Set theory and geometry in Hegel

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SET THEORY AND GEOMETRY IN HEGEL[1]

This paper investigates the subject matter of mathematics. More precisely, it is about the systematic dialectical determination of the quantitative and of mathematical mechanics in Hegel's *Encyclopädie der philosophischen Wissenschaften*. In modern terminology, the quantitative is akin to set theory and mathematical mechanics encompasses geometry. Mathematical definitions are strict and rigorous, but their genesis and the kind of abstractions involved in them are beyond mathematics. A dialectical perspective elucidates these issues.

A part of the previous literature on the merits of a Hegelian perspective on the mathematical focuses on the import of Hegelian insights for (specific) mathematical problems and paradoxes (e.g. Paterson 1994; 1997a; 1997b; 2000; 2002). Literature that elucidates, comments upon and expands Hegel's views on infinity is prominent among this group (cf. Ellsworth de Slade 1994; Lacroix 2000). Another part focuses on (Hegel's place in) the history of mathematics (e.g. Baer 1932; Paterson 1999; 2004/2005; Tóth 1972; Wolff 1986). Kol'man and Yanovskaya (1931) discuss the nature and extent of the influence of the Hegelian philosophy of mathematics on Marxism-Leninism and Fleischhacker (1982) uses Hegel's insights in his search for the object of mathematics. The current paper is unique in that it focuses on the dialectical determination of the conceptual foundations of set theory and of mathematical mechanics. Also, to my knowledge,

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I am the first to integrate the systematic dialectical determination of the quantitative with that of geometry.

I take Hegel's moments to be answers to the following questions: 1) how does a concept appear in total conceptual isolation? 2) How does it manifest itself in the world? And 3) how can the tension between a concept's pure form and the inhibitions to its manifestation be resolved? Hegel sometimes denotes these three perspectives by the Greek letters a, ß and ? respectively; I have done so consistently. Sometimes several gammas are needed before a full resolution is reached, from which a new opposition will generally arise. In my reconstruction of Hegel's presentation, some moments are brought to bear under a different heading than Hegel did or would have done. In those cases, the reason for doing so will be elaborated upon in the main text.

1. Hegel's Determination of the Quantitative

Since the quantitative is part of the Logic, its determination must begin with Being. The non-thought of a) Being immediately Becomes the non-thought of ß) Nothing. Thus, ?) Becoming is the first real thought in the presentation (Hegel 1969: 82-83, 1.1A-1.1C; Hegel 1930: §86-88; Carlson 2000: 11-12; Carlson 2003: 11-16).[2] The dynamism inherent in Becoming requires a further *static* determination of Being as ?) Presence – my translation of Hegel's 'Dasein' (Hegel 1969: 113, 1.1C; Hegel 1930: §89). This translation was chosen in order to stress that 'Dasein' is interpreted here as the whole of perception possible in the present. Thus, it moves along with Becoming. If it did not, that is, if its determination were to be fixed at some point, it is a) Something.

[2] When I refer to Hegel's *Wissenschaft der Logik*, I specify the page number in the Suhrkamp ('taschenbuch') edition of that work, as well as the segment, chapter, section and subsection, respectively. All references to the *Wissenschaft* are to the first book of the first part, so the part and the book in question need not be specified. Thus 1.1A means (part 1, book1,) segment 1, chapter 1, section A (The first chapter of Hegel 1969 has no subsections). In the case of the *Encyclopädie*, references to the §§ (notation: §#) suffice to enable comparisons to translations and to other editions than the Lasson edition usually referred to in this paper.

Because Something's determination is fixed, it can be left behind in the process of Becoming. When Something is left there-and-then, its ß) Other is here-and-now (Hegel 1969: 125-126, 1.2Ba; Hegel 1930: §91-93). But by Being here and now, this Other is itself Something. Hence Something and its Other are the same in the concept ?) One. In that the process by which Something Becomes its Other, which, taken as Something, again has an Other, is a *ceaseless* Becoming, there must be ?) Many Ones (Hegel 1930: §96-97).

