family $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ of sub σ -fields of $\mathcal{F}$ and it is P -complete. P is a σ -additive non-negative measure on the measurable space $(\Omega, \mathcal{F})$ representing a probability on $(\Omega, \mathcal{F})$ with $P(\Omega) = 1$. Finally, $W = (W_t, t \geq 0)$ denotes a one-dimensional $(\mathcal{F}_t)$ -adapted Brownian motion.

We assume that the exchange rate, S , follows GBM with drift and diffusion coefficients μS_t and $\sigma^2 S_t^2$, respectively:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

The target zone arrangement restricts the fluctuation potential of the exchange rate by imposing two boundaries upon the above stochastic process. At the lower boundary $\underline{S}$ and the upper boundary $\bar{S}$, with $0 < \underline{S} < \bar{S}$, interventions by monetary authorities will move the exchange rate back towards the centre of the target zone. Reflection here is assumed to be instantaneous and of infinitesimal size, i.e. we adopt the so-called reflection functions as defined in Skorokhod (1961). These reflection functions are the real right-continuous, non-negative and non-decreasing functions L_t and U_t that specify the cumulative amount of upward and downward reflection, respectively, and of which the points of growth are located at the reflecting boundaries. The resulting stochastic process, that will be referred to as RGBM, then emerges as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + dL_t - dU_t. \quad (1)$$

Three properties of Equation (1) are to be stressed here as we will extensively rely on them later when discussing option valuation. First, the increments of L_t and U_t are of infinitesimal magnitude or the reflection functions are continuous in S and t and thus are of finite variation. Also, the RGBM process is unique with S , L and U being uniquely determined by S_t and W_t , except perhaps on a

set of measure zero (see Skorokhod, 1961; Harrison and Reiman, 1981; Ikeda and Watanabe, 1981). Second, reflection takes place instantaneously, i.e. the speed of return from the boundaries is infinite. Formally, L_t and U_t have uncountably many points of increase in finite time on the boundaries, but the set of all such points has (Lebesgue) measure zero as discussed in more detail in, for instance, Harrison (1985).¹ The exchange rate thus spends no time on either of the boundaries which will be crucial for arbitrage pricing. Indeed, if the exchange rate were able to spend finite time on the lower (upper) boundary, the price subsequently would only be able to go up (down). Such perspective would allow investors to devise strategies that yield certain gains without initial investment.²

Third, Itô's lemma for the RGBM process in Equation (1) is a straightforward extension of its formulation under GBM. Indeed, L and U are $(\mathcal{F}_t)$ -measurable for all $t \geq 0$ and thus are adapted to $(\mathcal{F}_t)$ that in turn is generated by the Brownian motion process. Hence, S is composed of a local $(\mathcal{F}_t)$ -martingale with continuous sample paths, namely the Brownian motion, and three right-continuous $(\mathcal{F}_t)$ -adapted processes, namely the drift and reflection components. Thus, S is continuous and both the drift and reflection components have sample functions of bounded variation on any finite interval (see, for instance, Harrison and Reiman, 1981). As a result, for any function $f: R \rightarrow R$ that is dependent on S and t and that is twice continuously differentiable with the motion of S as given in Equation (1), Itô's lemma yields:

$$\begin{aligned} df(S_t, t) &= \frac{\partial f(S_t, t)}{\partial S_t} dS_t + \frac{\partial f(S_t, t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f(S_t, t)}{\partial S_t^2} (dS_t)^2, \\ df(S_t, t) &= \frac{\partial f(S_t, t)}{\partial S_t} S_t (\mu dt + \sigma dW_t + dL_t - dU_t) + \frac{\partial f(S_t, t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f(S_t, t)}{\partial S_t^2} \sigma^2 S_t^2 dt. \end{aligned} \quad (2)$$

Under GBM, the term $(dS_t)^2$ equals $\sigma^2 S_t^2 dt$ as $(dW_t)^2 = dt$ and $dt dt = dW_t dt = 0$. This result also carries over to RGBM due to the aforementioned bounded-variation nature of the reflection

¹For an interpretation of the reflection functions in terms of local time, we can refer to Ikeda and Watanabe (1981) and Harrison (1985).

²Alternative reflection mechanisms such as slow reflection (Revuz and Yor, 1994), delayed reflection (Skorokhod, 1961) and reflection at so-called sticky barriers (Karlin and Taylor, 1981) would then clearly not be acceptable for derivative pricing since the speed of return from the boundaries in these mechanisms is finite and thus positive time is spent on them.

components that guarantees that all additional multiplicative terms vanish, i.e. $dt dL_t = dt dU_t = dW_t dL_t = dW_t dU_t = dL_t dL_t = dU_t dU_t = dL_t dU_t = 0$. It is to be noted that the increments of the reflection functions remain present in the first right-hand side term in Equation (2).³

III The transition probability density function of RGBM

Applying Itô's lemma in Equation (2) to the transform $s_t \equiv \ln S_t$ yields:

$$ds_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t + dL_t - dU_t,$$

with reflection at $\underline{s} = \ln \underline{S}$ and $\bar{s} = \ln \bar{S}$.

