Perspectives on an Integrated Computer Learning Environment

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Chapter 3

Computer Tools for Cross-Disciplinary Work with Real Data

In this chapter I present exploratory case studies on working with real data in practical investigation tasks using the versatile computer learning environment Coach 6. The presented case studies are field experiments and usability studies that were part of my development work on the design and implementation of an integrated computer environment for learning mathematics and science in an inquiry-oriented approach. In the spiral model of case-based design of educational hardware and software they played an important role in testing and evaluating pre-release versions of tools of Coach 6 (cf., Section 1.4). But these studies served in fact many goals: (1) They were meant to gain insight in the needs of secondary school students for doing authentic inquiry work. (2) They helped me specify requirements for an integrated computer learning environment from a mathematical point of view. (3) They served to test the usability and scope of (prototypical) implementations of particular tools for collecting, processing and analyzing data. (4) They gave an impression of the potential of ICT regarding the realization of challenging, cross-disciplinary practical work in which secondary school students were engaged in activities such as experimenting, data collection, and data analysis in much the same way as scientists and practitioners.

I published about the field experiments and usability studies in conference proceedings and journals. In these papers I aimed to describe the potential of such a computer environment via sample Coach activities, I discussed obstacles encountered while designing and carrying out the activities, and I brought up envisioned solutions in the software environment. In all papers I gave examples of ICT-supported quantitative mathematical modeling activities using an integrated tool set. In this chapter I have selected some of these papers and briefly describe their contents. Although most papers contain examples of integrated use of more than one tool, they have been categorized according to the most eye-catching tool part. This led to the following sectioning of the chapter: (1) overview of activity types; (2) digital image and video analysis; (3) modeling; (4) combination of measurement with sensors, control of experiments, and video; and (5) combination of video analysis and modeling.
3.1 Overview of Activity Types

In several publications I gave an overview of the design and implementation of the integrated computer learning and multimedia authoring environment called Coach. Recent papers, on which the overview in this section has been based, are:


Coach 6 is a recent implementation of the STOLE concept of an integrated, tool-based environment suitable for an inquiry-oriented approach to science and technology learning, in which students and teachers work with multimedia-based activities and create such activities themselves by using a variety of resources. The STOLE concept, acronym for Scientific & Technical Open Learning Environment and visualized in Figure 3.1, was outlined in Section 1.2 (pp. 13–14).

![Figure 3.1](image)

Figure 3.1: Pictorial presentation of the STOLE concept for inquiry-based learning.

Software tools selected for STOLE were grouped into functional modules that combined instruments needed by a student at a certain stage of a scientific investigation. The student was central in this concept and it should be possible to adapt the environment to the students level and to the science curriculum in which the student and his or her teacher participate. The criteria for tool selection were the following:

- All the relevant tools (functions) for practical investigations must be included.
- Tools must be selected for functional use in investigations.
- The environment must be transparent for the data in all the modules in the software package. Users should not have to make conversions because of the format of data.
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- It must be possible to exchange data, models, and information between students within and outside school. The environment must also work with data coming from other resources.

- If a module in STOLE cannot be constructed, then it must be possible to work with other software packages and exchange data.

- It must be possible for publishers to deliver templates, data, models, and information from science textbooks that can be processed by students.

In this section there is no need to discuss the historical evolution of the STOLE concept; I only present the realization of this concept in Coach 6. Thus it suffices to repeat the one-sentence description of Coach as a single, activity-based, open computer working environment that is designed for the educational setting and that offers its users a versatile set of integrated tools for the study of natural phenomena, mathematics, science and technology. A closer look at the elements of this description reveals that the environment is meant to

- aid students in collecting, processing, and analyzing various types of data, to provide visualization and analysis tools, and to offer opportunities for creating computational models and animations;

- be universal and applicable in several curricula, in various instruction models, and at many levels of education, and be adjustable by teachers to their students’ abilities;

- transform mathematics and science lessons into learning activities in which students are deeply engaged in building up and practicing knowledge, experience, and skills. In other words, active learning is fostered;

- change the computer into an instrument that allows students to explore real-world phenomena, helps them develop deep understanding of mathematical, scientific, and technological concepts, and supports communication of students’ ideas and results; and

- involve students in similar activities to what scientists, engineers, and practitioners engage in and thus lead to authentic mathematics, science, and technology learning, in which various higher-order thinking skills like problem solving, critical thinking, creativity, and connecting contexts with fundamental concepts in mathematics and science are highly valued (See, for example, Chinn & Malhotra, 2002; Edelson, Gordon, & Pea, 1999; Edelson & Reiser, 2006; Roth et al., 2008).

Coach activities are mostly based around the selected tools for collecting, generating, processing, or analyzing data. Teachers can use ready-made activities or author new activities, and they can organize them in projects to structure the lesson materials (experiments) for their students. Activities typically contain components of various types:

- Texts with explanations and/or instructions of activities.

- Pictures to illustrate experiments, equipment, and context situations.
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- Video clips or digital images to illustrate phenomena or to use for measurements.
- Representations of measured data and computed results as graphs, tables, meters, or digital values;
- Models (text-based, equations-based, or graphical) to describe and simulate phenomena.
- Programs to control devices and systems, and to make mathematical computations.
- Animations to dynamically represent and interact with models of phenomena and with control programs, and to give visual feedback to measurements.
- Links to Internet sites and other external resources for students.

Basic descriptions of several types of activities and concrete examples are given in the following subsections, but at this point it is good to realize that all kinds of activities can be combined and are supported in a single computer working environment instead of a suite of separate programs.

3.1.1 Data Logging

In a data logging activity, that is, a measurement activity with sensors, students gain insight into setting up an experiment and collecting data with sensors that are connected through an interface with a computer or directly attached to a hand-held device. This understanding of doing an experiment helps students process and analyze the data, and interpret the graphical representations of the data. It helps even more because the data are dynamically linked with different representations such as graphs and tables during and after the measurement, and because this measurement can be replayed on the computer screen as many times as wanted. Experiments are quite easily set up (by dragging and dropping sensor icons on the virtual interface panel on the computer screen or by automatic sensor recognition) with a variety of interfaces supported and a large library of calibrated sensors (for temperature, light intensity, sound level, etc.) available. Experimenters can select an appropriate measurement method (time-based, event-based, or manual measurement, with or without triggering) and a useful measurement setting (duration and sample rate).

The above description could give the impression that students must do all the things mentioned to carry out a measurement activity, but this is not true. For young pupils, everything may be prearranged in such way that they cannot (accidentally) modify the activity. More experienced students may get access to measurement settings or the choice of sensors. Senior students may get maximum freedom of work for end-users, which is pretty close to the facilities for authors of COACH activities. This approach leads to a learning trajectory for doing measurements that works well in school practice (cf., Van den Berg & Ellermeijer, 2006). It is noted that the same strategy can be applied in other types of COACH activities.

Figure 3.2 shows a screenshot of the measurement (red dots) and the signal analysis (blue graph) of the voice sound ‘eh’ recorded with the €Sense interface, which is mostly used at primary school or by novice users. The diagrams show that the sound signal is well described by a sinusoidal signal that consists of four frequencies. A visual
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representation of the ESense interface is also present in the activity screen to make the experimental set-up clear to students. A text window is used for explanation and description of tasks.

A control activity offers several modes of programming with varying levels of difficulty. These modes are a manual control mode, an instruction mode, a microworld programming mode, and a free programming mode using the Coach language. They can be used to create and execute programs for automated measurements (e.g., an automated pH titration system in which a titrator is controlled while measuring with a pH sensor), for manipulation of measurement data (e.g., converting voltage signal from a sound sensor to decibels), for programming any phenomenon (mathematical, scientific, natural, artificial, or whatsoever), and for control of systems such as LEGO® models, such as in the computer microworld for manipulating a robot arm in Figure 3.3. Control activities give students the opportunity to create physical artifacts, such as vehicles, line-followers, and robots, and program them with interesting behavior. In this way a design project becomes a fun project and at the same time students learn a lot about the obstacles that technicians, engineers and technical designers have to overcome in similar work.

Figure 3.2: Measurement and signal analysis of voice sounds with the ESense interface.

3.1.2 Control

A control activity offers several modes of programming with varying levels of difficulty. These modes are a manual control mode, an instruction mode, a microworld programming mode, and a free programming mode using the Coach language. They can be used to create and execute programs for automated measurements (e.g., an automated pH titration system in which a titrator is controlled while measuring with a pH sensor), for manipulation of measurement data (e.g., converting voltage signal from a sound sensor to decibels), for programming any phenomenon (mathematical, scientific, natural, artificial, or whatsoever), and for control of systems such as LEGO® models, such as in the computer microworld for manipulating a robot arm in Figure 3.3. Control activities give students the opportunity to create physical artifacts, such as vehicles, line-followers, and robots, and program them with interesting behavior. In this way a design project becomes a fun project and at the same time students learn a lot about the obstacles that technicians, engineers and technical designers have to overcome in similar work.

Figure 3.3: A computer microworld for manipulating a robot arm.
3.1.3 Digital Image and Video Analysis

Figure 3.4: Video analysis of a $3 \frac{1}{2}$ forward somersault dive.

Figure 3.4 is a screen shot of a video analysis of the motion of a diver making a $3 \frac{1}{2}$ forward somersault dive from the 3 meter springboard. The upper-left window shows a still of the video clip in which the diver starts leaving the tuck position. The video clip originates from the website of the video measurement system ViMPS (www.physik.uni-mainz.de/Lehramt/ViMPS) and the below analysis (Heck, 2004c) follows the description of May and Kayser (2003). The open red dots mark locations of the head of the diver measured by point-clicking in selected frames of the video clip. The closed dark blue dots show the trajectory of the center of mass, which is estimated by mathematical analysis of the motion of the diver’s head. This analysis is based on the assumption that the motion can be decomposed as a parabola, representing more or less the motion of the center of mass under gravity, and a sinusoid, describing the rotational motion of the body in tuck position. Regression analysis on the basis of the least squares method of peeling-off functions (Foss, 1969) is the computational engine. In the lower-left window in Figure 3.4 are shown the $y-t$ diagram of the data and a quadratic function fit, which represents the trend of the motion of the head. In the next step one fits the residue to a sinusoid, that is, one subtracts the quadratic fit from the data and determines a sinusoidal regression curve. The results is shown in the lower-right diagram. The sum of the quadratic and sinusoidal fit describes the vertical position of the head of the diver quite well; see the corresponding curve in the
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upper-right diagram. A similar analysis can be carried out for the horizontal motion of the head, but in this case the trend function is linear. The linear and quadratic trend functions of the horizontal and vertical position of the head, respectively, can be used as coordinate functions of a certain imaginary point and displayed as overlay in the video clip. The trajectory of this imaginary point (closed blue dots in the video window) suggests that this point is close to the center of mass of the diver.

Secondary school students do not often get the opportunity to analyze phenomena that can be decomposed into parts, but many examples presented in this chapter and the previous one show that this is a missed opportunity. It severely limits the quantitative mathematical modeling to simple models and simple phenomena. With a little bit of extra effort, more interesting phenomena can be investigated by students. In this case for example, they may explore the rotational motion around the center of gravity and investigate how changes in the moment of inertia affect the rotational velocity of the diver when he opens his tuck position. This particular type of investigation actually brought a weakness in the video analysis tool to the fore: When polar coordinates are used for measuring rotational motion, one does not obtain a continuous graph; instead one is confronted with a graph that has jumps of size $2\pi$ (or $360^\circ$). One can repair this by mathematical processing of the data, but it would be much easier to have this automatically done upon request while collecting data. A simple algorithm can keep track of this, provided that rotational motion between two frames in a video clip is in reality not too big: if for two consecutive measured angles $\phi_n$ and $\phi_{n-1}$ holds that $\phi_n - \phi_{n-1} > \pi$, then one must subtract $2\pi$ from the measured angle $\phi_n$ (i.e., $\phi_n \leftarrow \phi_n - 2\pi$); if $\phi_{n-1} - \phi_n > \pi$, then one must add $2\pi$ to the measured angle $\phi_n$ (i.e., $\phi_n \leftarrow \phi_n + 2\pi$); otherwise, no adjustments to measured values are done. Such requirements and improvements of tools often originate from concrete activities in usability studies, field experiments, and classroom experiments. This is the reason why case-based design of educational software is often more effective than design from behind the software engineer’s desk.

Several mathematical tools come into play in video analysis activities for processing and analyzing measured data: numerical differentiation to obtain velocity and acceleration, curve fitting, formula manipulation in order to get formulas of interest, and graphing functions. But also the video analysis tools used in practical investigations are varied: video capture; adjustment of the quality of video clips and digital images; transformations of movies and images such as rotation, reflection and perspective correction; automated point tracking in video clips; usage of moving reference frames; and measurement tools for distance, angle, area, etc. In the discussion of case studies in this chapter, several of the video analysis tools and many of the issues of accuracy of webcams in 2D motion analysis identified by Page et al. (2008) pass in review.

I close this subsection by briefly lingering upon the educational benefits of video analysis in mathematics and science education. Reported educational benefits are often about the effects of video analysis, also referred to as Data Video and Video-Based Laboratory (VBL), on conceptual understanding of kinematics concepts and kinematics graph interpretation skills and on student motivation, attitude, and time-on-task (See, for example, Beichner, 1996, 1999; Escalada & Zollman, 1997; Larkin-Hein & Zollman, 2000; Koleza & Pappas, 2008; Zollman & Brungardt, 1995).

The first reported benefit of VBL is the positive effect of the feature of viewing a motion graph simultaneous with a video replay on student motivation, time on task,
and graph sense (Larkin-Hein & Zollman, 2000; McCullough, 2000; Zollmann & Brungardt, 1995). A simultaneous view of graphs and video means in Coach in the first place that the video clip on which one measures and the corresponding mathematical representations such as graphs and tables are always synchronized. Pointing at a graph or a table entry automatically shows the corresponding video frame and selecting a particular frame highlights the corresponding points in diagrams, when scanning mode is on. This makes scrubbing, that is, manually advancing or reversing a clip, an effective means to precisely identify and mark interesting events in the video clip and to relate them with graphical features. This supports students to transition between graphs and physical events and is probably the main reasons for the reported improvements in graph sense due to VBL. This does not mean that non-linked or semi-linked multiple representations could not play a role in student activities: Özgün-Koca (2008) reported that semi-linked representations could be as effective as fully linked representations in video analysis activities in mathematics classrooms.

Because the video tool of Coach offers the possibility of automated point tracking, simultaneous view of graphs and video can also mean direct feedback in the form of real-time graphing, similar as in measurements with sensors (data logging, also known as Microcomputer-Based Laboratory [MBL]), that is, the real-time pairing of events and the mathematical representations. In my opinion, this pairing of events and representations may clarify the use of a variable as a symbol for a variable object, in addition to the use of a variable as a placeholder and a polyvalent name (cf., Freudenthal, 1983; Heck, 2001). It amplifies the notion that a motion graph is not a static picture, but represents the position of a moving object through coordinates that change in time or derived quantities that are time-dependent. In other words, it amplifies the notion of a motion graph as a pictorial representation of a dynamic process. When video capture via a webcam and automated point tracking can be combined in a computer learning environment (not yet implemented in Coach), then the webcam would become just another sensor attached to the computer and the contributions of data logging to graph sense identified in many research studies will probably also hold for VBL (cf., Boujaoude & Jurdak, 2010; Lapp & Cyrus, 2000; Linn, Layman, & Nachmias, 1987; McRobbie & Thomas, 2000; Mokros & Tinker, 1987; Newton, 2000; Nicolaou, Nicolaidou, Zacharia, & Constantinou, 2007; Russel, Lucas, & McRobbie, 2003, 2004; Stylianou, Smith, & Kaput, 2005; Thornton & Sokoloff, 1990; Trumper, 1997). The same characteristics that explain the effectiveness of data logging in teaching and learning science—elimination of the drudgery of data collection and graph construction, use of multiple modalities in exploring phenomena, no need of formulas for graph construction, and the ease of use of the tools, amongst others—seem to apply obviously for the extended VBL as well, but I wrote ‘probably’ because two major lessons learned from micro-analytic studies such as carried out by Lindwall and Ivarsson (2004, 2010) in comparing data logging with simulation-based software in a predict-observe-explain (POE) approach are: (1) Details of the used technology and the designed activities strongly influence the learners’ actions and the learning outcomes. (2) With regard to the design of educational technology and the effectiveness of the technology in practice, the proof of the pudding is in the eating.

Another important feature of VBL is that it mimics a technique often used in movement science. This means that students can act in research projects like movement scientists: They record video clips of a motion in which they are interested with
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a webcam or digital video camcorder; they measure on these movies and analyze the collected data. I refer to (Gross, 1998; Heck, 2009a,b; Heck & Ellermeijer, 2009; Heck & Holleman, 2003; Heck & Van Dongen, 2008) for illustrative examples, which show that video and image measurement have great potential for studying everyday scenes of motion and for linking mathematics and science with the real world. The user must decide in such activities about many things, like: how to make the best video recording; how to enable calibration afterwards; what number of data points to measure; which coordinate system to use; in which frames of the video to measure; and so forth. Some advantages of video and image measurement (Laws & Pfister, 1998), compared to measurements with sensors and traditional laboratory work, are:

- It is an easy, fast, broadly applicable, and attractive method of collection data in practical work for students who grow up with video technology and who can devise in this way their own projects.

- Simple mouse clicking and automated point tracking replace the tedious work of manual recording of data and allow students to concentrate on the investigated phenomena. Data can be verified later on and, if necessary, be corrected.

- The experimental set-up is rather simple. This saves time, lowers costs, and takes away many of the practical issues that must be dealt with in a real experiment.

- Processes which are too difficult or impossible to measure with sensors can be studied. The range of projects has been increased now that affordable high speed cameras have become available (cf., Heck & Ellermeijer, 2009; Heck, Knobbe, Nijdam, Slooten, & Uylings, 2011; Heck & Uylings, 2010a, 2011; Heck, Uylings, & Kędzierska, 2010; Heck & Vonk, 2009; Koupil & Vícha, 2011; Mathavan, Jackson, & Parkin, 2009; Vollmer & Möllmann, 2011b).

Figure 3.5: Screen shot of a photogrammetry activity with image rectification and regression.

An example of a measurement on a digital image that is otherwise difficult to realize is shown in Figure 3.5, in which the shape of the Golden Gate Bridge is investigated. The digital image to the left is the original picture, but it is perspective distorted. Thus, data collection must be done in a modified image, in which a fronto-parallel view of the bridge has been realized. For details how to rectify the image (and video clips) I refer to (Heck & Uylings, 2006; Heck & Ellermeijer, 2009) and Section 3.2.2. An
important point that is illustrated by this particular example is that computer technology can also be used to raise mathematical or scientific questions: In the diagram to the right, several regression curves that successfully match the recorded position data of suspension cable are drawn, but which one is the best? On the basis of what criteria? In popular wording: tool use can turn brains on (Heck, 2007d).

3.1.4 Modeling and Simulation

In Section 2.3.2 I already discussed a classroom study in which the text-based modeling tool of COACH 5 had been used to mathematically model the main span of a bridge. In COACH 6, three types of model editors have been made available: a text-based, equations-based, and graphical editor. In Figure 3.6 is shown a graphical model of the main span of Golden Gate bridge that is based on the approximation of the suspension cable by $k_{\text{max}}$ straight line segments with horizontally equidistant joints. It leads to a parabolic approximation (Heck, 2007d; Heck & Holleman, 2002b) that matches well with the measurements of the shape of the main span through digital image measurement as shown in Figure 3.5. The meanings of the icons in the graphical model are similar to those of system dynamics-based software like STELLA and POWERSIM, which use a stock/flow metaphor for dynamically changing systems: There are icons for state variables (stocks, also called levels), rates of change (flows), auxiliary variables, constants, and information arrows. The graphical elements represent a computer model, which provides in many cases an iterative numerical solution of a system of differential equations. Research (e.g., Löhner, 2005) seems to indicate that graphical modeling offers students an easier-to-use and richer framework for understanding the structure of a dynamic system in comparison with text-based modeling, and that it allows students to build more complex models of high quality, because they can concentrate on qualitative specifications during initial stages of the modeling process and do quantitative, formula-based specifications at later stages. The use of the various modeling modes is flexible in the sense that a student can start the creation of a model in graphical mode and then switch to the text mode or equation mode to add to or change the model (See Figure 3.7). Once the model has been created, it is easy to modify its results by changing values of parameters. Students can use this to test a hypothesis and to compare empirical data with results obtained from a theoretical model (as shown the diagrams in in Figures 3.6 and 3.7).

Figure 3.6: A graphical model of the shape of the Golden Gate Bridge and hypothesis testing.
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As discussed in Section 2.8, the modeling tool of COACH 6 has been designed such that it combines a classical system dynamics approach with event-based modeling for processes that change abruptly. More examples of this hybrid structure will be discussed in subsequent sections.

I end this section with spending a few words on why computer modeling is considered an important component of an integrated computer environment for learning mathematics and science in an inquiry-oriented approach. First and above all, a modeling approach to teaching and learning mathematics and science may provide students knowledge about the nature of these fields, in which modeling is a key activity, and may help them learn and appreciate mathematics and science that assists (in combination with other abilities) in the construction and interpretation of scientific models. Like Van der Valk, Van Driel, and De Vos (2007), I consider a scientific model as a representation developed and underpinned by scientific principles for the purpose of analyzing a specified target, which may be a system, a phenomenon, a process, an object or an idea. The main purpose of modeling is usually to describe, predict, explain, or understand the target. The above working definition of a scientific model is in line with the one given by Schwarz and White (2006), except that they include the option of using more than one representation: A scientific model is broadly defined as “a set of representations, rules, and reasoning structures that allow one to generate predictions and explanations” (p. 166). These authors use the term scientific modeling to mean (p. 167): “the process used in much of modern science that involves (a) embodying key aspects of theory and data into a model (frequently a computer model), (b) evaluating that model using criteria such as accuracy and consistency, and (c) revising that model to accommodate new theoretical ideas or empirical findings.” I speak of a mathematical model if the focus is on the structural characteristics (rather than, for example, physical characteristics) of the target and, more importantly, the representations of the target come from the field of mathematics (e.g., it is a graph, a table, a formula, a function, an equation, a differential equation, a phase plot, etc.).

I was in my thesis work particularly interested in ICT-supported quantitative mathematical modeling. The adjective ‘quantitative’ emphasizes that the quality of a mathematical model is always determined by the answers to the following three questions: (1) How good does the model describe empirical data? (2) What is the predictive power of the model? and (3) Does the model lead to a better explanation or under-
standing of the empirical results? My focus on ICT support in the modeling process had several sources, but the main ones were: (1) ICT extends the scope of what students can explore and takes away the drudgery of computational work. (2) The authenticity of many a practical investigation is increased by ICT because it reflects the use of computers for modeling in scientific practice. (3) Mathematics and science curricula gradually put more emphasis on computer-based mathematical models and modeling. In the Dutch science curriculum, system dynamics-based graphical modeling has been advocated (cf., Savelsbergh, 2008). From this point of view, it is almost compulsory that a versatile integrated computer learning and authoring environment contains a modeling tool of this type, in addition to the facility for writing and running computer programs written in a programming language. The graphical, system dynamics-based modeling tool is expected to provide an easy-to-use graphical interface for specifying the structure of a system dynamics-based model of a system and the interaction between its objects, and for running the computer model in order to study the behavior of the system. It allows solving realistic problems that cannot be solved analytically, or at least not at the students’ level of mathematics and science knowledge. Students are provided a means to model complex, real-world phenomena and they may get in this way a broader, more complete view of models and modeling in scientific practice: It may give them, for example, a good impression of the efforts that professional modelers have to put in for obtaining valid and reliable results and for the obstacles that these modelers face and must overcome. Another benefit of using a graphical computer model in a modeling-oriented approach to mathematics and science education is that the model structure is easily modified, allowing tryouts of different modeling ideas. Last but not least, modeling can also be a fun activity for students, not related to classical school mathematics and science topics. I give one example of this kind.

Figure 3.8: Screen shot of a model and simulation of a cost-benefit analysis of a business.
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Figure 3.8 is a screen shot of a modeling activity of two vwo-students who constructed a computer model for the cost-benefit analysis of a pizzeria, of which the owner wanted to expand business but needed financial advice. The model was supposed to help answering the following questions of the owner of the pizzeria: Is it profitable to expand business? What will be the cost per pizza after expansion? How many people will probably come per day? How much money will be earned or lost one hundred days after the re-opening of business. Variables built into the computer model were, for example, the payment of the advisors, the fluctuating costs for ingredients of pizzas, random numbers of new customers and of persons who stop coming to the pizzeria, and rent. The imagination of the students seemed unlimited.

3.1.5 Animation

Figure 2.32 in Section 2.8 was a screen shot of a COACH activity in which an animation linked to the computer model was used to visualize the chaotic behavior of a ball bouncing on an oscillating platform. Animations are considered useful when: (1) data in graphs or in tables are expected not to be enough for students to fully understand the underlying principles of a phenomenon; and (2) they help motivate students to actively explore computer simulations by interactive change of conditions via sliders or buttons. An example of the first type of use is illustrated in Figure 3.9.

Figure 3.9: A manipulable, model-driven animation of the water level in a tank.

It is a screen shot of a modeling activity of a water tank, which is filled by a tap and (at the same time) depleted by a small drain or hole at the bottom of the tank. It contains a manipulable animation that mimics the one created in the Co-Lab project (Savelsbergh, Bell, Bossler, et al., 2003; Manlove, Lazonder, & De Jong, 2006), in the framework of collaborative, scientific discovery learning with computer simulations and animations, and in the environment developed by Estrada-Medina and Arenas-Sánchez (2006) to study the relation between accumulation of a quantity and its rate...
of change. The filling rate is adjustable by opening or closing the tap in the control panel. The depletion rate is given by Toricelli’s theorem. It depends on the size of the drain, which is also adjustable, and on the water height. In the screen shot, the diagrams illustrate that the water level first decreased until an equilibrium state was reached and hereafter increased when the drain size was halved after about 1200 seconds. All representations can be viewed simultaneously during execution of the model. The main idea behind this manipulable animation is that students can first interact with the realistic animation and explore the context situation before trying to fully understand the underlying model.

The above animation has been constructed with animated graphics objects like rectangles, ellipses, vectors, and pictures, which have been linked to model variables to control their positions or sizes. In other words, the computer model is the engine which contains all the rules and formulas governing the variables involved and which leads to data as input for the animation. Parameters in the model have been linked to sliders, which can be used before or during the simulation to set values.

If the author of an activity does not want to supply the model on which the animation is based, then (s)he can hide it and create a more traditional simulation and animation environment. In other words, an author can create a microworld for learning. I give two examples of such use of animation. The first one, shown in Figure 3.10, is about the use of the animation tool to construct graphical representations that are intermediate between the animated phenomenon and the conventional motion graphs.

![Figure 3.10: Various dynamic representations of an animation representing the 100 meter run of four girls.](image)

It is a kind of combination of trace graphs and discrete graphs like Doorman (2005) used for students’ exploration of the motion of a tropical storm and the dynamic icon representations and stamp graphs of Ploetzner and colleges (2008, 2009) for studying the motion of a runner. The realistic phenomenon is a 100 meter run of four runners on a track viewed from above. The screen shot is inspired by one of the learning activities of the study “Grafieken leren met de computer” (Kanselaar, Van Galen, Beemer, et
3.1. Overview of Activity Types

al., 1999), in which originally the software Measurement in Motion was used to measure on video clips. Many questions and tasks can be asked, ranging from “Which runner passes the 20 m line first?” to “Figure out who is running in which lane.” The screen shot contains various dynamic representations that are in practice not all used together in one activity; they have only been put together to show the technological possibilities for designing a hypothetical learning route that could lead to student understanding of kinematics graphs.