Bringing the concepts One and Many Ones to bear under γ) is a major digression from the treatment those terms get in the *Encyclopädie*. In that book, Hegel first introduces Being-for-self ('Fürsichsein') under γ) as the union of Something and Others (1930: §95). Next, under α) Hegel contends that any Being-for-self is One and under β) he clarifies that there must be Many Ones, which as self-contained units are Repulsive of one another (Hegel 1930: §§96-97). Then, under γ) he continues that Ones are not only self-contained units, but also Ones. As such, each One is conceptually the same as each Other One and they have a relation not of Repulsion, but of Attraction (Hegel 1930: §98).

The major problem with this treatment is that the concepts of Repulsion and Attraction are opposites. So instead of resolving a dialectical opposition under γ), on this occasion Hegel introduces one there. He rarely, if at all, does this in the rest of the *Encyclopädie*, so in order to keep this passage consistent with the rest of the presentation in this section, I juggled the α), β) and γ) around a little bit. This operation does not leave the meaning of the terms in question entirely unchanged.

As self-contained units, these Ones are distinguishable As Many through a relation of ß) Repulsion, but because everything is in a notorious state of flux through Becoming, it is not clear where Something ends and its Other begins. Hence the One is only limited through an arbitrary external reflection. In that this limit is arbitrary, one might think of many more Ones within the unit One. So, from the standpoint of Becoming the Many Ones have a relation of a) Attraction instead of Repulsion (Hegel 1930: §97-98).

As was alluded to above, Hegel regards the conceptual sameness of the Ones as the locus of Attraction instead of the indeterminacy of their Becoming. The difference between his and my treatment stems from my juggling around of a), ß) and ?). Also, the conceptual sameness of the Ones in Hegel's treatment seems to imply a regress towards only the one Presence, whereas I think the point should be that positing a One requires an arbitrary external reflection. This reading seems to be confirmed by the conceptual development towards subsequent moments such as Discrete and Continuous Magnitude.

With this we have entered the realm of ?) Quantity. Thus, Quantity is an arbitrary and therefore external reflection on Many Ones distinguishable through Repulsion, but divisible through Attraction (Hegel 1930: §99). Alberts, (1998), Fleischhacker (1982) and Dijkgraaf (2001) would all agree with this result. That is, like Hegel, these authors consider 'an external reflection on many distinguishable but divisible elements' an apt description of the object and nature of mathematics and mathematical abstractions (Alberts 1998: 20, 27-28; Baer 1932: 104; Dijkgraaf 2001: 7; Fleischhacker 1982: 16-17). Hegel's position is remarkable for, in his time, Quantity was conceived of as a property of things rather than an external relation between indeterminate, abstract elements.

In its moment of Attraction, Quantity is a a) Continuous Magnitude, but in that the Many Ones are distinguishable self-contained Units, it is ß) Discrete. A specified Quantity is a ?) Quantum. If this Quantum is taken in its moment of Attraction it is a a) Unit: a One that explicitly consists of Many Ones. Counting Units means expressing their ß) Amount. When an Amount is explicitly related to a Unit, the result is a ?) Number (Hegel 1930: §100-102).

With Number we shed the last vestiges of quality. A Number is an a) Intensive Magnitude that derives its meaning from the ß) Extensive Magnitude it excludes. That is, the Number 100 is one less than 101 and one more than 99. So an Intensive Magnitude must continually go beyond itself into its Extensive Magnitude to gain meaning. This, in turn, means that the Extensive Magnitude progresses towards a bad potential infinity, which is no longer quantitative (Hegel 1930: §103-104; Hegel 1969: 250-256, 2.2Ba-2.2Bb). However, this progression is the result of transfinite iterations of a successor function. In Hegel's *Wissenschaft der Logik*, this function is the locus of true ?) Quantitative Infinity (Hegel 1969: 260-264, 276-278, 2.2Ca-2.2Cc; Fleischhacker 1982: 143-147; Lacroix 2000: 311-315), but this term is absent from the *Encyclopädie*. I mention it here, because it helps to explain some of the conclusions of this paper.