The transition pdf for s is denoted by $q(s, t; s_0, t_0)$ and specifies the probability of attaining s at time t given that the process currently, i.e. at t_0 , is at the source point s_0 . This density function, see for instance Risken (1989), must satisfy the Fokker-Planck equation:

$$\frac{1}{2} \sigma^2 \frac{\partial^2 q(s, t; s_0, t_0)}{\partial s^2} - \left(\mu - \frac{1}{2} \sigma^2 \right) \frac{\partial q(s, t; s_0, t_0)}{\partial s} = \frac{\partial q(s, t; s_0, t_0)}{\partial t} \quad (3)$$

for $\underline{s} < s_0 < \bar{s}$, $\underline{s} < s < \bar{s}$ and $t > t_0$. Equation (3) is to be solved subject to an initial condition and two boundary conditions. As noted earlier, instantaneous reflection ensures that the two target zone boundaries have zero probability or equivalently that all probability mass is situated between them, i.e. $\int_{\underline{s}}^{\bar{s}} q(s, t; s_0, t_0) ds = 1$ and $\frac{\partial}{\partial t} \left[\int_{\underline{s}}^{\bar{s}} q(s, t; s_0, t_0) ds \right] = 0$. Plugging Equation (3) into the latter expression then yields the following two boundary conditions:

$$\lim_{s \downarrow \underline{s}} \left[\frac{1}{2} \sigma^2 \frac{\partial q(s, t; s_0, t_0)}{\partial s} - \left(\mu - \frac{1}{2} \sigma^2 \right) q(s, t; s_0, t_0) \right] = 0, \quad (4a)$$

$$\lim_{s \uparrow \bar{s}} \left[\frac{1}{2} \sigma^2 \frac{\partial q(s, t; s_0, t_0)}{\partial s} - \left(\mu - \frac{1}{2} \sigma^2 \right) q(s, t; s_0, t_0) \right] = 0. \quad (4b)$$

Finally, the initial condition for Equation (3) is:

$$\lim_{t \downarrow t_0} [q(s, t; s_0, t_0)] = \delta(s - s_0) \delta(t - t_0), \quad (5)$$

³We will come back to this property when giving a formal justification for the boundary behaviour of the option price.

in which $\delta(\cdot)$ denotes the Dirac delta function. This condition guarantees that all initial probability mass is located at the initial value and the initial point of time which by construction is the appropriate initial condition for processes based on Brownian motion.

The transition pdf $q(s, t; s_0, t_0)$ is the solution to the initial-boundary value problem in Equations (3)-(5) and an equivalent system has been solved in Veestraeten (2004). Adapting the solution in Veestraeten (2004) and transforming it in terms of RGBM for the exchange rate S with the initial point of time now set at t and the end of the prediction interval at T , i.e. $Q(S_T, T; S_t, t)$, gives:

$$\begin{aligned}
Q(S_T, T; S_t, t) = & \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{S_T \sigma \sqrt{2\pi(T-t)}} \exp[\gamma n (\ln \underline{S} - \ln \bar{S})] \right. \\
& \times \exp \left[-\frac{(\ln S_T + 2n(\ln \underline{S} - \ln \bar{S}) - \ln S_t - (\mu - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right] \Big\} \\
& + \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{S_T \sigma \sqrt{2\pi(T-t)}} \exp[-\gamma(n \ln \bar{S} - (n+1) \ln \underline{S} + \ln S_t)] \right. \\
& \times \exp \left[-\frac{(2n \ln \bar{S} - 2(n+1) \ln \underline{S} + \ln S_t + \ln S_T - (\mu - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right] \Big\} \\
& - \gamma \sum_{n=0}^{+\infty} \left\{ \frac{1}{S_T} \exp[\gamma(n \ln \bar{S} - (n+1) \ln \underline{S} + \ln S_T)] \right. \\
& \times \left(1 - \Phi \left[\frac{2n \ln \bar{S} - 2(n+1) \ln \underline{S} + \ln S_t + \ln S_T + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \right] \right) \Big\} \\
& + \gamma \sum_{n=0}^{+\infty} \left\{ \frac{1}{S_T} \exp[\gamma(n \ln \underline{S} - (n+1) \ln \bar{S} + \ln S_T)] \right. \\
& \times \Phi \left[\frac{2n \ln \underline{S} - 2(n+1) \ln \bar{S} + \ln S_t + \ln S_T + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \right] \Big\}, \tag{6}
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \frac{2(\mu - \frac{1}{2}\sigma^2)}{\sigma^2}, \\
\Phi[x] &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}y^2\right] dy.
\end{aligned}$$

It is fairly straightforward, albeit rather lengthy, to show that the integral of the transition pdf over its domain, i.e. $\int_{\underline{S}}^{\bar{S}} Q(S_T, T; S_t, t) dS_T$, returns unity. Or, the boundaries under RGBM indeed possess no probability mass, which as argued before is essential for arbitrage-free option valuation.

IV European option pricing under RGBM

This section values European call and put options via the risk-neutral valuation strategy.⁴ The risk-neutralized homologue of the density in Equation (6) is obtained by replacing the drift factor μ by its risk-neutral equivalent $r - r^*$ (see Garman and Kohlhagen, 1983), where r and r^* denote domestic and foreign risk-free interest rates. The price at time t of a call option with time to maturity $(T - t)$, exercise price K and the target zone boundaries $\underline{S}$ and $\bar{S}$ is given by:

$$C(S_t, T - t, \underline{S}, \bar{S}) = \exp[-r(T - t)] \int_{\underline{S}}^{\bar{S}} \max[0, S_T - K] Q(S_T, T; S_t, t)|_{\mu=r-r^*} dS_T. \quad (7)$$