The second example of using animation to design a simulation environment for exploring a phenomenon is shown in Figure 3.11: It is a screen shot of an activity designed for students to explore the physics of elastic 1D-collisions by setting masses, initial positions, and initial velocities before running the simulation, and to relate the motion of the balls to the shape of the kinetic energy, momentum, position, and velocity graphs (cf., Simpson, Hoyles, & Noss, 2005; De Jong, Martin, Zamarro, et al., 1999). The students can predict the graphs (within the software) before running the simulation and compare their predictions with the simulation results. They can explore the situation freely or work through various scenarios suggested in assignments (e.g., a hint to investigate the motion of the center of mass). Such simulations and animations are not only expected to promote understanding of physics, but in the end they may let students appreciate more the physics and mathematics that underpins the underlying model and handles all scenarios. Another reason for doing this kind of activities has been given by Araújo, Veit, and Moreira (2008): They reported improvement of kinematics graphs interpretation by computational modeling activities.

Figure 3.11: A manipulable animation of a 1D elastic collision.

Use of animation is not restricted to modeling activities. As a matter of fact, the dynamically represented quantities can come from various sources: from variables defined in a control program or microworld, and from data measured with sensors. For an example of an animation driven by measured data I refer to the animation of the Atwood machine presented by Kędzierska, Van Buuren, Ellermeijer, and Uylings (2009). Figure 3.12 shows a screen shot of a SimCalc-MathWorld lookalike activity (Kaput & Schorr, 2008): Here a microworld (lower left window), which was prepared by the author of the activity, is intended to be used by students to define a velocity graph (lower-right window), to determine the position-time graph (lower-middle window), and to create an animation (top window) of the corresponding motion of a bicyclist. The position of the cyclist in the animation window corresponds with the computed
position of the cyclist at a certain time and the arrow represents the velocity of the
cyclist at that moment. Students can explore kinematics graphs in the realistic context
of cycling provided by the microworld. In Section 3.4 I change the control program
such that the sound level measured by a sensor determines whether the cyclist moves
forward, constructing in this way a funny applause meter (Figure 3.39).

![Figure 3.12: Animation of a piecewise constant velocity function specified in a
microworld.](image)

It is a pleasant surprise that several secondary school students are able to create
tsophisticated animations and are keen on doing this. An example is the catching game
shown in Figure 3.13, which is an animation driven by a computer model that uses
events and randomization. My experience, also with computer modeling, is that if one
can appeal on the imagination of students, marvelous computer models, programs,
and animations can come out of their hands.

![Figure 3.13: Screen shot of a computer game implemented by a secondary school
student as a model-driven animation.](image)

### 3.2 Digital Image and Video Analysis

Several of my papers serve the following purpose: (1) illustrating the importance of
features like perspective correction and automated point tracking in a video analysis
3.2. Digital Image and Video Analysis

tool for doing small practical investigations; and (2) explore the attractiveness of high speed video clips in mathematics and science education. In Section 3.2.1, in which the elongation of a Coil Spring Slinky hanging under gravity is measured and mathematically modeled, I illustrate how a digital camera or a webcam can be used to measure a physical system in static equilibrium and I discuss some findings and requirements for an image analysis tool that follow from this usability study. In Section 3.2.2, a straightforward application of perspective correction in crime scene photogrammetry is presented and linked to other ways of using ICT in combination with digital images. Section 3.2.3, in which an investigation of moving coins on a horizontal table is discussed, exemplifies that perspective correction, automated point tracking, and the use of high speed video clips in a video analysis activity can play an important role in enhancing students’ understanding of physics of real phenomena.

3.2.1 Image Analysis of a Hanging Slinky

Reference

Introduction
In Section 2.3.1 I discussed a classroom experiment in which the video analysis tool of Coach 5 was used to collect data from a digital image, or more precisely from a video clip in which each frame is the same still image. It turned out that this unusual application of a video tool for image measurement caused some problems in practice: It resulted in more work and discomfort for students, teachers, and authors of activities. Identified drawbacks were:

- Some of the tools for movies often do not make sense for an image measurement. Think of time recordings, time calibration, moving origin, and playing speed.
- The linking between graphical and table representations and the image measurement is somewhat different. In image measurement, the ordering of marking points may be irrelevant and may not be connected with time.
- One cannot simply use a single picture; one must always duplicate it. This is space consuming.

The conclusion was quickly drawn that indeed a dedicated digital image analysis tool was needed in Coach. The design goals were kept simple for a prototypical version: (1) mimic as much as possible the tools from the video analysis environment; (2) remove or disable irrelevant tools from data video; and (3) add tools relevant for image measurement (e.g., a tool for counting objects in regions). The following four purposes of image measurement were distinguished: (1) recording coordinates of points, for example, points on a curve (the corresponding diagram and table have no time component); (2) working with strobe photographs (time is implicitly involved); (3) counting of objects in regions; and (4) measuring properties like distance, angle, area, shape. In terms of functionality, common formats of digital images (bitmap, jpeg, etc.) and simple modifications of digital images like change of brightness, contrast, and orientation (rotation and reflection), all for the purpose of making the measurement process easier, were expected to be supported.
In the released version (since 2005), only point-by-point data collection, that is, recording coordinates of one point after another, and measurement of one sequence of points were made available. Anticipated differences with video measurement were: (1) Tracing of measurement is always on. (2) No frame bar is needed because the link between a table entry or point in graph and the corresponding measured point can be made visible by highlighting. (3) Rows in a corresponding data table are linked to points on the image instead of frames in a video clip. (4) Measurement on strobe photographs must be specifically requested, because by default it is assumed that the variable time is not involved.

A variety of instructive experiments can be done using a low cost digital camera (a webcam, a camera of a mobile phone, etc.). Alternatively one can use digital images from Internet. Gil, Reisin, and Rodríguez (2006) presented some examples of the use of digital images for analyzing the shape of an object and a curve in space, such as the characterization of the geometry of a shadow, the trajectory of a water jet from a hose, catenaries, reflection caustics, beam deflections, and more. Foong and Lim (2010) discussed how the density of a watermelon floating in a pail of water can be estimated with different levels of simplification. Medeiros, Tavares, and Duarte (2009) described how one can create strobe-like photographs from a video recording with a low cost apparatus. Heck (2004e, 2008d) presented activities for mathematical modeling of shapes of eggs. In all examples, computer-based quantitative mathematical modeling was used to describe data measured on a digital image. In the usability study discussed in this section, a slinky hanging under gravity was modeled via basic principles of mathematics and physics, that is, it was not merely described by experimental mathematical modeling such as regression analysis. Modeling results were compared with measurements on a digital image to evaluate the quality of the model.

Usability Study
A Slinky is a toy consisting of a helical spring that stretches and can be used in physics education to enliven discussion of physics principles such as wave mechanics. Its behavior is sometimes counter-intuitive: For example, when a Slinky hanging under gravity in equilibrium is released, the top of the spring toy begins to fall, but the bottom remains absolutely stationary. Only when the entire Slinky has collapsed onto itself, the bottom begins to move. This is easily confirmed by video recording of this motion with a high speed camera. But also a Slinky hanging in static equilibrium under gravity is worth studying. Two questions that students can investigate are: (1) Where is the center of mass of a suspended Slinky? and (2) How does the elongation of a Slinky depend on its number of coils? These questions have been addressed in various journals for physics teachers (e.g., French, 1994 [corrected by Hosken, 1994]; Gluck, 2010; Newburgh & Andes, 1995; Sawicki, 2002; Serna & Joshi, 2011; Toepker, 2004). Digital image measurement is an outstanding example of a method to collect data for comparison between experiment and theory. It offers the opportunity to validate various mathematical models of a suspended Slinky.

Assume a Slinky of mass $M$ with a total of $N$ coils and undistorted length $L_0$ that is hung up on one of its endpoints such that the lowest coil does not rest on the floor and the Slinky is in static equilibrium, hanging under gravity. One very much simplified model for this situation is shown in Figure 3.14, taken and slightly modified from (Newburgh & Andes, 1995). It considers the Slinky as a system of $N$ weights, each of mass $m$ and thickness $\delta$, sequentially linked with massless springs
3.2. Digital Image and Video Analysis

with spring constant $k$. Thus: $M = Nm$ and $L_0 = N\delta$. Weights are numbered in the upward direction, starting with index 1, and distances between consecutive weights are denoted $y_i$.

\[ L = L_0 + \frac{1}{2}N(N - 1)u, \]

where $u$ is the displacement between the first and second coil, that is, $u = \Delta y_1$. Because of static equilibrium: $u = mg/k$. The two most important conclusions drawn from this mathematical model are: (1) The distance between two consecutive coils increases linearly in the upward direction. (2) The increase in length of a Slinky at rest to a Slinky hanging in static equilibrium under gravity depends quadratically on the number of coils.

The quality of the above model can be explored by comparing the mathematical predictions with empirical data obtained through digital image measurement. When this was done with the digital image measurement tool of COACH, an inconvenience of the (current) implementation of the tool became apparent: Because one cannot zoom in on a particular point of a digital image, one cannot benefit from the high resolution of the digital image for measurement purposes. Instead one must work with the resolution set by the dimensions of the image measurement window. This is why in Figure 3.15 the digital image has been rotated clockwise ninety degrees to get the image as large as possible in the direction one is interested in. This inconvenience is actually a consequence of the fact that technological progress surpasses development of a tool: Nowadays low cost cameras create high resolution images so that much detail is available in a digital image. It is therefore recommended that a digital image measurement tool provide zooming facility for collecting accurate data on high resolution images.

The Slinky used in the experiment had an undistorted length $L_0 = 0.0575 \text{ m}$, mass $M = 0.231 \text{ kg}$, a diameter of 0.069 m, and $N = 86$ coils. It follows that $\delta = 0.0006686 \text{ m}$ and $m = 0.002686 \text{ kg}$. The linear regression curve of $\Delta p_i$ plotted against the coil number $i$ with $\delta = 0.0006686$ gave $u = 0.00044 \text{ m}$. This value was used to get the best quadratic fit for the height of the $i$th coil: height = $0.00022(i - 1) - 0.04743$. Ideally the constant term in the right-hand side of the previous equation should be equal to zero. The graph of measured data in the middle frame compared with the quadratic fit suggests a stronger distortion near the bottom of the Slinky, that is, a larger spring constant near the ends of the Slinky. There can be various reasons for this: Heavy use
of the Slinky may have caused it, and one must also not forget that a certain minimum force is required to pull coils apart (cf., Gluck, 2010; Mak, 1987).

Intuitively, the center of mass of the Slinky hanging under gravity is expected by many a student to be halfway, that is, at the coil in the middle of the Slinky. This is wrong! The mathematical model allows a computation leading to the answer:

\[
\text{position of center of mass} = \left(\frac{N + 1}{3N}\right) L + \left(\frac{N - 1}{6N}\right) L_0.
\]

When the number of coils, \(N\), is large, then the above approximate answer is the same as the one published by Hosken (1994):

\[
\text{position of center of mass} \approx \frac{1}{3} L + \frac{1}{6} L_0.
\]

This answer differs from formulas in the literature that ignore the thickness of the Slinky. However, when the number of coils is sufficiently large, then \(L_0\) is small compared with \(L\) and the second term can be ignored. This kind of reasoning can be promoted in a modeling approach to mathematics and science education and is in my opinion reachable for secondary school students.

When the above mathematical model was applied to the Slinky used in the experiment, it was found that the displacement \(u\) between the first and second coil was less than the thickness of the Slinky. This means that the lower coils were not all parted in reality. This was also clearly visible in reality and in the digital image. The number of non-parted coils can be estimated as \(\frac{kL_0}{3}\). The height of the non-parted coils is twice as large as the elongation of the same number of coils would have been under that assumption of infinitely thin material. This leads to the following small correction for the previously found position of the center of mass: \(-\frac{\bar{m}}{6M}\), where \(\bar{m}\) and \(\bar{L}\) are the mass and length of the pack of non-parted coils, respectively.

### 3.2.2 Perspective Correction Applied in Crime Scene Photography

#### References


3.2. Digital Image and Video Analysis

Introduction
One of the identified problems with measurement on digital images and video clips is that the camera often cannot be oriented fronto-parallel to the plane of interest, with the consequence that Euclidean properties such as length, angles, and parallelism of lines are distorted. Computer vision provides methods for correcting this perspective distortion, that is, for transforming the digital image such that the geometrical relationships are those which would be seen had the photograph of the plane of interest been taken with the camera fronto-parallel to the plane. This process is called (image) rectification. Projective geometry lays the mathematical foundation of this discipline. See Figure 3.5 and Figure 2.25 for examples of screen shots containing a recorded and rectified digital image and video, respectively. The map between the perspective image and the world plane is a plane projective transformation, a 2D homography. Correcting perspective distortion is a matter of determining the eight degrees of freedom of the homography. The rectifying homography is computed from scene geometric information, specifically parallelism, angles between lines, and ratios of length along lines in different direction. A full description of the mathematics of image rectification of planes can be found in (Liebowitz, 2001; Liebowitz & Zisserman, 1998) and in many textbooks on computer vision (e.g., Hartley & Zisserman, 2004).

Figure 3.16: Perspective correction of the image of a tennis court.

I use Figure 3.16, taken from (Heck, 2004a), to exemplify in what way the perspective distortion of a plane can be corrected in COACH 6. The plane of interest is a tennis court’s ground surface. In reality, the outermost lines of the tennis court are two pairs of parallel lines; but this is not the case in the photograph of the tennis court. The image transformation that restores this property is determined by mapping the four corners of a rectangular part of the tennis court with a projective transformation to the corners of a rectangle in a new image. Actually, the orthogonality of two pairs of parallel lines (highlighted in the image on the left-hand side of Figure 3.16) determines the plane rectification up to an unknown aspect ratio. This scaling can be specified, if one wishes a rescaling to the real proportionality. In the image on the right-hand side of Figure 3.16, the length-width ratio is not in agreement with reality. For image measurement this does not matter as long as one appropriately calibrates length in both directions. Also note that the rectified image is not equal to the view from above. The basis of the rectification method is that any projective transformation can be uniquely written as a composition of three transformations,
Chapter 3. Computer Tools for Cross-Disciplinary Work with Real Data

namely, a perspective mapping $P$ that maps a quadrilateral into a parallelogram, an affine mapping $A$ that transforms the parallelogram into a rectangle, and finally a similarity transformation $S$ with isotropic scaling that rotates, translates, and scales the rectangle. For measurements in a rectified photograph or video clip it suffices to find appropriate transformations $P$ and $A$. These transformations can be computed from scene constraints, in this case the identification of a rectangular object in the plane of interest. Sometimes, the real world itself contains easily identifiable rectangular parts. In other circumstances, like in crime scene photogrammetry, one can deliberately place a rectangular calibration object in the scene (See Figure 3.18), or three circular perspective discs (Robinson, 2010, ch. 8). Occasionally one has to use one’s imagination to deduce a rectangular part in the plane of interest. This has been done in the video analysis activity illustrated in Figure 3.17.

Figure 3.17: Screen shot of a video analysis activity with students walking in a straight line toward the camera at constant speed.

In the lower-left corner is the self-recorded video clip of two girls walking in front of a building, in a fairly straight line in the direction of the camera. This video was used in the second lesson of the classroom experiment with pre-vocational secondary school students (Section 2.5) to illustrate that the plane of motion is preferably perpendicular to the camera direction. The instructional task was to determine at what speed the two girls walked. To this end, image rectification can be applied to one of the following planes: (1) the perspective correction of the imaginary vertical plane in which the student on the right-hand side walked; and (2) the rectification of the plane formed by the paving stones. The result of the first image rectification is the video clip shown in the upper-left corner of Figure 3.17. As vertices of the 4-gon that is in reality a
rectangle I chose the left shoe and the head of the student on the right-hand side in two separate video frames that are far apart from each other. The rectification of the footpath gives the alienating video clip in the lower-right corner. The rectified image of the vertical plane was used to measure the distance walked by the student. The data of this measurement are shown in the diagram in the upper-right corner, together with a straight line fit. Apparently the student walked at a constant speed. The slope of the line implies a walking speed of 4.8 km/hr, which is a value that makes sense. Students could validate this answer by using the other rectified video clip for an alternative measurement and in this way practice good research methodology.

**Usability Study and Adjoining Field Experiment**

Crime scene photography is a real-world context in which image rectification is applied (cf., Robinson, 2010, ch. 8). This also provides a nice context for secondary school students to study the mathematics of perspective drawing and imaging. A crime scene photographer includes a perspective grid or perspective discs in the scene so that this can be used afterwards to process the photograph and enable measurements in the digital image. There is not much required for a perspective grid: Minimally it is just a rectangle of known dimensions, but it is often a set of four square tiles with diagonal lines drawn on top arranged such that together they form a larger square. Figure 3.18 shows the self-made perspective grid and its use in rectifying the image of a fake crime scene set up in a workshop for secondary school teachers in mathematics.

![Image rectification of a 'crime scene' photograph on the basis of a perspective grid included in the scene.](image)

Figure 3.18: Image rectification of a ‘crime scene’ photograph on the basis of a perspective grid included in the scene.

Although the above screen shot gives the impression that the perspective grid is a square in the rectified image, it takes quite some effort to arrive at this result. When one places the vertices of the 4-gon used for correction of perspective distortion one by one at the four corners of the perspective grid, in each step the digital image is transformed. At the end of this process it usually has resulted in a new image in which the perspective grid is only a rectangle and not a square. One often has to rescale the 4-gon in several steps to arrive at a square perspective grid. In practice, students are able to do the required steps of image rectification, but they would be helped if the proportionality could be maintained automatically during the process of rectification, so that squares and circles in reality become squares and circles in the rectified image.
The original crime scene photograph can also be used in a geometrical construction carried out for the purpose of measuring distances in the scene. Figure 3.19 shows how grid extension and grid refinement can be applied in a dynamic geometry software package to specify a location of an object. The grid extension method is related to the distance point construction of Pêlerin (1505) and Alberti’s method for constructing perspective drawings (cf., De Rijk, 2004; Edgerton Jr., 1996). The experience of students and the teacher in a field experiment carried out in a Mathematics D class was that the mathematical exploration of a photograph of a fake crime scene in the school building was an exciting subject in which they learned much, but also had much fun. It can be the first step toward a profile project on perspective in art.

Figure 3.19: Grid extension and grid refinement of a crime scene photograph.

3.2.3 Using High Speed Video to Study Moving Coins

Reference

Introduction
Technological progress has led to the development of high speed cameras at low cost, consumer level (See Vollmer & Möllmann, 2011a, for a discussion of high speed video technology). These cameras provide students the opportunity to work directly with high-quality video data in cases where motion was in the past too quick for recording with a normal digital camera or webcam; for example, data collection of human and animal locomotion or gathering of motion data in sports. They allow students to carry out authentic activities in which they record video clips or image sequences of rather fast motions and use them for a detailed investigation of a real-world phenomenon. In addition to watching a video in slow motion, one can also take measurements on a high speed video clip. In this case, automated tracking of a reference point on an object in motion is a convenient, if not indispensable feature of a video analysis system.

Various methods exist for the automated tracking of reference points on moving objects in a video. They can be based on image recognition techniques and the geo-
3.2. Digital Image and Video Analysis

metric properties of the objects of interest (Riera et al., 2003; Salumbides et al., 2002) or on color discrimination (Bach & Trantham, 2007). The second method of point tracking has been implemented in Coach as a DirectShow transformation filter, which can be thought of as a program that connects a source, for example, a video clip or a streaming video of a webcam, with the data video window, while at the same time passing the recorded position data to the computer application. The algorithm used in the tracking filter consists of two parts: (1) finding the best match of a given model template in a subsequent frame, that is, locating the area that most resembles the specified area; and (2) limiting the search area to reduce computing time or to avoid ambiguity. The template that is tracked and that is selected by the user at the start of the tracking process is an area bounded by a circle with a user-specified radius. The comparison of an area with the model template is based on pixel intensities in the three channels of the RGB color model. More specifically, the sum of the squared differences of intensities between the image and model template is minimized in the search algorithm. This method has been selected for template matching after testing various commonly used algorithms on sample videos. The search area for the subsequent frame is a rectangle of user specified dimensions, centered on the position that is found for the current frame (See, for example, Figure 2.12 and Figure 3.20). Despite its simplicity, the tracking filter works well in many practical situations, and the need for special illumination of the moving objects or other special scene preparation is limited. It is mostly restricted to avoidance of: (1) severe distortion such as rotation and deformation of the detected reference area; (2) accidental change in the coloring of the moving objects because of changes in the incident light; and (3) occlusion of the reference point, that is, temporary disappearance of the reference point within the view field (when it reappears again, then it is not guaranteed that the point is always automatically found again). For example, the reference points in the scene shown in Figure 2.12 (a video analysis of the leg motion of a student on an exercise bike in a fitness center) were a label on the shorts of the cyclist, a black dot made with a whiteboard marker on the knee joint, and a simple homemade marker placed at the ankle. The normal lighting of the fitness center was used, and the video was created under these circumstances by a CCD webcam with a speed of 30 fps. There were no problems with the point tracking algorithm.

Figure 3.20: Screen shot of a video analysis activity on weightlifting, in which data have been collected via automated point tracking.

Another example of point tracking is shown in Figure 3.20. The video clip on the left-hand side comes from the television broadcasting of the women’s weightlifting
competition at the Sidney Olympic Games in 2000. It shows Tara Nott failing in her third attempt in the clean and jerk to lift 105 kg. The motion of the weights at the end of the barbell are tracked and the heights are plotted in the diagram on the right-hand side. The students’ task in a field experiment is to relate the graphs with what goes wrong in the clean phase of the lift. The point tracking works surprisingly well in this case too, after experimenting with some search areas of various size.

On the basis of usability studies and field experiments, the following features of automated point tracking were implemented in COACH:

- Automated point tracking can be combined with manual video measurement.
- One can select a moving coordinate system and track its origin.
- When automated point tracking fails, one can interrupt it, move to the first frame where the tracking algorithm failed, reinitialize the model template or change its settings and the dimensions of the search area, and continue automated point tracking from this frame on, using the new settings.
- Point tracking can be combined with other video filters that have been implemented such as change of brightness and contrast, and correction of perspective distortion.

Field Experiment
Having appropriate video analysis tools designed for educational purposes at their disposal, students can explore many real-world phenomena at almost any level of detail (See, for example, Koupil & Vícha, 2011; Mathavan, Jackson, & Parkin, 2009; Vollmer & Möllmann, 2011b). A challenging example is finding a good explanation of the motion of money in the following experiment: Six coins of the same denomination are lined up, at equal distances, against a wooden stick. This stick, making a circular movement, pushes the coins over a flat piece of paper on a horizontal table. The origin of this circle is at one side of the row of coins (in this case, near the upper left corner of paper in the pictures in Figure 3.21 taken from a high speed video recording of the experiment). The stick stops and the coins leave the stick (Figure 3.21(a)), having a velocity perpendicular to the stick. Once the coins have stopped moving they seem to lie on a parabola; at least Figure 3.21(c) suggests this conclusion. The students’ task is to validate and explain this.

Figure 3.21: Frames from a video-recorded experiment with moving money. (a) Coins leaving the wooden stick. (b) Moving coins. (c) Final positions of the coins.
3.2. Digital Image and Video Analysis

The mathematics and physics needed for the explanation of the phenomenon is not beyond high school level. It might go in the following way and lay the foundation of experimental verification: At the point when the coins leave the stick, friction comes into play. For each coin, the work done by the constant frictional force equals the kinetic energy of the coin at the moment that it loses contact with the stick. This implies that the distance traveled by each coin is proportional to the square of its velocity when it loses contact with the stick. Because the stick makes a circular movement, this velocity is proportional to the radial distance of the coin to the center of rotation. This means that the distance traveled by a coin is proportional to the square of its radial distance to the center of rotation. In other words, one may expect that the coins, once they have stopped moving, lie on a parabola.

But when this mathematical model of the motion of the coins was validated by a video analysis of an experiment with real coins, some unexpected problems popped up. The first assumption that caused a problem was the statement: “The stick stops, and the coins leave the stick, having a velocity perpendicular to the stick.” Figure 3.22 is a screen shot of the activity in which the motion of the coins in the rectified video clip was tracked. The coordinate system had been rotated such that the $x$-axis had been aligned with the wooden stick when it stopped moving and the coins were leaving the stick. Please note that the $x$-coordinate axis shown in Figure 3.22 is not aligned anymore with the wooden stick because the frame shows the coins in the final position and the wooden stick had been moved a little bit further after the coins had left it (One cannot manually stop the stick at once). The track of the sixth coin is shown in the diagram window on the right-hand side and it is clearly not a vertical line. This means that the velocity at the moment when the coin lost contact was not perpendicular to the stick. A close look at the high speed video and measurement of distances of coins to the rotation center revealed that, while the stick had been rotating and in contact with the coins, the coins had in reality moved along the stick. In other words, the coins had a non-zero horizontal velocity component when they lost contact with the wooden stick. An open question, which could be discussed in classroom, remains why and how this motion of the coins along the rotating stick took place.

![Figure 3.22: Tracking of coins in a video analysis activity and the measured orbit of the sixth coin.](image-url)
The above surprising experimental finding also raises questions about the following statement: “Once the coins have stopped moving they seem to lie on a parabola.” The first thing done was to orientate the coordinate system in the video window such that the sixth coin moved in a straight vertical line. Figure 3.23 shows clearly that this horizontal axis was not aligned to the orientation of the stick at the moment that the coins lost contact. This new choice of coordinate system had been driven by mathematics, and less by physics or the experiment. The intended achievement was only that the coins moved in a vertical line. Because this coordinate system in Figure 3.23 is only rotated about 2.5 degrees with respect to the paper sheet, one can understand that the human eye cannot distinguish between the motion with respect to this coordinate system and the motion with respect to the paper sheet. The coins only appeared to be moving parallel to the edge of the paper, but in reality they moved perpendicular to a slightly rotated coordinate system. The diagram to the right in Figure 3.23 shows the final positions of the six coins in the new coordinate system. The coins still seemed to lie on a parabola and regression analysis supported this view.

The case study continued with a detailed analysis of position-time and velocity-time graphs (Numerical derivative were computed via automatic spline smoothing to get the best results). The interested reader is referred to the full paper of Heck and Vonk (2009). Here, I only repeat the conclusions of the authors:

“The experiment with moving money was simple and it confirmed at first sight the predictions made on the basis of mathematics and physics taught at high school level. Taking a closer look with the help of appropriate ICT tools brought up some peculiarities, initiating a search for a more satisfactory explanation. A more thorough investigation of the phenomenon by high speed video analysis and a sound reasoning based on physics and mathematics that was still at high school level brought measurements and theory again into agreement.

As in every good physics experiment at high school, the phenomenon under investigation and the modeling of it offered students and teachers great opportunities to discuss physics concepts, scientific methods of study, and the importance of experimental validation in science. The physics of both uniform and rotational motion was present in the study of moving coins. Interpreting motion graphs, numerical differentiation, and regression techniques were some of the mathematical aspects that came into play in the discussion of the problem situation. In conclusion, physics and mathematics were natural ingredients in the students process of coming to grips with the problem posed.”
This case study also illustrates again that the perception of the quality of a mathematical model depends to a large extent on the accuracy that the modeler requires for his or her description, prediction, explanation, and understanding of the phenomenon.

3.3 Modeling

One of the most important areas for mathematical modeling in chemistry education is the study of chemical kinetics. In Section 3.3.1 I discuss a system dynamics approach of chemical reaction systems, possibilities and obstacles of current graphical modeling systems like STELLA and POWERSIM in this context, and the design of a new graphical element in the modeling tool of COACH that makes it easier to explore chemical kinetics in this way.