So what have we achieved? On the basis of quality alone we were unable to make qualitative distinctions, so we entered the realm of Quantity, which is governed by external reflections on sets of elements. The elements are arbitrarily chosen Units One and the sets are Amounts of them expressed through Number in an Intensive Magnitude that has its ultimate meaning in the bad potential infinity that develops in its Extensive Magnitude. So just as quality is not sufficient to understand the absolute, so is Quantity. So we need both. That is, we need a qualitative Quantum: ?) Measure (Hegel 1930: §106-107; Carlson 2002: 110). In that Measure reinternalizes quality, its subsequent development goes beyond mathematics proper, so I will not discuss Measure any further and turn to the philosophy of nature instead.

2. Hegel's Determination of mathematical mechanics

Geometry is part of the philosophy of nature, so its determination proceeds from the universal principle Space. [3] Space is the universal principle of all material observables and hence of the natural sciences, simply because all observables are Spatial. The ß) Point is the limit of the infiniteness of Space, which in turn is posited in the a) Spatial Dimensions. The determination of the ?) Line, ?) Plane and ?) Distinct Space is the result of step-by-step unlimiting of the Point into first one, than two and finally all three Spatial Dimensions (Hegel 1930: §§254-256; Paterson 2004/2005: 18, 29, 33-34).

[3] As a universal principle that is not, like Being, itself part of an opposition, Space is in neither of the categories α), β), or γ).

When you confine yourself to a Distinct part of Space you may see Something Become its Other: Becoming in action. Our perception of ?) Time is predicated upon this. It is observed Becoming (»das angeschaute Werden«) (Hegel 1930: §258; cf. Inwood 1992: 295). To Hegel, Time consists of the three a) Temporal Dimensions past, present and future (Hegel 1930: §259). The present is the span of Time needed to perceive. The present Something was shaped by a series of changes in the past and will change into countless Others in the future.

ß) Now, by contrast, is entirely changeless. It is not a span of Time. This means it all but disappears from Time altogether. While Time is observed Becoming, ?) Place is observed Presence. To observe Presence requires Distinct Space and the concept Now. The latter ensures that Presence instead of Becoming is observed. As such, Place is the union of determinate Distinct Space, here, and Now (Hegel 1930: §§259-260). In other words: it is the Spatial Now. The spatial location of a Place is arbitrary, because Distinct Space is only an arbitrarily fenced of part of Space. This means that every Place is the same as every Other Place. Given Time, this allows Many Ones to roam freely between different Places. Since Now is infinitesimally short, it immediately passes over into Time and in Time the Ones change their Place, thus constituting ?) Motion. Motion then, is the Becoming of the natural realm as a whole. So, Motion is the Becoming of everything spatial, that is, of ?) Matter. Matter therefore is the actual Presence (and not only the observed Presence) of the natural realm (Hegel 1930: §261). These points can only be phrased like this when 'Dasein' is interpreted as the whole of the present.

3. How This Dialectic Reflects on Mathematics

Now that the quantitative and mathematical mechanics have been dialectically determined, the question what insights can be gained from such a perspective can be dealt with.

To begin with, the exhibition of the quantitative above solves what Baer calls 'the problem of Number theory'. To apprehend Numbers with the tools of mathematics, we need to have those tools first. But to build up these tools, especially induction, we first need to have all natural numbers. So this is circular. In mathematics this problem is 'solved' by assuming a 'one' and a successor function that increases the assumed one by one. Of course the full set of natural numbers can be built up this way, but it does not explain the origin of the mathematical mindset.