Plugging the risk-neutralized transition pdf into Equation (7), assuming $\underline{S}_T \leq K \leq \bar{S}_T$, re-arranging and simplifying yields the following expression for the call option price when domestic and foreign interest rates differ:

$$\begin{aligned} C(S_t, T - t, \underline{S}, \bar{S}, r \neq r^*) &= S_t \exp[-r^*(T - t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n d_1 (\ln \underline{S} - \ln \bar{S})] (\Phi[q_{1,n}] - \Phi[q_{2,n}]) \right\} \\ &- K \exp[-r(T - t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n d_2 (\ln \underline{S} - \ln \bar{S})] \left(\Phi[q_{1,n} - \sigma\sqrt{T-t}] - \Phi[q_{2,n} - \sigma\sqrt{T-t}] \right) \right\} \\ &+ d_3^{-1} S_t^{-d_3} \exp[-r^*(T - t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[d_1 ((n+1) \ln \underline{S} - n \ln \bar{S})] (\Phi[q_{3,n}] - \Phi[q_{4,n}]) \right\} \\ &+ \exp[-r(T - t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[d_2 (n \ln \bar{S} - (n+1) \ln \underline{S})] \right. \\ &\quad \times \left. \left\{ \bar{S}^{d_3} (d_3^{-1} d_2 - K \bar{S}^{-1}) \Phi[q_{3,n} + d_3 \sigma\sqrt{T-t}] + d_3^{-1} K^{d_3} \Phi[q_{4,n} + d_3 \sigma\sqrt{T-t}] \right\} \right\} \\ &+ \exp[-r(T - t)] \sum_{n=0}^{+\infty} \left\{ \exp[d_2 (n \ln \bar{S} - (n+1) \ln \underline{S})] \left\{ \bar{S}^{d_3} (K \bar{S}^{-1} - d_3^{-1} d_2) - d_3^{-1} K^{d_3} \right\} \right\}, \end{aligned} \quad (8)$$

with

$$\begin{aligned} d_1 &= \frac{2(r - r^* + \frac{1}{2}\sigma^2)}{\sigma^2}, \\ d_2 &= \frac{2(r - r^* - \frac{1}{2}\sigma^2)}{\sigma^2}, \\ d_3 &= \frac{2(r - r^*)}{\sigma^2}, \end{aligned}$$

⁴Complete derivations of all results in this article can be obtained from the author upon simple demand.

and

$$\begin{aligned}
q_{1,n} &= \frac{\ln S_t - \ln K - 2n (\ln \bar{S} - \ln \underline{S}) + (r - r^* + \frac{1}{2}\sigma^2) (T - t)}{\sigma\sqrt{T-t}}, \\
q_{2,n} &= \frac{\ln S_t - (2n+1) \ln \bar{S} + 2n \ln \underline{S} + (r - r^* + \frac{1}{2}\sigma^2) (T - t)}{\sigma\sqrt{T-t}}, \\
q_{3,n} &= \frac{\ln S_t + (2n+1) \ln \bar{S} - 2(n+1) \ln \underline{S} - (r - r^* + \frac{1}{2}\sigma^2) (T - t)}{\sigma\sqrt{T-t}}, \\
q_{4,n} &= \frac{\ln S_t + \ln K + 2n \ln \bar{S} - 2(n+1) \ln \underline{S} - (r - r^* + \frac{1}{2}\sigma^2) (T - t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

The pricing formula in Equation (8) requires a non-zero interest rate differential in order to prevent divisions by zero. In the case of identical domestic and foreign interest rates, l'Hôpital's rule allows us to express the limit of Equation (8) as:

$$\begin{aligned}
C(S_t, T-t, \underline{S}, \bar{S}, r=r^*) &= S_t \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n(\ln \underline{S} - \ln \bar{S})] (\Phi[q_{1,n}] - \Phi[q_{2,n}]) \right\} \\
&- K \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n(\ln \bar{S} - \ln \underline{S})] \left(\Phi[q_{1,n} - \sigma\sqrt{T-t}] - \Phi[q_{2,n} - \sigma\sqrt{T-t}] \right) \right\} \\
&- \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[(n+1) \ln \underline{S} - n \ln \bar{S}] \left\{ \Phi[q_{3,n}] \left(q_{3,n} \sigma\sqrt{T-t} - 1 + K\bar{S}^{-1} \right) \right. \right. \\
&\quad \left. \left. - \Phi[q_{4,n}] q_{4,n} \sigma\sqrt{T-t} \right\} \right\} \\
&- \sigma\sqrt{T-t} \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[(n+1) \ln \underline{S} - n \ln \bar{S}] (\phi[q_{3,n}] - \phi[q_{4,n}]) \right\} \\
&+ \exp[-r(T-t)] \sum_{n=0}^{+\infty} \left\{ \exp[(n+1) \ln \underline{S} - n \ln \bar{S}] \left(K\bar{S}^{-1} + \ln \bar{S} - \ln K - 1 \right) \right\},
\end{aligned} \tag{9}$$

with

$$\phi[y] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}y^2\right].$$

The valuation formulas in Equations (8) and (9) are analytic closed-form expressions that consist of infinite sums. Despite their complex appearance, workability is guaranteed as convergence to the limiting option price occurs extremely fast. This is illustrated in Table 1 where convergence for realistic

parameter set-ups requires a value of n of not more than 4 (in absolute value).⁵

Table 1: around here

As RGBM superimposes reflecting boundaries upon GBM, removing the boundaries gives the GBM, i.e. the unbounded-domain, price of Garman and Kohlhagen (1983). Indeed, the limit for $\underline{S} \rightarrow 0$ and $\overline{S} \rightarrow +\infty$ yields:

$$C(S_t, T-t) = S_t \exp[-r^*(T-t)] \Phi[q_5] - K \exp[-r(T-t)] \Phi[q_5 - \sigma\sqrt{T-t}], \quad (10)$$

with

$$q_5 = \frac{\ln S_t - \ln K + (r - r^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The familiar put-call parity also holds within exchange rate target zones as reflection keeps the currency option contract alive until the maturity date such that the investment strategies that underlie this parity can also be developed when reflecting boundaries exist.⁶ The value of the European put option, $P(S_t, T-t, \underline{S}, \overline{S})$, therefore emerges as:

$$P(S_t, T-t, \underline{S}, \overline{S}) = C(S_t, T-t, \underline{S}, \overline{S}) - S_t \exp[-r^*(T-t)] + K \exp[-r(T-t)]. \quad (11)$$

Target zones have a strong impact on option pricing as will be illustrated in Fig. 1. The dotted lines depict the GBM option prices of Equation (10), whereas the solid nonlinear lines specify the RGBM option prices of Equation (8). The horizontal solid lines represent the maximum and minimum RGBM option prices as the fluctuation limits for the exchange rate also create a band for the option price.