Another area of chemical education at secondary school in which an integrated computer learning environment could play an important role is acid-base reactions and titration. Here, (automatic) data collection with pH sensors is used in combination with computer modeling of titration curves. The main ideas of the designed activities, described in Section 3.3.2 are that students get a modern perspective on computer use in the chemistry laboratory and that the computer models help them to enhance their understanding of buffer solutions.

In Section 3.3.3 I present a case study in which a model for the moon-earth dynamical system, yielding the tidal movement on an all-ocean world, was implemented in the graphical, system dynamics-based modeling environment of COACH 6. The tool was also used to implement a mathematical model of the tidal behavior of the well-known tidal river Thames as a function of the tides of the North Sea. Using data sources on Internet with tidal data, the harmonic analysis for various ports was reconstructed with advanced data analysis techniques implemented in COACH 6 for sinusoidal regression.

3.3.1 Modeling Chemical Kinetics Graphically

Reference

Introduction
In literature on chemistry education it has often been suggested that students, at high school level and beyond, can benefit in their studies of chemical kinetics from computer supported activities. Proposed activities range from computer-assisted instruction, data logging, use of dedicated packages, modeling with computer algebra systems and other scientific computing environments, use of a graphical calculator and a spreadsheet program to computer simulations and system dynamics-based computer modeling. All of these approaches attempt to make the concepts related to chemical equilibrium and chemical kinetics accessible or comprehensive for students, for example by giving students first-hand experience with reactions through laboratory work or by simulating and visualizing the reaction dynamics and the dynamic nature of chemical equilibrium, and they attempt to help to remediate alternative conceptions of students. I discuss the methodology, strengths, and weaknesses of the implementation of graphical, system dynamics-based software for modeling chemical kinetics.
I propose an extension of classical graphical modeling that brings it closer to how chemists think about chemical reactions and that could make it easier for students to investigate chemical kinetics, especially in cases of non-trivial reaction mechanisms. Illustrative examples are given. The proposed approach has also useful applications in other subject areas.

**Graphical System Dynamics-Based Modeling of Chemical Kinetics**

I discuss the methodology, strengths, and weaknesses of the implementation of classical graphical system dynamics modeling software for modeling chemical kinetics through the following two examples: (1) a unimolecular chemical equilibrium system; and (2) the Michaelis-Menten reaction mechanism.

As a prototypical example of using a graphical system dynamics-based model for studying chemical kinetics I use the following cis-trans isomerization:

\[

cis-\text{Mo(CO)}_4[P(n-\text{Bu})_3]_2 \overset{k_{1f}}{\rightleftharpoons} \text{trans-} \text{Mo(CO)}_4[P(n-\text{Bu})_3]_2.
\]

![Figure 3.24: Screen shot of the graphical model of cis-trans isomerization, and concentration-time graphs in a simulation starting from pure cis-\text{Mo(CO)}_4[P(n-\text{Bu})_3]_2 at 80°C. Graphs created in a previous simulation at 85°C are also displayed, in gray color at the background.](image)

Figure 3.24 is a screen shot of the graphical model implemented in the modeling tool of CoACH 6. Four types of variables are present in this graphical model and they are differently iconified: (1) a parameter (temperature \(T\)); (2) auxiliary variables (reaction rate constants \(k_{1f}, k_{1r}\)); (3) levels, that is, aggregated amounts that change over time (concentrations \([\text{cis}], [\text{trans}]\)); and (4) flows, which determine the dynamics of the reaction system (rates of change of concentrations \(r_{1f}, r_{1r}\)). Information arrows indicate dependencies between these variables: For example, the arrows from \(T\) to \(k_{1f}\) and \(k_{1r}\) indicate that the author of the model wanted to express that the forward rate constant \(k_{1f}\) and the reverse rate constant \(k_{1r}\) both depend on the temperature \(T\) at
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which the reaction takes place. The following mathematical expressions were derived from (Bengali & Mooney, 2003) and used in the simulation:

\[
 k_{1f} = T \cdot 10^{8.87 - \frac{5195}{T}}, \quad k_{1r} = T \cdot 10^{8.78 - \frac{5394}{T}},
\]

where temperature \( T \) has been specified here in Kelvin (instead of \( ^\circ C \)) and rate constants are in \( s^{-1} \).

The model window in the screen shot in Figure 3.24 illustrates what graphical modeling is all about: An author (curriculum designer, teacher, or student) literally draws variables representing physical quantities or mathematical entities and the relations between them. This contrasts with computer simulations where only the parameters of a given model can be altered and not the underlying model. The graphical model can be considered as a representation at conceptual level of the system dynamics, where physical flows represent rates of changes and information arrows indicate dependencies between quantities. Once the sketch of the model has been made, the details of a model, that is, the algebraic formulas needed to build up the system of equations, can be filled in by clicking on the icons and be hidden again. The general picture of the model is considered most important for understanding. In this particular example, the graphical model almost literally presents a chemical equilibrium.

A graphical, system dynamics-based model generally corresponds in mathematical terms with a system of differential equations or finite difference equations. Under the assumption that only elementary, unimolecular reaction steps are involved, the graphical model of Figure 3.24 represents the following coupled differential equations for the rate of change in the concentrations of the two species involved:

\[
 \frac{d[\text{cis}]}{dt} = -r_{1f} + r_{1r}, \quad \frac{d[\text{trans}]}{dt} = r_{1f} - r_{1r},
\]

where \( r_{1f} = k_{1f} \cdot [\text{cis}] \), \( r_{1r} = k_{1r} \cdot [\text{trans}] \). But the graphical model represents in fact more: It also represents an automatically generated computer program that solves this system numerically and allows the user to simulate the behavior of the modeled reaction system and to interpret the modeling results.

In the context of chemical kinetics, students are immediately confronted in a simulation of a reaction system with potential alternative conceptions. In the example of cis-trans isomerization, the visualization in Figure 3.24 allows for instance a student to observe in the diagram with the concentration profiles that (1) it takes time before the equilibrium is reached; and (2) at equilibrium, the concentrations of cis- and trans-complexes are not necessarily equal. More alternative conceptions about chemical equilibrium can be addressed by looking at rate-time and net rate-time graphs of the chemical equilibrium. Some of the points that a student could notice in the graphs on the right-hand side of Figure 3.24 are:

- The rate of the forward reaction decreases with time until equilibrium is reached (and not to completion).
- The rate of the reverse reaction increases with time until equilibrium is reached.
- A system in equilibrium does not mean that the reactions ceased. It only means that the net rate of concentrations is zero and that the forward and reverse reaction rates are equal.
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Executable models offer students the opportunity to observe the effects of changing the model or, less dramatically, of changing the parameter values and initial conditions. For example, by changing the initial concentrations of the cis- and trans-complex it can easily be verified that the system will always reach the same equilibrium concentrations, no matter what the starting concentrations are. In fact, this kind of exploration has already been anticipated in the model shown in Figure 3.24: The introduction of temperature $T$, the change of which has been simplified by the incorporation of a corresponding slider in the activity, was motivated by the wish to investigate the effect of temperature on the reaction system. Figure 3.24 shows the results of a simulation of the isomerization at $80^\circ C$. The graphs of the previous simulation at a higher temperature of $85^\circ C$ are shown in gray at the background to support easy comparison. A student could discover from the graphs that changing the temperature for this chemical reaction

- does not necessarily mean that the concentrations at equilibrium are affected;
- only affects the time needed for the system to reach equilibrium; and
- may change the absolute magnitudes of the forward and reverse rates, also at equilibrium.

The graphical modeling of chemical kinetics illustrated by the example of cis-trans isomerization is rather simple. Other examples of reaction systems that can be dealt with in this way are all unimolecular (cf., Chonacki 2004; Kosinsky 2001; Ricci and van Doren 1997; Steffen and Holt 1993). Any other type of reaction system would lead for stoichiometric reasons to a disconnected, from chemical point of view incomprehensible graphical model. Consider, for example, the gas-phase oxidation of nitric oxide: $2\text{NO} + \text{O}_2 \rightarrow 2\text{NO}_2$. Although there exist more than one mechanism that leads to third-order reaction kinetics (Tsukuhara, Ishida, & Mayumi, 1999), the simplest one, namely a termolecular reaction mechanism, would already lead in a classical graphical system dynamics-based modeling approach to a disconnected picture, because one could not simply draw physical flow arrows from reactants toward products. The reason is that the meaning of the graphical modeling tool, which is based on the level-flow model in which the sum of inflows in a level variable is by definition equal to the sum of outflows of this level variable (the so-called ‘principle of flow balance’), does not lead then to the correct coupled differential equations. In other words, if both an arrow from $\text{[NO]}$ toward $\text{[NO}_2]$ and an arrow from $\text{[O}_2]$ toward $\text{[NO}_2]$ were drawn, this would mean that the increase in concentration of NO$_2$ over time is equal to the sum of the decrease in concentration of NO over time and the decrease in concentration of O$_2$ over time. This is from chemical point of view incorrect for the given reaction.

In fact, due to the selected graphical modeling approach of level-flow diagrams, which is based on a metaphor of water tanks and valves, the diagrams for bi- and termolecular chemical reactions are inevitably disconnected. Forrester, the founder of the system dynamics and level-flow modeling approach in the context of socio-economic systems, was aware of this limitation and wrote (Forrester, 1961, p. 70):

"It should be noted that flow rates transport the content of one level to another. Therefore, the levels within one network must all have the same kind of content. Inflows and outflows connecting to a level must transport
the same kind of items that are stored in the level. Items of one type must not flow into levels that store another type. For example, the network of materials deals only with material and accounts for the transport of the material from one inventory to another. Items of one type must not flow into levels that store another type."

Clearly, chemical reactions do not meet this principle of material consistency in the structure of a graphical model that is written in terms of levels interconnected by rates of flow: In a bimolecular reaction, two molecules may react to result in one molecule; that’s chemistry! On the other hand, it must be stressed that the problem only lies in the translation of the graphical model into the coupled differential equations that describe the kinetics of the chemical reaction.

The fact that the conventions of a classical graphical, system dynamics-based modeling tool, which state how the coupled differential equations or difference equations are to be generated from the graphical representation, are inconvenient for chemical kinetics comes even more to the fore when complex chemical reaction networks are modeled instead of elementary reactions. The following example, which is the simplest (Michaelis-Menten and Briggs-Haldane) mechanism for a two-step enzyme-catalyzed reaction, illustrate this:

\[
E + S \xrightleftharpoons{\frac{k_{1f}}{k_{1r}}} ES \xrightarrow{k_{2f}} E + P
\]

where E, S, ES, and P are the unbound enzyme, substrate, intermediate enzyme-substrate, and product, respectively. One of the things students learn from or need to accept in this mechanism is that a species can simultaneously be involved in more than one reaction: The intermediate enzyme-substrate can both form a product as well as the original substrate. All reaction steps are considered as elementary reactions. See (Bruist, 1998) and (Halkides & Herman, 2007) for simulations of the reaction system with a spreadsheet, and (Mulquiney & Kuchel, 2003) for simulations with a computer algebra system. A steady-state approximation is used in most cases to simplify the algebraic and computational work. A graphical model that represents this enzyme-catalyzed reaction is shown in Figure 3.25. This graphical model, which Lee and Yang (2008) implemented in POWERSIM (www.powersim.no), does not reflect anymore the underlying reaction mechanism.

Figure 3.25: A model of \( E + S \xrightleftharpoons{\frac{k_{1f}}{k_{1r}}} ES \xrightarrow{k_{2f}} E + P \) in POWERSIM.

This example makes clear that a standard, rather simple chemical reaction network already leads to a graphical model in which the chemical reaction mechanism is
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obscured by the spaghetti (and meatballs) tangle of arrows and boxes. When the reaction mechanism of the enzyme-catalyzed reaction becomes more complicated, the corresponding graphical model that represents the chemical kinetics readily gets snarled up, to put it mildly.

**Improved Graphical Modeling of Chemical Kinetics**

The problem that graphical models based on the traditional level-flow metaphor do not present a clear overview of the chemical reaction mechanism except for simple unimolecular reaction systems is known and suggestions for improvement have been made. For example, the key ideas of chemical kinetics and thermodynamics have been expressed in a bond graph approach (Cellier 1991, ch. 9) and the level-flow metaphor has been replaced in (Elhamadi, 2005; LeFèvre, 2002, 2004) by the so-called kinetic process metaphor, which was inspired by graphical models of biochemical reaction networks and metabolic pathway systems. But these alternatives for and extensions of the traditional level-flow metaphor are at the level of system dynamics specialists and they are too complicated for use in chemistry education at high school level or first-year undergraduate level.

I present a simpler graphical approach to modeling of chemical reactions that covers the basics of chemical kinetics. It is based on a graph theoretic description of reaction kinetics that is similar to the oriented species-reaction graph introduced in (Craciun & Feinberg, 2006) and the directed bipartite graph of a reaction network developed by Vol’pert and Ivanova (1987) [See also (Vol’pert & Hudjaev, 1985, ch. 12)], more thoroughly analyzed in (Ermakov & Goldstein, 2002; Mincheva & Roussel, 2007), and for example implemented in a computer simulation and visualization environment for metabolic engineering (Qeli, 2007). The graphical approach will be exemplified by the termolecular gas-phase oxidation of nitric oxide used before, $2\text{NO} + \text{O}_2 \xrightarrow{k} 2\text{NO}_2$.

For a thorough description of the improved graphical approach, I refer to the paper. In short, the approach is based on the principle that a chemical reaction network can be represented by a directed graph in which the set of vertices is partitioned into two sets, namely a set of species nodes and a set of reaction nodes. This directed bipartite graph is called a kinetic graph because it also incorporated information about the kinetics of the chemical reaction. The kinetic graph of the reaction $2\text{NO} + \text{O}_2 \xrightarrow{k} 2\text{NO}_2$ is shown in Figure 3.26.

![Figure 3.26: The kinetic graph of the reaction $2\text{NO} + \text{O}_2 \xrightarrow{k} 2\text{NO}_2$.](image)

The kinetic graph of a chemical reaction network clearly suggests how the classical level-flow formalism of graphical, system dynamics-based modeling tools could be extended to function well for chemical reaction networks: A graphical icon for a reaction, say an Erlenmeyer flask symbol, must be added to the formalism and then levels can represent concentrations of species involved in the reaction network, provided that flows are between level icons and Erlenmeyer flask symbols. Inflows of an Erlenmeyer flask symbol originate from reactants and outflows of an Erlenmeyer flask
symbol point at products in the chemical reaction that is symbolized by the Erlenmeyer flask. The Erlenmeyer flask symbol also represents the dynamics of the levels connected with it via the stoichiometry of the reaction: The Erlenmeyer flask symbol is linked to a formula for the reaction rate, which depends on the kinetic coefficient, the concentrations of reactants and their stoichiometric coefficients. The stoichiometric coefficients determine the formulas for the inflows and outflows of the Erlenmeyer flask symbol.

The improved graphical modeling of chemical reaction, based on kinetic graphs, leads to much clearer visual representations of chemical reaction networks for the following reasons:

- Levels, flows, and process elements give a visual overview of the reaction mechanism.
- The stoichiometry of a reaction already determines the formulas for the inflows and outflows so that there is no need to use information arrows from the reaction node toward these flows.

An illustrative example, implemented in COACH 6, is the two-step enzyme-catalyzed reaction $E + S \overset{k_{1f}}{\rightleftharpoons} ES \overset{k_{2r}}{\rightarrow} E + P$. The screenshot of the COACH 6 model of the enzyme-catalyzed reaction shown in Figure 3.27 reflects the underlying reaction mechanism much better than the POWERSIM model shown in Figure 3.25.

![Figure 3.27: Screenshot of the graphical model of the E + S → ES → E + P network.](image)

In a qualitative or semi-quantitative approach to chemical equilibrium phenomena there is hardly any other instructional strategy than applying Le Châtelier’s Principle or reasoning with the Equilibrium Law to explain how a system in equilibrium responds to an external perturbation such as addition of a reactant, depletion of a product, change in pressure or temperature, and so on. A quantitative approach seems more suitable for discussing how chemical equilibrium is reached or how it changes when conditions change. This holds especially when a modeling and simulation environment offers tools to interactively change conditions during a simulation and allows an easy implementation of event-handling such as response to a sudden change in concentration, temperature, and so forth. Solomonidou and Stavridou (2001) pointed at the potential of computer simulations and animations to help students construct appropriate conceptions about Le Châtelier’s Principle and the Equilibrium Law.
Interactive change of initial conditions and event-handling of sudden changes during a simulation run have been implemented in the Coach 6 environment. I exemplify this with the equilibrium shift of a gas mixture of hydrogen, iodine, and hydrogen as a response to a sudden change in hydrogen concentration and temperature. The reaction system under consideration is $\text{H}_2 + \text{I}_2 \xrightleftharpoons[k_{1r}]^{k_{1f}} 2\text{HI}$, where second-order rate kinetics is assumed given by the following coupled differential equations for the rate of change in the concentrations of the three species involved:

$$\frac{d[H_2]}{dt} = -r_{1f} + r_{1r}, \quad \frac{d[I_2]}{dt} = -r_{1f} + r_{1r}, \quad \frac{d[HI]}{dt} = r_{1f} - r_{1r},$$

(3.1)

where $r_{1f} = k_{1f} \cdot [H_2] \cdot [I_2]$, $r_{1r} = k_{1r} \cdot [HI]^2$, and the Arrhenius equations for the rate constants are given (Graven, 1956) for temperature $T$ (in $^\circ K$) by $k_r = 7.18 \times 10^{12} \times e^{-24775/T}$ and $k_f = 1.23 \times 10^{12} \times e^{-20646/T}$. It follows from these equations that the forward gas phase reaction is exothermic. Figure 3.28 is a screen shot of a simulation run based on this kinetic model.

![Figure 3.28: Screen shot of the graphical model of the H$_2$ + I$_2$ $\xrightleftharpoons[k_{1r}]^{k_{1f}}$ 2HI equilibrium reaction and a simulation with user interaction and an event during model execution.](image)

Figure 3.28 shows a simulation run starting with only a nonzero concentration of HI at a temperature $T = 721^\circ K$. After 6000 seconds the concentration of H$_2$ is suddenly raised by 0.002 M, which has an immediate effect on the concentration time course. This sudden change is realized in the graphical model by introduction of an event (iconized by the thunderbolt symbol). The code behind this event icon is very simple: 

Once $t > 6000$ then $[H_2] := [H_2] + 0.002$. The effect is that the equilibrium which was almost established is shifted right to less dissociation of hydrogen iodine. After a new equilibrium has been established, the user has pressed about 12000 seconds after the start of the reaction the button in the control panel to cause a sudden raise in temperature of 50$^\circ K$. The effect is that the equilibrium shifts to the left, that is, more hydrogen iodide dissociates again. This is in agreement with Le Châtelier’s principle that states that increasing the temperature will shift the equilibrium to the left because the forward reaction is exothermic. Although the kinetic and thermodynamic approaches to chemical equilibrium phenomena are of different nature, results obtained by either method complement each other.
3.3. Modeling

It is worth mentioning that the improved graphical modeling approach also has applications beyond chemical kinetics. This aspect is important in education because it would most probably not be worth the effort to add new elements to a general purpose graphical modeling tool if they were only relevant for a small part of the science curriculum. Students and teachers have to use their time effectively and economically. Much is won when students and teachers can use one and the same modeling environment for many science subjects. Then they have ample opportunities to grow into their roles of knowledgeable and skilled modelers of natural phenomena.

The following example illustrates that quantitative pharmacokinetic models can be conveniently treated through the new graphical formalism. Figure 3.29 shows a graphical model and simulation of the pharmacokinetics of the metabolism of ecstasy in the human body [taken from the instructional materials “Swilling, Shooting, and Swallowing” for Mathematics D, see also (Heck, 2007a-c)]. The improved graphical modeling approach provides a connected diagram that indicates the flow of the pharmacon in the body over time.

One final remark about computer-based modeling of chemical kinetics has to do with the effectiveness of numerical methods for solving differential equations. When realistic values are chosen for reaction rate constants in chemical reaction mechanism, popular fixed-time step methods such as the Euler method and the 2nd or 4th order Runge-Kutta methods, in which the solver steps blindly from \( t \) to \( t + dt \), require such a small time step \( dt \) that the simulation may take hours to complete. In these cases one needs a numeric algorithm that is more appropriate to solve a so-called stiff system of differential equations. The special purpose package KSINSIM developed by Barshop, Wrenn, and Frieden (1983) for example uses a Gear-type method to solve differential equations numerically. This is an example of an adaptive time step solver, which automatically changes in each step of the algorithm the time step \( dt \) in order to maintain the error of the numerical solution prescribed by the user. It is recommended that a versatile computer learning environment that includes a modeling tool has various numerical solvers for differential equations at its disposal. An explicit adaptive Runge-Kutta method such as the Fehlberg method (1970) or the Cash-Karp method (1990) will be suitable for general purpose; for stiff systems of differential equations, an implicit adaptive Runge-Kutta method based on the Radau or Lobatto quadrature formulas (cf., Hairer & Wanner, 2010) can be recommended.
3.3.2 ICT-Supported Study of Acid-Base Titration Curves

Reference

Introduction
The topic of acid-base reactions is a regular component of many chemistry curricula that requires integrated understanding of various areas of introductory chemistry. Many students have considerable difficulties understanding the concepts and processes involved. It has been suggested and confirmed by research that students may benefit from computer-supported activities such as data logging, simulation, and modeling. Different methods of using computer acquisition and modeling to examine acid-base titration are reviewed and in particular I discuss how a versatile, integrated computer learning environment like COACH can be successfully applied to this end. This environment integrates, amongst other things, tools for data logging, control, and modeling: Using an inexpensive step-motor buret, automated pH measurement in acid-base titration can be realized and measured data can then be compared graphically and in tabular form with data computed via the (text-based or graphical) system dynamics-based modeling tool. I present concrete examples, taken from an in-service teacher training course developed during the Socrates Comenius project ‘IT for US—Information Technology for Understanding Science’ (www.itforus.oeiiikt.waw.pl), and culminating in the student practical investigation of analysis of acid in soft drinks.

Data Logging with an Automated pH Titration System
Traditionally, in a laboratory experiment of acid-base titration performed with a buret-pH meter system, a student delivers small increments, dropwise at times, from a buret, records pH as a function of titrant volume with a pH meter and constructs the pH against volume plot for further analysis to determine the unknown concentration of the sample or the ionization constant of the substance in the sample. The data analysis can be done by pencil-and-paper, with the help of a graphing calculator, or via a spreadsheet. However, doing both the data logging and data analysis in a single computer environment has the following pedagogical advantages: It eliminates tedious and repetitious operations, allowing the students to concentrate on data analysis, on design of experiments, and on comparison of measured data with predictions from a chemical model. In addition, use of an automated pH titration system offers students the opportunity to work directly with high-quality data in much the same way as chemists do in the laboratory. They learn then how a computer is not only used to measure data via sensors, but also can simultaneously control addition of reagents. Last but not least, in an integrated computer environment students can process and analyze their collected data and report their findings. All this is not still in the future at secondary school level, but already available (e.g., Witteck & Eilks, 2006).

Figure 3.30 is a screen shot of a COACH activity, in which an inexpensive step-motor buret is used to control accurately titration of a strong acid with a strong base. The activity is shown in the format as it was prepared for students’ use by the author of the activity. This author could have been a developer of curriculum materials or a teacher. This screen shot shows several multimedia components: a picture of the equipment and the experimental set-up, text frames for explanations and instructions,
3.3. Modeling

a frame with the initial computer program to control the step-motor buret, a table window, and a graph window. Multiple representations are used here because it was thought that one representation would become too complex to show all information and that use of more than one representation would offer a source of referential accuracy by providing redundancy. For example, it was considered helpful for a learner to describe the setting of an experiment partly in textual format and partly in a pictorial format (because a picture may indeed say more than many words). Here, the experimental set-up of a titration with an automated pH titration system is not only described in a text window, but also shown in a separate picture. Instructions for setting up the experiment are in the screen shot of the COACH activity also divided over several text windows for reasons of clarity. Multiple representations are used here to provide complementary information, which is one of the pedagogical functions of multiple representations distinguished by Ainsworth (1999) in her functional taxonomy of multiple representations. In the elaborated version known as the DeFT framework (Ainsworth, 2006, 2008) she considered in addition the design parameters that are unique to learning with more than one representation (Think of the number and forms of representations, distribution of information, support for translation between representations, sequencing of representations) and the cognitive tasks that a learner must undertake when interacting with multiple representations (For example, understanding representations and their relations with the domain, being able to translate between representations, knowing what representations are appropriate for what tasks and being able to construct or create them). Well-known principles of multimedia learning (Mayer, 2009) can also guide the authoring of the student activities.

Figure 3.30: Screen shot of a COACH activity on titration with an automated pH titration system.
Mathematical Modeling of Acid-Base Titration

Typically, an acid-base titration curve is dissected into two segments, namely before and after the equivalence point has been reached. In addition, separate calculations are done for the different segments in one of the following three ways:

1. Via approximate formulas that hold at certain stages of titration.

2. Via exact or approximate methods to compute the pH for a given titrant volume. This leads to the problem of solving a polynomial equation in $[\text{H}_2\text{S}_{\text{lv}}]^+$, where the solvent is denoted by $\text{H}_{\text{S}_{\text{lv}}}$ and its autoionization is $2\text{H}_{\text{S}_{\text{lv}}} \rightleftharpoons \text{H}_2\text{S}_{\text{lv}}^+ + \text{S}_{\text{lv}}^-$. 

3. As the inverse of the progress curve, that is, the graph showing the titrant volume as a function of pH, determined with or without making approximations. In this case one computes for a given pH what the corresponding titrant volume should be.

In the first method, which is still the common textbook treatment, separate approximate formulas are used for the starting point of the titration curve, for the equivalence point, and for points in between. Autoionization of the solvent is in most cases ignored. A drawback of the first method is that many a student gets confused by all those mathematical formulas and has no clear idea why one has to consider these distinct stages in titration and under what simplifying assumptions the formulas can be applied safely. For example, in case of a titration of a monoprotic weak acid with a strong base, when an approximate formula for the starting point is applied with a very small acid concentration, one gets a much too high pH value, which may even indicate a basic solution instead of an acidic solution. When the so-called Henderson-Hasselbalch equation is applied after addition of the base, but with very small titrant volumes, addition of the base leads in the computation first to a smaller pH before it rises again, which is of course very strange. These problems can only be solved by more robust mathematical methods. But these methods may be beyond the algebraic skills of many a student. In that case, a computer algebra system possibly gives a helping hand.

Explicit expressions for the shape of several aqueous acid-base titration curves have been published before by Gordus (1991) and De Levie (1993, 1996) under the assumption that autoionization of the solvent can be ignored. The Gröbner basis technique (Heck, 1997), which has been implemented in computer algebra systems like MAPLE or MATHEMATICA, can also be used to derive the required polynomial equation in $[\text{H}_2\text{S}_{\text{lv}}]^+$ from the system of polynomial equations obtained from the following three condition: (1) charge balance; (2) mass balance; and (3) chemical equilibrium. For the titration of a weak monoprotic acid with a strong monoprotic base, the final result is that the $[\text{H}_2\text{S}_{\text{lv}}]^+$ satisfies a third degree polynomial, which simplifies to a simpler polynomial of degree 2 when autoionization is ignored. The Gröbner basis technique can also be applied to derive an explicit expression of the progress curve (graph of titrant volume against pH, i.e., the inverse of the titration curve). One easily gets a formula that is equivalent to the following formula published by De Levie (2001a):

$$
\frac{V_b}{V_a} = \frac{K_w}{[\text{H}^+]} + \frac{K_a}{[\text{H}^+]} C_a - \frac{[\text{H}^+] + K_w}{C_b + [\text{H}^+] - \frac{K_w}{[\text{H}^+]}}.
$$
where \([H^+]\) stands for \([H_2Slv^+]\). Thus, given pH and therefore given \([H^+]\), one can use the above formula to compute a corresponding titrant volume \(V_b\). The computed pair (pH, \(V_b\)) can be used either to plot the progress curve (graph of \(V_b\) as function of pH) or the titration curve (graph of pH as function of \(V_b\)). Generalizations to titration of mixtures of acids or bases, to the polyprotic case, to general solvents, and to titration of bases with acids are possible. The same computer algebra supported approach can be extended to complexation, precipitation, and redox equilibria (De Levie, 2001b).