The exhibition above offers a solution to this problem by showing that a reflection on the realm of quality automatically leads the presentation into the realm of Quantity. This happens because the attempt at truly qualitative distinctions fails, which leaves us with an external reflection on sets of arbitrarily chosen elements One. Hegel was ahead of his time in recognizing this as the subject matter of mathematics. In that the elements One must eventually Become an Other One, they form the Many Discrete Ones upon which the natural numbers are based. In that the elements are arbitrarily chosen and divisible, this set is further determined as the Continuum R.

Thus, for Hegel the quantitative is only the necessary external reflection on the realm of quality, without which neither can exist. Modern mathematics, by contrast, does not usually make true statements, but correct ones of the form: 'in a world where Euclid's axioms are true….'. Not all modern mathematicians care whether this world actually exists. Given the axioms, they create self-contained and basically tautological truths. Hence, the notion that the quantitative is the condition of existence of the qualitative must appear alien to them.

On the other hand, Hegel determined the quantitative somewhat presciently as the realm of external reflection on sets of elements and from this perspective there is considerable scope for studying quantitative relations on their own account. But without the qualitative there would be no quantitative and hence there would be no Numbers and no mathematics. According to Kol'man and Yanovskaya this insight is one of the greatest merits of Hegelian philosophy in the field of mathematics (1931: 2, 5).

The second thing that is clarified by the systematic dialectical exhibition is the use of ordinal and cardinal Numbers. If you count the Amount of elements in a finite set you have to begin somewhere, so while counting, you implicitly call One element the first, One the second etc. But if the Numeration is complete, we have arrived at an Intensive Magnitude (Baer 1932: 115).

Within that Intensive Magnitude it no longer matters which element we counted first and which second, so the arbitrary distinctions between the elements that were created by the form of the series (e.g. 1,2,3,…,n) of the concomitant Extensive Magnitude have now disappeared. Every element in a set of size n may be the n^{th} element. Then the size of the set (n) is a cardinal Number. This cardinal Number is the Intensive Magnitude of the finite set. As such the size of the set is itself a Unit: it expresses the Number of elements it contains while denying them autonomy.

Hegel’s point about the relationship between Intensive and Extensive Magnitudes not only elucidates the use of finite ordinal and cardinal Numbers, but also that of infinite ones. If a set is expanded with all subsets contained within it, this is equivalent to raising 2 to the power of the number of elements in that set. By analogy, since the Intensive Magnitude of the denumerable infinite set of all natural numbers N is defined as $\aleph_0$, it contains $2^{\aleph_0}$ subsets, so the size or Intensive Magnitude of the power set of N, P(N), that is of the continuum R, is $2^{\aleph_0}$ and that of P(R) is 2 to the power of $2^{\aleph_0}$, etc. (Horsten 2004: 26). It can be proven that each set thus obtained is of a higher order of infinity than the previous set. This implies that infinite cardinal Numbers may themselves be ranked in a well ordering. So just as transfinite iterations of a successor function lead a finite Intensive Magnitude into the bad potential infinity associated with its Extensive Magnitude, transfinite iterations of the power operation lead an infinite Intensive Magnitude into the ‘worst’ potential infinity associated with the size of the class of all sets V. In short: even for an infinite Intensive Magnitude there exists an Extensive Magnitude through which it gains meaning.

»[S]ince the power set of N […], contains an enormous Amount of infinite sets as elements that therefore must be seen as complete, ‘finished’, limited objects«, this way of thinking implies »the existence of an enormous amount and enormously big *actually infinite sets*« (Horsten 2004: 27, my translation). This fact, together with the fact that even infinite Numbers can be ordered to fit Hegel’s conceptual apparatus, at least partially dispenses with the ‘badness’ of Hegel’s bad potential infinity. Because the founding father of set theory, Georg Cantor, was born after Hegel’s death, Hegel cannot possibly have been aware of these points.

Finally, we may say that it would be misguided to think of the Line as the result of the Movement of a Point or of the Plane as resulting from a moving Line. This is because Time and Motion presuppose Distinct Space. But the real novelty with regard to the determination of mathematical mechanics in this paper stems from my interpretation of ‘Dasein’ as the whole of perception in the present.

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