Figure 1: around here

⁵The prices for the unrestricted-domain model of Garman and Kohlhagen (1983) in Table 1 will be used below to discuss the impact of applying the latter model also to target zone exchange rates.

⁶This constitutes a crucial difference to pricing of knockout options where the put-call parity does not hold as argued in Kunitomo and Ikeda (1992). In fact, knockout options are nullified when the underlying asset reaches the boundaries and option pricing thus turns into a stopping-time problem. Since the point of time at which the option may be cancelled is unknown, the strategies that yield the put-call parity then cannot be applied.

Four properties of target zone currency option pricing are to be noted. First, the most striking feature of Fig. 1 is the S -shaped option price function that tangentially nears the boundaries of its band. In fact, absence of arbitrage opportunities requires the option to have zero movement upon reflection. Non-zero movement would indeed allow for predictable gains if the option were bought just prior to intervention. Or, the first derivative of the price function to the exchange rate is to equal zero at the boundaries. This requirement can be illustrated within the following simple formal argument. Itô's lemma in Equation (2) applies both just before as well as at reflection as argued in, for instance, Ikeda and Watanabe (1981). Using Equation (2) for the option price then specifies its instantaneous change just before and at reflection at the lower boundary, respectively, as:

$$\begin{aligned}
dC(S_t, T-t, \underline{S}, \bar{S}) &= \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t (\mu dt + \sigma dW_t) - \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial (T-t)} dt \\
&\quad + \frac{1}{2} \frac{\partial^2 C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t^2} \sigma^2 S_t^2 dt, \text{ and} \\
dC(S_t, T-t, \underline{S}, \bar{S}) &= \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t (\mu dt + \sigma dW_t + dL_t) - \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial (T-t)} dt \\
&\quad + \frac{1}{2} \frac{\partial^2 C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t^2} \sigma^2 S_t^2 dt,
\end{aligned} \tag{12}$$

where we use the property that the reflection function L only increases upon reflection. Similarly, at the upper boundary the following expressions must apply:

$$\begin{aligned}
dC(S_t, T-t, \underline{S}, \bar{S}) &= \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t (\mu dt + \sigma dW_t) - \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial (T-t)} dt \\
&\quad + \frac{1}{2} \frac{\partial^2 C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t^2} \sigma^2 S_t^2 dt, \text{ and} \\
dC(S_t, T-t, \underline{S}, \bar{S}) &= \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t (\mu dt + \sigma dW_t - dU_t) - \frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial (T-t)} dt \\
&\quad + \frac{1}{2} \frac{\partial^2 C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t^2} \sigma^2 S_t^2 dt.
\end{aligned} \tag{13}$$

Precluding predictable profits upon reflection requires the instantaneous change in the option price just prior to and at reflection to be identical. The expressions in Equations (12) and (13) then yield

the following two conditions that are to hold upon reflection at the lower and upper boundaries, respectively:

$$\begin{aligned}\lim_{S_t \downarrow \underline{S}} \left[\frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t dL_t \right] &= 0, \\ \lim_{S_t \uparrow \bar{S}} \left[\frac{\partial C(S_t, T-t, \underline{S}, \bar{S})}{\partial S_t} S_t dU_t \right] &= 0.\end{aligned}$$

As the increments in the reflection functions are strictly positive upon reflection and since the exchange rate is always larger than zero, we indeed obtain the aforementioned two derivative conditions that are also prominently present in Fig. 1.

Second, Fig. 1 shows that RGBM prices typically fall below GBM prices as the upper boundary restricts the moneyness region and the likelihood of reaching it. However, and this may seem surprising at first sight, the panels also reveal that RGBM prices can exceed GBM prices. In fact, the lower boundary can create additional value by preventing the exchange rate from moving far or farther below the exercise price and by pushing it upwards again. The resulting higher likelihood of gathering intrinsic value may then even surpass the loss of value caused by the presence of the upper boundary. This effect must be larger for exercise prices that are closer to the lower boundary as shown in panels (a) to (c).

Third, widening the target zone brings RGBM prices closer to GBM prices as illustrated across panels (d) to (f). Expanding the moneyness region by lifting the upper limit steps up RGBM prices, widens the band for the option price and the price function starts more and more to resemble the unbounded-domain valuation function.

Fourth, using the Garman and Kohlhagen (1983) model also for valuing options on currencies that actually evolve within a target zone can result in severe overpricing.⁷ For instance, in the 25%-wide target zone of panel (b) overpricing by GBM quickly surpasses 100%. Also the RGBM and GBM prices

⁷ As mentioned earlier, underpricing is also possible. However, the required tight range of parameter values of exchange rates and exercise prices that both have to be (very) near to the lower target zone limit makes this possibility of probably limited practical relevance.

in Table 1 confirm this substantial degree of mispricing for the 25%-wide band. Given that the width of the latter zone can be seen as large by historical standards, the documented degree of overpricing must even be seen as rather conservative since narrower bands further constrain the moneyness region for RGBM options. The RGBM model thus is of considerable economic value as erroneously applying the Garman and Kohlhagen (1983) model also to target zone exchange rates generally generates (severe) overpricing. This has important implications for exchange rate risk management. Indeed, the higher-than-justified cost of hedging could well depress demand for currency options and as such create larger exposure to exchange-rate risk.