The Gröbner package of a modern computer algebra system can support the derivation of general mathematical formulas required for acid-base reactions. In my opinion, this approach is feasible for chemistry students at university level. For secondary school students however, it focuses in my opinion too much on mathematical concepts and techniques instead of on chemical concepts. At this level, students will learn more when they first determine experimentally in laboratory sessions some titration curves to get a feeling for what a titration curve is, how it is created, what is its shape in general, and how it can be interpreted. After this practical experience students can use computer models to explore titration curves in more detail without having the need to calculate themselves pH by mathematical formulas that are actually meaningless to them. With the computer models they can compare, for example, experimental titration curves with theoretical curves, determine ionization constants from real data, and explore buffer solutions.

**Computer Modeling of Acid-Base Titration**

The possibility of using system dynamics-based modeling software for the calculation of titration curves and for the comparison of real data with modeling results seems to have been overlooked or not promoted in literature about chemical education. I think that the main reason is that when one concentrates on mathematical formulas of titration, system dynamics does not easily come into mind because this mathematical field is linked with differential equations. However, system dynamics-based modeling software programs are nothing more, nor less than computer programs that can be used to specify relationships between variables and to compute how these relationships evolve. I illustrate this by a graphical computer model implemented in COACH 6 that computes the progress curve and the titration curve for titration of a diprotic acid with an aqueous strong monoprotic base. The calculation of the progress curves for a titration of a sample of volume \(V_s\) and concentration \(C_s\) with a titrant volume \(V_t\) and titrant concentration \(C_t\) is based on Equation (8) of (De Levie, 2001b), stated in the following general form, with details hidden in functions \(F_s\) and \(F_t\) of \([H^+]\) and coefficients \(K_a\):

\[
V_t = -V_s \cdot \frac{[H^+] - [OH^-] + \sum F_s C_s}{[H^+] - [OH^-] + \sum F_t C_t}.
\]

Here \(F = -F_a = F_b\) for each species participating in the titration (index ‘a’ refers to acid and ‘b’ to base). The summation is only needed for mixtures, where each component contributes its own \(F\) and its own concentration \(C\).

In case of titration of a weak acid with a strong monoprotic base, the above formula specializes to the following formula that can be used to compute the progress curve (\(V_t\) as a function of pH):

\[
V_t = -V_s \cdot \frac{[H^+] - [OH^-] - F_a C_a}{[H^+] - [OH^-] + F_b C_b},
\]

and

\[
V_b = -V_s \cdot \frac{[H^+] - [OH^-] - F_a C_a}{[H^+] - [OH^-] + F_b C_b}.
\]
where \( F_b = 1 \) and \( F_a \) is given by any of the below formulas (De Levie, 2001a,b):

<table>
<thead>
<tr>
<th>Proticity of Acid</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monoprotic</td>
<td>( \frac{K_a}{[H^+] + K_a} )</td>
</tr>
<tr>
<td>Diprotic</td>
<td>( \frac{K_{a1}[H^+] + 2K_{a1}K_{a2}}{[H^+]^2 + K_{a1}[H^+] + K_{a1}K_{a2}} )</td>
</tr>
<tr>
<td>Triprotic</td>
<td>( \frac{K_{a1}[H^+]^2 + 2K_{a1}K_{a2}[H^+] + 3K_{a1}K_{a2}K_{a3}}{[H^+]^3 + K_{a1}[H^+]^2 + K_{a1}K_{a2}[H^+] + K_{a1}K_{a2}K_{a3}} )</td>
</tr>
</tbody>
</table>

Figure 3.31 is a screen shot of a graphical computer model that uses the above formulas to compute the progress curve \( V_b \) as function of pH. The corresponding titration curve (pH as function of \( V_b \)) is drawn in the diagram next to the computer model. To make things more concrete, I took the example of Breneman and Parker (1991): the titration of sulfurous acid (\( pK_{a1} = 1.91, pK_{a2} = 7.18 \) at 25°C) with a strong base (sodium hydroxide) and with concentrations \( C_a = C_b = 0.100 \text{M} \).

Figure 3.31: Screen shot of a graphical model of diprotic acid–monoprotic strong base titration.

Note that in the above graphical computer model, time is not the independent variable, but pH. The only thing that happens when one runs the above model is that pH is repeatedly incremented with a user-defined step size \( \text{d}p\text{H} \). The chemical relevance of the value of parameters in the titration model is rapidly visualized: For example, a student can easily investigate the relation between the shape of the titration curve and the values of ionization constants of the acid and base involved.

A Practical Investigation: Analysis of Acids in Cola Drinks

As an example of a practical investigation for upper-level secondary school students I briefly describe results of an analysis of acids in various Cola drinks. Phosphoric acid is a common ingredient in Cola drinks; it provides a taste that is both sweet and sour, but it does not compete with other flavors. There is some variability in both the amount and composition of this acid in Cola drinks. The composition is affected by the following equilibrium of this weak aqueous triprotic acid: \( \text{H}_3\text{PO}_4 + \text{H}_2\text{O} \rightleftharpoons \text{H}_2\text{PO}_4^- + \text{H}_3\text{O}^+ \), \( \text{H}_2\text{PO}_4^- + \text{H}_2\text{O} \rightleftharpoons \text{HPO}_4^{2-} + \text{H}_3\text{O}^+ \), and \( \text{HPO}_4^{2-} + \text{H}_2\text{O} \rightleftharpoons \text{PO}_4^{3-} + \text{H}_3\text{O}^+ \), with equilibrium constants \( K_{a1}, K_{a2}, \) and \( K_{a3} \), respectively. The acidic mixture of a
3.3. Modeling

A sample of fresh Coca-Cola Light (also known as Diet Coke) is investigated through titration with a strong base (0.1M sodium hydroxide). Citric acid, which is also triprotic, is another common ingredient of soft drinks. It is certainly present in Coca-Cola Light because the product specification of the bottle mentions the presence of the nutritive acid E330 (The indication E338 reveals the presence of phosphoric acid). Since the drink is not de-carbonated, the diprotic carbonic acid will also be present in the sample of 30 mL. There are three unknown parameters in the model, namely the concentrations of carbonic acid, phosphoric acid, and citric acid. By the method of trial and improvement a reasonable 3-tuple of parameter values can be found so that the titration curve computed with the model matches the measured titration curve. By the way, the measurement was carried out with the step-motor buret adding slowly the strong base to the sample while recording the pH of the solution. Figure 3.32 shows a screen shot with the graphical model and a graph window with the computed (blue line) and measured (red line) titration curves. The model is useful for understanding the titration of complex solutions, with multiple polyprotic acids and bases.

![Figure 3.32: Screen shot of a graphical model of titration of Coca-Cola Light with a strong base and a comparison of the modeling results with a measured titration curve.](image)

Classroom experience with students in their final year of pre-university education (age 17-18 years) was that they liked this kind of practical investigations and got a good idea what the purpose of titration is, how polyprotic acids and bases can be recognized, how one can deal with mixtures of acids or bases in titration, and how titration is done in the reality of a chemistry laboratory. The concepts of acid-base chemistry, pH, and titration came to life with this type of student activities.

These positive effects of practical work on students’ understanding of acid-base reactions and titration did not occur as a matter of course, but were more or less orchestrated by the chemistry teacher. To ensure that students not just did their experiments and reported their results without much reflection, it was considered wise to provide students with guidelines for the report and to emphasize the expected and required quality of both experimentation and reporting. Figure 3.33 illustrates that the quality of the titration experiment indeed differed from one student team to another. This was not a big issue, at least when students paid attention to the quality of their experimental results, when they could figure out what went wrong during the experiments, and could formulate or make improvements. Students’ guidelines for the reports, which concerned (1) the theoretical background of the experiment and (2) the discussion of the obtained results and the conclusions drawn, turned out to help improve the quality of the students’ learning and reporting process, and they
increased the satisfaction with which both the chemistry teacher and the students looked back at the practical investigation.

Figure 3.33: Titration curves obtained by two student teams. The quality of diagram on the left-hand side is good, but the diagram on the right-hand side was obtained under weaker experimental conditions: The speed of the step-motor buret was too fast and the rotation speed of the magnetic stir bar in the beaker was too slow.

3.3.3 Modeling of Tidal Movement

References


Introduction
Although the focus in most case studies in my education research was on quantitative mathematical modeling and therefore the descriptive and predictive quality of models was highly valued, I did not shun unrealistic models when they were part of an instructional approach aiming at understanding. Modeling tidal movement was an example of this type of work. A model for the moon-earth dynamical system with the moon in the equatorial plane that yields the tidal movement on an all-ocean world was expected to give students insight in the origin and behavior of tides. The fact that one does not live on such a world does not matter; more important is the basic understanding of the phenomenon that follows from it. This over-simplified, all-ocean model with the moon in the equatorial plane explains the basic period behavior of the tidal movement, the so-called M2 tide produced by the moon. A similar model including the sun as well, with semidiurnal M2 and S2 tides produced by the moon and the sun, respectively, yields interference effects as neap and spring tides when sun, moon, and earth are in line (during new and full moon) and the lunar tide and the solar tide are reinforcing each other. The presence of land not only makes high and low water observational, it also changes the local phases and amplitudes in a dramatic way, both at sea and over land, because ‘land gets in the way’ of the moving water. The (almost) periodic tidal movement is only recognizable when many higher harmonics like M4, S4, and other tidal constituents are produced. To describe and accurately predict tidal effects at seaports harmonic analysis is indispensable. To study the change in phase and
amplitude as the tide runs up from a seaport into a convergent estuary, the behavior of a tidal wave must be modeled.

**Graphical Modeling of the All-Ocean Equilibrium Tide**

I start with describing the lunar tide caused by the effect of gravity in the earth-moon system with the moon in the equatorial plane. Note that the earth and the moon rotate about a common center just inside the earth’s surface. On the earth’s surface, there is an imbalance between the centripetal (inward) and centrifugal (outward) accelerations. The centrifugal acceleration is the same everywhere on the earth, but the gravitational force due to the moon varies over the surface of the earth. This results in the tidal force. The so-called equilibrium tide is that which would result from the tidal force if the earth were completely covered by water and responded instantly to the changing forces and there were no friction. As a result of the tidal force, the equilibrium tide has two bulges, one toward the moon and another on the opposite side. Thus one notices two high and two low water per lunar day. At the side nearest to the moon, the gravitational force wins from the centrifugal force while on the opposite side of the earth the centrifugal force is dominant. This suggests the widespread misconception (cf., Viiri, 2000) that high water should be the result of the moon’s gravitational pull. But how could the very small radial component of the tidal acceleration (in the order of $10^{-7}$ times the earth’s acceleration of gravity) pull this off against the earth’s acceleration of gravity? Obviously it will disappear completely in the balance of the dominant normal (or buoyant) force and gravity. Actually, the equilibrium tide is determined by the remaining tangential components of the tidal force, called the tractive force. The mathematical formula for the tangential component of the tidal acceleration of a point on the surface of the earth is given by:

$$a_\varphi = -\frac{3Gm_M}{2r^3} \cdot R \sin(2\varphi),$$

where $G$ is the gravitational constant, $R$ is the radius of the earth, $m_M$ is the mass of the moon, $r$ is the distance between the centers of these two bodies, and $\varphi$ is the rotation angle. The tractive force is maximal for $\varphi = 45^\circ$ and $\varphi = 135^\circ$. With respect to the moon, the angle $\varphi$ varies with a period given by the relative angular speed: $\varphi = \omega t = 360t/T$, where $T$ is the period determined by the motion of the earth and the moon.

Assuming that, for an equatorial sea channel, the work done on the water fully originates from the tidal acceleration, then one gets the following equality of differentials: $a_\varphi R d\varphi = g\, dh$, where $g$ is the acceleration of gravity and $h$ is the water level. This immediately results in:

$$\frac{dh}{dt} = \frac{d\varphi}{dt} \frac{dh}{d\varphi} = \frac{dh}{d\varphi} = \omega \frac{a_\varphi R}{g}.$$

Together with the above expression for $a_\varphi$, this equation can now be used in a system dynamics-based graphical model as shown on the left-hand side of Figure 3.34. This over-simplified, all-ocean model turns out to give an M2 tidal amplitude of 27 cm. A similar model including the sun as well yields interference effects as neap and spring tides. The right-hand side of Figure 3.35 illustrates this.
Harmonic Analysis of Tides by Students

Because of the geographic position and the size of the Netherlands every inhabitant is familiar with tides. In the North Sea, the tides are semidiurnal, that is, the level rises and falls cyclically on a twice-daily basis, and the graph of a tidal motion, measured or computed for a coastal place or an oil platform, is periodic with the two low waters of each tidal day being almost equal in height and with the same behavior of the two high waters. A periodic function can under certain conditions be mathematically described with sums of sine functions, but tidal curves are always approximated in Dutch mathematics textbooks by a single sine function. Students search on Internet for tidal data of a coastal town on a certain day or the yearly average, and then they try to match the data found with a good sine fit using graphical software or a graphing calculator. It is true that limitations of this simple mathematical model are briefly discussed and that in particular the asymmetry between low and high tide is pointed at, but textbook authors do not go further than a short reference to harmonic analysis and a pointer for background information on Internet. They suggest students to explore tidal motion further in practical work or a research project, but they do not give any clue of how to do this with a chance of success. A citation of De Lange (2000, p 25), in a paper on studying tidal motion in the classroom, is hardly encouraging: “The students don’t have the tools to find a better way of coping with the lack of symmetry.
of the real graph.” But, when students have access to a state-of-the-art signal analysis tool, then it allows them to come up with a more realistic description of tidal motion. Examples show that the parametric high-resolution methods of signal analysis that were added to COACH 6 can indeed help in accomplishing and understanding tidal harmonic analysis.

The first example concerns the predicted tidal data at Flushing from May 21 till May 23, 2006 (the black dots in Figure 3.36). By the least squares method of peeling-off functions (Foss, 1969) using sinusoidal fits, a student can determine an approximation of the tidal data with sum of two sinuoids: She or he determines first the best sine fit, subtracts it from the data, that is, computes the residue, and then determines the sine fit of the residue; the sum of these two sine fits is a reasonable approximation of the tidal data. This method is illustrated in Figure 3.36, in which COACH was used. A similar approximation would have been obtained with the implemented Prony method for spectral analysis (Mackisack, Osborne, & Smyth, 1994; Smyth, 2000).

Figure 3.36: Predicted tides at Flushing from May 21 till May 23, 2006, and two approximations of the tidal curve.

Tidal currents are not everywhere on earth of the same type: There are coastal areas with a diurnal cycle, that is, with a period of approximately 24 hours, and there are locations on earth where one has two cycles per day, but the two high waters and the two low waters have marked differences in their heights. Official tide tables for almost all coastal areas on earth can be found on Internet. This enables students to find places and time periods for which the tidal model of a single sinusoid works well and those for which more sinusoidal terms, called harmonic constituents, are needed for a good description of the tidal data. In COACH, this tidal harmonic analysis is best done with the the R-ESPRIT method (Mahata, 2003). For example, if one models the tidal data for Sewells Point, Hampton Roads in Virginia, from March 23 till March 25, 2006 (retrieved from the website www.tidesandcurrents.noaa.gov of the National Oceanic & Atmospheric Administration in the United States of America) with eight sinusoidal functions, then one gets the following formula (with the automatically chosen snapshot dimension equal to 84):

$$41.11 + 35.12 \sin(28.982t - 75.637) + 6.38 \sin(30.002t + 6.794)$$
$$+ 4.97 \sin(15.052t - 152.698) + 4.95 \sin(13.946t + 78.075)$$
$$+ 0.45 \sin(0.200t + 106.750) + 0.96 \sin(27.990t + 92.143)$$
$$+ 0.02 \sin(31.472t - 24.033) + 0.02 \sin(33.216t + 103.765)$$,
where \( t \) is the time (in hours) from the beginning of 2005 and the speed of each constituent (if you wish, the frequency of each sine function) has the in tidal analysis commonly used unit of degrees per hour. The standard deviation turns out to be about 8 cm. One can extend this model to 16 harmonic constituents by applying the same spectral analysis to the difference of the predicted tides and the approximation already found. In Table 3.1, the five most important contributions in the COACH model are compared with literature values of the harmonic constituents. One may call this agreement between the COACH model and the literature values astonishing.

The labels M2 and S2 belong to the harmonic constituents that are linked with the motion of the moon around the earth and the motion of the earth around the sun, respectively, and which cause the semiidiurnal tide. The N2 constituent takes into account the effect that the orbit of the moon around the earth is in reality not a circle but an ellipse. The diurnal constituents K1 and O1 take into account (amongst other things) the inclination of the earth’s equatorial plane with respect to the plane of the moon’s orbit. Most (if not all) harmonic constituents can be related to astronomical phenomena and therefore a tidal prediction is often referred to as astronomical tide.

In practice, the number of constituents needed for accurate tidal prediction and the amplitude, speed and phase of each harmonic constituent are often determined from tidal records of three consecutive years. As a matter of fact, the speeds are always fixed and only the amplitudes and phases of the strongest tidal constituents that have propagated to a point of interest must be determined by regression methods. Short term tidal constituents (say for data periods up to one month) are also determined by the Fourier harmonic analysis method. More examples can be found in the references given at the beginning of Section 3.3.3.

<table>
<thead>
<tr>
<th>Coach model</th>
<th>Literature data</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed (({\circ}^{\circ}/h))</td>
<td>amplitude (cm)</td>
</tr>
<tr>
<td>28.982</td>
<td>35.12</td>
</tr>
<tr>
<td>28.438</td>
<td>7.70</td>
</tr>
<tr>
<td>30.002</td>
<td>6.38</td>
</tr>
<tr>
<td>15.052</td>
<td>4.97</td>
</tr>
<tr>
<td>13.946</td>
<td>4.95</td>
</tr>
</tbody>
</table>

Table 3.1: Comparison of the principle harmonic constituents at Sewells Point (Virginia) from March 23 till March 25, 2006, found in COACH and provided by the National Oceanic & Atmospheric Administration in the USA.

Tidal analysis becomes more interesting for students when they can investigate tidal motion closer at home. Figure 3.37 is a COACH screen shot of a tidal analysis of two consecutive days at Europahaven, namely May 24-25, 2005. It shows that the phenomenon of double low water (also known as agger), which means that the low water consists of two minima separated by a relatively small elevation, can be modeled well by harmonic analysis. Actually the model consists of a fundamental tidal speed and overtides, that is, harmonic constituents with speeds that are an exact multiple of the fundamental constituent.

In order to get results with the R-ESPRIT method that are in good agreement with recorded data or official tidal predictions, the choice of the snapshot dimension (one of the two parameters in R-ESPRIT) is important. One must determine this parameter by trial and improvement when the automatic value does not work well.
3.3. Modeling

The Implemented Signal Analysis Tools

Tidal analysis is the third application of a sinusoidal regression tool that allows curve fitting with sums of two or more sinusoid; gait analysis was the first application mentioned in Section 2.4, and an example of speech analysis was shown in Section 3.1.1. It seems time to discuss the choice of the high-resolution, spectral estimation tools implemented in COACH and motivate the design choices.

Signal analysis can display the frequency spectrum of a data set presenting a periodic signal of the form

\[ y(t) = \gamma + \sum \rho_j \sin(\omega_j t + \phi_j). \]

Peaks in the frequency spectrum correspond with frequency components \(\omega_j\) (Actually, in COACH it is written as \(\omega_j = 2\pi f_j\) and the \(f_j\)'s are called the frequencies in the spectrum). Once the frequency components of a periodic signal have been found, the other parameters (amplitude \(\rho_j\) and phase \(\phi_j\)) can be estimated by linear regression methods. The design choice for COACH 6, after thorough evaluation of the educational usability of many methods of spectral analysis (Heck, 2004d) on the basis of sample data sets and computer simulations, was to provide four methods: (1) Fourier transform; (2) linear prediction; (3) R-ESPRIT (Mahata, 2003); and (4) the symmetry-adapted, modified Prony method described in detail by Smyth (2000). These four methods differ fundamentally.

Fourier transform and the linear prediction method were already available in version 4 of COACH. Fourier transform is a nonparametric method, in which the estimate of the power spectral density (PSD) is made directly from the signal itself. In this case, the periodogram is used as an estimate of the PSD. The linear prediction method is a parametric method in which the signal whose PSD one wants to estimate is assumed to be output of a linear system driven by white noise. The linear prediction method is an example of an autoregressive (AR) method in which one first estimates the parameters (coefficients) of the linear system that hypothetically ‘generates’ the signal. These parameters are then used to generate an artificial spectrum, called the pseudospectrum. The linear prediction method was originally introduced to allow students to carry out speech analysis. It was mainly kept for backward compatibility reasons.

R-ESPRIT and Prony are subspace methods, which are known to be best suited for line spectra, that is, for spectra of sinusoidal signals, and are effective in the detection of sinusoids buried in noise when the noise-to-signal ratios are high. The methods are called subspace methods because the autocorrelation matrix of the signal depends on a subspace that is the direct sum of the noise subspace and the signal subspace. They generate frequency component estimates for a signal based on an analysis of eigenvalues and eigenvectors of the autocorrelation matrix of the signal.
The R-ESPRIT method is a state-of-the-art subspace method that depends on two parameters. Firstly, the number of frequencies determines the order of the sinusoidal model of the signal. The second parameter in the R-ESPRIT method, the snapshot dimension, is linked to the construction of the autocorrelation matrix of the signal. To form the autocorrelation matrix of the signal, the data set is broken into small overlapping fragments or snapshots of fixed length using a sliding window technique known as spatial smoothing. There exists no simple criterion for an automatic choice of the best snapshot dimension. In Coach 6, the spectral analysis is done for all values less than 100 and the best snapshot dimension in this range is selected. Higher snapshot dimensions can only be set manually in the program.

A Prony method is a parametric method based on the autoregressive-moving average (ARMA) model of the signal and generates frequency component estimates for a signal based on a direct estimation of the parameters in this model. An analysis of eigenvalues and eigenvectors of a matrix plays again an important role here. The symmetry-adapted, modified Prony method described in detail by Smyth (2000) has been implemented in Coach 6. There is in this Prony method only one parameter, namely the model order, which equals the number of frequencies that one wants to estimate. In R-ESPRIT and Prony, a pseudospectrum is generated after estimation of the parameters in the model of the signal. Peaks in this pseudospectrum correspond with frequencies components in the signal, but have no other meaning. The strengths of the four signal analysis methods complement each other, so it is advantageous to have all of them available in a computer learning environment for inquiry-oriented mathematics and science education. There exist many more signal analysis methods, but the usability in education at secondary school level makes it necessary to select only a few signal analysis methods and to find herein a good balance between simplicity and range of applicability. Table 3.2 lists the strengths and weaknesses of the implemented methods.

An additional example illustrates the application of the high-resolution methods. In Figure 3.38 are shown on the left-hand side the data points of a recorded sound signal consisting of two close frequencies created by two tuning forks. The effect that one would hear is known as sound beats. In the diagram is also the graph of the approximation found by the R-ESPRIT method, the pseudospectrum of which is shown in the middle. The frequencies found are 383 and 440 Hz, which is in excellent agreement with the tuning forks actually used in the sound experiment. The graph of the approximation is most convincing evidence of the quality of the spectral analysis. This result could not be obtained with the method of Fast Fourier Transform or by other low resolution signal analysis techniques.

Figure 3.38: Spectral analysis of sound beats.
### 3.3. Modeling

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fourier transform</strong></td>
<td></td>
</tr>
<tr>
<td>• Faster than other methods like linear prediction or R-ESPRIT, especially when the oscillation consists of pure tones.</td>
<td>• Very poor frequency resolution when few data points are available.</td>
</tr>
<tr>
<td>• Very good at determining intensities in the spectrum.</td>
<td>• Increased data size improves resolution, but it does not improve performance regarding the variance of the frequency estimator.</td>
</tr>
<tr>
<td>• Asymptotically efficient, i.e., as sample size goes to infinity the Cramér-Rao bound is approached.</td>
<td>• The noise-to-signal ratio must be small in order to get good estimates.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linear prediction</strong></td>
<td></td>
</tr>
<tr>
<td>• Good frequency resolution, even if there are few data points.</td>
<td>• The more accurate the frequencies are determined, the less accurate are the intensities.</td>
</tr>
<tr>
<td>• Best model order can be automatically estimated via Akaike’s Final Prediction Error Criterion (Akaike, 1969).</td>
<td>• Poor settings may lead to a spectrum that lacks important details.</td>
</tr>
<tr>
<td>• By using lower quality for the coefficients, the envelope of the spectrum (i.e., formants) can be gained. This is, for example, useful for determining resonant frequencies.</td>
<td>• Accuracy is lower for high-order models, long data records, and large noise-to-signal ratio (causing line-splitting or extraneous peaks in the spectrum).</td>
</tr>
<tr>
<td>• Asymptotically efficient, i.e., as sample size goes to infinity the Cramér-Rao bound is approached.</td>
<td>• Peak locations may slightly depend on initial phase of a noisy sinusoidal signal.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R-ESPRIT</strong></td>
<td></td>
</tr>
<tr>
<td>• Excellent resolution (R-ESPRIT outperforms other methods).</td>
<td>• One must determine the order of the model; no criterion implemented.</td>
</tr>
<tr>
<td>• Works in general also good with small snapshot dimensions.</td>
<td>• No criterion for selection of snapshot dimension and occasionally high values are needed for good performance.</td>
</tr>
<tr>
<td>• Allows large noise-to-signal ratios provided that the sample size is large enough to get good results.</td>
<td>• Computationally less efficient than Fourier transform for large data sets or large snapshot dimensions.</td>
</tr>
<tr>
<td>• Less sensitive to noise-to-signal ratio in case of close frequencies than other methods (this is why it is also called a super-resolution method).</td>
<td>• Increased data size improves resolution, but it does not improve performance regarding the variance of the frequency estimator.</td>
</tr>
<tr>
<td>• Asymptotically efficient, i.e., as sample size goes to infinity the Cramér-Rao bound is approached.</td>
<td>• Biased frequencies estimates may occur depending on the estimation of the autocorrelation matrix.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The symmetry-adapted, modified Prony method</strong></td>
<td></td>
</tr>
<tr>
<td>• Resolution as good as autoregressive methods, sometimes better.</td>
<td>• One must determine the order of the model; no criterion implemented.</td>
</tr>
<tr>
<td>• The only parameter needed is the model order, i.e., the number of frequencies.</td>
<td>• Fewer frequencies than asked for when the model in ill-chosen.</td>
</tr>
<tr>
<td>• Allows incorporation of zero frequency to deal with a constant shift of the data.</td>
<td>• Computationally less efficient than Fourier transform for large data sets or large number of frequencies.</td>
</tr>
<tr>
<td>• Symmetry adaptation improves frequency estimation.</td>
<td>• Requires the computation of complex roots of a polynomial.</td>
</tr>
<tr>
<td>• Asymptotically efficient, i.e., as sample size goes to infinity the Cramér-Rao bound is approached.</td>
<td>• High decimal precision is often needed in case of large model orders to avoid numerical problems.</td>
</tr>
</tbody>
</table>

Table 3.2: Strengths and weaknesses of four signal analysis tools.