We proceed by deriving the option delta or hedge ratio, i.e. the first derivative of the option price to the underlying exchange rate. This ratio is of vital importance to adequate management of exchange-rate exposure. The hedge ratio for the call option price in Equation (8) is:⁸

$$\begin{aligned}
\frac{\partial C(S_t, T-t, \underline{S}, \bar{S}, r \neq r^*)}{\partial S_t} = & \exp[-r^*(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[nd_1(\ln \underline{S} - \ln \bar{S})] \left(\Phi[q_{1,n}] + \frac{\phi[q_{1,n}]}{\sigma\sqrt{T-t}} \right. \right. \\
& \left. \left. - \Phi[q_{2,n}] - \frac{\phi[q_{2,n}]}{\sigma\sqrt{T-t}} \right) \right\} \\
& - KS_t^{-1} \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[nd_2(\ln \underline{S} - \ln \bar{S})] \left(\frac{\phi[q_{1,n} - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} - \frac{\phi[q_{2,n} - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} \right) \right\} \\
& - S_t^{-d_3-1} \exp[-r^*(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[d_1((n+1)\ln \underline{S} - n\ln \bar{S})] \left(\Phi[q_{3,n}] - d_3^{-1} \frac{\phi[q_{3,n}]}{\sigma\sqrt{T-t}} \right. \right. \\
& \left. \left. - \Phi[q_{4,n}] + d_3^{-1} \frac{\phi[q_{4,n}]}{\sigma\sqrt{T-t}} \right) \right\} \\
& + S_t^{-1} \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[d_2(n\ln \bar{S} - (n+1)\ln \underline{S})] \right. \\
& \left. \times \left(\bar{S}^{d_3} \left(d_3^{-1}d_2 - K\bar{S}^{-1} \right) \frac{\phi[q_{3,n} + d_3\sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} + d_3^{-1}K^{d_3} \frac{\phi[q_{4,n} + d_3\sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} \right) \right\}. \tag{14}
\end{aligned}$$

⁸The hedge ratios for put options can be obtained from their call option homologues via the derivative of the put-call parity in Equation (11).

Identical interest rates at home and abroad call for careful evaluation of the limit of Equation (14)

and this gives:

$$\begin{aligned} \frac{\partial C(S_t, T-t, \underline{S}, \bar{S}, r=r^*)}{\partial S_t} = & \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n(\ln \underline{S} - \ln \bar{S})] \left(\Phi[q_{1,n}] + \frac{\phi[q_{1,n}]}{\sigma\sqrt{T-t}} \right. \right. \\ & \left. \left. - \Phi[q_{2,n}] - \frac{\phi[q_{2,n}]}{\sigma\sqrt{T-t}} \right) \right\} \\ & - K S_t^{-1} \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[n(\ln \bar{S} - \ln \underline{S})] \left(\frac{\phi[q_{1,n} - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} - \frac{\phi[q_{2,n} - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} \right) \right\} \\ & - S_t^{-1} \exp[-r(T-t)] \sum_{n=-\infty}^{+\infty} \left\{ \exp[(n+1)\ln \underline{S} - n\ln \bar{S}] \left(\Phi[q_{3,n}] - \frac{\phi[q_{3,n}]}{\sigma\sqrt{T-t}} (1 - K\bar{S}^{-1}) - \Phi[q_{4,n}] \right) \right\}. \end{aligned}$$

As required, the limit of Equation (14) for $\underline{S}$ and $\bar{S}$ going to 0 and $+\infty$, respectively, yields the hedge ratio of Garman and Kohlhagen (1983):

$$\frac{\partial C(S_t, T-t)}{\partial S_t} = \exp[-r^*(T-t)] \Phi[q_5]. \quad (15)$$

Fig. 2 illustrates hedge ratios for RGBM and GBM options. The dotted curves correspond to the hedge ratio for the unbounded process that increases from 0 for $S_t \rightarrow 0$ in Equation (15) to $\exp[-r^*(T-t)]$ for $S_t \rightarrow +\infty$. The target zone hedge ratio on the contrary has a hump shape with two minima at zero as required by the abovementioned no-arbitrage condition for the option price function.

Figure 2: around here

Target zone hedge ratios never exceed values for the unbounded process and mostly fall well below those levels. Panels (d) to (f) also illustrate that the target zone hedge ratios for growing band width coincide more and more with the GBM hedge ratio in Equation (15) although the upper boundary keeps dragging the hedge ratio in Equation (14) to zero. In sum, hedge ratios tend to differ considerably between the two valuation approaches such that employing the Garman and Kohlhagen (1983) model when actually a target zone is present can create substantial errors in hedging.

V One-sided target zones

This section specializes the above results for set-ups in which monetary authorities defend a single lower or upper boundary. For instance, countries with sizeable foreign-currency denominated public and/or private sector debt may want to keep their currency from depreciating beyond some level in view of keeping the domestic-currency value of foreign-currency debt under control. On the other hand, countries may for reasons of international competitiveness actively intervene in foreign-exchange markets to prevent appreciations of their currency beyond a certain level whilst at the same time not curbing depreciations. The substantial interventions by Japanese monetary authorities in the yen-dollar market in 2002-2004 are often quoted in this respect (see Ito, 2005; Hillebrand and Schnabl, 2008). Since 6 September 2011, the Swiss National Bank stands ready to buy foreign currency in unlimited amounts to keep the Swiss Franc (CHF) from falling below CHF 1.20 per Euro in view of limiting the deflationary impact that the appreciations of 2010-2012 brought (Swiss National Bank, 2012).

We first discuss call prices and hedge ratios when only a lower boundary applies and subsequently turn to the slightly more complex set-up of a sole upper boundary. We will also show that the relation between RGBM and GBM prices now is unambiguous in the sense that a sole lower (upper) boundary causes RGBM prices to be larger than or equal to (smaller than or equal to) GBM prices.

A sole lower boundary

The call option price when only a lower reflecting boundary at $\underline{S}$ is imposed will be denoted by $C(S_t, T - t, \underline{S})$ and arises as the limit of Equation (8) for $\bar{S} \rightarrow +\infty$:⁹

$$\begin{aligned} C(S_t, T - t, \underline{S}, r \neq r^*) &= S_t \exp[-r^*(T - t)] \Phi[q_5] - K \exp[-r(T - t)] \Phi[q_5 - \sigma\sqrt{T - t}] \\ &\quad - d_3^{-1} K^{d_3} \underline{S}^{-d_2} \exp[-r(T - t)] \left\{ 1 - \Phi[q_6 + d_3\sigma\sqrt{T - t}] \right\} \\ &\quad + d_3^{-1} S_t^{-d_3} \underline{S}^{d_1} \exp[-r^*(T - t)] \{1 - \Phi[q_6]\}, \end{aligned} \quad (16)$$

with

$$q_6 = \frac{\ln S_t + \ln K - 2 \ln \underline{S} - \left(r - r^* + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}.$$

For a zero interest rate differential, the following expression applies:

$$\begin{aligned} C(S_t, T - t, \underline{S}, r = r^*) &= S_t \exp[-r(T - t)] \Phi[q_5] - K \exp[-r(T - t)] \Phi[q_5 - \sigma\sqrt{T - t}] \\ &\quad - \sigma\sqrt{T - t} \underline{S} \exp[-r(T - t)] (q_6 (1 - \Phi[q_6]) - \phi[q_6]). \end{aligned} \quad (17)$$

The first and second terms in both Equations (16) and (17) specify the Garman and Kohlhagen (1983) price and the remaining terms represent the non-negative price effect of the lower boundary.

The existence of the lower boundary indeed can generate additional value when compared with the GBM case as reflection may increase the likelihood that the option ultimately ends in the money. The

RGBM option price will exceed the GBM price provided that the distance of the exchange rate versus the lower boundary and the exercise price is not too large. Otherwise, RGBM and GBM prices will

be indistinguishable. Formally, we know that $\lim_{\underline{S} \downarrow 0} [C(S_t, T - t, \underline{S})] = C(S_t, T - t)$ and it is easy to see

that $\frac{\partial C(S_t, T - t, \underline{S})}{\partial \underline{S}} \geq 0$. Or, $C(S_t, T - t, \underline{S})$ can increase in $\underline{S}$ as raising the lower boundary yields more

scope for reflection and thus may increase the potential for the option to end in the money, which

⁹Veestraeten (2008) reports the call option price when a lower barrier restricts the fluctuation range of the stock price. Again, put option prices and their hedge ratios can be obtained via the put-call parity in Equation (11).

must have a non-negative effect on its price. Hence, for decreasing $\underline{S}$, $C(S_t, T-t, \underline{S})$ must approach $C(S_t, T-t)$ from above and thus $C(S_t, T-t, \underline{S}) \geq C(S_t, T-t)$ for all values of $\underline{S}$.

The hedge ratio in the case of a sole lower boundary is specified as follows for a non-zero interest rate differential:

$$\begin{aligned} \frac{\partial C(S_t, T-t, \underline{S}, r \neq r^*)}{\partial S_t} &= \exp[-r^*(T-t)] \left(\Phi[q_5] + \frac{\phi[q_5]}{\sigma\sqrt{T-t}} \right) \\ &- K S_t^{-1} \exp[-r(T-t)] \frac{\phi[q_5 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} + S_t^{-d_3-1} \underline{S}^{d_1} \exp[-r^*(T-t)] \left(\Phi[q_6] - d_3^{-1} \frac{\phi[q_6]}{\sigma\sqrt{T-t}} - 1 \right) \\ &+ d_3^{-1} S_t^{-1} K^{d_3} \underline{S}^{-d_2} \exp[-r(T-t)] \frac{\phi[q_6 + d_3\sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}}, \end{aligned}$$

and in the case of identical domestic and foreign interest rates it equals:

$$\begin{aligned} \frac{\partial C(S_t, T-t, \underline{S}, r = r^*)}{\partial S_t} &= \exp[-r(T-t)] \left(\Phi[q_5] + \frac{\phi[q_5]}{\sigma\sqrt{T-t}} \right) \\ &- K S_t^{-1} \exp[-r(T-t)] \frac{\phi[q_5 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} + S_t^{-1} \underline{S} \exp[-r(T-t)] (\Phi[q_6] - 1). \end{aligned}$$