**Tidal Waves in Convergent Estuaries**

Returning to the subject of tidal movement, I would like to mention that Gastel, Heck, and Uylings (2007) also explored how far one would get with studying the change in
phase and amplitude of tidal waves as the tide runs up from a seaport into a convergent estuary, when a system dynamics-based modeling tool was used. The underlying thought was that, in case one designs a computer environment for inquiry-oriented mathematics and science education, teachers and instructional designers should at least be supported in doing more advanced inquiry activities themselves. In other words, one wants to get an idea of how large the research space is within which the environment can be fruitfully used. The authors were able to model the tidal behavior of the English river Thames as a function of the tides of the North Sea. They managed to model the river such that the amplitudes and the phases of the tidal movement turned out to correspond reasonably well with empirical data. Their calculations confirmed that the amplitude almost linearly increases during the first 20 km, reaches a maximum at 33 km from the mouth (actually at the Tidal Barrier at 40 km), and then decreases again. In other words, the quality of the model was remarkably good, given all assumptions made. The interested reader is referred to the original paper. Clearly this level of modeling is beyond what can be expected from secondary school students.

3.4 Data Logging, Control, and Video Combined

In Section 3.1.5 (p. 114), where I described the possibility of creating animations driven by computer models and programs, I promised the creation of a funny applause meter in which the measurement of the sound level by a microphone is element of the control program that determines whether the cyclist moves forward. I keep my promise with Figure 3.39. It is an appetizer for a section about the use of various combinations of data logging, control, and video.

Figure 3.39: Screen shot of a Coach activity in which data logging, control, and computer animation are combined to create a funny applause meter: During the computer simulation, the cyclist moves forward when the sound level is high enough.
3.4. Data Logging, Control, and Video Combined

Two examples of inquiry activities for students in which data logging was combined with synchronized video recording are presented: (1) part of a profile research project of a secondary school student in which muscle activity during walking was explored; and (2) a field experiment in which vertical jumping was explored using a force plate and a webcam. The third example, a field experiment in which the pupil light reflex in a human eye was explored, combines data logging, control, and video in one activity.

3.4.1 Gait Analysis via Electromyography

Reference

Introduction
Caroline van Dongen, a secondary school student preparing for medical studies, collected and analyzed in her profile research project gait data in much the same way movement scientists do, namely, via recording and measurement of motions with a video tool and via electromyography, that is, measurement of muscle activity. The authenticity of the student project was rooted in: (1) the use of inexpensive tools (a webcam, a simple EMG/ECG set and Coach software), which were on the one hand fit for educational practice and on the other hand in essence close to the techniques used by biomechanists; and (2) the student’s use of the theoretical framework, nomenclature, and research methods of practitioners. That is to say, the student conducted many aspects of motion analysis herself: She formulated research questions about a self-selected gait pattern, searched and studied background information (amongst which articles and fragments of gait analysis books), designed and carried out experiments, processed, analyzed and interpreted collected data, and finally contributed to a joint paper (Heck & Van Dongen, 2008).

The main research question in the profile project of Caroline van Dongen was: “What is the course of human gait?” This was specialized as follows:

1. “Which phases are distinguished in the gait cycle?”
2. “What muscle activity happens during gait?”
3. “How do bones and joints make gait possible?”

The first and third subquestions were addressed by video measurement with a webcam of the planar motion of the leg around the hip and knee joint during normal walk on a motorized treadmill in a sports center; see Figure 2.11 in Section 2.4.2 (p. 54) for a picture taken from the part of her research report in which she analyzed a hip-knee cyclogram of normal walk. In this subsection, I discuss the student’s exploration of muscle activity during normal walk via dynamic electromyography (EMG).

Electromyography by Secondary School Students

In case of measurement of activity of superficial muscles, surface electrodes are placed on the skin surface to detect the electric activity responsible for contraction of muscles (Perry & Burnfield, 2010). EMG recording is rather difficult because correct placement of electrodes is critical. Processing and interpreting an EMG for a muscle is also not easy. Nevertheless, the profile research project of Caroline van Dongen and other
secondary school students showed that they were able to get empirical results that were qualitatively in agreement with results from research literature (e.g., Rose & Gamble, 2006, p. 40). It is extremely useful in the experimental setting that COACH allows simultaneous data logging and video capture. The video clip and the measured data are synchronized: This means that pointing with the mouse at a point on the graph or at a table entry automatically shows the corresponding video frame in the video clip and that selecting a particular frame highlights the corresponding points in diagrams, when scanning mode is on. Then, scrubbing reveals that peaks in the clearly periodic EMG signals are consistently linked to certain gait events.

This research project offered the student the opportunity to personally experience the challenges faced in gait analysis. Although the student found it difficult to process and interpret the recorded EMG data, she managed to read off from the EMG signal when muscle activity was on and off in various gait patterns. She also managed (See Figure 3.41) to interpret the processed signal in terms of phases and events in the gait cycle, simply by reasoning about what muscle groups are involved in producing a particular body part movement. A concrete example shows what processing and analysis of EMG data actually meant in practice.

The student studied the motion of the lower limb during normal walking and running at various speeds. The motion was filmed from the side by means of a webcam recording at a speed of 30 frames per second while the subject was walking on a motor-driven treadmill. EMG data of four muscle groups were obtained at various gait speeds: the gluteus maximus, the hamstring, the quadriceps femoris, and the gastrocnemius. The last muscle group is the big muscle group at the back of the lower leg. Surface electrodes were placed in a bipolar arrangement, that is, two electrodes were placed over a muscle about 2 cm apart and a third electrode was placed at an electrically neutral site (in this case, the ankle). The still from the recorded video on the left-hand side of Figure 3.40 illustrates the locations used for the EMG of the gastrocnemius, following the recommendations of the SENIAM project (SENIAM is an acronym for Surface Electromyography for the Non-Invasive Assessment of Muscles; www.seniam.org). A sample rate of 900 Hz was used. The right-hand side of Figure 3.40 shows the graph of the raw EMG signal of the gastrocnemius, recorded for walking at a speed of 2.5 km hr\(^{-1}\). The EMG signal is clearly periodic, which indicates that the recording was successful. Scrubbing, when the scanning mode in COACH is on, reveals that the maxima and minima are consistently linked to certain gait events.

Figure 3.41 shows the graph of the processed EMG signal during one gait cycle: Processing means here that the absolute value of the raw signal was taken with respect to the zero level of the signal (so-called full-wave rectification) and that the rectified signal was smoothed via a moving average (A 9-point moving average was used). These techniques exemplify the use of basic mathematics in biomechanics practice. Peaks in the diagram correspond with muscle activity and corresponding gait events, which have been found by scanning the graph while watching the corresponding video frames, have been marked in the diagram. The most general conclusion of the student, based on the processed EMG diagram, was that the gastrocnemius is predominantly active during stance phase. This is in agreement with on-off diagrams shown in research literature (e.g., Rose & Gamble, 2006, p. 40). A deeper analysis went as follows: Peak values are at terminal stance and when the foot is taken off the ground. This can be understood because, during this part of the stride, the muscle first undergoes
3.4. Data Logging, Control, and Video Combined

eccentric contraction, which controls the forward movement of the tibia over the fixed foot. Thereafter the gastrocnemius undergoes a concentric contraction which initiates plantar flexion, which begins the toe-off phase. At terminal swing, the muscle might be used during foot descent in order to provide stability to the ankle joint in preparation for foot contact. Beginning at foot strike and ending at midstance, the gastrocnemius is probably used for shock absorption, stability, and progression of the limb.

![Figure 3.40: Recorded raw EMG signal of the gastrocnemius for normal walking.](image)

![Figure 3.41: Processed EMG signal of the gastrocnemius for normal walking.](image)

In summary, the practical investigation discussed in this subsection is an illustrative example of how students can collect both qualitative and quantitative real-time data of the human gait by means of an inexpensive digital camera. Using a rather simple EMG set, they can simultaneously capture a motion on video and record the muscle activity. Processing and interpreting an EMG for a muscle or muscle group is not easy. But, from the EMG signal, students can read off when muscle activity is on and off. They can often interpret the EMG signal in terms of phases and events in the gait cycle, simply by reasoning about what muscle groups are involved in producing a particular body part movement. The authenticity of the student activities is realized by the opportunity for high school students to investigate a real-world motion in much the same way as movement scientists would do. Students can learn how physics, mathematics, and biology come together in such work.

3.4.2 Exploring Standing Vertical Jumps

Reference
Introduction

Mathematics and physics play an important role in movement science and sports science: They are fundamental in modeling and analysis of all kinds of motions. In this field experiment, the main focus was on the human performance in standing vertical jumps and on various experimental ways to determine the vertical height of a jump. The work is similar to the analysis of standing vertical jumps done by Linthorne (2001) using a force plate and dedicated software, but it differs in the sense that in the presented field experiment synchronized data logging and video capturing was utilized and the complete analysis took place in a single computer environment. Two types of jumps were analyzed: (1) the squat jump, in which a jumper starts from a stationary semi-squatted position and then vigorously extends the knees and hips to jump vertically up off the ground; and (2) the countermovement jump, in which a jumper makes a preliminary downward movement by flexing the knees and hips before a vigorous extension. The sequence of action for each type of standing jump is schematically shown in Figure 3.42.

Figure 3.42: Sequence of actions in (a) a squat jump and (b) a countermovement jump. The indicated jumper’s center of mass moves only in the vertical direction.

One warning must be made in advance with respect to the image sequences in Figure 3.42: The suggestion is made that take-off happens when the legs are fully extended. In reality this is not true. Another surprising issue may be that in biomechanics literature more than one meaningful definition of vertical jump height can be found. Scholz, Bobbert, and Soest (2006) distinguished the following three definitions. A common definition is the height gained by the body center of mass in the airborne phase only. Under this definition and under the assumption that air resistance can be ignored and that the relative positions of body segments do not change during flight, jumping height is determined by the take-off velocity, which is the velocity of the center of mass at the instant of toe-off. Somewhat problematic in this definition is that it is in practice not easy to determine the center of mass of a jumper: It is not a fixed position within the body, but changes in response to relative changes in position of body segments. A segmental model is needed to calculate the center of mass during the jump. A second meaningful definition of vertical jump height is the total height gained by the center of mass during the jump, including the height gained during the push-off phase. Thirdly, vertical jump height could be defined with respect to the ground floor. In the below analysis, I use the first definition. In this case, flight height can be operationalized in terms of take-off velocity, \( v_{\text{take-off}} \), which is in turn determined by the kinetic energy imparted to the jumper’s body by the muscles. The height gained by the body center of mass in the airborne phase, \( \Delta h_{\text{air}} \), can be obtained by applying the law of conservation of mechanical energy to the flight phase of the jump between the instant of take-off and the instant the jumper reaches the peak of the jump and has a vertical velocity equal to zero. In the absence of air resistance, one can easily derive the following formula:

\[
\Delta h_{\text{air}} = \frac{v_{\text{take-off}}^2}{2g},
\]
where \( g \) is the acceleration of gravity. Thus, the flight height of a jump can be calculated once the take-off velocity is known. The difficulty lies in the determination of the take-off velocity: Video analysis or the use of a force plate, possibly in combination with synchronized video capture, can help out.

As Aragón-Vargas (2000), Linthorne (2001), and Moir (2008) pointed out, other methods can be used to calculate vertical jump heights. For example, the time in air, \( t_{\text{air}} \), can be used to estimate the vertical jump height when air resistance can be ignored. The formula can be easily derived as follows: In the airborne phase there is only a constant acceleration \(-g\) of the moving body. Because the take-off velocity is assumed to be in magnitude the same as the velocity at foot-strike and only in opposite direction, the following equation holds: \(-g = (v_{\text{foot-strike}} - v_{\text{take-off}})/t_{\text{air}} = -2v_{\text{take-off}}/t_{\text{air}}\). Thus: \( v_{\text{take-off}} = g t_{\text{air}} / 2 \). Substitution in the previous expression for flight height gives:

\[
\Delta h_{\text{air}} = \frac{g t_{\text{air}}}{8}.
\]

The difficulty lies in the determination of the duration of the airborne phase: video analysis or the use of a force plate can again help out. Moir (2008) reported that the method that uses the take-off velocity is preferred when a force plate is used. Henceforth, I only discuss the use of a force plate in combination with the use of synchronized video capturing, provided by the COACH hardware and software, following the approach of Linthorne (2001). It also illustrates the requirements of a versatile set of data analysis tools for an integrated open computer environment for mathematics and science education. In particular, it illustrates the application of numerical differentiation and integration.

**The Squat Jump**

The simplest case is the squat jump with the hands at the hips and with leg motion as the dominant factor in the jump motion. Figure 3.43 is a screen shot of a COACH activity in which the vertical ground reaction force \( F \) has been recorded through the force plate with a simultaneous video recording through a webcam. This feature of synchronized measurement and video recording allows easy determination of correspondences between graphical features and vertical jump events; see Figure 3.44. In the diagram is also shown the constant body weight \( M \), recorded at the jumper’s initial stationary position on the force plate (\( \approx 517.5 \text{ N} \approx 53 \text{ kg} \)).

![Figure 3.43: The vertical ground reaction force trace from a subject performing a squat jump.](image-url)
Figure 3.44: The force-time graph of a squat jump with annotated events and jump phases.

The impulse-momentum method offers a possibility to determine the impulse of the body at take-off and knowledge about the jumper’s body weight leads then to the take-off velocity. The impulse-momentum method is based on Newton’s second law of motion, which can be written as an equation of the integral of the propulsion force over time, known as the impulse due to the force, and the change in the momentum of the body from the start of the jump motion to the take-off. Application of this equation to the push-off phase of the standing vertical jump, starting from when the jumper is in stationary position ($v_{\text{start}} = 0$ at $t_{\text{start}}$) through to the instant of take-off (at $t_{\text{take-off}}$) and with the propulsion force equal to $F_{\text{GRF}} - M = F_{\text{GRF}} - mg$, where $F_{\text{GRF}}$ is the ground reaction force measured with the force plate, $m$ is the jumper’s body weight in kg and $M$ is the body weight in N, leads to:

$$\int_{t_{\text{start}}}^{t_{\text{take-off}}} (F_{\text{GRF}} - M) \, dt = mv_{\text{take-off}} - mv_{\text{start}}.$$ 

When this is worked out for a stationary starting position one gets:

$$\int_{t_{\text{start}}}^{t_{\text{take-off}}} F_{\text{GRF}} \, dt - Mt_{\text{push-off}} = mv_{\text{take-off}},$$

where $t_{\text{push-off}} = t_{\text{take-off}} - t_{\text{start}}$ is the time needed to move the body from the start of the standing jump to the take-off. The left-hand side of the last equation consists of the impulse due to the ground reaction force and the impulse due to the jumper’s body weight, which both can be determined from the force-time graph. The area under the force-time graph can be determined in COACH through the area tool from the data analysis facility (See Figure 3.45), which gives an estimate of 0.248s for the duration of the push-off phase, an impulse due to ground reaction force of about 219.6Ns and an impulse due to body weight of 128.3Ns. These data would lead to a take-off velocity of 1.7 m/s for a jumper with a body weight of 53 kg. Using the relationship between jump height and take-off velocity, this would correspond with a flight height of 14.7 cm. The relationship between jump height and the duration of the airborne phase gives then a flight duration of 0.347s, which is very close to the measured and observed flight time of 0.35s. The latter empirical value of flight time would correspond with a flight height of 15 cm. Although the qualitative interpretation of the force-time graphs in relation to the jumper’s motion is considered more relevant for science education
than the numerical evaluation (cf., Cross, 1999b), it is nice when measurements and theoretical results are so consistent.

Figure 3.45: Application of the area tool in COACH to the recorded force-time graph in order to determine the impulse due to the reaction ground force for the squat jump.

The velocity-time graph can be constructed in COACH through application of the integration tool from the data analysis facility to the propulsion force and dividing this function by the jumper’s body weight. Figure 3.46 shows the result for the recorded squat jump. Now the maximum velocity of the body center of mass is close to 1.85 m/s.

Figure 3.46: Application of the integration tool in COACH to the propulsion force-time graph in order to determine the impulse due to the reaction ground force for the squat jump.

The height-time graph can be constructed in COACH through application of the integration tool from the data analysis facility to the velocity. Figure 3.47 shows the result for the recorded squat jump. Now the maximum height of the body center of mass with respect to the initial height is close to 37 cm and the height in the final jumper’s upright position is close to 12 cm. This might mislead to the conclusion that the flight height is expected to be close to 25 cm, which is higher than the height predicted earlier from the take-off velocity. The error made is that the flight height must be determined from computed heights between the instant of take-off and the instant of landing. If one does so, taking the average value of 23.5 cm of the heights at take-off and landing, a flight height of 13.5 cm is found. This value is close to the previously determined flight height.
Chapter 3. Computer Tools for Cross-Disciplinary Work with Real Data

Figure 3.47: Application of the integration tool in Coach to the velocity-time graph in order to determine the vertical height of the body center of mass during the squat jump.

Some of the findings calculated from measurements with the force plate, such as the results about take-off velocity, flight time and jump height, can be verified through video analysis of the squat jump. Figure 3.48 is a screen shot of a video analysis activity in which the position of the hip joint during the squat jump is measured by automated point tracking. The velocity curve is determined by numerical differentiation of the position data. In scanning mode one can estimate the vertical position of the hip joint with respect to the selected frame of reference at the instant of take-off and when the highest position is reached: The values are 79 cm and 93 cm. At both instances, the legs are almost stretched and thus the video analysis leads to a flight height of 14 cm. This is close to values that were determined by other methods. Assuming that the vertical velocity of the hip joint at the instant of take-off is equal to the take-off velocity of the body center of mass would lead to a take-off velocity of 1.75 m/s. Again this value is close to the ones reported earlier on the basis of measurements with a force plate.

Even more interesting is the comparison of the height measurement in the video analysis and the computed height based on the measurements with the force platform: Figure 3.49 illustrates a very good match. It is always nice when empirical results and theoretical results match so well. Students can also learn in this way that it is useful to analyze data with various methods and compare the results. They can also explore other vertical jumps like the countermovement jump or a running jump, and compare the human performance. I only exemplify this for the countermovement jump.

Figure 3.48: Video analysis of the hip joint motion during a squat jump.
3.4. Data Logging, Control, and Video Combined

The Countermovement Jump

In the same manner as for the squat jump one can obtain the propulsion force-time graph, the velocity-time graph, and the height-time graph with the help of the data analysis tools of COACH for a countermovement jump. The results are shown in Figure 3.50. In comparison with the squat jump there is now an extra phase at the beginning of the jump in which the jumper moves from an upright standing position downward by flexing at the knees and hips before vigorous extension of knees and hips again to jump off the force platform. Such a pre-stretch enhances the jump performance. This can also be seen in the velocity-time graph in which the maximum velocity is greater than in the squat jump presented earlier in this paper. Scanning the vertical height-time graph would also reveal a higher flight height (20 cm) than in the squat jump (15 cm). Linthorne (2001) pointed out that the superiority in performance of the countermovement jump lies in the fact that the ground reaction force at the start of the upward phase is already much greater than body weight. The jumper thus performs more muscle work early in the upward phase of the jump than in the squat jump, and so the jumper reaches a higher take-off velocity. Such comparison studies would give students a good impression of how movement scientists think and work.

Figure 3.50: The vertical ground reaction force trace of a countermovement jump and a further analysis of motion data.
3.4.3 Measuring the Pupil Light Reflex

Reference

The Field Experiment
The pupil light reflex is the change of pupil area in response to a change of light. Constriction of the pupil of the eye happens quickly and automatically in response to a step-increase of light intensity. But a pupil cannot instantaneously reach its new size when the level of illumination is suddenly increased: There is a delay in reaction time (200-500ms) and hereafter constriction can be approximated reasonably by an exponential decay function. Hereafter, in case the step-increase was not too large, the pupil re-dilates slowly almost back to its original size. This response is called pupil escape. If, however, the increase in light intensity is very large, the pupil simply constricts without re-dilation, a response named pupil capture. Pupil escape and capture have been successfully modeled via nonlinear delay-differential equations (cf., Bressloff, Wood, & Howarth, 1996).

The pupil light reflex affects both eyes, even if only one eye is stimulated. This fact is used in the following experiment: The right eye of a person is stimulated by an oscillatory light stimulus via a bicycle lamp, which is periodically switched on and off, and the pupil motion is recorded at the same time with a webcam operating at a frame rate of 60 fps. A Coachlab interface connected with the computer is used to control the lamp and to measure the light intensity in the test tube. At the beginning of the experiment, the test tube with the lamp is held in front of the webcam so that, if necessary, the recorded video and the light intensity measurement can be resynchronized by matching the first increase of light intensity with the first time that the bicycle lamp is switched on. Hereafter the lamp is put in front of the right eye and the data collection continues. Figure 3.51 shows the screen shot of the Coach activity.

Note that manual synchronization of the video capturing process and the data logging afterwards can be required because of technological weakness of low cost webcams. The timings visible in the screen shot reveal that this was in fact done. The lower-left window is a visual representation of the Coachlab interface with the connected light sensor and lamp. The upper-right window shows the control program to switch the lamp on and off. The upper-left window contains the recorded video and the lower-left window shows the graph of the measured light intensity during the experiment. All windows are linked: so by comparing the step-increases of the light intensity with the frames in which the pupil constricts, one can find the delay in reaction time. The measured reaction time is 0.25 s.

The recorded video is used to measure the diameter of the left pupil during the second phase of the experiment. Automated point tracking makes this measurement an easy task: Two opposite points at the boundary of the pupil are selected in the starting frame and the coordinates of these points are automatically recorded in subsequent frames. These recordings are used to compute the pupil diameter at any time during the second phase of the experiment. Figure 3.52 is a screen shot of the video analysis activity. The graph of the measured pupil diameter is shown in the right window.
3.5 Video Analysis and Modeling Combined

This diagram also contains the graph of the sinusoidal regression curve that fits the data. The period of the pupil oscillation is 4.3s, which is in agreement with the period of the measured light intensity.

Figure 3.51: Screen shot of a COACH activity that combines data logging, synchronized video capturing and control of the experiment.

Figure 3.52: Screen shot of the video analysis of the pupil light reflex.

3.5 Video Analysis and Modeling Combined

The main focus in my research and development work described in this thesis was ICT-supported quantitative mathematical modeling in the context of mathematics and science education. I explored activities in which secondary school students can develop mathematical models in order to come to grips with natural phenomena and to interpret real data. By real data I mean data collected by the students in experiments.
In this section I consider practical work in which data were collected through video analysis and computer models were developed via the graphical, system dynamics-based modeling tool of Coach 6. I report on seven field experiments, which can be categorized into two groups. The first three field experiments—modeling the motion of a yoyo, a falling shuttlecock, and the decay of beer foam in a glass—were simple experiments that students can do in class and for which they are expected to be able to develop mathematical models that give a fair description of empirical data. In these field experiments, the usability of video analysis and graphical modeling tools was also a point of attention. The last four field experiments—modeling a falling chain (related to bungee jumping), sprinting, bouncing gaits, and the giant circle on the high bar—were more closely connected to the subject of how ICT can contribute to the realization of challenging authentic student research projects. In most of these case studies, motion analysis of sports movements was used to collect and analyze motion data, and computer models were developed with the graphical modeling tool in Coach, which were at a level of sophistication close to that of models created by sports scientists. One could say that I explored in the last four case studies the frontiers of research that students and teachers can do with a computer environment that is primarily designed for mathematics, science, and technology education at primary and secondary school level.

3.5.1 Modeling the Motion of a Yoyo

Reference

Introduction
All secondary school students investigate the motion of an object falling freely under gravity. A challenging extension is obtained when one ties a cord to this object and wraps it around an axle, that is, when one make a kind of yoyo out of it. The constrained coupling of translation and rotation as well as the limited length of the cord make the motion much more interesting. Students can collect experimental data of the yoyo going up and down via video measurement. Mathematics and physics help them to describe and understand the motion. With suitable system dynamics-based modeling software they can create their model(s) and compare with reality.

Video Analysis of a Downward Yoyo Motion
Figure 3.53 is a screen shot of a video analysis activity in which the downward motion of a self-made yoyo captured by a webcam was analyzed and mathematically modeled. Because of the unusual size of this object, most students do not believe their eyes when they look at the yoyo winding up and down slowly. One sees them thinking “How can this yoyo move so slowly and still go all the way up? What trick is behind this?” No trick at all! Mathematics and physics help students describe and understand the motion. One is referred to the original paper for details of the algebraic modeling of the unwinding yoyo based on fundamental physics concepts.

The first step in the investigation was to collect data of the yoyo that was winding down and up again. The position of the point near the rim of the disk and marked by a sticker (P1) was measured in a slightly moving coordinate frame whose origin was
the hand of the teacher holding the end of the cord of the yoyo. Automated point tracking was used to collect coordinates of points of interest: The red boxes show the areas that the software searched in the current frame of the video clip for the origin and for the point marked by the sticker. In the upper-right diagram, the horizontal position and the vertical position of the point P1 were plotted against time. This was combined in the diagram with a sinusoidal fit of the horizontal displacement of the yoyo, due to an unintentional pendular motion of the yoyo, and a quadratic function fit of the vertical position during the first phase in which the yoyo unwinds. These trend curves were used as coordinate functions of a computed point that is displayed in the video clip (P2): It turned out to be close to the position of the axle during the unwinding phase of the yoyo.

![Figure 3.53: Screen shot of a video analysis activity about the motion of a yoyo.](image)

**Experimental Modeling**

The motion of the point P1 near the rim of the disk is a superposition of a translational and rotational motion. I consider it important that secondary school students are exposed during their physics education with several examples of motions that can be best explored by decomposing the movements in two or more components. For an object freely falling under gravity, the acceleration of the object is constant, and thus the distance traveled by a free-falling object is quadratic in elapsed time after release of the object. A first guess could be to check whether the displacement of the center of mass located at the axle of the yoyo: This motivated the quadratic trend curve of the vertical position of the point P1. The rotational motion would then be an accelerated circular motion around the axle and the vertical projection of this motion would be an accelerated cycloid. For this viewpoint it would be wise to plot the vertical position of the point P1 against the square of time. Then the graph can be described as the sum of a straight-line approximation and a sinusoidal approximation of the residue. This
is shown in the lower-left window of Figure 3.53. The lower-right window contains the graph of the calculated orbit of the point P1 while the yoyo unwinds.

A System Dynamics Approach

A physical interpretation of the above description of the unwinding yoyo is as follows: The center of mass moves downward with a constant acceleration that is numerically much less than the acceleration due to gravity for a free falling object. The point P1 rotates around the axle with an angular velocity that increases linearly in time. It is an exercise in trigonometry to compute from the vertical displacement of the axle and its radius how much rotation already has taken place and to determine the coordinates of the point P1. This has been applied in the modeling activity shown in Figure 3.54 below, which was created with the graphical system dynamics-based modeling tool of Coach 6. The diagram in Figure 3.54 shows a phenomenal match between the measured vertical position of the point P1 near the rim of the yoyo and the computed y-value.

Figure 3.54: Screen shot of a modeling activity about the motion of a yoyo.

The meanings of the icons in the graphical model are similar to those of other system dynamics-based software such as Stella (Steed, 1992): There are icons for state variables (levels or stocks), flows, auxiliary variables, constants, and connectors. What is special about the modeling tool of Coach 6 is the possibility of handling discrete events in a rather easy way. In this particular case, one has to update the velocity and acceleration at the following discrete events: Firstly, when the end of the cord is reached and the yoyo starts rotating around the turning point, and secondly when the turn is completed and the yoyo continues winding upward. However, the details have been hidden in the model shown in Figure 3.54: the ‘Events’ box on the left in the graphical model contains the hidden subsystem that takes care of the change of direction from downward to upward motion of the yoyo. The ‘Ycoord’ box on the right contains the subsystem that computes the coordinates of point P1 at any time \( t \). The ‘a0’ box down in the middle contains the subsystem that computes the constant acceleration determined by the physical dimensions of the yoyo, its mass and the acceleration of gravity. All three subsystems are closed in Figure 3.54 to give the simple, but still complete graphical model of the motion of the yoyo. Figure 3.55 shows the model with all subsystems opened; details can be found in the paper.