A sole upper boundary

$C(S_t, T-t, \bar{S})$ is the price of the call option when only an upper target zone boundary exists and it emerges as the limit of the two-boundary price in Equation (8) for $\underline{S} \rightarrow 0$:

$$\begin{aligned} C(S_t, T-t, \bar{S}, r \neq r^*) &= S_t \exp[-r^*(T-t)] \{ \Phi[q_5] - \Phi[q_7] \} - K \exp[-r(T-t)] \Phi[q_5 - \sigma\sqrt{T-t}] \\ &+ d_3^{-1} S_t^{-d_3} \bar{S}^{d_1} \exp[-r^*(T-t)] \{ \Phi[q_8] - \Phi[q_9] \} + d_3^{-1} K^{d_3} \bar{S}^{-d_2} \exp[-r(T-t)] \Phi[q_9 + d_3\sigma\sqrt{T-t}] \\ &+ d_3^{-1} d_2 \bar{S} \exp[-r(T-t)] \Phi[q_7 - \sigma\sqrt{T-t}], \end{aligned} \tag{18}$$

with

$$\begin{aligned} q_7 &= \frac{\ln S_t - \ln \bar{S} + (r - r^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ q_8 &= \frac{\ln S_t - \ln \bar{S} - (r - r^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ q_9 &= \frac{\ln S_t + \ln K - 2 \ln \bar{S} - (r - r^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

For a zero interest rate differential, the call option price is:

$$\begin{aligned}
C(S_t, T-t, \bar{S}, r=r^*) &= S_t \exp[-r(T-t)] \{\Phi[q_5] - \Phi[q_7]\} - K \exp[-r(T-t)] \Phi[q_5 - \sigma\sqrt{T-t}] \\
&- \bar{S} \exp[-r(T-t)] \left(\Phi[q_8] \left(q_8 \sigma \sqrt{T-t} - 1 \right) - \Phi[q_9] q_9 \sigma \sqrt{T-t} \right) \\
&+ \sigma \sqrt{T-t} \bar{S} \exp[-r(T-t)] (\phi[q_9] - \phi[q_8]).
\end{aligned} \tag{19}$$

Equations (18) and (19) are slightly more complex than the pricing formulas when a sole lower boundary applies. This is due to the fact that the upper boundary enters pricing not only through the conditional density function, but now also emerges as the upper integration limit in Equation (7).¹⁰

The prices in Equations (18) and (19) cannot exceed the unbounded-domain price in Equation (10), i.e. $C(S_t, T-t, \bar{S}) \leq C(S_t, T-t)$. This is due to the fact that the upper boundary reduces the likelihood for the option to end in the money and this effect will be noticeable as long as the exchange rate is not too far from the exercise price. Formally, we have $\lim_{\bar{S} \uparrow +\infty} [C(S_t, T-t, \bar{S})] = C(S_t, T-t)$ and $\frac{\partial C(S_t, T-t, \bar{S})}{\partial \bar{S}} \geq 0$ as the RGBM price increases in the ceiling or at least does not decrease in the latter. Indeed, a higher upper boundary raises the potential for the call to end in the money and as a result the RGBM price will near the unbounded-domain price from below.

The hedge ratio when the target zone is characterized by the presence of a sole upper boundary is

¹⁰The risk-neutral valuation Equation (7) in the case of a sole lower boundary can be written as $C(S_t, T-t, \underline{S}) = \exp[-r(T-t)] \int_K^{+\infty} (S_T - K) Q(S_T, T; S_t, t)|_{\mu=r-r^*} dS_T$, whereas the price under a sole upper boundary is given by $C(S_t, T-t, \bar{S}) = \exp[-r(T-t)] \int_K^{\bar{S}} (S_T - K) Q(S_T, T; S_t, t)|_{\mu=r-r^*} dS_T$.

given by:

$$\begin{aligned}
\frac{\partial C(S_t, T-t, \bar{S}, r \neq r^*)}{\partial S_t} &= \exp[-r^*(T-t)] \left(\Phi[q_5] + \frac{\phi[q_5]}{\sigma\sqrt{T-t}} - \Phi[q_7] - \frac{\phi[q_7]}{\sigma\sqrt{T-t}} \right) \\
&- K S_t^{-1} \exp[-r(T-t)] \left(\frac{\phi[q_5 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} - \frac{\phi[q_7 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} \right) \\
&- S_t^{-d_3-1} \bar{S}^{d_1} \exp[-r^*(T-t)] \left(\Phi[q_8] - d_3^{-1} \frac{\phi[q_8]}{\sigma\sqrt{T-t}} - \Phi[q_9] + d_3^{-1} \frac{\phi[q_9]}{\sigma\sqrt{T-t}} \right) \\
&+ S_t^{-1} \bar{S}^{-d_2} \exp[-r(T-t)] \left(\bar{S}^{d_3} (d_3^{-1} d_2 - K \bar{S}^{-1}) \frac{\phi[q_8 + d_3\sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} + d_3^{-1} K^{d_3} \frac{\phi[q_9 + d_3\sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} \right).
\end{aligned}$$

Finally, specializing this relation for a zero interest rate differential yields:

$$\begin{aligned}
\frac{\partial C(S_t, T-t, \bar{S}, r = r^*)}{\partial S_t} &= \exp[-r(T-t)] \left(\Phi[q_5] + \frac{\phi[q_5]}{\sigma\sqrt{T-t}} - \Phi[q_7] - \frac{\phi[q_7]}{\sigma\sqrt{T-t}} \right) \\
&- K S_t^{-1} \exp[-r(T-t)] \left(\frac{\phi[q_5 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} - \frac{\phi[q_7 - \sigma\sqrt{T-t}]}{\sigma\sqrt{T-t}} + \frac{\phi[q_8]}{\sigma\sqrt{T-t}} \right) \\
&- S_t^{-1} \bar{S} \exp[-r(T-t)] \left(\Phi[q_8] - \frac{\phi[q_8]}{\sigma\sqrt{T-t}} - \Phi[q_9] \right).
\end{aligned}$$

VI Conclusions

This article studies currency option pricing when the fluctuation range of the exchange rate is constrained by a target zone arrangement. Valuation of such options requires careful specification of the boundary behaviour. It must be ascertained that the exchange rate can spend no finite time on the boundaries of the target zone or instantaneous reflection upon intervention is required. In fact, if the exchange rate were able to actually spend finite time on either of the boundaries, it could subsequently only move in one direction. Investment strategies with certain profits would be enabled and this would rule out arbitrage pricing.

We therefore superimpose instantaneous and infinitesimal reflection upon the familiar stochastic framework of Geometric Brownian Motion (GBM). This process is accordingly termed Reflected Geometric Brownian Motion (RGBM). Risk-neutral valuation subsequently allows us to obtain European

call and put option prices and their hedge ratios for two-sided target zones. As required, our pricing equations reduce to the GBM prices of Garman and Kohlhagen (1983) when evaluating the limit for infinitely wide target zones. Despite the added complexity in taking account of reflection, the pricing relations continue to be analytic closed-form expressions. They contain infinite terms that, however, converge extremely fast such that accuracy and practicability are guaranteed. We subsequently specialize results for set-ups in which monetary authorities maintain either a sole upper or a sole lower boundary. Such schemes arise when countries in order to, for instance, safeguard international competitiveness combat appreciations beyond a certain level without however limiting depreciations as is the case in Switzerland since December 2011.

We illustrate that the presence of exchange rate target zones strongly affects option prices and hedge ratios such that our results, next to their theoretical appeal, also have strong practical and economic implications. In fact, neglecting target zones in currency option valuation by erroneously applying the no-boundary GBM model of Garman and Kohlhagen (1983) easily results in overpricing by more than 100%. Such overpricing could then well depress demand for hedging and as such lead to excessive and potentially extremely costly exposure to foreign-exchange risk.

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Table 1: RGBM and Garman-Kohlhagen (GK) call option prices.*

n	Time to maturity (in years)				
	0.25	0.5	0.75	1	1.5
$\sigma = 0.1$					
0	0.2546607	0.3487049	0.3681804	0.3677916	0.3545504
-1 .. + 1	0.2683349	0.4287606	0.5098498	0.5534654	0.5939744
-2 .. + 2	0.2683349	0.4287606	0.5098498	0.5534654	0.5939775
-3 .. + 3	0.2683349	0.4287606	0.5098498	0.5534654	0.5939775
-4 .. + 4	0.2683349	0.4287606	0.5098498	0.5534654	0.5939775
:	:	:	:	:	:
-25 .. + 25	0.2683349	0.4287606	0.5098498	0.5534654	0.5939775
<i>GK</i>	<i>0.2734595</i>	<i>0.4886122</i>	<i>0.6678295</i>	<i>0.8262633</i>	<i>1.1039209</i>
$\sigma = 0.15$					
0	0.3325188	0.3468956	0.3406567	0.3322912	0.3121554
-1 .. + 1	0.4146038	0.5141157	0.5463655	0.5591067	0.5609685
-2 .. + 2	0.4146038	0.5141158	0.5463749	0.5592178	0.5622520
-3 .. + 3	0.4146038	0.5141158	0.5463749	0.5592178	0.5622521
-4 .. + 4	0.4146038	0.5141158	0.5463749	0.5592178	0.5622521
:	:	:	:	:	:
-25 .. + 25	0.4146038	0.5141158	0.5463749	0.5592178	0.5622521
<i>GK</i>	<i>0.4842911</i>	<i>0.7927349</i>	<i>1.0393322</i>	<i>1.2517274</i>	<i>1.6136443</i>
$\sigma = 0.2$					
0	0.3396400	0.3338812	0.3229624	0.3094384	0.2812416
-1 .. + 1	0.4834677	0.5358969	0.5470465	0.5449896	0.5272086
-2 .. + 2	0.4834677	0.5359390	0.5477143	0.5476107	0.5373947
-3 .. + 3	0.4834677	0.5359390	0.5477143	0.5476110	0.5374150
-4 .. + 4	0.4834677	0.5359390	0.5477143	0.5476110	0.5374150
:	:	:	:	:	:
-25 .. + 25	0.4834677	0.5359390	0.5477143	0.5476110	0.5374150
<i>GK</i>	<i>0.6993484</i>	<i>1.0981516</i>	<i>1.4107735</i>	<i>1.6764377</i>	<i>2.1222563</i>

* Parameter values: $S_t = 22$, $\underline{S} = 20$, $\bar{S} = 25$, $K = 22.5$, $r = 0.05$ and $r^* = 0.03$.

Figure 1: Call option prices under RGBM (solid lines) and GBM (dotted lines) for $r = 0.06$, $r^* = 0.03$, $\underline{S} = 0.15$ and $(T - t) = 0.5$.

(a) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 20.5$

(b) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 22.5$

(c) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 24$

(d) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 20.5$

(e) $\underline{S} = 19$, $\bar{S} = 26$ and $K = 20.5$

(f) $\underline{S} = 15$, $\bar{S} = 30$ and $K = 20.5$

Figure 2: Hedge ratios under RGBM (solid lines) and GBM (dotted lines) for $r = 0.06$, $r^* = 0.03$, $\mathbf{S} = 0.15$ and $(T - t) = 0.5$.

(a) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 20.5$

(b) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 22.5$

(c) $\underline{S} = 20$, $\bar{S} = 25$ and $K = 24$

(d) $\underline{S} = 19$, $\bar{S} = 26$ and $K = 20.5$

(e) $\underline{S} = 17$, $\bar{S} = 28$ and $K = 20.5$

(f) $\underline{S} = 15$, $\bar{S} = 30$ and $K = 20.5$