By creating open or closed subsystems of a model, an instructor can focus on the overall structure of the model and zoom in on any subsystem for more details. In other words, the information can be shown at different levels of complexity. The use of subsystems in graphical modeling is a kind of use of multiple representations to pro-
vide complementary information (cf., the DeFT framework of Ainsworth, 2006): One representation would become too complex to show all information; it seems better to change the representation by a single mouse-click and to instantly alter the information level of the graphical model. This is an example of how principles of multimedia learning, use of multiple representations, and user interface design play an important role in the design and usability of a computer learning environment.

Figure 3.55: Screen shot of the modeling window with open subsystems.

3.5.2 High Speed Video Analysis of a Falling Shuttlecock

Reference

Introduction
Casio Computer Co., Ltd., brought in 2008 high speed video to the consumer level with the release of the EXILIM Pro EX-F1® and the EX-FH20® digital camera. The EX-F1 point-and-shoot camera can shoot up to 60 six-megapixel photos per second and capture movies at up to 1200 frames per second. All this, for a price of about US $1000 at the time of introduction and with an ease of operation that allows high school students to learn operate the camera within ten minutes. The EX-FH20 is a more compact, user-friendlier, and cheaper high speed camera that can still shoot up to 40 photos per second and capture up to 1000 fps. Yearly, new camera models appeared and prices have gone down to about US $250-300. Technological progress has actually led to the development of high speed cameras at such low cost, consumer level (See, Vollmer & Möllmann, 2011a, for a discussion of the technology of high speed cameras).

In this paper I intended to illustrate that with the advent of such high speed video technology at the consumer level, or at least at a level that schools can afford such cameras for use in a science lab, video analysis in education has reached a next stage of effectiveness in understanding science. Students now have the opportunity to work directly with high-quality video data in cases where motion was in the past too quick for recording with a normal digital camera or webcam, for instance, data collection of human and animal locomotion or motion in sports. It allows students to carry out authentic activities in which they record video clips or image sequences of rather fast
motions and use them for a detailed investigation of a real-world phenomenon (cf., Koupil & Vícha, 2011; Mathavan, Jackson, & Parkin, 2009; Vollmer & Möllmann, 2011b). They can work with software tools to measure on movies or image sequences of real phenomena, to analyze collected data, to build and simulate computer models, and to compare results from computer simulations with the obtained video data. In this paper I also wanted to point out that for this kind of practical investigative work it is convenient that the video analysis system which is used by the student provides tools for perspective correction and tracking of points of interest.

As showcase example I chose a vertical fall experiment where the effects of air resistance are important and measurable. This topic is not new in education: In the popular experiment of dropping coffee filters, balls, party balloons, or paper cones (references in the original paper), students investigate the movement of an object released at a certain height and they determine the influence of weight, size and shape of the falling object on its motion. The intended audience of students (age 15-16 yrs.) carries out such experiments as practical classroom work using various data collection techniques. In these experiments, which normally take two lessons, it is almost always stated without support that the drag force acting on the falling objects is approximately proportional to the square of their velocity. Students then use this to explain how a constant terminal velocity is reached.

Many of the objects used in the previously mentioned experiments exhibit a scholastic and artificial character. For a follow-up investigation to be carried out one or two years later, I selected the motion of a sports object, namely, a badminton shuttlecock. It is known from classroom experience that investigating drag forces in a setting of real-life sports constitutes a more interesting challenge to a high school student.

Another reason for choosing this object is that in the past studies have been published which used the vertical fall of a badminton shuttlecock to investigate various models of air resistance: Peastrel, Lynch, and Armenti (1980) measured the times required for a shuttlecock to fall given distances (up to almost 10 meters) and they compared their data with the predictions of several models of air resistance. Their least-squares analysis of distance-time data favored a resistive force quadratic in velocity. In an attempt to conclusively determine whether the drag force on a feather shuttlecock in vertical fall is proportional to the velocity or the velocity squared, McCready (2005) recorded parts of the motion of a shuttlecock (immediately after the start of the vertical fall, after a fall of 1.33 m, and after a fall of 1.88 m) with a high speed camera that could record 250 frames per second and she analyzed her video clips with the VideoPoint video analysis software. She concluded that the resolution of the camera and the fact that she could only record parts of the motion of the shuttlecock instead of a complete trajectory made it impossible to find conclusive evidence of the value or the nature of the resistive force.

Technology at school level has improved to such an extent that a successful aerodynamics study is currently within reach of high school students. This study of the motion of the free falling shuttlecock served as a pilot case of practical investigative work intended for pre-university students. Next to the Casio high speed camera, the Coach 6 computer learning environment for video analysis and graphical modeling was used. The simulation results from the computer model were compared with the video data. This case study was also done to get an idea whether this work is feasible in upper secondary education and how much time it would cost students. Conclusion
was that senior high school students (age 17-18) are likely able to find in reasonable time strong support for the quadratic model of air drag of the shuttlecock from this experiment. The results were in agreement with results of Cooke (1992, 1999, 2002) and of a more recent research study of Chen, Pan, and Chen (2009), which illustrates that current technology contributes to the realization of students’ practical investigative work that is pretty close to the level of experiments carried out by sports scientists. On the basis of classroom experiences for many years with video recording, video analysis, and graphical modeling, and on the basis of this pilot study it seems fair to estimate that students in a pre-university stream can investigate the free fall of a shuttlecock in one afternoon, provided that they are already familiar with the techniques. For novices the estimated amount of time must be doubled.

**Video Analysis and Modeling of a Free Falling Shuttlecock**

In the experiment, a commonly available synthetic badminton shuttlecock was dropped from a height of approximately 4.5 m and its motion was recorded with the Casio EXILIM Pro EX-F1 digital camera at a resolution of 384×512 and a frame rate of 300 fps. No special arrangements such as extra light sources were set up. Image rectification was applied to the recorded movie clip and data for the vertical position of the shuttlecock during the free fall were collected via automated point tracking. The velocity of the shuttlecock was obtained by the built-in, quintic penalized spline-smoothing based algorithm (Ramsay & Silverman, 2005) for computing numerical derivatives. Such advanced algorithms are needed in a computer learning environment if one does not want to be confronted in students’ activities with technical obstacles such as noise dominating data after numerical differentiation with finite-difference formulas. Numerical differentiation will be further discussed in Section 3.5.5.

The velocity-time graph indicates that the falling shuttlecock reached after a short time a constant velocity. In Figure 3.56, this information about the restricted range in which the velocity is fairly constant has been used to perform a straight-line fit of the position-time graph. This method, which was advocated by Gluck (2003), gave a value of 4.75 m/s for the terminal velocity.

![Figure 3.56: Vertical position–time graph (magenta curve) and velocity–time graph (green curve) of the falling shuttlecock. The linear function fit of the position–time graph shows the approach to linear motion after about 0.8 second.](image)

The modeling of the motion of the free falling shuttlecock aimed to examine which of the two models of air resistance, linear or quadratic drag, best describes the collected data. In fact, students must realize that they better model the velocity instead of the
position of the shuttlecock, because Newton’s second law of motion can be considered in this case as a first-order differential equation in the velocity \( v \):

\[
\frac{dv}{dt} = \begin{cases} 
  g \left[ \frac{v}{v_T} - 1 \right] & \text{for linear drag,} \\
  g \left[ \left( \frac{v}{v_T} \right)^2 - 1 \right] & \text{for quadratic drag,}
\end{cases}
\]

\( g \) is the acceleration of gravity, and \( v_T \) is the terminal velocity of the shuttlecock. In both cases, the differential equation can be solved analytically; this even holds for a sum of linear and quadratic drag. This makes it possible to apply nonlinear least-squares curve fitting to the empirical data: for example, modified exponential curve fitting for the linear drag model. The results (best estimate and standard deviation \( \sigma \)) are shown in Table 3.3. Clearly the model based on quadratic drag described the empirical data better than the model with linear drag.

<table>
<thead>
<tr>
<th>model</th>
<th>( v_T ) (m/s)</th>
<th>( \sigma ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>-7.36</td>
<td>0.08</td>
</tr>
<tr>
<td>quadratic</td>
<td>-5.13</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 3.3: Least-squares fit of terminal velocity for linear and quadratic drag force.

But solving the differential equations analytically is beyond the mathematics level of secondary school students. They could use instead a (graphical) modeling tool to numerically solve the differential equations and compare modeling results with empirical results. In the graphical, system dynamics-based modeling tool of Coach 6, the equation of motion for both cases could be represented as in Figure 3.57; only the mathematical formula behind the icon for the drag force would differ for the two cases. This graphical model can be considered as a representation at conceptual level of the system dynamics, where relation arrows indicate dependencies between quantities. Pre-university students seem to quickly grasp this idea.

For both the linear and quadratic drag model, one finds by trial and improvement the best value for the proportionality constant \( k \) so that the computer position and velocity curves visually match best with the empirical data. Figure 3.58 and
3.5. Video Analysis and Modeling Combined

Figure 3.59 show the best simulation results for the position–time graph and the corresponding velocity–time graph for the linear and quadratic drag model, respectively. The experimental quantities of position and velocity are displayed in these diagrams as background graphs. The figures clearly illustrate that a much better match between modeling results and empirical data was obtained for the quadratic drag model than for the linear drag model. The parameter choice of the proportionality constant $k$ in the quadratic drag model corresponded with the drag coefficient $C_d = 0.47$, which is in good agreement with reported literature values (Cooke, 1992; 1999) ranging from 0.48 to 0.53, and led to the terminal velocity of $-5.2 \text{ m/s}$, which was not far away from the velocity found through experimentation.

The most important message coming from this field experiment is in my opinion that, supported by appropriate ICT tools, students can examine a drag model, instead of simply assume or accept a given drag model. It promotes a critical attitude, in which questioning and examining statements is considered normal.

3.5.3 Modeling the Decay of Beer Foam in a Glass

References

**Introduction**
Exponential growth and decay are mathematical models that can be applied to many physical phenomena. Common examples of exponential decay in mathematics and physics textbooks are cooling processes, radioactive decay, and discharge of a capacitor. In this paper I described a fascinating example from real life for secondary school students to investigate with inexpensive means, namely, the collapse of the head of a beer poured into a glass. This example, which illustrates an entire investigation in a Predict-Observe-Explain (POE) instructional approach to practical work, was part of an e-class on continuous dynamical systems for the optional Mathematics D program in the Netherlands. An e-class can be summarized as web supported instruction in a blended learning approach; for more details about the e-class concept, I refer to the descriptions of the e-classes on dynamical systems (Heck, Houwing & de Beurs, 2009; Heck, Houwing, Val, et al., 2009). The e-class about continuous dynamical systems also emphasized doing real experiments regarding phenomena that can be mathematically modeled by differential equations. The experiments had to be simple enough so that students could do them outside school. An example is the collapse of beer foam, which can be modeled by an exponential decay model. When also the liquid content of the glass is taken into account a linear system of differential equations comes into play. All mathematics needed is part of the Dutch pre-university mathematics curriculum.

**Video Analysis of the Head of a Beer**
The experiment is simple and similar to the one described by Leike (2002) and by Hackbarth (2006): a natural beer pour into a glass with height measurements for beer and foam. Figure 3.60 shows a screen shot of a video analysis of the height of the beer (BeerH, defined as the height from the inside bottom of the glass to the beer/foam interface) and of the head of an alcohol-free beer (WetFoamH, defined as the height from the beer/foam interface to the top of the foam). The movie has been recorded within Coach via a webcam capturing at a speed of 1 frame per 4 seconds. It is an example of low speed video technology, which is more popular in fields like astronomy and biology. Some of the beer foam sticks to the side of the glass and this makes it difficult to get accurate measurements of the height of the head of the beer through automated point tracking. Nevertheless, this noisy data set will do for the purpose of the analysis of the experimental results.

![Figure 3.60: Screen shot of a video analysis of the head of an alcohol-free beer.](image-url)
3.5. Video Analysis and Modeling Combined

The exponential decay model of beer foam corresponds with the following ordinary differential equation: $\text{WetFoamH}'(t) = -\frac{1}{\tau} \cdot \text{WetFoamH}(t)$, where $\tau$ is a time constant called the lifetime of the beer foam. In the diagram to the right in Figure 3.60, the best exponential decay curve for the measured head of the beer is shown, using the formula $\text{WetFoamH}(0) \cdot \exp(-t/\tau)$, where $\text{WetFoamH}(0)$ is the initial foam height at time $t = 0$. The graph matches the data well, except at the early stage, and the lifetime in this experiment is 131 seconds with initial foam height equal to 8.7 cm. In connection with this exponential decay, an exponentially decelerating growth curve fits the measured height (BeerH) of the beer. It is expressed by a function of the following type $\text{BeerH}(t) = c - a \cdot \exp(-b \cdot t)$, where the best parameter values were $a = 3.7 \text{ (cm)}$, $b = 0.04 \text{ (L/s)}$, and $c = 7 \text{ (cm)}$.

The processes that play a role in the foam formation and retention are not simple and form the trick of the trade of beer brewing. The key physical events of foam formation, retention and adhesion are as follows (Lewis & Bamforth, 2006): (1) bubble formation; (2) creaming; (3) drainage; (4) disproportionation; and (5) lacing. Dale, West, Eade, et al. (1999) found that a simple exponential law models the first phase of the decay of beer foam mass, which may take about 300 seconds (depending amongst other things on the type and temperature of the beer and the manner of pouring the beer into the glass) and accounts for 85-90% loss of foam mass. But as drainage proceeds, the foam structure changes from spherical to polyhedral bubble shapes and it shrinks. The concentration of polypeptide material in the foam increases and the foam coarsens through disproportionation (i.e., diffusion of gas from a small bubble to a larger adjacent bubble). This consolidation phase can also be modeled by exponential decay (Dale et al., 1999). Thus, the beer foam collapse is usually better modeled through a bi-exponential model that consists of a fast and slow decay. Regression analysis on the basis of the least squares method of peeling-off functions (Foss, 1969) can be used successfully for finding suitable parameter values for the bi-exponential model (Heck, 2009b).

Refinement of the exponential model of the decay of the head of a beer will only be necessary when one wants to put the dots on the i’s and add step-by-step more details to the model with the purpose of getting a better match between model and reality. This progressive aspect is common in quantitative mathematical modeling: First one simplifies the situation to such an extent that a simple model can be constructed. Hereafter one evaluates this model, preferably by comparing it with experimental data, and one adapts it if necessary. Adaptation of the model normally means that one makes the model more complicated by taking more factors, which cannot really be neglected, into account or by undoing some earlier simplifications. In my opinion, students should get ample opportunity to experience through investigative work the nature of modeling. I strongly believe that by looking at various models of one and the same phenomenon a critical attitude of students is promoted. I illustrate this through further modeling of beer foam collapse.

The beer model can be improved by separating the wet part of the beer foam from the dry part of the foam, and by distinguishing between the contribution of drainage and condensation to the beer foam collapse. I follow Hackbarth (2006), who introduced dry foam height (DryFoamH) by $\text{DryFoamH} = \text{WetFoamH} - (\text{FinalBeerH} - \text{BeerH})$, where FinalBeerH is the height of the beer in the glass after the head of the beer has disappeared (assuming that one does not take a drink). Hackbarth (2006) showed that
the dry foam height can be mathematically modeled well through exponential decay over the whole time period. The diagram in Figure 3.61 illustrates that this also held for the recorded pour and decay of the beer in the experiment. The lifetime of dry foam was 153 seconds, which is as expected more than the lifetime found for wet foam. The initial dry foam height was 6.7 cm. In combination with the bi-exponential model of the decay of the wet part of the beer foam, I obtained a tri-exponential model for the decay of the head of the beer that described the measured decay in the experiment very well (Heck, 2009b).

Besides the above experimental modeling, one can also model all parts of the decay process via differential equations and numerically solve these equations with a computer modeling tool. I used the graphical, system dynamics-based modeling tool of Coach 6 for this purpose. It also illustrates an example of the use of the extension of the graphical system dynamics-based modeling discussed in Section 3.3.1, in which chemical kinetics was modeled graphically. Below I discuss two models used for modeling the empirical data obtained in the experiment.

In the first model, the exponential decay of beer foam and the increase in height of the liquid beer are combined into the following system of equations:

\[
\frac{d}{dt} \text{WetFoamH}(t) = -\text{Collapse}(t), \quad \frac{d}{dt} \text{BeerH}(t) = f \cdot \text{Collapse}(t),
\]

\[
\text{Collapse}(t) = \frac{1}{\tau} \cdot \text{WetFoamH}(t),
\]

where \(f\) is the conversion factor that relates an amount of beer foam to an equivalent amount of liquid beer. Figure 3.62 is a screen shot of the corresponding graphical model, in which the Erlenmeyer flask icon links the outflow of the foam height and the inflow of the beer height. The diagram were obtained for parameter values \(\tau = 100\) (s), \(f = 0.15\), WetFoamH(0) = 10 (cm), BeerH(0) = 5.65 (cm).

A better match between experimental data and results from a theoretical model can be obtained by separating wet and dry foam height, just like Hackbarth (2006) did, and by assuming a nonconstant life time of the drainage. The differential equations of the second model are:
3.5. Video Analysis and Modeling Combined

Figure 3.62: Screen shot of a coupled model of foam height and beer height.

\[
\frac{d}{dt}\text{DryFoamH}(t) = -\text{DryDecay}(t), \quad \frac{d}{dt}\text{WetFoamH}(t) = -\text{WetDrainage}(t),
\]

\[
\frac{d}{dt}\text{BeerH}(t) = f \cdot \text{DryDecay}(t) + \text{WetDrainage}(t)
\]

\[
\text{DryDecay}(t) = \frac{1}{\tau_{\text{dry}}} \cdot \text{DryFoamH}(t), \quad \text{WetDrainage}(t) = \frac{1}{\tau_{\text{wet}}} \cdot \text{WetFoamH}(t).
\]

Figure 3.63 is a screen shot of the corresponding graphical model, in which the following sigmoid function was chosen after trial and improvement for the lifetime of the drainage process:

\[
\tau_{\text{wet}} = 0.1 + \frac{99.9}{1 + 1.2e^{(2 - 0.05t)}}.
\]

A very good match between the computed model results and the empirical data was obtained. Figure 3.63 hardly leaves space for doubt. This makes the increase of the liquid beer height also mathematically well modeled.

Figure 3.63: Screen shot of a coupled model of foam height and beer height with distinguished dry and wet parts of the foam, and with a time-dependent lifetime of the drainage of the beer foam.

### 3.5.4 Understanding the Motion of a Falling Chain

**Reference**


**Introduction**

Changing mass phenomena like the motion of a falling chain, the behavior of a falling elastic bar or spring, and the motion of a bungee jumper surprise many a physicist. In this paper, an analogue of the first phase of bungee jumping, when the bungee
jumper falls down, but the bungee rope is still slack was discussed, namely, the fall of an object connected to a hanging chain. In instructional materials, this phase is often considered a free fall, but when the mass of the cord is taken into account, the object reaches acceleration greater than \( g \). This result is contrary to the usual experience with free falling objects and therefore hard to believe by many a person, even by an experienced physicist. But experiments do reveal the truth and students can do this supported by ICT tools. In this paper, colleagues and I reported on an authentic research project done by secondary school students and used their work to discuss how measurements with sensors, video analysis of self-recorded high speed video clips, and computer modeling allowed studying physics of bungee jumping and the motion of a falling object connected to a hanging chain.

**Authentic Inquiry by Secondary School Students**

The bungee jumping project of Niek Dubbelaar and Remco Brantjes in 2003, at that time secondary school students at the Bonhoeffercollege in Castricum, can be considered as an example of authentic inquiry by students according to the characteristics present in Section 1.1 (p. 10). The work of the student researchers resembled the work of scientists and practitioners regarding affective and cognitive processes in authentic inquiry. To begin with, there were: (1) the students’ intrinsic motivation to select the subject of their investigation; and (2) the generation of their own research question(s).

The students wrote in their report that they teamed up to investigate the physics of bungee jumping, triggered by their own interest and the article of Menz (1993) on www.bungee.com. In particular, they were intrigued by the alleged greater-than-\( g \) acceleration of a bungee jumper. The students formulated the following research question: “How large is the acceleration in a bungee jump and to what degree is this acceleration influenced by the relative mass of the rope and the jumper?”. They hypothesized that the acceleration would be greater than \( g \) and that this effect would be more dramatic if the rope was relatively heavy as compared with the jumper.

In order to find quantitative support for their hypothesis, the students designed an experiment in which they recorded with a webcam the motion of a dropped scale model (an Action Man toy figure) and of dropped wooden blocks of various weight attached to ropes of various stiffness, and they collected position-time data through video measurements. The students realized that working with a scale model of a bungee jumper or a wooden block is not the same as investigating real bungee jumping, but that it would provide them with enough information on what happens in reality and could lead to a good understanding. This indicates a research design in which procedures to address a problem are determined, variables to investigate are selected, control of variables is thought of, and measurements are planned. Note that not the measured position is the variable of interest, but a derived quantity, namely, acceleration. The students had learned in previous video analysis activities that adequate numerical derivatives could be obtained from measured position data.

After some trials, the students realized that the stiffness of the rope plays only a minor role and that the mass ratio between rope and objects needs to be large enough to notice an outstanding result. Therefore they repeated the experiment with objects of larger mass ratio. This is another aspect of authentic inquiry: Researchers are responsible for detecting flaws in their experimental set-up and must decide how to adjust their original plans. In this case, the students decided to change the mass ratio and to concentrate on the moment when the object has fallen a distance equal to the
rest length of the elastic (because they had observed that the acceleration is greatest at this point of the motion). They collected more data and determined the graph of the acceleration at this moment as a function of the mass ratio of elastic and block. It is shown in Figure 3.64, together with the graph of the following theoretical result from (Kagan & Kott, 1996): \[ a = g \left( 1 + \frac{\mu(4 + \mu)}{8} \right) \]
where \( \mu \) is the mass ratio of the elastic and the wooden block. The students noted that these graphs were alike, with the theoretical values just a bit higher. They attributed the difference mainly to the development of heat during the motion. Again, this indicates a behavior of the students that resembles the attitude of competent researchers, namely, the habit of comparing own results with work of others and trying to explain differences by scientific reasoning.

![Figure 3.64: Graphical display of experimental results and fit (red) and computed values (purple).](image)

The student researchers not only investigated the acceleration of a bungee jumper by quantitative methods, but also tried to explain the greater-than-\( g \) acceleration both on qualitative, theory-driven grounds, as well as through a more detailed mathematical model of the phenomenon. The students followed the approach of Kagan and Kott (1996), in which it was noted that here one does not deal with a falling rigid body, but instead with a system consisting of the falling object and the rope. The main point here is that the students’ work contained clear signs of their understanding that observation, experiment, and theory are related tools for scientific thinking.

Not knowing that a Dutch secondary school physics teacher had written around the same time about an experimental verification of the physics of bungee jumping (Biezeveld, 2003), the students wrote an article about their work that was published in the Journal of the Dutch Physics Society (Dubbelaar & Brantjes, 2003). It triggered quite a number of reactions in the journal and for almost a year on Internet. There were complaints about the quality of physics teaching in the Netherlands, arguing that obviously(!) \( a \leq g \) and that the students’ work proved that the level of physics education in the Netherlands had decreased in the last decades. The editorial commentary was subtle, but to the point: “The students who wrote the paper may consider it a compliment that scepticism overcame professional physicists and physics teachers. That’s how (or maybe it is just the point that) experienced intuition can be wrong.” In other words, the students’ work revealed that in practice, scientists are not as rationally, non-algorithmic thinking and having an open mind as commonly pictured in literature about science and scientists. In the same issue of the journal, Pasveer and De Muynck (2003) agreed with the findings of the students and they explained that
physics intuition is easily fooled, as everyone is taught the Galilean paradigm of the
motion of constant masses, according to which every acceleration must be produced by
a force. A launched rocket and a falling chain or slinky are important counterexamples
to this line of thought. Actually, believing the statement \( a > g \) means giving up or
generalizing the law \( F = ma \).

**Video Analysis and Computer Modeling**
The students’ work led to an in-service training module on bungee jumping, which
was developed in the framework of the European project Information Technology for
Understanding Science. All teaching and learning activities, which can be downloaded
from the project’s website (www.itforus.oeiizk.waw.pl), are based on the use COACH for
data logging, video analysis, and for computer modeling, simulation, and animation.
With the advent of high speed video technology it has become possible to experimen-
tally verify a theoretical model of the fall of an object connected to a hanging chain.
The experiment is simple: Two identical wooden blocks are dropped at the same time
from a height of a couple of meters. One block is in free fall and the other block is
chained. The chained block touches the ground earlier than the block that is in free
fall, which can be observed with the naked eye and can be recorded with a common
camcorder. This implies that the chained block must have acceleration greater than
the acceleration of free fall (assuming that air drag can be ignored in the experiment).
The motion of the blocks was recorded with a high speed camera at a speed of 300 fps
(a video clip is available at http://stacks.iop.org/physed/45/63/mmedia). The veri-
tical position of the blocks can be automatically measured in the video analysis tool of
COACH via automated point tracking. Manual data collection would be too time con-
suming. Figure 3.65 shows the graphs of the measured distances of the blocks, relative
to the points where they were released (i.e., a coordinate system has been selected
with a positive vertical coordinate in the downward direction), and the velocity-time
graphs of the blocks. These graphs have been obtained with a numerical differenti-
ation algorithm that is based on a penalized quintic spline smoothing technique (for
details about numerical differentiation algorithms in COACH, the reader is referred to
Section 3.5.5). The blue velocity-time graph, which is almost a straight line, belongs
to the free falling block. The red graphs, where the cross hairs in scanning mode meet,
belong to the chained block that has already traveled at the selected moment a greater
distance than the free falling object.

![Figure 3.65: Video analysis of two dropped blocks. The red position and velocity-time graphs relate to the chained block and the blue curves belong to the free falling block.](image)

Kagan and Kott (1996) based their theoretical modeling of a falling object con-
nected to a hanging chain on the conservation law of energy. Pasveer and De Muynck
(2003) applied the following equation of motion: \( \sum F = p' \), where the left-hand side is the sum of forces \( F \) acting on the object and the right-hand side is the derivative of the momentum \( p \) of the moving object. However, they did not reproduce the main result of Kagan and Kott (1996), namely, the exact formula of the acceleration of the object during the fall. Colleagues and I resolved this in the presented paper. However, the focus is here on the computer modeling based on the analysis of the phenomenon based on mathematics and physics. Then it suffices to realize that one does not deal with a falling rigid body, but instead with an object of changing mass. Therefore, the traditional form of Newton's second law \( F = ma \) is not suitable here and should be replaced by the following generalized form: \( \sum F = p'_{\text{obj}} = m'_{\text{obj}}v_{\text{obj}} + m_{\text{obj}}a_{\text{obj}} \), where \( m_{\text{obj}}, v_{\text{obj}}, a_{\text{obj}}, \) and \( p_{\text{obj}} \) represent the mass of the object (changing in time), the velocity, acceleration, and momentum of the object, respectively, and \( F \) represents a force acting on the object. The most interesting object is in this case the wooden block together with its attached chain. The sketch of the experimental setting shown in Figure 3.66 illustrates that the moving part on the right-hand side diminishes during the fall because part of the chain "moves" to the left-hand side. This implies \( m'_{\text{obj}} < 0 \). Because \( \sum F = m_{\text{obj}}g \), when only gravitational force is taken into account, and \( v > 0 \) in the direction of motion, \( a > g \) must hold!

Figure 3.66: Sketch of the block of mass \( M \) attached to a chain of length \( L \) that has already fallen a distance \( y \) and is traveling at a speed \( v \) and acceleration \( a \).

Let \( m \) denote the mass of the chain, then the moving mass \( m_{\text{obj}} \) of the block plus the chain on the right-hand side in the sketch shown in Figure 3.66 can be expressed in mathematical terms as: \( m_{\text{obj}} = M + \frac{1}{2}(L-y)m/L \). Differentiation leads to the following equation: \( m'_{\text{obj}} = \frac{1}{2}mv_{\text{obj}}/L \). A complication is the question what to take here as velocity of the object consisting of a wooden block and part of the chain. \( v_{\text{obj}} \) denotes the velocity with which the mass leaves the moving system. In this case, this velocity therefore almost instantaneously decreases from \( v \) to 0 and is taken to be the speed of the bend. Taking the viewpoint of Biezeveld (2003), the free side of the bend falls with speed \( v \), the fixed side of the bend hangs still, and the bend, where links of the chain in motion come to rest, moves at speed \( u = \frac{1}{2}v \). Thus: \( v_{\text{obj}} = \frac{1}{2}v \), where \( v \) is the velocity of the wooden block. This motivates the choice of the formula \( g + \frac{1}{2}m'_{\text{obj}}v/m_{\text{obj}} \) behind the inflow \( a \) in the graphical, system dynamics-based model shown in Figure 3.67. In this figure are also shown the position and velocity-time graphs of a simulation run and the graph of the ratio \( a/g \), which increases while the chained block is falling. Parameter values have been chosen such that the model-based graphs for the chained block are in good agreement with the graphs obtained through measurements. Prediction and measurement match very well: The time that
the chained block needed in theory to reach its lowest position for the given masses and chain length is equal to the measured time and to the time found in a simulation run within an error margin of 1%!

The computer model was also used to create an animation of the motion of the chained and free falling block. The tool windows on the right-hand side of Figure 3.67 are a slider and an animation window that displays the simulation results as animations where model variables are presented as animated graphics objects. A student can interact with the model and the animation through a slider bar, that is, select the value of the mass of the chain before the start of the simulation and also during the model run. Animation allows students to concentrate on understanding a phenomenon with the help of simulations before going into the details of how the simulations have been implemented by means of computer models.

Figure 3.67: Screen shot of a COACH activity in which a graphical model implements the motion of (1) a chained block; and (2) a free falling block. The position and velocity-time graphs of a simulation run have been plotted. Parameter values have been chosen such that the calculated plots for the chained block match well with the measured data shown as background point plots. The graphical model is connected with a slider and animation window.

3.5.5 Models of Sprinting

References
3.5. Video Analysis and Modeling Combined

Introduction
A biomechanical study of sprinting is an interesting task for students who have a background in mechanics and calculus. These students can work with real data and do practical investigations similar to the way sports scientists do research. Student research activities are viable when the students are familiar with tools to collect and work with data from sensors and video recordings, and with modeling tools for comparing simulation and experimental results. The cited papers describe a multi-purpose system, named COACH, that offers students and teachers a versatile integrated set of tools for learning, doing, and teaching mathematics and science in a computer-based inquiry approach. Automated tracking of reference points and correction of perspective distortion in videos, state-of-the-art algorithms for data smoothing and numerical differentiation, and graphical system dynamics based modeling, are some of the built-in techniques that are suitable for motion analysis. Their implementation and their application in student activities involving models of running are discussed. In this section I only focus on: (1) data smoothing and numerical differentiation of noisy data, because these methods play an important role in motion analysis; (2) mathematical models of running, illustrating that different types of models exist that describe empirical data quite well; and (3) a field experiment with secondary school students.

Data Smoothing and Numerical Differentiation
Derived quantities such as speed and acceleration are often needed in motion analysis. The associated numerical differentiation methods must be sophisticated enough to deal with noise in the signal. Otherwise, numerical derivatives might become noisy and useless, especially when divided difference formulas for the first and second derivative are applied (See Figure 3.68). Popular data smoothing techniques such as moving averages, Savitsky-Golay filtering, and spline smoothing can reduce the noise in a signal. Most of these methods are available in COACH, but special attention has been paid to the the spline smoothing technique of equidistant data. COACH provides a generalized cross-validatory penalized quintic spline smoothing technique. The method is outlined in the following; details can be found in (Green & Silverman, 1994; Heckman & Ramsay, 2000; Ramsay & Silverman, 2005).

Given a set of equidistant abscissae \( t_1, \ldots, t_n \) with corresponding ordinates \( y_1, \ldots, y_n \), a penalized spline of order \( 2m \) (with \( 2m \leq n \)) is a piecewise polynomial in \( \hat{y} \) of degree \( 2m - 1 \) that minimizes the penalized residual sum of squares

\[
C_\lambda(t, y) = \sum_{i=1}^{n} |\hat{y}(t_i) - y(t_i)|^2 + \lambda \int_{t_1}^{t_n} (\hat{y}^{(m)}(t))^2 dt,
\]

where \( \lambda \) is a suitable penalization or smoothing parameter. The first \( 2m - 2 \) derivatives of each local polynomial of the spline are continuous at each value of \( t_i \). A remarkable theorem (de Boor, 2001) is that the spline function that minimizes \( C_\lambda(t, y) \) is a natural spline, that is, the derivatives at the corners \( \hat{y}^{(j)}(t_1) \) and \( \hat{y}^{(j)}(t_n) \) are zero for \( j = m, \ldots, 2(m - 1) \). The first term in the above expression is the usual sum of squared errors that is used for fitting data by a spline. The second term, the integrated square of the \( m \)th order derivative, is a measure of the roughness of the data. The parameter \( \lambda \) is the smoothing parameter that controls the trade-off between fitting the data by minimizing the residual sum of squares and minimizing the roughness of the approximation. For \( \lambda = 0 \) only fitting of the data matters, so an interpolating spline
through the data points will be obtained. As $\lambda$ increases, more emphasis is placed on penalizing roughness. When $\lambda \to \infty$, only roughness matters and the standard least squares fit of the data using a single polynomial of degree $m - 1$ will be obtained.

There exist data-driven methods for automatic selection of the parameter $\lambda$. The generalized cross-validation criterion developed by Craven and Wahba (1979) has been implemented in COACH. Although the generalized cross-validation method is reliable, it offers only a reasonable smoothing parameter for exploratory work. It inevitably involves judgment to select a suitable value of $\lambda$. For this reason a user can set the value of $\lambda$.

The most common choice of the order $m$ in the penalized spline smoothing is $m = 2$, which results in a cubic smoothing spline. This was already implemented in COACH 5, following the approach of Reinsch (1967). But it has as disadvantage that second derivatives are by definition zero at the corners of the data set. This is an unwanted side effect in many a physics context: Acceleration is usually not approaching zero at the beginning or the end of the movement of an object. Such erratic behavior in numerical differentiation is referred to as endpoint error. In general, one is advised to let $m$ be equal to two plus the highest order derivative that is required so that one can control the curvature of the highest order derivative. In a physics context one often wants the second derivative or acceleration function and this would mean an advised choice of $m = 4$. In COACH 6, the choice became $m = 3$, that is, penalized quintic spline smoothing has been implemented to reduce the endpoint error. Other reasons to reconsider the choice of the order $m$ were that biomechanical studies (for example, Vint & Hinrichs, 1996; Walker, 1998) support the choice of $m = 3$ as being most suitable for obtaining the best possible numerical derivatives in motion analysis, and that infinite smoothing leads to a quadratic regression curve with which students are familiar. Figure 3.68 illustrates the superiority of this differentiation method, when applied to the drop of a golf ball (Vaughan, 1982), with the smoothing parameter $\lambda = 4.9 \times 10^{-6}$ automatically obtained via the GCV criterion, in comparison with the three-point difference method. The data come from the Biomechanical Data Resources provided by the International Society of Biomechanics at www.isbweb.org/data/
Another comparison of numerical differentiation techniques uses the modification by Lanshammar (1982) of the raw angular displacement data set of (Pezzack, Norman, & Winter, 1977). This data set is used in the biomechanics community for benchmark purposes. The screen shot in Figure 3.69 shows the graphs of the digitized reproduction of the analog acceleration data, the acceleration computed through a three-point difference formula, and the acceleration computed through penalized quintic spline smoothing with the smoothing parameter $\lambda = 3.9 \times 10^{-8}$ automatically obtained via the GCV criterion. It is obvious which of the differentiation methods is best.

**Figure 3.69: Acceleration curves for biomechanical benchmark data.**

### Mathematical Modeling of Running

Two types of models of running are discussed: (1) kinematic models based on Newton’s second law of motion; and (2) kinetic models based on an energy balance.

**The Kinematic Approach**

All models of this kind are based on $a(t) = F_{\text{propulsive}}(t) - F_{\text{resistive}}(t)$, where $a(t)$ is the acceleration of the runner at time $t$, $F_{\text{propulsive}}$ is the horizontal component of the propulsive force per unit mass (the normalized force at time $t$ that the runner to be in motion), and $F_{\text{resistive}}$ is the force per unit mass that the runner has to overcome at time $t$. I adopt the common approach in biomechanics and normalize quantities to body mass. Keller (1973, 1974) assumed in his model of competitive running that the propulsive term is constant for sprinting and that the dominant resistive effects from running mainly result from frictional losses within the body of the runner and can be modeled by a term linear in speed. This model, in which air friction is neglected, can be written for a runner who is initially at rest as $v'(t) = F - v/\tau$, $v(0) = 0$, where $F$ is the constant force per unit mass that the runner exerts in the horizontal direction, $v$ is the velocity of the runner, and $\tau$ is a time constant. This initial value problem can be solved analytically: $v(t) = F\tau(1 - e^{-t/\tau})$. The product $F\tau$ equals the maximum speed of the sprinter. The distance $D(t)$ covered in time $t$ is found by integrating the solution: $D(t) = F\tau^2(t/\tau + e^{-t/\tau} - 1)$. For small values of $t$, $e^{-t/\tau} \approx 1 - t/\tau + \frac{1}{2}(t/\tau)^2$ and this leads to the following approximate formula valid at the start of a sprint: $D(t) \approx \frac{1}{2}F\tau^2$. Using this approximation, a good estimate of the parameter $F$ can be obtained from a plot of the distance covered against time for the start of the sprint. In combination with the maximum speed $F\tau$, this value provides an estimate of the parameter $\tau$. These estimates of $F$ and $\tau$ can be used as initial parameter values in
an iterative regression method for the distance-time data, using the exact solution of the initial value problem in $v(t)$ as a regression function. Parameter values found in this manner can be compared with the values obtained by linear regression in an acceleration-speed data plot, or by cubic polynomial regression with the following alternative approximate relation valid at the start of a sprint: 

$$D(t) \approx \frac{F(t)}{2} + \frac{F(t)}{6}. $$

It is interesting and instructive to discuss with students which regression method is most reliable. It illustrates that mathematical formulas are not just a hobby of the mathematics or physics teacher, but that they play in these models an important role in the determination of good estimates for the model parameters.

Tibshirani (1997) removed the unrealistic assumption that a sprinter applies a constant maximum force for the duration of a race. He chose a linear decrease of the propulsive force in time. The equation of motion is: 

$$v'(t) = F(t) - ct - v(t)/τ, $$

for some constant $c$. This model can also be solved analytically: 

$$v(t) = ke^{-t/τ} - ct + k, $$

where $k = Fτ + cr^2$. In the Tibshirani model of sprinting, mathematical formulas again play an important role in the determination of good estimates for the model parameters.

In this case study secondary school students used the graphical, system dynamics-based modeling tool of Coach 6 in a sports motion context to create representations of quantities and relationships where analytic solutions of equations were too difficult with respect to the students’ mathematics abilities or simply impossible. The need for computer models is quickly felt when one wants to take several factors into account. For example, the effect of a head- or tailwind and air resistance is a drag force whose empirical form is given by (Frohlich, 1985) 

$$F_{\text{drag}} = \frac{1}{2} \rho C_d A (v - w)^2, $$

where $w$ represents the wind speed (positive and negative values indicate tailwind and headwind, respectively), $\rho$ is the air density, $C_d$ is the drag coefficient, and $A$ is the cross-sectional area of the sprinter. This effect on sprint times and other effects such as altitude, air pressure, temperature, humidity, curvature of lanes at the sprint track can also be easily incorporated in computer models.

Figure 3.70 is a screen shot of the analysis of the sprint of Carl Lewis in the 100 m final at the 1987 IAAF World Championships in Athletics in Rome using the wind-adapted Keller model. The graphical model in the upper left corner of the screen shot is a visual representation of the following initial value problem $D'(t) = v(t)$, 

$$v'(t) = F_{\text{propulsive}}(t) - F_{\text{resistive}}(t) - F_{\text{drag}}(t), D(0) = v(0) = 0, $$

where $F_{\text{propulsive}}(t) = F$, $F_{\text{resistive}}(t) = v(t)/τ$, and $F_{\text{drag}}(t) = c(v(t) - w)^2$. It is assumed that the wind speed $w$ was constant during the run. The race was run with a +0.95 m/s tailwind. The table in the upper-right corner lists the split times of the sprint of Carl Lewis (Moravec, Ruzicka, Susanka, et al., 1988). The time was corrected for the relatively slow reaction time of 0.196 s at the start. Speed and acceleration data, which are shown in the diagrams as data points, were computed numerically from the split times through generalized cross-validatory penalized quintic spline smoothing. They are referred to as the measured speed and acceleration. Figure 3.70 shows the computed curves that match the measured quantities. The values of the model parameters are $F = 9.2$ N/kg, $τ = 1.343 s$, and $c = 0.00375 m^{-1}$.

The Kinetic Approach

All models of this kind are based on an energy balance. Many forms of energy besides mechanical energy play a role in human motion, including elastic energy, thermal energy, and chemical energy. In muscles, chemical energy is transformed into mechanical
energy. Mechanical energy degrades partly to thermal energy for concentric muscle contraction (when the muscle is shortened). It can also be stored in tendons and muscles as elastic energy when there is eccentric muscle contraction (the muscle is extended). A power balance model based on a supply-demand approach describes the energetics of many human motions, including running and other endurance sports (cf., Van Ingen Schenau & Cavanagh, 1990). The sum of all rates of flow of energy into and out of the human body equals the rate of change of energy of the human body. The power balance method, which relates power production and power dissipation, can be written as

$$P = P_f + E'(t) + H'(t),$$

where $P$ represents the power which in principle might be used to perform movements, $P_f$ is the power loss due to friction, $E'(t)$ is the rate of change of the external mechanical energy (kinetic, rotational, and potential energy of the body), and $H'(t)$ is the power loss due to degradation of mechanical energy into thermal energy. The power production $P$ is usually the sum of the power production by the aerobic ($P_{\text{aerobic}}$) and anaerobic ($P_{\text{anaerobic}}$) energy production systems:

$$P = P_{\text{aerobic}} + P_{\text{anaerobic}}.$$  

Ward-Smith (1985) assumed that $P_{\text{aerobic}} = R(1 - e^{-\lambda t})$ and $P_{\text{anaerobic}} = P_{\text{max}} e^{-\lambda t}$, with $R$ the maximum aerobic power, $P_{\text{max}}$ the maximum anaerobic power at $t = 0$, and $\lambda$ a constant. In addition, he modeled the thermal power as $H'(t) = \alpha v$, for a constant $\alpha$. The friction term and the rate of change of the external mechanical energy were given in this model by:

$$P_f = F_{\text{drag}} v = \frac{1}{2} \rho C_d A v (v - w)^2$$

and $E'(t) = \frac{1}{2} v^2 t = v \cdot v'$. All equations together lead to:

$$\frac{dv}{dt} = \frac{1}{v} \left[ R(1 - e^{-\lambda t}) + P_{\text{max}} e^{-\lambda t} - \alpha v - Kv(v - w)^2 \right], \quad v(0) = 0,$$

where $K = \frac{1}{2} \rho C_d A$. There is only one problem with this differential equation: The initial velocity $v(0) = 0$ gives an infinite acceleration. Therefore it is better rewritten...
as an initial value problem for the kinetic energy:

$$\frac{d}{dt} \left( \frac{1}{2} v^2 \right) = R(1 - e^{-\lambda t}) + P_{\text{max}} e^{-\lambda t} - \alpha v - K v(v - w)^2, \quad v(0) = 0.$$  

The 1987 sprint of Carl Lewis was also analyzed with the Ward-Smith model expressed in the above differential equation. The parameters for the simulation shown in Fig. 3.71 were $R = 25 \text{ W/kg}$, $\lambda = 0.03 \text{ s}^{-1}$, $P_{\text{max}} = 48.5 \text{ W/kg}$, $\alpha = 3.5 \text{ J kg}^{-1} \text{ m}^{-1}$, $K = 0.002 \text{ m}^{-1}$, and $w = +0.95 \text{ m/s}$. Although the two models of sprinting are qualitatively different, they lead both to results that are in good agreement with measured data. Sports scientists apparently make their choice of models on other grounds.

Figure 3.71: Simulation of the Ward-Smith model for the sprint of Carl Lewis at the 1987 World Championships in Athletics. A comparison of theory and measurement.

Kinematic and kinetic models of sprinting can be used to discuss hypothetical questions such as “How fast could Usain Bolt have run in the 100 m final at the Beijing Olympics 2008?” A recent analysis of Eriksen and colleagues (2009), which was based on data smoothing, numerical differentiation, and extrapolation, reported that a race time somewhere between 9.55 and 9.61 s could have been realized. The wind-adapted Keller model, with parameter values chosen such that the measured split times for the first 80 m are in agreement with the computed split time, leads to a similar race time estimation of 9.60 s. This model accurately predicts or apparently affirms the world record of Usain Bolt at the 2009 World Championships in Athletics in Berlin (9.58 s). Helene and Yamashita (2010) used the Tibshirani model of sprinting to compute biomechanical quantities such as maximum force, maximum power, and the total mechanical energy produced by Bolt in both races. To their own surprise, they came to the conclusion that all of these values were smaller in 2009 than in 2008. Although the world record was broken in Berlin, Bolt’s physical performance was not better than a year before in Beijing.

The last paragraph clearly illustrates the purpose of quantitative mathematical modeling: to describe and predict data, and to explain or contribute to understanding of a phenomenon. In my opinion, pre-university students should get ample opportunity to work in this way with mathematical models. Theory and experiments are expected to supplement each other in the student activities. I take the view that modeling is not just the understanding of the (computer) model with the hope and expectation that nothing went wrong during the theoretical work; it includes understanding of the underlying physics principles and of the assumptions made in the modeling process, as well as validation of the model on the basis of experiments. The latter point is in my
opinion essential in good mathematics and science education. The words of the Nobel Prize winner Martinus Veltman (cited by Mols, 2003)—“If one removes experiments, physics becomes religion. Then the facts do not count anymore, but the opinions of someone who was appointed pope”—also hold for science education.

**Student Investigations**

Although it is interesting to study sprinting models through performances of top-class athletes, there is nothing more compelling than for students to investigate their own running performances. This work can be realized in various ways. In the field experiment, videos of sprints were taken during a one-day visit of upper secondary school students to the Faculty of Human Movement Science of the VU University Amsterdam and to the University Sports Center of the University of Amsterdam. The purpose of this visit was to give the students an impression of what movement scientists do and to introduce them into the field of exercise physiology and measurement of energy capacity. Some students had the opportunity to measure their oxygen consumption in a bicycle ergometer test in the laboratory. At the sports center, the main focus was on research methods to collect motion data. Students measured their heart rates during running at various intensity levels, and assessed their anaerobic capacity via the Wingate cycle ergometer test (cf., Inbar, Bar-Or, & Skinner, 1996), in which they pedaled at maximal speed for 30s against a high braking force. The students also used a high speed camera to record a soccer kick and the take-off phase of a sprint, and used a webcam to record 25 m sprints of other students. Back at school they analyzed their videos and compared the measured and derived physical quantities with simulation results from some of the models discussed before.

Figure 3.72 shows a screen shot of a video analysis of a 25 m sprint of a 16 year old female student running near the wall of the sports hall. The upper-left corner of the screen shot immediately reveals a problem when one wants to measure the position of the sprinter in the recorded video clip: perspective distortion to the experimental setting. A front view of the plane of motion is actually needed. Image rectification (See Section 3.2.2) is more or less possible by using the wall as rectangular object in the plane of motion. The result is shown in the upper-right corner of Figure 3.72. The video was also flipped horizontally to get a motion from left to right, which matches a traditional coordinate system with a positive axis pointing to the right. Because the sprint took about 5s and the frame rate of the recorded video was 30 fps, manual data collection was too time consuming, error prone, and tedious, and therefore the data collection was done by automated tracking of the sprinter. The result in the lower-left window in Figure 3.72 shows that this method is applicable even under non-optimal conditions but might lead to noisy data. In this case, the penalized spline smoothing method, which was outlined before, helped to obtain a suitable speed-time curve (by manual selection of an appropriate penalization). The speed-time data plot in the lower-right window in Figure 3.72 was approximated by an exponential regression curve according to the Keller model with a nonzero initial velocity: \( v(t) = 6.38(1 - e^{-0.862t}) + 1.25 \). The time constant \( \tau \) was about 1.2s, the initial speed 1.25 m/s, and the maximum speed 7.6 m/s. These values correspond to a horizontal propulsive force \( F \) per unit mass of 6.1 N/kg in the Keller model. These values can be used for the parameters in a computer model. A comparison of the Keller model with measured data points is shown in Fig. 3.73.
The descriptive power of the Keller model seems to be good for the measured sprint data. When it was used to compute the expected result of a 100m sprint by this student, the race time of 14s seemed unrealistic for someone who is not expected to be able to maintain the maximum speed reached after 20m for another 80m. The Tibshirani model predicts a more realistic race time of 15.4s. It would have been interesting to compare this theoretical result with the result of a real 100m sprint by the same student. In other words, the predictive power of the Keller model is not so strong; the Tibshirani model seems to lead to more realistic results. It is important that high school students get acquainted with this kind of working and reasoning with (computer) models.

More details of the initial motion of a sprinting student can be found in the video of the first 6 m using a high speed camera with a frame rate of 300 fps. In this case point
tracking becomes indispensable for collecting kinematical data. Figure 3.74 shows the distance and speed-time graphs obtained. The speed curve, which was determined by penalized spline smoothing, is the more interesting curve: increasing and decreasing parts of the graph correspond with propulsion during foot contact and the flight phase when both feet are airborne, respectively. Differences in consecutive increases in the speed-time graph indicate that the propulsive force of the right leg is stronger than that of the left leg. This difference might be related to the left- or right-handedness of the student, but it may also have to do with search of a good balance during take-off. Anyway, the speed curve is closely related to real motion. The goal of this part of the investigative work was to help students understand that a graph is a means rather than an end in itself.

Figure 3.74: Video analysis of the first 6 m of the sprint of a student recorded at a frame rate of 300 fps and the distance and speed curves.

3.5.6 Modeling Bouncing Gaits

References

Introduction
In this case study, human body motions such as bouncing on a jumping stick, hopping, and making kangaroo jumps were explored through video analysis and computer modeling. Some mathematical models of these motions using basic biomechanical principles were developed and modeling results were compared with experimental data obtained from video measurements. Three motions were discussed in details: (1) vertical bouncing on a jumping stick; (2) hopping upward; and (3) hopping forward like a kangaroo. Highlight was the application of the model of a planar inverted spring-mass system to kangaroo jumping. This rather simple model turned out to work qualitatively and quantitatively well for the complex motions of hopping, skipping, and running at moderate speeds, that is, in so-called bouncing gaits. The examples of video analysis and modeling activities presented in the paper give a good impression of the potential of the subject of human gait for student practical investigations or
profile projects and as a context for applied mathematics and physics at secondary and undergraduate level. They also give an impression of how close one can get to contemporary biomechanical research.

**Vertical Bouncing on a Jumping Stick**

This part of the paper was based on an item in a 2008 nationwide secondary physics computer-based examination in the Netherlands for pre-university students. The subject was video analysis and modeling of vertical bouncing on a pneumatic jumping stick. In the case study it served as source of inspiration for describing some natural gaits of humans and animals with an inverted spring-mass model.

The rather clear situation of a periodic motion of a person on a jumping stick (See the left-hand side of Figure 3.75) can be described well with a model based on simple mathematics and physics. The quality of the chosen model can be evaluated by comparing the model results with data acquired through video analysis of the motion. A schematic drawing of the one-dimensional spring-mass model is shown in the right part of Figure 3.75. In this model, the mass of the spring is ignored and, for convenience, the period of one jump is divided into two phases, namely the aerial phase (with aerial time $t_a$) and the contact phase (with contact time $t_c$). Furthermore it is assumed that the jumper is able to vertically jump on the stick without changing posture and that the body center is near the hand supports. At landing, the jumping stick has rest length $L$. It is further assumed that during aerial phase only the gravitational force $F_g = -mg$, where $m$ is the body mass and $g$ is the acceleration of gravity, plays a role and that during contact phase also the spring force $F_s = C(L - y)$ must be taken into account, that is, for heights $y \leq L$ the linear elastic motion of the spring depends on a spring stiffness $C$ (for heights $y > L$, one may take $C = 0$).

![Figure 3.75: The student with his jumping stick and the corresponding spring-mass model.](image)

The dynamics of the spring-mass system is now determined by a second order differential equation and two initial conditions: $a = y'' = (F_g + F_s)/m = F/m$, with $y(0) = y_0$ and $y'(0) = v_0$. This can be rewritten as a system of first-order differential equations: $v = y'$, $a = v' = (F_g + F_s)/m$, $y(0) = y_0$ and $v(0) = v_0$. This system of equations can be easily implemented in a graphical, system dynamics-based modeling tool. The left-hand window in Figure 3.76 shows a graphical model in COACH that numerically solves the system of equations. The diagram in the middle shows the graph of the computed height and the point plot of the vertical heights measured in a digital video recorded while the student was vertically bouncing on his jumping stick. Simulation results match well with the measured data for suitable
3.5. Video Analysis and Modeling Combined

parameter values. The computer model gave the following values for quantities of interest: $t_c = 0.27\text{s}$, $t_a = 0.31\text{s}$, and $C = 12.7\text{kN/m}$. The measured data suggest that a sinusoidal regression curve would also describe the data quite well, and indeed it does from mathematical point of view. But the spring-mass model is considered better than the experimental modeling via regression because it is based on physics laws. According to this model, a sinusoidal displacement during contact phase is followed by a parabolic aerial phase. This kind of judgment of the quality of a model is what students should learn.

Figure 3.76: A graphical model implementing the one-dimensional spring-mass model and the results of a simulation run compared with data obtained from a video analysis of a movie clip.

Hopping Upward

One may truly wonder whether the previous one-dimensional spring-mass model is a good model for human hopping in the upward direction without the use of a device. The proof of the pudding is in the eating. So, for the purpose of data collection, students went to the Sports Center of the University of Amsterdam in order to hop in vertical and forward direction on a motorized treadmill. Motions were recorded with a high speed camera at a speed of 300 frames per second so that as many details as needed could be observed and a rather high time resolution was assured.

The exact solution of the one-dimensional spring-mass model can be determined in the hope and expectation that one gets in this way more insight in the motion of the body center during contact phase of upward hopping. The stance leg (in this case actually both legs) is in this case modeled as a massless, linear spring with stiffness $C$ and rest length $L$. This is the starting point of mimicking gait models of Blickhan (1989), McMahon and Cheng (1990), and many other biomechanical scientists. For ease of computing, time $t = 0$ can be set equal to the moment that the leg makes first contact with the treadmill (In a video measurement in COACH one can easily calibrate time in this way) and the landing speed can be assumed equal to $-v$. The vertical position of the body center has been chosen to be the same as the position of the hip joint of the hopping student (See Figure 3.77). Under the assumption that only gravitational force and spring force play a role, Newton’s second law of motion and Hooke’s law of elasticity lead to the following equation of motion for the height during contact phase: $my'' = -mg + C(L - y)$, $y(0) = L$, $y'(0) = -v$. Let $u = y - L$ be the displacement during ground contact. Then the equation of motion can be rewritten as follows: $u'' + \omega^2 u = -g$, $u(0) = 0$, $u'(0) = -v$, where the natural spring frequency $\omega$ is given by $\omega^2 = C/m$. This equation can be solved analytically and the solution is a sinusoidal function. This solution can be used to find a relationship between contact time $t_c$, landing speed $v$, and frequency $\omega$: All one has to realize is that halfway contact
time the stance leg is maximally bent and the speed of the body center is equal to zero. In other words, the motion during contact phase depends on three out of the following four factors: (1) the acceleration of gravity \( g \); (2) the natural frequency \( \omega \) of the spring-mass system; (3) the take-off and landing speed \( v \); and (4) the contact time \( t_c \). Because of the definition of the natural frequency one can exchange this factor by the spring constant \( C \) (stiffness), provided that the body weight \( m \) is known. To conclude, by using exact mathematical methods one can make grounded statements about a bodily motion and investigate the dependencies of determining factors. Relationships between parameters in a mathematical model may also be used to find an initial parameter value in a computer model that can subsequently be improved.

![Figure 3.77: Video analysis of an upward hopping girl on a motorized treadmill that is not turned on.](image)

The exact solution of displacement can also be used to estimate the natural frequency and the landing speed on the basis of measurements. Figure 3.77 is a screen shot of a video analysis using Coach. The sine curve matching best the data points can be obtained with the data analysis tools. The following best parameter values were found in this way: \( \omega = 21.25 \text{ Hz} \) and \( v = 2.5 \text{ m/s} \). This landing speed is only a little bit greater than the speed of 2.3 m/s obtained by numerical differentiation of the measured data. The estimated values give a contact time of 0.17s, which is also only a little bit less than the measured contact time of 0.19s in the video clip (For this precision one needs a high speed camera). The take-off speed can be used to compute the duration of the aerial phase (cf., Section 3.4.2): \( t_a = \frac{2v}{g} \), under the assumption that the aerial motion depends only on gravity. This gives \( t_a \approx 0.47s \) and the estimated flight time deviates little from the measured flight time of 0.40s.

In short, the one-dimensional spring-mass model applied to a person hopping upward using no special device leads to model results that are in good agreement with results obtained from video analysis measurement on recorded movie clips. The agreement between model results and measured data gets even better when not a sinusoidal regression curve for the measured data during stance phase is made, but instead one tries to find the best values for parameters in the spring-mass model (by the method of trial and improvement). For the hopping girl in Figure 3.77 a very good match between model and video analysis was found, which holds for nine consecutive hops, using \( C = 19 \text{ kN/m} \) and \( v = 1.95 \text{ m/s} \). This value of the stiffness is in good agreement with values found in the literature (Farley et al., 1991). From these values one obtains the following results: \( \omega = 18.9 \text{ Hz} \), \( t_c = 0.19s \), and \( t_a = 0.40s \). Whereas aerial and contact time were almost equal for hopping on a jumping stick, they differ for human hopping without a device. Flight time is about twice the contact time. The stride
period \((T)\) is the sum of the contact time and the flight time. The video analysis gives a stride period of 0.59 s, which corresponds with a hopping frequency \((f = 1/T)\) of about 1.7 Hz.

The graphical computer model in COACH can be extended to include the expressions for gravitational energy (with respect to the height taken equal to the spring-leg length), the spring energy, the kinetic energy of the system, and the total energy of the system. A student can then diagrammatically examine the different forms of energy during the motion and examine that conservation of energy holds for the model system. An animation can be used to investigate the influence of particular parameters. Figure 3.78 shows how such an activity screen could look like.

![Figure 3.78: Screen shot of a COACH activity consisting of a computer model and an animation of a vertical spring-mass system, in which measured hip heights are compared with computed results and energies are computed to examine forms of energy and the law of conservation of energy.](image)

**Hopping Forward Like a Kangaroo**

What should set the seal on the work is the application of a planar inverted spring-mass model to human double-legged forward hopping, that is, to a motion resembling kangaroo jumping. The model is in this case two-dimensional. So to start with, the one-dimensional spring-mass model of upward hopping is extended. The new model contains besides the kinematical variables \((y, v_y, a_y, F_y)\) also the variables \((x, v_x, a_x, F_x)\), as it were doubled. Quantities like speed and force are decomposed in the \(x\)- and \(y\)-direction for the contact phase. The aerial phase is modeled as a parabolic trajectory of an object bouncing from the floor with center of mass at a certain height and an angle of take-off velocity \(\beta\), and then goes through air with only gravity force acting on the object (It is assumed that there is no air drag). Another new parameter
that comes into play is the leg angle of attack $\alpha$, when the leg makes ground contact. These angles are most easily defined when one selects the stance point as the origin of the coordinate system during contact phase, with the positive x-axis in the direction of motion and the positive $y$-axis in the upward direction, and when we assume that the stance leg lands at time $t = 0$: $\tan \alpha = -y(0)/x(0)$ and $\tan \beta = v/u$, where $u$ is the horizontal landing speed (equal to the speed of the motorized treadmill when the gait is on such device) and $-v$ is the landing and take-off speed. The planar inverted spring-mass model for bouncing gaits such as hopping and running is schematized in Figure 3.79.

Figure 3.79: Spring-mass model for forward hopping and running. A point mass $m$ supported by a massless spring with rest length $L$, spring stiffness $C$, leg angle of attack $\alpha$, vertical landing velocity $-v$, and horizontal landing velocity $u$.

The leg angle of attack and the angle of take-off velocity are not necessarily equal. It is not difficult to determine these angles in a recorded high speed video clip because COACH, like any professional video analysis tool, provides its user a digital ruler, a digital protractor, and graphs of position and velocity. Notice that one cannot freely change the parameters $\alpha$ and $\beta$ in a computer model for given values of leg length and landing velocity if the model must be periodic. After all, the leg angle of attack and the leg angle of take-off must be equal for a periodic motion. Under the given circumstances the following condition can be used to distinguish between aerial and contact phase: when $y \leq L \sin \alpha$, there is ground contact and the leg can be considered as a linear spring with stiffness $C$. This spring force during ground contact must be decomposed into horizontal and vertical components in order to derive the equations of motion. After moderate algebraic manipulation one obtains the following initial value problem from Newton’s second law of motion:

$$
\begin{align*}
\ddot{x} &= -\omega_s^2 x + \frac{g\lambda x}{\sqrt{x^2 + y^2}}, \\
\ddot{y} &= -g - \omega_s^2 y + \frac{g\lambda y}{\sqrt{x^2 + y^2}}, \\
x(0) &= -L \cos \alpha, \\
x'(0) &= u, \\
y(0) &= L \sin \alpha, \\
y'(0) &= -v.
\end{align*}
$$

Here the parameters $\omega_s = \sqrt{C/m}$, $\omega_p = \sqrt{g/L}$, and $\lambda = \omega_s^2/\omega_p^2$ have been introduced. The main difference between vertical and forward hopping is that the leg length, parameterized by $\lambda$, comes seriously into play in the modeling of the body motion during ground contact. These equations of motion can also be derived through the Euler-Lagrange formalism with the Lagrangian of the planar spring-mass, in Cartesian coordinates, equal to $\mathcal{L} = \frac{1}{2}m(x'^2 + y'^2) - \frac{1}{2}C(\sqrt{x^2 + y^2} - L)^2 - mgy$. The main application of the analytical methods like the one discussed in this section is that it allows investigating the influence of gait parameters on the body motion and researching dependencies between the various parameters in the mathematical model.
3.5. Video Analysis and Modeling Combined

The equations of motion of the spring-mass model of forward hopping can be implemented in the graphical modeling tool of COACH, although it is admittedly beyond the ability level of many a student. One of the problems is how to deal with the moving coordinate system from one stance point to the other. The fact that COACH has been designed as a hybrid system that combines a classical system dynamics approach with event-based modeling for processes that change abruptly helps solve this problem. As triggering condition for the landing of a hopping person one can use the Boolean expression \( y \leq y_{\text{start}} \land y' < 0 \land x + x_{\text{start}} > 0 \), where the initial conditions \( x_{\text{start}} \) and \( y_{\text{start}} \) are given by \(-L \cos \alpha\) and \(L \sin \alpha\), respectively. The event is defined behind the red upper-left icon (landing) in the graphical model in Figure 3.80. The idea behind the event handling is that one starts with a coordinate system of which the origin coincides with the first stance point. In the variable \( x_0 \) one stores the current value of the \( x \)-coordinate of the origin of the moving coordinate frame. Each time when the event of touch down occurs, the variable \( x_0 \) is refreshed with the \( x \)-coordinate of the new stance point via the assignment \( x_0 := x_0 + L \cos \alpha \).

In order to evaluate the suitability of the planar inverted spring-mass model for human hopping like a kangaroo model results were compared with experimental results obtained by motion analysis. To this end, the motion of a hopping girl on a motorized treadmill going at a speed of 3 km/h was recorded on video and the height-time graph was created via automated point tracking of a hip joint marker. This measurement has been used as a background graph in the modeling activity to find by trial and improvement appropriate parameter values. It was quite tricky to set the values such that the spring-mass model runs periodically for a long time: The value of the leg angle of attack had to match with other parameter values in order that the take-off velocity equals the landing velocity. But the reward was great: Figure 3.80 illustrates a perfect match. The upper-right \( y-x \) diagram illustrates the periodicity of the simulated motion. The parameter values found for the hopping girl weighing 53 kg were: \( C = 28 \text{kN/m} \), \( u = 0.84 \text{m/s} \), \( v = 1.95 \text{m/s} \), \( L = 0.91 \text{m} \), \( \alpha = 86.0003 \). Other computed
quantities were: \( \omega_s = 23.0 \text{ Hz}, \omega_p = 10.8 \text{ Hz}, t_a = 0.40 \text{ s}, t_c = 0.20 \text{ s} \). What the results indicate is that leg stiffness and spring frequency are greater in hopping forward like a kangaroo than in hopping upward. Yet this had little or no effect on aerial and contact time. All this is not so very strange considering the low speed of the treadmill. Apparently, only leg stiffness must be increased to maintain a short contact phase.

**Epilogue**

Mathematical models of the following bouncing gaits were constructed in this paper: bouncing on a jumping stick, hopping, and making kangaroo jumps. This seems very ambitious because such vivid motions are at first sight not easily modeled. The pushing-off and landing of a vertically hopping person savor strongly of the motion of an extending and compressing inverted spring-mass system. All sorts of models of this type are used in biomechanical studies. But how simple or complex must such a mathematical model be to describe reality to a reasonable extent? Can students at secondary and undergraduate level with modest knowledge of mathematics and physics actually do such investigations?

It is striking that a relatively simple spring-mass model describes bouncing gait patterns so well. Bullimore and Burn (2007) confirmed that the model allows prediction of several gait characteristics such as contact time, vertical momentum, and stride length. But they also noticed that it often overestimates the horizontal ground reaction force, the flight time, and the change of mechanical energy of the body center. Geyer (2005) successfully adapted the spring-mass model to walking and running gait patterns. The actual power of the mathematical models is that they help researchers investigate various aspects of bouncing gaits, such as step frequencies, forces, stability of gait patterns, costs of energy, and so forth, and compare gait patterns.

The second question was whether students at secondary and undergraduate level with modest knowledge of mathematics and physics can actually do such investigations. My experience is that students with a keen interest and good ability in mathematics and physics can master the modeling process. Less gifted students are still expected to be able to grasp the one-dimensional inverted spring-mass model, which was after all used in a nationwide examination. More importantly, all students can do with modest means biomechanical research in much the same way as motion scientists and they can practice herein mathematical knowledge and skills such as graph comprehension, numerical differentiation and integration, data processing and analysis, regression, and so on. They can develop the critical attitude that is necessary for successful modeling of natural phenomena. For this it is very important that the students can compare the results of computer models with real data, preferably collected in an earlier measurement activity. Confrontation of a model with reality turns graphical modeling not only into a fun way of learning, but it also makes it exciting, challenging, and concrete work. It is joyful when experiment, model, and theory are in good agreement, as was the case in this field experiment.

### 3.5.7 Exploring the Giant Circle on the High Bar

**Reference**

3.5. Video Analysis and Modeling Combined

Introduction
Daan Knobbe and Nic Nijdam were secondary school students with artistic gymnastics as a hobby. They jointly investigated the mechanics of the gymnastics swing around the high bar. They not only did this to meet the curriculum requirement of carrying out a large research project, but also to satisfy their curiosity regarding the sports science subject. With a high speed camera they recorded the motion of a backward giant circle on the high bar. Hereafter they analyzed their data both quantitatively and qualitatively. I present the results of their research work, which resembled the practice of sports scientists, and discuss the authenticity of this project and the role of ICT. This means that I go into questions like “What did the students actually do in their project and what could they have done more or better?” “How did ICT contribute to the realization and the success of the sports science project of the secondary school students?” and “To what extent did the students’ work resemble scientific practice?”

The Backward Giant Circle on a High Bar
The high bar is one of the six pieces of apparatus used in Men’s Artistic Gymnastics competitions. A high bar routine consists of a number of circling skills, flight elements, and a dismount. The backward giant circle is used to link the circling techniques and to provide the required flight and rotation for flight elements and the dismount. It is an artistic gymnastics element in which the gymnast passes through a handstand position above the bar and fully rotates around it, without releasing the bar at any time, without bending arms and knees, and with only smooth changes in hip and shoulder joint angles that have little influence on the aesthetic execution (cf., www.fedintgym.com). This at least is the description of a traditional backward giant circle in which the gymnast extends his body close to the highest point of the circle, maintains it in the downward phase and flexes the shoulder and hip joints during the early upward phase; see Figure 3.81.

Figure 3.81: Graphics sequence of the traditional technique of a backward giant circle, in which the gymnast circles backwards (anti-clockwise direction) from a handstand on the high bar through 360° [after (Hiley & Yealdon, 2001) and (Tsuchiya, Murata, & Fukunaga, 2004)].

The students teamed up to investigate the mechanics of the backward giant circle on the high bar, triggered by their own hobby and interests. They were given printed copies of relevant literature (e.g., Hiley & Yealdon, 2001) to put them on the way. Their physics teacher and gym teacher advised them during their research. The students formulated the following main purpose of their study: “We investigate the influence of
the shoulder and hip joint angles on the angular velocity in a backward giant circle on
the high bar and we investigate how a gymnast can optimize these and other factors
in his performance in order to achieve the highest possible angular velocity and still
perform well in the eyes of the gymnastics jury.”

**Video Analysis of the Backward Giant Circle**

In this paragraph I discuss the students’ experimental design, their data analysis, and
their use of ICT. In order to get a better impression of the students’ performance and
to assess their results, I have also redone the video analysis of one giant circle in a
more advanced way.

The Students’ Experimental Design

For the purpose of exploring the influence of the shoulder and hip joint angles on
the angular velocity in a backward giant circle with both quantitative and qualitative
methods, the student researchers designed an experiment in which they could collect
position, angle, velocity, and time data for a backward giant circle on a high bar
through video measurement and in which they could use video tools for analyzing
video clips of various types of swing motions. They did this as follows. The location
of their experiments was the practice room of their gymnastics club where the high
bar apparatus could be used on a quiet moment during daytime. A performance of
several subsequent giant circles was required for a good analysis of a backward giant
circle in which a gymnast just swings about the bar without the goal of increasing
or decreasing the angular velocity at the highest point above the bar. In order to be
able to make several full swings after another, the students used a training tool for the
apparatus that reduced the friction when the gymnast circled about the bar, made it
easier to do a full swing, and simplified the control over experimental conditions. The
student researchers used a point-and-shoot high speed camera operating at a frame
rate of 120 fps to get enough data for a quantitative video analysis and to get at
the same time a sufficient resolution quality of the video clip. Ideally, the camera for
recording the giant circle would have been positioned perpendicular to the plane of
motion, at a distance and height that reduce perspective distortions, and with the
camera oriented such that the x- and y-axis are aligned horizontally and vertically,
respectively. In reality, these experimental conditions were too difficult to arrange and
the students had to use the perspective correction tool in the video analysis program
to handle perspective distortion.

Before recording body movements, a researcher must at least have an idea of what
(s)he will do with the video clips. The questions that the student researchers asked
themselves already give a clue of the type of body segments model of a gymnast
performing a backward giant circle that they had in mind. The rigid 3-segments model
is shown in Figure 3.82 (translated from the students’ report). To make it easier to
measure positions of wrist, elbow, shoulder, hip, knee, and ankle, the students attached
markers to the gymnast’s skin and shorts over the right body side joints. The markers
over elbow en knee joint were attached only to verify whether the arms and legs were
kept straight during the giant circle, but were not used in the video measurement
and in the biomechanical analysis of the motion. Body orientation was defined as the
angle between the vertical and a line from the bar to the body center of gravity. The
body orientation angle, shoulder angle, and hip angle were estimated on the basis of
displacement data using the convention illustrated in Figure 3.82.
Figure 3.82: Body segments model of a gymnast performing a backward giant circle. White dots represent the wrist, elbow, shoulder, hip, and ankle markers, and the location of the body center of gravity (CG).

**Video Analysis by the Student Researchers**

The students used the video analysis tool of Coach to measure the distance of the body center of gravity and to compute its velocity during the giant circle. They used a fixed position for the body center of gravity, slightly above the hip joint as shown in Figure 3.82. To calibrate distance, they used the known gymnast’s height and the distance of the high bar at rest to the upper face of the landing mat.

The student determined the joint angle-rotation angle graphs in the following way. They utilized the open source video analysis software Kinovea (www.kinovea.org) to collect data about shoulder and hip joint angles for certain predetermined body rotation angles per giant circle. The students decided to do measurements only at body rotation angles $0^\circ$, $45^\circ$, $90^\circ$, $135^\circ$, $180^\circ$, $225^\circ$, $270^\circ$, and $315^\circ$ for the 6th up to and including the 10th giant circle in a sequence of 14 consecutive giant swings on the high bar. The picture on the right-hand side of Figure 3.81 gives an impression of a subject’s body configuration at these rotation angles.

Using an Excel spreadsheet, the students averaged the measurements for the 5 consecutive giant circles in order to deal with small changes between giant circles. In this way, they obtained a data plot of 8 points, in which the shoulder and hip joint angle were plotted against the body rotation angle. In order to get line graphs, they utilized the demo version of the Igor scientific graphing and data analysis program (www.wavemetrics.com) for curve fitting. They used a sinusoidal regression type. Figure 3.83 shows the graphs obtained by the students.

Figure 3.83: The shoulder and hip joint angle plotted against the rotation angle of the body center of gravity for a backward giant circle on the high bar.

The students qualitatively analyzed their graphs and they explored whether a gymnast can return to handstand when his shoulder joint angle gets small during the movement. They concluded that only a free hip swing to handstand on the high bar would be possible; but this is not considered a giant circle. The motion analysis of the
student researchers continued with reflection about other factors that influence the performance of a backward giant circle on the high bar but are under less control of the individual gymnast. Using qualitative arguments, based on fundamental biomechanics principles about angular motion such as moment of inertia, angular velocity, and angular impulse, and based on formulas for these quantities in simple cases, the students came to the conclusion that gymnasts can swing faster about the high bar when they are less heavy and have a shorter distance between their body center of gravity and the bar. In other words they looked into classical rotational mechanics and applied it to the sports motion.

The behavior of the student researchers in their video analysis of the backward giant circle on the high bar resembled the attitude of scientists in that they tried to explain the effects of various factors on the gymnast’s performance by scientific reasoning and also by quantitative analysis, if possible. Their data collection, processing, and analysis were based on the same methods that sports scientists use in practice and references to research literature were made. The students also reflected on their research design and the obtained results, looked for alternative methods and explanations of phenomena, and made suggestions for further investigations in their detailed research report. This is what professionals do in practice, too. In the choice of ICT, the student researchers had the same attitude as many scientists: They used whatever tools were available to them and seemed useful for their research aims.

**Video Analysis Based on a Computed Body Center of Gravity**

In reality, the body center of gravity is not always located at the same body point: Due to changes in body configuration, the location changes and may even be outside the body (as shown in Figure 3.84). In a more advanced video analysis one would compute the location of the body center of gravity during the giant circle from the recorded position of the markers and from anthropometrical data for human body segments. It is more work, but the procedure is standard (See, for example, Robertson, Caldwell, Hamill, et al., 2004) and the benefit is an improved quantitative video analysis of the motion. I used the same five-segment model of the gymnast (two upper and low extremities, plus the head and torso as one component) as Townend (1993) and utilized anthropometrical data as presented in (Robertson et al., 2004) to predict the location of the body center of gravity during the giant circle.

Figure 3.84 is a screen shot of the video analysis of one giant circle (actually the 6th in the uninterrupted sequence of 14 giant circles performed by the gymnast) based on the described methodology. The upper-left window in the screen shot shows a still from the recorded video clip in which the gymnast has bent his shoulder and hip joints in the upswing phase of the giant circle. The points overlaying the video clip are the measured positions of the shoulder, hip and ankle joints, and the computed position of the body center of gravity with respect to the moving reference frame that has the right hand of the gymnast as its origin. Because the spatial and time data collected in the video clips and the corresponding mathematical representations like graphs and tables are synchronized in COACH, pointing at a graph automatically shows the corresponding video frame, and selecting a particular frame highlights the corresponding points in diagrams, when scanning mode is on. This makes scrubbing, that is, manually advancing or reversing a clip, an effective means to identify and mark interesting events in the video clip and to relate them with graphical features. This feature is much used in motion analysis. In the diagrams of Figure 3.84 are
3.5. Video Analysis and Modeling Combined

shown sinusoidal regression curves for the data, the formulas of which have been used in computer modeling of the motion to be discussed later.

Figure 3.84: Screen shot of a video analysis activity about one backward giant circle.

Going in clockwise direction through the above screen shot, the first diagram consists of the distance-time graph of the body center of gravity and a curve obtained by approximating the data with a sum of sine functions. In this particular case, a sum of two sine functions (plus a constant) gives a remarkably good approximation. The distance-time graph of the body center of gravity can be understood in the following way. Initially, when the gymnast is in a handstand above the bar, his body is not fully extended and the distance of the body center of gravity is not maximal. During the first phase of the down phase, this quantity actually increases to a peak value when the body is close to horizontal orientation. Hereafter the distance of the body center of gravity to the bar decreases to a (local) minimum when the gymnast passes under the bar. His hip joint is maximally hyper-extended at this time. In the first phase of the upswing the gymnast closes his hip joint angle. First his body configuration becomes more extended, with the effect that the distance of the body center of gravity to the bar increases to a local maximum when his body is close to full extension. But as the upswing phase continues, the gymnast’s shoulder and hip joint angles close more, with the effect that the distance of the body center of gravity to the bar decreases to a minimum when the gymnast is already in the second phase of the upswing and is going to pass the bar soon in handstand. The minimum distance of the body center of gravity coincides more or less with the minimum of the shoulder joint angle. The gymnast in the last part of the upswing extends the shoulder joint again so that the distance of the body center of gravity to the bar increases.

The rotation angle-time graph and the angular velocity-time graph are displayed together in the second diagram, in which the left vertical axis is used for the measured rotation angle and the right vertical axis is used for the angular velocity. The time profile of the angular velocity, which was determined via an advanced numerical method
for computing smooth derivatives, can be understood in the following way. Angular
velocity increases in the downswing as the gymnast falls from the handstand position
above the bar and reaches a maximum value shortly before the gymnast passes below
the bar when the hip joint is maximally hyper-extended. The angular phase velocity
decreases when the gymnast starts moving to a more extended body configuration and
progresses to the upswing. If the gymnast were a rigid body swinging about the bar,
then his angular would decrease monotonically in the upswing. The angular velocity-
time graph reveals that this assumption is not true: The gymnast opens and closes his
shoulder and hip joint angles. The changes in body configuration during the upswing
result in a second small peak in the graph due to closing of the shoulder and hip joint
angles. The local maximum coincides with maximal flexion at the hip joint.

The hip joint angle-time graph and the shoulder joint angle-time graph show clearly
when the gymnast flexed and extended his body, and they reveal that the gymnast
did not perform optimally. The relationships between shoulder joint angle, hip joint
angle, and distance of the body center of gravity, and certainly the special body con-
figurations during the backward giant circle become more visible when these quantities
are plotted against the rotation angle. This has been done in the lower-left diagram
of Figure 3.84. These graphs are in good agreement with the graphs found by the
students (Figure 3.83) and graphs found in the research literature (e.g., Tsuchiya,
Murata, & Fukunaga, 2004; Sevrez, Berton, Rao, & Bootsma, 2009).

Modeling and Simulating the Backward Giant Circle
By use of basic principles of physics and simple models of the gymnast-apparatus
system one can already analyze the backward giant circle. For example, Townend
(1993) was able to estimate the magnitude of the reaction force experienced by the
arms and shoulders of the gymnast via a simple body model: He found a reaction
force of about four times the gymnast’s body weight. It is clear that a gymnast
has to be strong, as does the apparatus. Taking away simplifying assumptions, such
as rigidity of the gymnast, no friction, and inelasticity of the bar, complicates the
dynamics of the motion enormously and makes it difficult to construct a computer
program that simulates the gymnast’s motion so that there is good agreement between
empirical data and modeling results. Such advanced models would be beyond the
abilities of secondary school students (and probably also beyond the level of physics
undergraduates). But students could explore simpler models under supervision of a
knowledgeable physics teacher.

Here I considered a model of the gymnast as a one-segment body consisting of a
slim, uniform, circular cylinder of variable length \(r\) and mass \(m\) attached to a rigid
axis of rotation and subject only to gravity. Note that this model is merely a math-
ematical model, which has little resemblance with reality, but nevertheless serves the
purpose of analyzing the swing motion, and note that it may be a source of inspira-
ion for studying more complicated models. The equation of motion is written as
\((I\omega)' = mgr\sin\theta\), where \(\theta\) is the angle of rotation, \(\omega\) is the angular velocity, and \(I\)
the moment of inertia given by \(I = \frac{4}{3}mr^2\). Using the rules of differentiation and after
further algebraic manipulation, one gets the following system of differential equations:
\(\theta' = \omega, \omega' = (g\sin\theta - \frac{8}{3}r\omega)/(4r)\). When input for initial values and data for \(r\) are
available, or when \(r\) is modeled by a given mathematical function, this system of dif-
ferential equations can be solved numerically. Considering the movement as a periodic
motion, the function can be described as a sum of two sinusoids (This already gives a
reasonable approximation). This was done with the graphical, system dynamics-based modeling tool of Coach and the model results were compared with the data obtained from a video analysis of the recorded motion of the backward giant swing performed by one of the student researchers. The results are shown in Figure 3.85. The background graphs came from the video analysis activity described before.

3.5. Video Analysis and Modeling Combined

![Graphical model](image)

Figure 3.85: A graphical model implementing the system of differential equations driven by the sinusoidal model of the distance $r$. The diagrams on the right-hand side show the calculated graphs of the rotation angle and angular velocity, respectively, together with graphs of experimental data.

The phase of the backward giant circle where the model results actually differ from the empirical data is the upswing phase, when the gymnast changes his configuration by strongly bending his shoulder and hip joints. Obviously our model of the gymnast as a circular cylinder fails under these circumstances. But overall, the quality of this rather simple model can be considered as remarkably good.

Once a computer model has been accepted as useful one can utilize it to examine what happens when the gymnast changes the timing of his movements. In this way, one can search for a more optimal performance of a swing technique and learn about possible improvements in techniques. In more advanced computer models, researchers use multiple-segment models of the human body, include elasticity of the bar and friction, and they use joint angle histories or models of applied torques as driving quantities for computer simulations. The interested reader is referred to the biomechanics research literature (e.g., Arampatzis & Brüggeman, 1998; Sevrez et al., 2009; Yeadon & Hiley, 2000).